delta-k

Journal of the Mathematics Council of the Alberta Teachers' Association

Volume 58, Number 1 Fall 2023

Problem Solving for All Levels

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom or a scholarly focus. They may include

- letters to the editor;
- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the National Council of Teachers of Mathematics (NCTM).

Suggestions for Writers

- *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
- If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
- Peer-reviewed articles are normally 8–10 pages in length.
- All manuscripts should be submitted electronically (in Microsoft Word format) and double-spaced. All pages should be numbered.
- The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identity to the reviewers.
- Pictures or illustrations should be submitted as separate files (such as JPEG or GIF) and clearly labelled. Their placement should be indicated in the text of the article. A caption and photo credit should accompany each.
- All sources should be properly referenced. Entries in the reference list and in-text citations should be formatted consistently, using the author–date system.
- If any student work is included, please provide a signed consent form from the student's parent/guardian allowing publication in the journal. The editor will provide this form on request.
- Send manuscripts and inquiries to the editor: Lorelei Boschman, c/o Medicine Hat College, Division of Arts and Education, 299 College Drive SE, Medicine Hat, AB T1A 3Y6; e-mail lboschman@mhc.ab.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-k

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From the Editors' Desk

Lorelei Boschman and Barbara O'Connor

Problem solving. Is that not what we do all day long when we creatively solve life's problems as they present themselves? How we approach these tasks is embedded in our histories, cultures and present contexts. We learn to approach problems in a variety of ways from experience and exposure to different perspectives. The concept of problem solving extends beyond math's realm and is being recognized as a 21st century essential skill. The process we work through with students for problem solving in our mathematics classes has far-reaching benefits as our students take these brain processes and transferable skills with them into other interdisciplinary subjects and life in general.

The content of our first 2023 *delta-K* issue offers a breadth of articles that speak to problem solving across a variety of contexts, and from shorter moments to multiple tasks involving more in-depth critical thinking skills.

Initially, in the theoretical research paper, Lu, Rajasekharan and Khan begin their exploration from a culturally responsive stance. They share how to partner ethnomathematics—aspects of culture and mathematics—with variation theory, using kolam drawings. Next, two rich conversation starters will help you think about mathematics more deeply. If you were not able to make it to the May 2023 Spring Symposium, read Patricia Shoemaker's review of the book *Dragon Curve: A Mathematical Journey* by Alicia Burdess (past MCATA executive member). Moreover, in the first of several posts, Darin Trufyn (MCATA executive member) invites us to think about the uses of common assessments to assist learning.

We are once again sharing a couple of math contests for you and your students. Enjoy the problem solving *extraordinaire* (answer keys are provided)! To continue the problem-solving aspect, preservice teachers at Medicine Hat College offer two more examples with their rich mathematical tasks, Sweet Tooth Snack Shop (Grades 2–3) and Candy Bar Creation (Grades 5–6). You can use these to demonstrate the open-ended problem-solving aspect that is so beneficial for students. These rich tasks can be adapted to low-floor, high-ceiling questions for multiple levels and grades.

Finally, there are also two website highlights. The first is *Nerdles: The Daily Numbers Games.* Links to the various levels (Grades 1–12) of these exciting daily challenges are included for both you and your students. Also check out Graham Fletcher's *three act tasks*. These are an excellent math routine for elementary students and are ready to use! We are hoping to hear more of Graham's problem-solving ideas at our Fall 2023 Conference in Calgary.

As you continue teaching mathematics, and students in general, know you are valued. By integrating problem solving into your math classes and classroom life, you are giving your students practice at a lifelong skill that will serve them well. Your mathematical mindset makes a difference!

Towards a Meaningful Partnership between Variation Theory (VT) and Ethnomathematics (EM) with the Support of Kolams in Mathematics Teacher Education

Olivia Lu, Sreedevi Rajasekharan and Steven Khan

Abstract

In this theoretical paper, we argue that Variation Theory (VT) and Ethnomathematics (EM) form a necessary and mutually enriching partnership that simultaneously advances multiple goals such as those aligned with equitable outcomes for diverse learners, cultural relevance, cultural respect and humanizing mathematics education. We argue that the external horizon of the object of learning is a critical aspect of the ongoing evolution of relevance and motivational for learning mathematics. The lives of mathematical objects of learning that are embedded in cultural practices should maintain a balanced relationship between their ongoing cultural aspects and mathematical aspects. Variation theory also provides an important frame for the design of systematically sequenced and structured patterns of critical discernments that attend to the pedagogical dimension of ethnomathematics within schools and other settings. We illustrate our ideas through the example of the practice of kolam drawing, which originates in Tamil Nadu, India, and has been previously explored in the EM literature.

"As I have explored the works of variation theory, I see the potential and importance of intertwining this concept to introduce cultural learning within mathematics classrooms. I hope that cultural significance gains traction in mathematics, not only to show respect but as a way to support and promote students' learning. If variation theory can offer openings for students' diverse ways of learning and knowing, there should be opportunities to extend their learning of mathematics beyond the classroom."

—*Olivia Lu*

"My experience with kolam drawings started as a young bride whose husband worked in a small town called Karur in Tamil Nadu, India. It was customary for the ladies of every Tamil household to get up early in the morning, sweep the front courtyard of the house with coconut palm leaf stalk brooms—which are recyclable—and sprinkle water to purify the swept area of the courtyard before drawing the traditional kolam designs. The purpose of sprinkling water is to get the rice flour stuck to the ground. People who have cows at home used to mix cow dung with water, and cement the front courtyard with the mixture, before making the kolams. In Kerala, we have a tradition of creating beautiful patterns on the floor, known as aniyal, using wet rice ground that is thickened with resin from chopped okra.

Kolams are drawn in front of Hindu households as a sign of welcome. I took lessons on kolam drawings from my friends, village women who lacked the opportunity to pursue higher education because of their gender and family traditions. Kolam drawings require mental concentration and prowess and are well connected with self-discipline (Rajasekharan, pers. experience; Ascher, 2002). I used to practice the kolam drawings many times on paper before putting them on the floor. I remember putting the dots in a particular sequence, and this was done with precision without a ruler."

—*Sreedevi Rajasekharan*

"I have used sand drawings, including kolams, in my work with elementary teachers for more than a decade (Khan 2010, 3). They are always surprised at how much effort it takes to make even simple images that are aesthetically pleasing and feel great satisfaction when they make increasingly intricate patterns. I am always careful to try to situate these traditions within their contexts and cultures. However, in math classes and literature, where cultural artefacts or practices form the basis for instruction, I have often found the cultural aspects to be rapidly dropped, and never returned to, as the mathematics takes precedence."

—*Steven Khan*

The work of teaching mathematics in schools is complex and challenging (Potari 2012). In addition to ambitious goals of high achievement, mathematical proficiency and equity for all learners, there are new goals related to humanizing, cultural responsiveness, and socio-emotional competence based on emerging consensus around how people learn. This consensus foregrounds the critical and necessary roles of contexts and cultural situatedness (National Academies of Science, Engineering and Medicine 2018). At the same time, research on theories of learning reveals swarms of discourses (Davis and Francis 2021) that intersect, interact, evolve, compete for attention and influence approaches to, and beliefs about, effective teaching.

A critical convergent insight from neuroscience,

The lives of mathematical objects of learning that are embedded in cultural practices should maintain a balanced relationship between their ongoing cultural aspects and mathematical aspects.

economics and education research is that attentional working memory (ie, bandwidth scarcity) is a critical (ie, non-volitional) limiting factor for learning (and learning to teach). This can be severely attenuated by factors such as physical, emotional and economic stress due to poverty, racism, trauma and marginalization (Mullainathan and Shafir 2014; Verschelden and Pasquerella 2017) but can be ameliorated by careful design considerations such as limiting unnecessary distractors, chunking into smaller pieces and providing immediate corrective or evaluative feedback within a nonjudgmental and growth-oriented environment. This environment offers repeated and increasingly elaborative experiences, which elicit positive affective responses (Figure 1). These findings are consistent with the design principles used in certain types of digital games and puzzles (Khan 2019), mathematics curriculum resource partners (Preciado-Babb et al 2015) as well as the *universal design for learning* (UDL) frameworks (Lambert 2021; Takacs et al 2021).

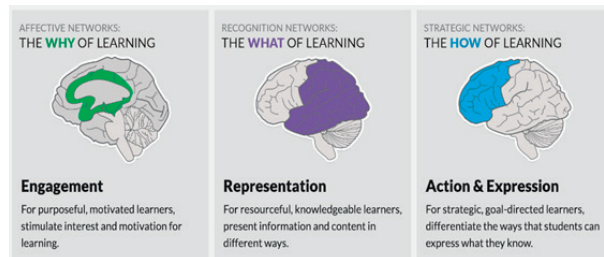

Figure 1. *Three networks influencing UDL principles—affective, recognition and strategic (Takacs et al 2021, 33).*

In this paper, we bring together our understanding of the curricular and pedagogical design implications of variation theory (VT) and ethnomathematics (EM) in an attempt to create a novel partnership between them that we believe will be useful to preservice and inservice teachers who seek to include the cultural life-worlds of learners and their communities into their classrooms in meaningful, impactful and respectful ways while also advancing and remaining consistent with effective principles for learning mathematics. Our literature search and review found many articles over the last three decades that drew on ethnomathematics, or culturally responsive pedagogical practices, and those that drew on variation theory. We could find no work that *explicitly* partnered the two frameworks, though we acknowledge that in the description and design of some classroom ethnomathematics activities, we saw evidence of practices consistent with the broad principles of variation theory. Likewise, in some work drawing on variation theory, we saw evidence of the wider cultural context of learners and communities being drawn in without explicit reference to ethnomathematics or culturally responsive pedagogy.

Why Partnering?

The idea of *partnering* draws on the work of the Math Minds project (which can be accessed at www.structuringinquiry.com) as well as work on multispecies' flourishing (Khan 2020), both of which share an emphasis on an enactivist understanding of structural coupling and owe a debt to Indigenous ways of knowing, thinking and being in relation to the other-than-human relatives—including mathematical concepts. The former involves working with teachers on partnering with well-ravelled resources that are consistent with principles of learning to improve teaching. They argue and present examples from JUMP math lessons (see jumpmath.org) that systematically use structured and clearly sequenced patterns of variation in critical discernments that are sustained over multiple lessons and years. The work, however, presents as acultural and is focused on critical discernments of mathematical concepts and associated linking logics.[1]

Our work starts from a position of cultural responsiveness (Matthews et al 2022; Seda and Brown 2021) and is consistent with recent findings from the National Research Council (1994) on *how people learn*. These findings suggest that contexts for learning matter: they are essential for learning—more so for students from nondominant cultures in education systems—and are not merely a backdrop or prop for learning. We take a more critical decolonial, antiracist and eco-communal approach that attempts to trouble the capitalist-colonialist essentialist understanding of *resources* as something that is just *there* to be taken and exploited for individual or corporate profit. Indeed, the language of *resource* in education is one that we find especially problematic in the context of the Truth and Reconciliation Commission's calls to action and moves to decolonize the university/curricula.

Variation theory also provides an important frame for the design of systematically sequenced and structured patterns of critical discernments that attend to the pedagogical dimension of ethnomathematics within schools and other settings

Intentionally coupling EM with VT allows us to develop a strategy for working towards the ambitious goals of modern curricula and schooling in ways that feel honest, have a high probability of success, and honour the dignity and integrity of the learners we work with, the communities, heritages and cultures to which they belong and contribute, and, who too must be invited into renewed vivifying partnerships with schools and education as they continue to transform each other. Thus, extending the work of Math Minds, we propose keeping cultural ideas central and find ethnomathematics a valuable partner, along with variation theory, in thinking about the design of learning experiences. We should note that there are other potential theoretical and pedagogical partners with either member of this coupling, but they are not our focus in this paper.

As variation theory concentrates on the cognitive and design components, and ethnomathematics focuses on the sociocultural components and human practices in multispecies worlds, we can couple aspects of working memory (from variation theory) with ethnomathematics and/or the cultural connections of ethnomathematics with variation theory. We will provide an example of kolam drawing to focus attention on how the use of variation theory and ethnomathematics might inform mathematics education and teacher practice.

A Brief Introduction to Ethnomathematics

Following Lubis et al (2019), we take ethnomathematics (EM) as expressing the reciprocal relationship between mathematics and/in culture. It can be considered a "culturally specific practice performed by one cultural group seeking to make sense of another, often by reference to a specific conceptualization of mathematics" (Peralta 2020). Katz (1994) pointed out that mathematical ideas have grown out of the needs of various cultures around the world, and it is important that students in western countries are exposed to the mathematical practices in different cultures. The current Eurocentric challenges in mathematics education can be relieved through applications of ethnomathematics, as ethnomathematics creates opportunities to recognize the contribution of non-western approaches to mathematics and explore beyond the traditional framework of mathematical thinking (Hall 2007). People all over the world have developed several mathematical methods consistent with their interests, religious beliefs, aesthetic or recreational goals/purposes. Some cultures use arts and designs rich in symmetry, proportions and transformations as part of their daily rituals, and *kolam* is one such example.

Ethnomathematics helps students understand the cross-curricular applications and identify mathematics in real-life situations. In the mathematics education context, ethnomathematics can introduce mathematical perspectives that focus on bringing diversity into the classroom through local knowledge and the idea that mathematics appears anywhere (Peralta 2020). A multicultural approach to teaching mathematics can also guide students to comprehend the subject in an academic setting (ie, in the classroom) and in an informal way, outside traditional classrooms. According to Uy (2013), humanizing mathematics lessons helps to include all students and boost their confidence levels, promote holistic learning of mathematics and acknowledge the existence of the *other* within mathematics. The practices of ethnomathematics have the ability to provide math educators with crucial resources that connect dominant and nondominant forms of knowledge in ethnomathematics. The recognition of mathematics within cultural practices in conjunction with the discovery of different ways of thinking can be brought together as two perspectives of ethnomathematics (Peralta 2020).

. . . research on theories of learning reveals swarms of discourses (Davis and Francis 2021) that intersect, interact, evolve, compete for attention and influence approaches to, and beliefs about, effective teaching. A critical convergent insight from neuroscience, economics and education research is that attentional working memory (ie, bandwidth scarcity) is a critical (ie, non-volitional) limiting factor for learning (and learning to teach).

Ethnomathematics can serve as a bridge that connects the theoretical aspect of mathematics with lived experiences. The increasing diversity of the population in Canada/North America has led to an emphasis on including learners' familial and cultural curricular assets in the formal curriculum. Ethnomathematics helps to overcome learning difficulties (Orey and Rosa 2007). Orey and Rosa (2007) elaborated on their assumption regarding the origin of modern mathematics within different communities: "Much of what we call modern mathematics came about as diverse cultural groups sought to resolve unique problems such as exploration, colonization, communications, and construction of railroads, census data, space travel, and other problem-solving techniques that arose from specific communities" (p 11).

Cultural variables have influenced students' consideration of how they understand the world and interpret their experiences as well as those of others. In other words, culture influences the ways we gather and utilize our own mathematical knowledge (National Research Council 1994). Ethnomathematics helps students appreciate the contributions of their own culture and others (D'Ambrosio 1990; Joseph 1991). As Freire (1986, 1998) mentioned, students are not containers to be filled with information, rather, teaching must involve the creation of knowledge and the transference of information. Ethnomathematics is communal in the sense that all students in the class are significant and stay connected to their roots so that they develop resistance to harassment or domination. They are equipped with the ability to engage in important concepts within mathematics, thus linking mathematics with its contexts.

However, incorporating ethnomathematics approaches has its limitations. One limitation is that ethnomathematics can privilege socio-cultural aspects over cognitive aspects of mathematics teaching and learning. The progressive nature of learning is frequently overlooked when introducing the link between mathematics learning and culture. In addition, the common method of practice and conceptualization for ethnomathematics is accomplished by only retaining the cultural aspects that are *deemed* relevant to the mathematical topics of interest. In this way, the mathematical portions are *extracted* from those cultural practices, often losing the meaning behind the tradition shortly after it is introduced. As a result, the appropriate realistic applications of ethnomathematics practices should incorporate various significances (traditional, ecological, familial, historical, theological and holistic) continuously throughout lessons—hence our semiotic signal in choice of the signifier *partner* rather than *resource*.[2]

A Brief Introduction to Variation Theory

Variation theory (VT) is a theoretical framework of learning and experience described by Ference Marton and several others (Marton et al 2004). The *variation theory of learning* (Marton 2014) focuses on the need for learners to notice/discern critical aspects of the object of learning. There is a core sequences of patterns of variation and invariance are required; these are called *critical discernments*. While developing these critical discernments, there are four necessary conditions of learning in VT to consider: contrast, separation, fusion and generalization. We have summarized our understanding of the elements of variation theory in a series of concept maps below, which are also available online.[3]

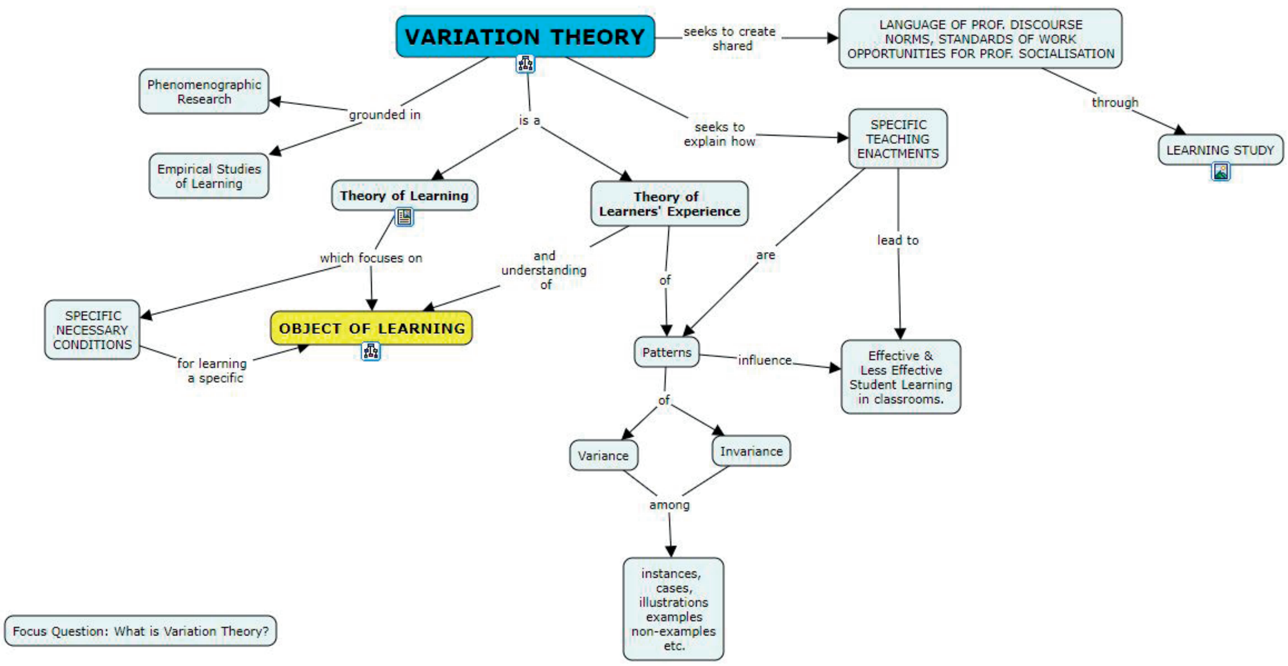

Figure 2. *Concept map on theories of learning and experience to attain the object of learning. See https://cmapscloud.ihmc.us/viewer/cmap/1Q2TCGFCL-D1G01X-4CP.*

concept of the *object of learning* (OOL) that is situated in a context concentrating on what *is* learned and what the students are *expected* to learn. In the framework of VT, the learners are drawn to contrasting by observing how something changes or is different, allowing them to gain the ability to generalize concepts and fuse various aspects (Handy 2021). Marton and Booth (1997) defined learning as the advancement of experiencing something in a new or different way, particularly at a moment (or moments) where there are differences in the structure of awareness. For Runesson (2005), "learning is defined as a change in the way something is seen, experienced or understood" (p 70). In order to draw awareness to changes,

In mathematics education, elements of the VT framework can provide practical guidance for teachers in designing mathematical lessons/tasks that can enhance the mathematical understanding of the learners (Handy 2021; Watson and Mason 2006). Jing et al (2017) suggested that only a handful of studies are available on the effects of variation theory when used to inform teaching on students' outcomes. Donovan et al (1999) pointed out that in order to develop competence in a specific area of inquiry, students must understand facts and ideas in the context of a theoretical basis, and the knowledge must be organised in ways that facilitate recovery and application. Our attempt is to provide mathematics teaching and learning

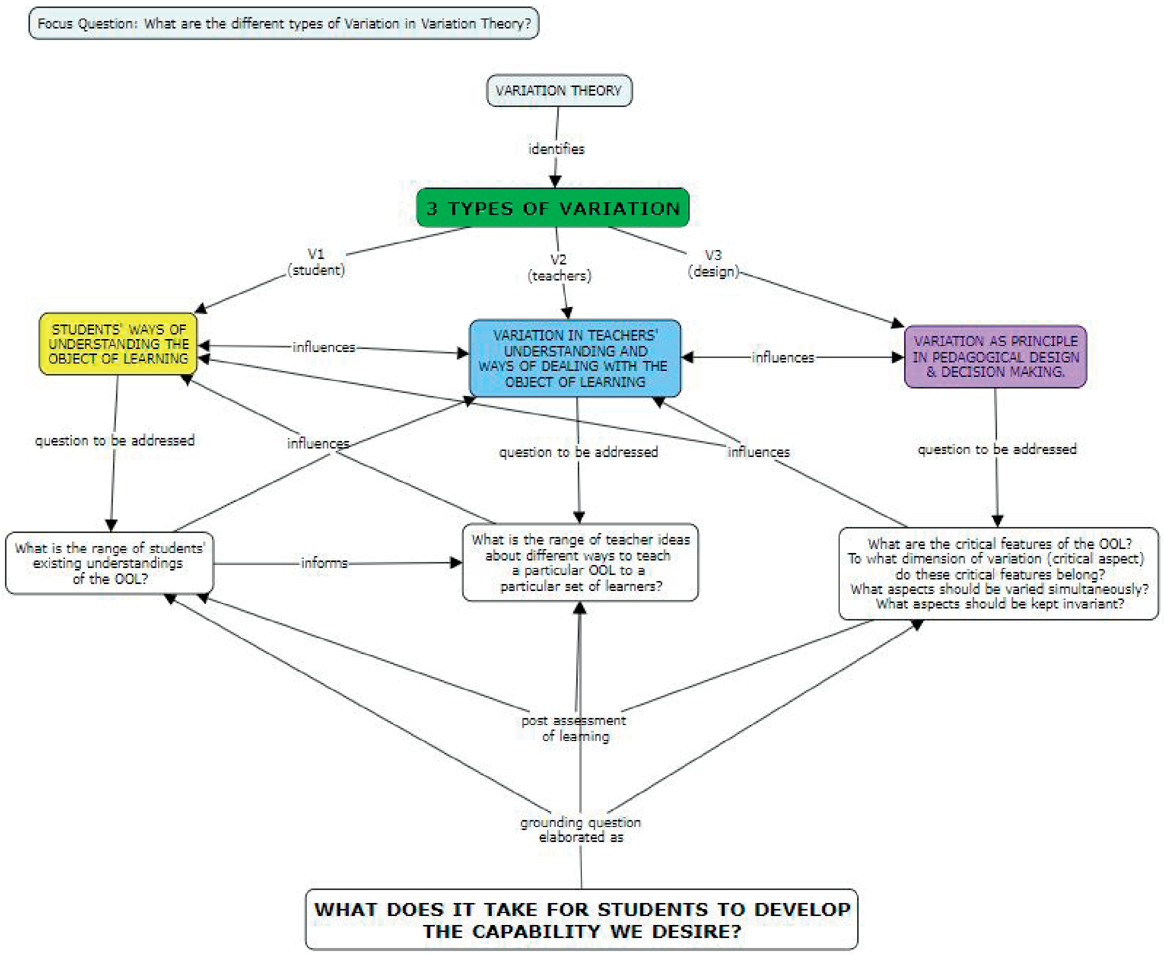

Figure 3. *The three types of variation within variation theory. See https://cmapscloud.ihmc.us/viewer/cmap/1Q2TCGFCL-1TDZZ2J-4CG.*

opportunities that can help learners experience mathematics.

In the traditional mathematics classroom, students memorize a formula or algorithm, work through problems individually or in groups, or they take a test to demonstrate their understanding, thus helping teachers evaluate them. The question here is: How can we empower learners with meaningful experiences in mathematics? In the educational context, the learning experience is the interactions that connect a learner with the matter being learned (Leung 2010). Leung further identified three categories of understanding—of mathematical concepts by students—based on research conducted in a primary mathematics classroom in Hong Kong: (1) creating/discovering mathematical knowledge beyond the present level, (2) shaping new mathematical knowledge and (3) reshaping prior mathematical knowledge.

Ethnomathematics helps students understand the cross-curricular applications and identify mathematics in real-life situations.

Ethnomathematics is present in the cultural practices of Indigenous Peoples, and such practices have helped in preserving their cultural identity (Pradhan et al 2021). It is important that teachers understand how mathematical knowledge is related to various cultures in classrooms with cultural diversity, and how socio-cultural factors influence the academic achievement of students (Haghi et al 2013). The mathematics practices of the classroom should bring meaning to reignite mathematics knowledge, which can be explored through a multitude of activities and concepts, including the fascinating designs, techniques or patterns such as kolams.

A Background on Kolams

Kolams are drawn using loose powder-like material (ie, rice flour, chalk powder, rock powder, ground rice, sand, etc) that connects and surrounds predetermined dots with lines. The practice of drawing kolams is generally performed on a wet surface so that the design stays in place for a long time. Usually, the dots are connected by lines to make a pattern, or loops are drawn around dots to create a design. Kolam drawings originated in Tamil Nadu, India, from women's practices and rituals performed in the early mornings before sunrise. Typically, the rice flour used in making kolams can feed many types of animals, like ants, birds and squirrels; therefore, this land art is a symbol of harmonious living with nature. Kolam designs are recursive in nature and start off as simple motifs that form a complex structure by repeating the subunits. There is a synchronisation of yoga in the practice of drawing kolams. The health benefits range from improved blood circulation to meditative effects on the mind and strengthening the body. Kolams are drawn to channel positive energy to one's home/office, and they have a calming effect on the mind and body as an individual prepares for and faces the hardships in store for the day.

The Potential of Kolams in Mathematics Classrooms

The art of kolams can create opportunities for students to recognize and build the context of mathematics applications outside of mathematics (Chenulu 2007). Kolams can be used in educational applications of key mathematics concepts that include—but are not limited to—probability, counting, patterns, symmetry, fractions, geometry, graph theory, algebraic thinking and spatial analysis/awareness.

Kolams provide and introduce a holistic environment in which students' lived experience can be integrated by focusing on the individual as a whole. Kolams have the potential to not only incorporate an invitational approach into the mathematics classroom, but the lessons learned can also expand beyond the classroom as lifelong learning, respecting the environment and resources, the cycle of life, and spiritual connections with the body, mind, soul and nature.

How Kolams Exemplify VT and EM

We see potential in providing an example of approaching the use of both variation theory and ethnomathematics in mathematics education through the practices of kolam drawing as a learning sequence for teachers. When approaching the study of kolams through an ethnomathematics lens, the mathematical aspects overshadow the cultural aspects. The ethnomathematics perspective observes kolams as a resource since we can extract and recognize the mathematics portions from the cultural practice to make connections within the classroom (Perlata 2020). Additionally, we notice the capacity to use kolams in the mathematics classrooms as a compelling example of partnering variation theory with ethnomathematics.

Kolams are exemplary in joining the perspectives of ethnomathematics and variation theory, and the practice brings forth several mathematical concepts in a constructive interconnected fashion. The mathematical concepts expressed through kolam practices can provide opportunities for variation theory in student learning in making connections with critical discernments and allow for focus on the *internal and external horizons of the object of learning* (OOL). The object of learning acquires meaning through its external horizon in variation theory approaches (Lo 2012) and generally relates to the cultural aspects, but this link is not made within teacher education since there is limited time and experience for most teachers in the exploration of *external horizons that are truly external/different to their own*.

Figure 4. *Variation and developmental sequence in holding and dropping the rice flour.*

Figure 5. *The variation (rounded and slanting dots) from different methods of holding the flour.*

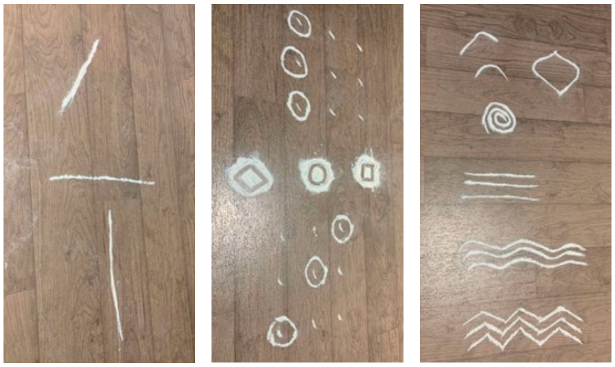

Figure 6. *Sequencing of different dots, simple lines and curves used in complex kolam designs.*

Tracing the design, as shown in Figure 6, can be used to initiate an interest of students in feeling the resources and partnering with them.

Figure 7. *Dots, circles and basic shapes/tiles are part of traditional kolam designs.*

In Figure 7, the image on the left shows designs that are traced on the rice flour. While this type of tracing does not form part of kolam designs, the authors consider it a strategy that can be adopted in elementary mathematics classrooms to help create an interest in the subject.

Figure 8. *Variation in the skill level of the 1:3:1 dots kolam designs.*

Every Kolam design starts with dots laid in a specific ratio, where straight lines, curves and loops are drawn around the grid dots based on the grid pattern and the number of grid dots. The ratio determines the size/dimension of the geometric kolam design. In Figure 8, the dots are laid in a 1:3:1 pattern, connected by a stroke that runs around each dot.

Figure 9. *Variation of beginner patterns created using 2:3:2 kolam designs.*

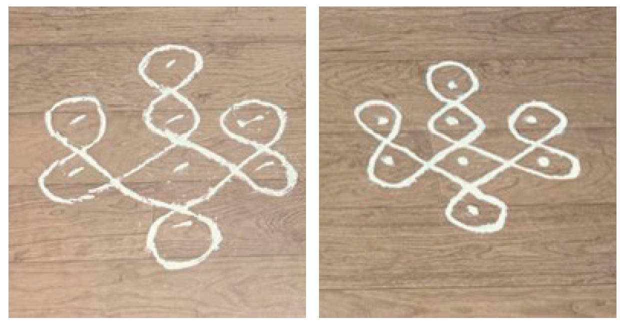

Figure 10. *Variation in skill levels shown in 1:3:3:1, traditional kolam lamp (wick) design.*

Discernments and Drawing Explanation VT and EM

Using the framework of variation theory, we would like to probe the prelived and postlived teaching as well as the learning experiences of mathematics educators with their students on partnering culturally responsive instructional designs (ie, kolams) with particular objects of learning. The object of learning is the practice of kolam drawing which has critical features (discernments) related to how to hold the flour, the type of forms that can be created (dots, lines and curves), and the cultural meanings related to the practice. The intention of engaging with this object of learning is to connect students to their physical, visual and spatial senses towards meaning making and familiarizing for a better understanding of mathematics concepts (eg, pattern recognition, shapes and ratios). Students' lived experience can be brought to the surface in their learning with the internal and external horizons as components of variation theory, which can in turn be directly indicated with the learning.

From the variation theory framework, the key elements of critical discernment can draw attention to and emphasize the multiple characteristics, movements, techniques and types of kolam drawings. The critical discernments developed through kolam drawing practice can be creatively curated with opportunities to relate to mathematical understandings and/or holistic learning concepts and integrated into the classroom. These types of discernments may include choosing the material or style when creating a design. The technique and methods may vary based on the selection of various types of flour and sand. The process of drawing kolams includes discernment of actions, such as performing body and hand movements to hold or pour the material (ie, tracing or dropping, as shown in Figure 4). Prior to drawing and selecting the method of drawing, the pattern type must be considered with several points of critical discernment. This may include the number of dots, types of dots (straight or round), organisation of dots (from center to the outer formation) and types of lines (straight, looped, curved or parallel). Also, critical discernments must be taken into account for the overall design pattern (odd number sequence pattern, connecting dot pattern, etc), finding a route from the starting point to completion and maintaining the connection with consistent lines to account for each dot with overall sequencing.

Katz (1994) pointed out that mathematical ideas have grown out of the needs of various cultures around the world, and it is important that students in western countries are exposed to the mathematical practices in different cultures.

In producing the final kolam drawing, critical aesthetic discernments are made in reflection, contemplation and admiration of the artwork done. Comparison with past drawings and/or the drawings of others provides further opportunities for discernment. These individual and peer comparisons can involve critical discernment regarding learning from different or similar techniques and patterns, reflecting on the preplanned process in design and observing the differences in actions that could have been made for improvement.

This paper has demonstrated the critical discernments at play. The practices shown in the images of introducing and addressing the critical discernments in the stages of kolam drawings do not have to be followed using the same procedure or organization of step-by-step instruction. In our trials, we only offer an example of some possible critical discernments at work to produce kolam designs. Figure 4 illustrates three methods of holding the rice flour: (1) flour between the index finger and thumb, (2) flour pinched by all fingers and dropped with the middle finger and (3) flour in a fist dropped from the pinky opening of the hand. In Figure 5, the types of dots and their placements are shown in the variation of circular/rounded dots and slanted dots, resulting from the third and first methods of holding, respectively. Figure 6 presents multiple variations and sequencing of different circles, simple lines and curves that produce a kolam design only when they are incorporated together. There are tracing methods of designs with classic dot and line circles that start with material on the surface and then finger drawing the designs. Additionally, there are basic types of lines (horizontal, vertical or diagonal) and complex lines (rounded, spiral, zigzag and parallel). Figure 7 highlights the variation of the variety of shapes through the tracing and line drawing methods and common kolam shapes inspired by polypad tiles (available at https://mathigon.org/polypad#patterns).

The intention of engaging with this object of learning is to connect students to their physical, visual and spatial senses towards meaning making and familiarizing for a better understanding of mathematics concepts (eg, pattern recognition, shapes and ratios).

Figures 8–10 show that there is a range of complexity in kolam dot line drawings. The kolam first starts with one of the simplest designs, which is a 1:3:1 design, then with 2:3:2 designs in star and flower patterns, and finally with a more complex design of the traditional lamp kolam pattern. In slowly progressing the aspects that make up a kolam design, critical discernments can draw attention to the areas of improvement and importance for students, so they can learn to make a complex kolam design. Throughout the paper and practice, the experience level of kolam designs is represented in side-by-side images, valuing both beginner and advanced learners and demonstrating the variation in direction and precision based on skill level.

The final product of the project is displayed in Figure 11 below. These designs were taken to the natural land surface, honouring the traditional ways of practicing kolam drawing outside in the natural environment. The designs below are the final works of the practice done in previous figures and compiled into one image to demonstrate variation in skill level, symmetry and the social aspect of drawing kolams.

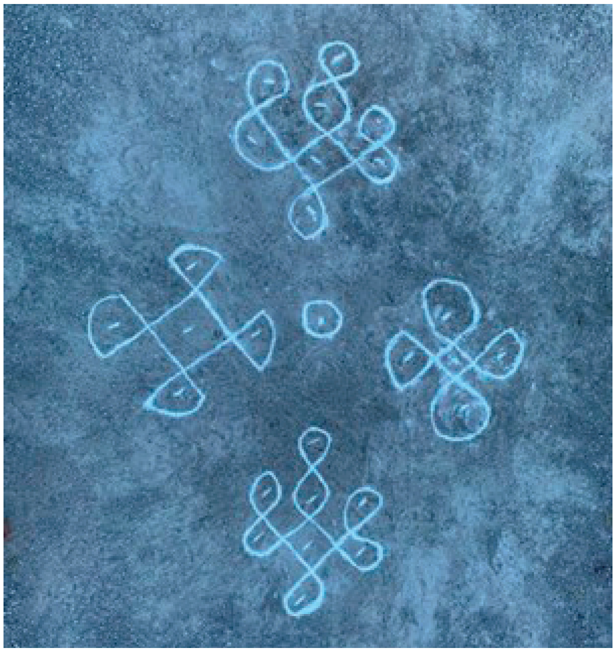

Figure 11. *Holistic realistic practice in demonstrating the final kolam drawings (with variations in skill level and symmetry).*

Conclusion

Partnering ideas from variation theory and ethnomathematics, we believe, has value for teachers of mathematics, education researchers and curriculum designers. The chief value is a reliable approach to developing critical awarenesses while not succumbing to the tendency to reduce the cultural practice only to its mathematically interesting aspects but to continue to situate both the practice and mathematics as living, evolving aspects of human cultures consistent with mythopoetic (Khan 2010) and multispecies flourishing (Khan 2020) frameworks.

Another value is in keeping the cognitive and cultural aspects of practices together in a respectful, non-resource-extractive or appropriating way. Kolam drawing provides an accessible yet sufficiently challenging and mathematically rich starting point for exemplification and extension of these ideas for teachers and can provide several points of entry for making critical mathematical and cultural discernments while honouring multicultural traditions in mathematics classrooms.

Notes

1. Note we are not saying that this aculturality is *bad*, merely that this is how that work is presented. It is a statement of fact and not a value judgement.

2. A more comprehensive analysis of EM and some of the arguments within the field can be found in Khan (2008).

3. To access the concept maps that extend and deepen the understanding of the OOL, the internal and external horizons of the OOL and the core concept of critical features (or discernments in the Math Minds rendering, please refer to https://cmapscloud.ihmc.us:443/rid=1Q2TCGFCL-BHCW6F-4CJ/Object%20of%20Learning%20in%20Variation%20Theory.cmap (accessed September 29, 2023).

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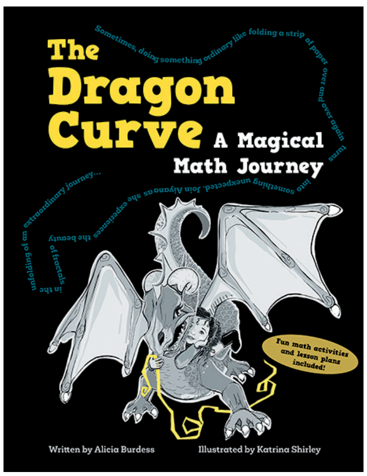
The Beauty and Strength of a Mathematical Story: Alicia Burdess' *The Dragon Curve: A Mathematical Journey*

Reviewed by Patricia Shoemaker



It is not enough to teach a set of rules and hope that students understand math that way; they have to play with it and be a part of it. As math educators, we cannot perpetuate the cycle of believing that you are only good at math if you can follow a set of rules set out by someone else. Where is the beauty of math in that? And how are students supposed to fall in love with a beautiful discipline? As educators and members of the older generation, we should want all children to believe that they can do math and that it is beautiful, before they are in university; otherwise, they'll spend their adulthood believing math is a set of algorithms and rules to follow.

Enter Alicia Burdess' book *The Dragon Curve*.

The Dragon Curve: A Mathematical Journey is a captivating exploration of the beauty and complexity of mathematics through the lens of one of its most intriguing creations—the dragon curve. This book is an accessible and engaging adventure for both students and enthusiasts of mathematics of all ages who want to dive into the fascinating world of fractals, as well as those who just want to read a great story.

Burdess provides a clear and concise exploration of the dragon curve and its underlying mathematical concepts, such as fractal geometry, recursion and self-similarity, in a way that will enthrall all. With the help of vivid illustrations, the author takes the reader on a journey through the various stages of the dragon curve's development, from its simple beginnings to its intricate and beautiful final form.

Burdess' writing style is lively and engaging, making even complex mathematical concepts easy to understand and appreciate.

Throughout the book, Burdess' storytelling interweaves human elements with the mathematical concepts being explored. This makes the book an enjoyable and personal read and a valuable educational resource. There is something to learn for students at every level: those in kindergarten can enjoy a story with math they can understand, while calculus students can explore the complexities of a fractal world.

Overall, *The Dragon Curve: A Mathematical Journey* is a highly recommended book for anyone interested in mathematics, whether as a student or as a curious enthusiast. With its engaging writing style and beautiful illustrations, this book is a *must read* for anyone who wants to deepen their understanding and appreciation of the fascinating and beautiful world of fractals and mathematics.

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Formative Assessment: Reflecting on a Common Assessment Experience

Darin Trufyn

Assessments and math instruction often become one and the same. With many teachers voicing concerns about learning loss over the last few years, I wonder what a *common formative assessment* could do to support these concerns. We know that formative assessments provide various information regarding student misconceptions; however, could we provide students with content that we as teachers could then reflect upon?

At this point, you are probably wondering why the title of this article includes the word *common*. Over the years, I would often notice how we as teachers use the information that we receive from provincial achievement test (PAT) results. I have been reflecting on what the typical start of the school year looks like for me: I would often return to my school after a nice summer break, eagerly await my assigned teaching load, build various lessons and assessments, reflect on student performance from the previous year end summative assessment (or PAT results), then leverage all this information to frame my yearly growth plan. Does any of this sound familiar?

A few years ago, Alberta Education decided to include two distinct parts in Grades 6 and 9 mathematical provincial achievement tests: one involving the use of a calculator and the other with no calculator being used. It was after the first year of implementation that I started to wonder how this new calculator-free assessment section might support further insight into how concepts are being taught or the experience that we as teachers leverage when supporting transitions between the grades for our students in junior high. At first, removing technology seems odd, especially given all the different types of technology currently available to students.

However, much can be learned from the way that Alberta Education breaks down the reporting categories for this section of the PAT. In Grade 6, the reporting category for Part A consists of questions taken from only the number strand: addition, subtraction, multiplication and division. When we look at the language that Alberta Education uses for part A for Grade 9 students, we see that things become a little more complex. According to Alberta Education, this part of the assessment is designed to assess "students' foundational skills and fluency in mental math, estimation, algebra, square roots, exponent laws, and arithmetic operations on rational numbers" (p 1). But what exactly does this mean and how can we begin to contextualize this increased level of detail for students moving through their junior high experience in mathematics?

In the next issue of *delta-K*, I will discuss how I implemented this type of formative assessment—stay tuned!

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Darin Trufyn has spent the majority of his career teaching Grades 8–12 mathematics from three different curriculums. Darin has been with the Edmonton Catholic School Division for the past 25 years, and he currently holds the position of secondary math consultant, supporting Grades 7–9. His passion is all things spatial. He resides in St Albert with his wife, daughter, and Maggie (their dog).

Alberta High School Mathematics Competition 2020-2021 Part I

1. If $\frac{2^a}{4^b} = 16$, then $a - 2b$ is equal to

   (A) 1/16 (B) 1/4 (C) 4 (D) 8 (E) 16

2. How many integers x satisfy the inequality $\frac{x^2-61}{x^2-16} \le 0$?

   (A) 3 (B) 4 (C) 6 (D) 8 (E) infinitely many

3. Two rectangles have the same area. In the first rectangle the ratio of the long edge to the short edge is 3 : 1 and in the second rectangle these edges are in a ratio of 4 : 3. Then the ratio of the perimeter of the first rectangle to the perimeter of the second rectangle is

   (A) 2 : 3 (B) 8 : 7 (C) 7 : 4 (D) 13 : 6 (E) not uniquely determined

4. $1a5$ and $6b9$ are two 3-digit numbers, where a and b are the middle digits of the two numbers. If $6b9 - 1a5 = 454$ and $6b9$ is divisible by 9, then $a + b$ is equal to

   (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

5. A number which reads the same backwards as forwards is called a palindrome. For example, 13231 is a palindrome, since backwards it also reads 13231. How many palindromes are between 10 and 2020?

   (A) 99 (B) 100 (C) 101 (D) 110 (E) 120

6. Let f be the function such that $f(x) = ax^9 + bx^5 + cx^3 - 7$, where a, b, and c are real numbers. If $f(-9) = 9$ then $f(9)$ is equal to

   (A) -23 (B) -9 (C) -5 (D) 23 (E) not sufficient information

7. A doctor's office has a row of chairs, two of which are already occupied. Ruby and Ollie come in and want to sit somewhere in the row, in adjacent chairs, that is, beside each other without a gap. Because of social distancing, there must be at least two empty chairs between them and the persons already sitting. How many chairs at least must the row contain so that Ruby and Ollie can always find a place to sit together, regardless of the location of the occupied chairs?

   (A) 10 (B) 11 (C) 12 (D) 13 (E) 14

8. The number of positive integer divisors of $10! = 10 \cdot 9 \cdot 8 \cdots 2 \cdot 1$ that have a remainder of 1 when divided by 3 is

(A) 14 (B)18 (C) 24 (D) 26 (E) 28

9. For a positive integer N, let $S(N)$ denote the sum of the digits of N. Consider the sequence $a_1 = 2$, $a_n = (S(a_{n-1}))^2 - 1$ for $n \geq 2$. The term a_{2020} is equal to

(A) 2 (B) 3 (C) 8 (D) 63 (E) 80

10. In how many ways can a team of 10 competitors be divided into three groups of two and one group of four (the order of the groups and also the order of the competitors in each individual group does not matter)?

(A) 1575 (B) 3150 (C) 6300 (D) 18900 (E) none of these

11. For how many integers $1 \leq n \leq 100$ is n^n a perfect square?

(A) 10 (B) 50 (C) 55 (D) 60 (E) none of these

12. A right triangle has an angle of 60° and the hypotenuse has length 2 cm. An ant walks around the outside of the triangle always maintaining a distance of exactly 1 cm to the nearest point on the triangle, and ending where it started. The distance (in cm) travelled by the ant is

(A) $3+\sqrt{3}+\pi$ (B) $3+\sqrt{3}+3\pi/2$ (C) $3+\sqrt{3}+2\pi$ (D) $3+\sqrt{3}+5\pi/2$ (E) none of these

13. Jane has 200 marbles of various colours. She splits them into groups of the same colour, and she observes that each group has a different number of marbles. The maximum number of groups she can have is

(A) 19 (B) 21 (C) 23 (D) 24 (E) none of these

14. The roots of a cubic polynomial $p(x)$ are the squares of the three solutions of the equation $x^3 - 3x + 1 = 0$. If $p(0) = -4$, then $p(4)$ is equal to

(A) -12 (B) -3 (C) 0 (D) 3 (E) 12

15. The number of integers x between 10 and 99 (inclusive) which have the property that the remainder when x^3 is divided by 100 is equal to the cube of the units digit of x is

(A) 9 (B) 11 (C)13 (D)14 (E) 17

16. Three equal smaller circles are all externally tangent to each other and also internally tangent to a larger circle of radius 10. The radius of one of the smaller circles is

(A) $6(\sqrt{3}-1)$ (B) $10(2\sqrt{3}-3)$ (C)$4(3\sqrt{3}-4)$ (D)$7(5\sqrt{3}-8)$ (E) none of these

Alberta High School Mathematics Competition 2020-2021 Solutions (Part I)

1. If $\dfrac{2^a}{4^b} = 16$, then $a - 2b$ is equal to

   (A) 1/16 (B) 1/4 (C) 4 (D) 8 (E) 16

   Solution:

   Since $\dfrac{2^a}{4^b} = \dfrac{2^a}{2^{2b}} = 2^{a-2b}$ and $16 = 2^4$, we have $a - 2b = 4$. The answer is (C).

2. How many integers x satisfy the inequality $\dfrac{x^2 - 61}{x^2 - 16} \le 0$?

   (A) 3 (B) 4 (C) 6 (D) 8 (E) infinitely many

   Solution:
   We should have $16 < x^2 \le 61$, hence x lies in $[-\sqrt{61}, -4) \cup (4, \sqrt{61}]$. Since x is an integer, it may be $\pm 5, \pm 6, \pm 7$. The answer is (C).

3. Two rectangles have the same area. In the first rectangle the ratio of the long edge to the short edge is $3:1$ and in the second rectangle these edges are in a ratio of $4:3$. Then the ratio of the perimeter of the first rectangle to the perimeter of the second rectangle is

   (A) $2:3$ (B) $8:7$ (C) $7:4$ (D) $13:6$ (E) not uniquely determined

   Solution:
   We can let the first rectangle be 3 by 1, and the second rectangle be $4k$ by $3k$ for some k. The areas are the same, so $3 = 12k^2$, so $k = 1/2$. Then the ratio of the perimeter of the first rectangle to the perimeter of the second rectangle is $(3+1+3+1):(2+3/2+2+3/2)$, that is 8:7. The answer is (B).

4. $1a5$ and $6b9$ are two 3-digit numbers, where a and b are the middle digits of the two numbers. If $6b9 - 1a5 = 454$ and $6b9$ is divisible by 9, then $a + b$ is equal to

   (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

   Solution:

   The rule for divisibility by 9 says that a number is divisible by 9 if the sum of its digits is divisible by 9. Hence, $6 + b + 9 = 15 + b$ is divisible by 9. Therefore $b = 3$ and hence $6b9 = 639$.

   The equality $639 - 1a5 = 454$ gives $a = 8$. Therefore $a = 8, b = 3$ and $a + b = 11$. The answer is (E).

5. A number which reads the same backwards as forwards is called a palindrome. For example, 13231 is a palindrome, since backwards it also reads 13231. How many palindromes are between 10 and 2020?

   (A) 99 (B) 100 (C) 101 (D) 110 (E) 120

   Solution:
   There are 9 two digit palindromes $(11, 22, \ldots, 99)$ and 90 three digit palindromes (all numbers of the form aba, where a and b are digits and $a \ne 0$). Any four digit palindrome smaller than 2020 must have the form $1aa1$ or 2002, hence there are 11 such palindromes. Therefore the number of required palindromes is 9+90+11=110. The answer is (D).

6. Let f be the function such that $f(x) = ax^9 + bx^5 + cx^3 - 7$, where a, b, and c are real numbers. If $f(-9) = 9$ then $f(9)$ is equal to

(A) -23 (B) -9 (C) -5 (D) 23 (E) not sufficient information

Solution:

$$f(9) = a \cdot 9^9 + b \cdot 9^5 + c \cdot 9^3 - 7 \tag{1}$$

$$f(-9) = 9 = a \cdot (-9)^9 + b \cdot (-9)^5 + c \cdot (-9)^3 - 7 \tag{2}$$

Adding equations (1) and (2), we obtain

$$f(9) + f(-9) = f(9) + 9 = -14.$$

From this,

$$f(9) = -14 - 9 = -23.$$

The answer is (A).

7. A doctor's office has a row of chairs, two of which are already occupied. Ruby and Ollie come in and want to sit somewhere in the row, in adjacent chairs, that is, beside each other without a gap. Because of social distancing, there must be at least two empty chairs between them and the persons already sitting. How many chairs at least must the row contain so that Ruby and Ollie can always find a place to sit together, regardless of the location of the occupied chairs?

(A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Solution:
Let $c_1, c_2, c_3, c_4, \ldots$ denote the chairs from left to right. The most unfavourable scenario is when c_4 and c_{10} are occupied. In this case Ruby and Ollie can not sit on the left of c_4 nor between c_4 and c_{10}. They should occupy c_{13} and c_{14}. Therefore to assure the social distancing the row must contain at least 14 chairs. The answer is (E)

8. The number of positive integer divisors of $10! = 10 \cdot 9 \cdot 8 \cdot \cdots \cdot 2 \cdot 1$ that have a remainder of 1 when divided by 3 is

(A) 14 (B)18 (C) 24 (D) 26 (E) 28

Solution:
Since $10! = 2^8 3^4 5^2 7$ the divisors of 10! which have remainder 1 when divided by 3 have the format $2^a 5^b 7^c$, where $0 \le a \le 8, 0 \le b \le 2, 0 \le c \le 1$ and $a + b$ is even. There are $2(5 \cdot 2 + 4 \cdot 1) = 28$ solutions. The answer is (E).

9. For a positive integer N, let $S(N)$ denote the sum of the digits of N. Consider the sequence $a_1 = 2$, $a_n = (S(a_{n-1}))^2 - 1$ for $n \ge 2$. The term a_{2020} is equal to

(A) 2 (B) 3 (C) 8 (D) 63 (E) 80

Solution:
The sequence is $\{2, 3, 8, 63, 80, \mathbf{63}, \mathbf{80}, \ldots\}$ and repeats with a period of 2. For even values of $n \ge 4$ we have $a_n = 63$, hence $a_{2020} = 63$. The answer is (D).

10. In how many ways can a team of 10 competitors be divided into three groups of two and one group of four (the order of the groups and also the order of the competitors in each individual group does not matter)?

(A) 1575 (B) 3150 (C) 6300 (D) 18900 (E) none of these

Solution:
There are $\binom{10}{2}$ ways to choose a group of two from the 10 competitors, then $\binom{8}{2}$ ways to choose a second group of two from the remaining eight, then $\binom{6}{2}$ ways to choose the third group of two from the remaining six. This leaves just four competitors left, who form the group of four. The order doesn't matter, so we must divide by 3! to eliminate duplications, and the requested number of choices is

$$\frac{1}{3!}\binom{10}{2}\binom{8}{2}\binom{6}{2} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{6\cdot 2\cdot 2\cdot 2} = 3150.$$

The answer is (B).

11. For how many integers $1 \le n \le 100$ is n^n a perfect square?

(A) 10 (B) 50 (C) 55 (D) 60 (E) none of these

Solution:
If n is even, then n^n is a perfect square. So all 50 even choices work.

If n is odd, then n^n is a perfect square exactly when n is a perfect square. Thus, only $n \in \{1^2, 3^2, 5^2, 7^2, 9^2\}$ work in the odd case. Therefore there are 55 integers having the required property. The answer is (C).

12. A right triangle has an angle of 60° and the hypotenuse has length 2 cm. An ant walks around the outside of the triangle always maintaining a distance of exactly 1 cm to the nearest point on the triangle, and ending where it started. The distance (in cm) travelled by the ant is

(A) $3+\sqrt{3}+\pi$ (B) $3+\sqrt{3}+3\pi/2$ (C) $3+\sqrt{3}+2\pi$ (D) $3+\sqrt{3}+5\pi/2$ (E) none of these

Solution:
The sides of the triangle are of lengths 2, 1 and $\sqrt{3}$ cm. The ant must trace the sides of the triangle and three arcs around each corner of lengths $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$. Hence the ant travels a distance of

$$2+1+\sqrt{3}+\frac{\pi}{2}+\frac{2\pi}{3}+\frac{5\pi}{6} = 3+\sqrt{3}+2\pi$$

cm. The answer is (C).

13. Jane has 200 marbles of various colours. She splits them into groups of the same colour, and she observes that each group has a different number of marbles. The maximum number of groups she can have is

(A) 19 (B) 21 (C) 23 (D) 24 (E) none of these

Solution:
Let n be the number of groups, and let us order the groups increasingly by the number of marbles.

The first group has at least 1 marble, the second group has at least 2 marbles,..., the last group has at least n marbles.

In total she has at least $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ marbles. Therefore,

$$\frac{n(n+1)}{2} \le 200$$

By inspection we see that $n = 20$ is too large, but $n = 19$ is the maximal solution.

$n = 19$ is an acceptable solution, as she could have groups of $1, 2, 3, .., 18$ and 29 marbles. The answer is (A).

14. The roots of a cubic polynomial $p(x)$ are the squares of the three solutions of the equation $x^3 - 3x + 1 = 0$. If $p(0) = -4$, then $p(4)$ is equal to

(A) -12 (B) -3 (C) 0 (D) 3 (E) 12

Solution:
Let $q(x) = x^3 - 3x + 1 = (x-a)(x-b)(x-c)$. Then $p(x) = k(x-a^2)(x-b^2)(x-c^2)$ and

$$p(x^2) = k(x^2-a^2)(x^2-b^2)(x^2-c^2) = k(x-a)(x-b)(x-c)(x+a)(x+b)(x+c) = -kq(x)q(-x).$$

Since $p(0) = -4$ and $q(0) = 1$, one obtains $k = 4$ and hence $p(x^2) = -4q(x)q(-x)$. Now

$$p(4) = -4q(2)q(-2) = -4 \cdot 3 \cdot (-1) = 12$$

The answer is (E).

15. The number of integers x between 10 and 99 (inclusive) which have the property that the remainder when x^3 is divided by 100 is equal to the cube of the units digit of x is

(A) 9 (B) 11 (C)13 (D)14 (E) 17

Solution:
If $x = 10a + b$ then $x^3 = 1000a^3 + 300a^2b + 30ab^2 + b^3$. The remainder when x^3 is divided by 100 is equal to the cube b^3 of the units digit of x if and only if 100 is a divisor of $30ab^2$ and $b^3 < 100$, which means that 10 is a divisor of ab^2 and $b \le 4$. This happens when $a \in \{1,2,3,4,5,6,7,8,9\}$ and $b = 0$, or if $a = 5$ and $b \in \{2,4\}$. Therefore there are 11 integers in total. The answer is (B).

16. Three equal smaller circles are all externally tangent to each other and also internally tangent to a larger circle of radius 10. The radius of one of the smaller circles is

(A) $6(\sqrt{3}-1)$ (B) $10(2\sqrt{3}-3)$ (C)$4(3\sqrt{3}-4)$ (D)$7(5\sqrt{3}-8)$ (E) none of these

Solution:
Let r be the radius of a smaller circle. The centres of the smaller circles are the vertices of an equilateral triangle of side $2r$. The height of this equilateral triangle is $r\sqrt{3}$. The centre of the larger circle is the same as the centre of the equilateral triangle, which is located twice as far from each vertex as from the opposite edge. Thus the radius of the larger circle is $r + \frac{2r\sqrt{3}}{3}$. Now,

$$r + \frac{2r\sqrt{3}}{3} = 10 \iff r = \frac{30}{2\sqrt{3}+3} = 10(2\sqrt{3}-3).$$

The answer is (B).

Alberta High School Mathematics Competition 2020-2021 Part II

Problem 1. Find all positive integers n such that n^2+n+19 is the square of an integer.

Problem 2. Ajooni and Sagal live beside a lake that has a 10 km long circular path around it. One day they start from their house at the same time, Ajooni biking around the lake in one direction, and Sagal walking around the lake in the same direction. Ajooni's biking speed is b km/hour, while Sagal's walking speed is w km/hour, where $b > w$ are positive integers. So Ajooni goes faster than Sagal and will pass him over and over, if they keep going around the lake. They agree to stop whenever they both arrive back at their house at the same time.

(a) Prove that eventually Ajooni and Sagal will meet back at their house, regardless of the values of b and w.

(b) Suppose that $b = 15$ and $w = 6$, and that Ajooni and Sagal leave their house at 9 AM. At what time do they meet back at their house and stop going around the lake?

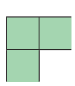
Problem 3. Show that for any positive integer n the number

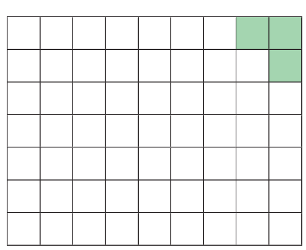
$$\underbrace{111\ldots1}_{3^n \text{ digits}}$$

consisting of 3^n 1's, is divisible by 3^n.

Problem 4. Let B be a point on the segment AC such that $B \neq A$ and $BA < BC$. Point M is on the perpendicular bisector of AC such that $\angle AMB$ is as large as possible. Find $\angle BMC$.

Problem 5.
The numbers $1, 2, \ldots, 63$ are placed on a 7 by 9 grid randomly, one number in each little square. Prove that you can always draw five L-shaped trominos on the grid so that no two of them overlap (but are allowed to touch along an edge), and so that, for each of the five trominos, the sum of the three numbers in it is at least 78. (An L-shaped tromino is a polygon made of three squares of the grid, connected edge-to-edge and having the shape of an L, but in any orientation. An example of a L-shaped tromino on the grid is given below.)

L-shaped tromino: 

A 7 by 9 grid: 

Alberta High School Mathematics Competition 2020-2021 Solutions (Part II)

Problem 1. Find all positive integers n such that $n^2 + n + 19$ is the square of an integer.

Solution:
$n^2 + n + 19$ is the square of an integer exactly if there exists some integer $k \geq 1$ such that

$$n^2 + n + 19 = (n + k)^2.$$

This is equivalent to

$$2nk + k^2 = n + 19. \tag{1}$$

Now, since $2nk > n$ we get $k^2 < 19$ thus $k \leq 4$ and hence k equals $1, 2, 3$ or 4. If you substitute these values of k in equation (1) and solve for n you obtain n equals $2, 5$ or 18.

Alternate Solution:
Let m, n be positive integers, $m > n$, such that $n^2 + n + 19 = m^2$. We have

$$n^2 + n + 19 = m^2 \iff 4n^2 + 4n + 76 = 4m^2 \iff 75 = (2m)^2 - (2n+1)^2 = (2m + 2n + 1)(2m - 2n - 1)$$

Since $m > n$ are positive, both $2m + 2n + 1$ and $2m - 2n - 1$ must be positive. Moreover, $2m + 2n + 1 > 2m - 2n - 1$.

Hence we may have the following possibilities:

Case 1.

$$\begin{cases} 2m & +2n & +1 & = 75 \\ 2m & -2n & -1 & = 1 \end{cases}$$

hence $4m = 76 \Rightarrow m = 19 \Rightarrow n = \frac{74-38}{2} = 18$.

Case 2.

$$\begin{cases} 2m & +2n & +1 & = 25 \\ 2m & -2n & -1 & = 3 \end{cases}$$

hence $4m = 28 \Rightarrow m = 7 \Rightarrow n = \frac{24-14}{2} = 5$.

Case 3.

$$\begin{cases} 2m & +2n & +1 & = 15 \\ 2m & -2n & -1 & = 5 \end{cases}$$

hence $4m = 20 \Rightarrow m = 5 \Rightarrow n = \frac{14-10}{2} = 2$.

Problem 2. Ajooni and Sagal live beside a lake that has a 10 km long circular path around it. One day they start from their house at the same time, Ajooni biking around the lake in one direction, and Sagal walking around the lake in the same direction. Ajooni's biking speed is b km/hour, while Sagal's walking speed is w km/hour, where $b > w$ are positive integers. So Ajooni goes faster than Sagal and will pass him over and over, if they keep going around the lake. They agree to stop whenever they both arrive back at their house at the same time.

(a) Prove that eventually Ajooni and Sagal will meet back at their house, regardless of the values of b and w.

(b) Suppose that $b = 15$ and $w = 6$, and that Ajooni and Sagal leave their house at 9 AM. At what time do they meet back at their house and stop going around the lake?

Solution:

(a) If Ajooni bikes m times around the lake, she takes $10m/b$ hours, and if Sagal walks n times around the lake, he takes $10n/w$ hours. For them to meet back home, we want $10m/b = 10n/w$ where m and n are positive integers. This equation has integer solutions regardless of the values of b and w, for example $m = b, n = w$, and thus Ajooni and Sagal will meet back home.

(b) For $b = 15, w = 6$ we get $m/15 = n/6 \iff m/5 = n/2$. The smallest positive integers m, n which verify this equation are $m = 5, n = 2$. Hence Ajooni and Sagal are meeting after $(10 \cdot 5)/15 = (10 \cdot 2)/6 = 10/3 = 3\frac{1}{3}$ hours, so they meet back home at 12:20 PM.

Alternate Solution:

(a) Let $r = b/w$. Consider any two consecutive occasions when Ajooni and Sagal meet. Ajooni will have gone 10 km further than Sagal during this time. But Ajooni would also have travelled r times further than Sagal during this time, so these 10 extra km must equal $r-1$ times the distance Sagal travelled. Thus Sagal must travel

$$\frac{10}{r-1} = \frac{10w}{b-w}$$

km between any two consecutive meetings. Since $b-w$ is a positive integer, on the $(b-w)$th time that Ajooni and Sagal meet after leaving home, Sagal will have travelled a total of $10w$ km. This is an integer multiple of the distance around the lake, so (if Ajooni and Sagal haven't already met back home) this meeting must happen back at home.

(b) When $b = 15$ and $w = 6$, from part (a) Sagal will travel $60/(15-6) = 20/3$ km between any two consecutive meetings. The smallest positive integer k so that $20k/3$ is an integer multiple of 10 is $k = 3$. This means that the first time that Ajooni and Sagal will meet back home will be at the third time they meet on the path, at which point Sagal will have walked 20 km, or exactly twice around the lake. This takes him $\frac{20}{6} = 3\frac{1}{3}$ hours, so they meet back home at 12:20 PM.

Problem 3. Show that for any positive integer n the number

$$\underbrace{111\ldots1}_{3^n \text{ digits}}$$

consisting of 3^n 1's, is divisible by 3^n.

Solution:

We prove the statement of the problem by induction on n. For simplicity let

$$A_n = \underbrace{111\ldots1}_{3^n \text{ digits}}.$$

If $n = 1$, $A_1 = 111 = 3 \times 37$, which is divisible by 3.
Now, assume that A_n is divisible by 3^n for some positive integer n and let us show that A_{n+1} is divisible by 3^{n+1}. We have

$$\begin{aligned} A_{n+1} &= \underbrace{111\ldots1}_{3^{n+1} \text{ digits}} \\ &= \underbrace{111\ldots1}_{3^n \text{ digits}}\ \underbrace{000\ldots0}_{2\cdot3^n \text{ digits}} + \underbrace{111\ldots1}_{3^n \text{ digits}}\ \underbrace{000\ldots0}_{3^n \text{ digits}} + \underbrace{111\ldots1}_{3^n \text{ digits}} \\ &= A_n \cdot \left(1\ \underbrace{000\ldots0}_{3^n-1 \text{digits}}\ 1\ \underbrace{000\ldots0}_{3^n-1 \text{digits}}\ 1\right). \end{aligned}$$

Since the number $1\ \underbrace{000\ldots0}_{3^n-1 \text{digits}}\ 1\ \underbrace{000\ldots0}_{3^n-1 \text{digits}}\ 1$ is divisible by 3 (the sum of its digits is 3) and A_n is divisible by 3^n one obtains that A_{n+1} is divisible by 3^{n+1}, and the claim follows.

Alternate Solution:

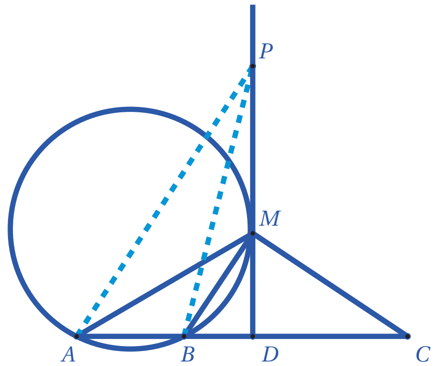
Since $A_n = \frac{10^{3^n}-1}{9}$ it suffices to show that 3^{n+2} divides into $10^{3^n}-1$, which can be proved by using mathematical induction, using the factorization

$$10^{3^{n+1}} - 1 = (10^{3^n})^3 - 1 = (10^{3^n} - 1)[(10^{3^n})^2 + 10^{3^n} + 1].$$

Problem 4. Let B be a point on the segment AC such that $B \neq A$ and $BA < BC$. Point M is on the perpendicular bisector of AC such that $\angle AMB$ is as large as possible. Find $\angle BMC$.

Solution:

Let D be the midpoint of AC. A circle passing through A, B, of radius $AB/2 + BD$, is tangent to the perpendicular bisector AC at M.



For any point P on the perpendicular bisector, on the same side of AC as M, $P \neq M$, we have $\angle APB < \frac{\overset{\frown}{AB}}{2} = \angle AMB$, hence M is the point on the perpendicular bisector for which $\angle AMB$ is as large as possible. Now, since $\angle BMD = \angle MAB = \frac{\overset{\frown}{MB}}{2}$ and $\angle CMD = \angle DMA$, on obtains

$$\angle BMC = \angle BMD + \angle DMC = \angle MAD + \angle AMD = 90^\circ.$$

Alternate Solution:
Let M be a point on the perpendicular bisector of AC and denote $AD = a$, $BD = b$, $\angle AMD = \alpha$, $\angle BMD = \beta$ and $MD = m$. Then we have $\angle AMB = \alpha - \beta$ and $\angle BMC = \alpha + \beta$. Since $\tan\alpha = \frac{a}{m}$, $\tan\beta = \frac{b}{m}$ one obtains

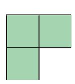
$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{\frac{a}{m} - \frac{b}{m}}{1 + \frac{ab}{m^2}} = \frac{a-b}{m + \frac{ab}{m}} = \frac{a-b}{(\sqrt{m} - \frac{\sqrt{ab}}{\sqrt{m}})^2 + 2\sqrt{ab}}$$

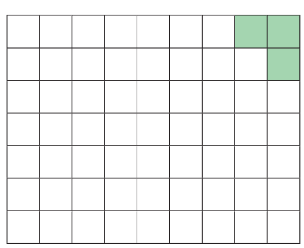
We conclude that $\tan(\alpha - \beta)$, and hence $\alpha - \beta$, has maximum value only when $m = \sqrt{ab}$, or equivalently $\tan\alpha\tan\beta = 1$, i.e., $\sin\alpha\sin\beta = \cos\alpha\cos\beta$. Consequently

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = 0$$

and hence $\angle BMC = \alpha + \beta = 90^\circ$.

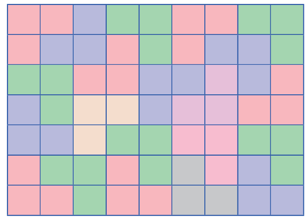
Problem 5.
The numbers $1, 2, \ldots, 63$ are placed on a 7 by 9 grid randomly, one number in each little square. Prove that you can always draw five L-shaped trominos on the grid so that no two of them overlap (but are allowed to touch along an edge), and so that, for each of the five trominos, the sum of the three numbers in it is at least 78. (An L-shaped tromino is a polygon made of three squares of the grid, connected edge-to-edge and having the shape of an L, but in any orientation. An example of a L-shaped tromino on the grid is given below.)

L-shaped tromino: 

A 7 by 9 grid: 

Solution:

The 7×9 grid can be tiled with 21 L-shaped trominos in different ways; one way is shown in the following figure:



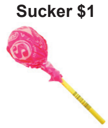
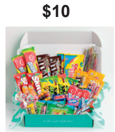
We will prove that among these 21 trominos at least five have number sum of at least 78. Let $s_1, s_2, \ldots, s_{21}$ be the sums of the three numbers contained in each of these trominos, where we may assume that $s_1 \geq s_2 \geq \ldots \geq s_{21}$. Since the smallest numbers that can occur in the 17 sums s_5 to s_{21} are $1, 2, \ldots 51$, we have $s_5 + s_6 + \cdots + s_{21} \geq 1 + 2 + 3 + \cdots + 50 + 51 = 1326$. If we assume that $s_5 < 78$ then $s_5 + s_6 + \cdots + s_{21} < 78 \times 17 = 1326$, which is a contradiction. Hence $s_1 \geq s_2 \geq s_3 \geq s_4 \geq s_5 \geq 78$.

Grades 2–3 Rich Task: Sweet Tooth Snack Shop

Charlee Ukalchuk, Torrie Jantz, Anna Taylor and Karlie Terlson

One day, you notice a brand new store opening up right down the street from your house! You walk down there to find a big sign saying Sweet Tooth Snacks! You decide to go inside and check it out!

Sweet Tooth Snacks Menu and Prices

Bubble Gum $1 	**Chocolate Bar $1** 	**Maynard Candies $2** 
Sucker $1 	**Chips $5** 	**Wagon Wheel $3** 
Sweets Combo Ultra $10 	**Chocolate Supreme Combo $12** 	**Sweets Supreme Combo$15** 
Chips and Juice Box $6 	**Juice Box $5** 	**Oreos and Rice Krispies $8** 

1) Choosing at least three snacks, how much would it cost for you to buy everything you wanted in the store?

Challenge: choose at least one combo snack!

What snacks do you buy?	How much does this snack cost?

What is the total cost of all your snacks?

2) Your dad gives you $15 to go buy snacks at the store. You want to buy as many snacks as you can with your $15, hopefully using all of it. You have to buy at least three snacks.

What snacks do you buy?

What snacks do you buy?	How much does this snack cost?

What is the total cost of all your snacks?

3) You tell your friend Sarah about the Sweet Tooth Snacks store. Sarah goes to the store and spends $10. Sarah bought more than two snacks.

What could she have bought?

What snacks does Sarah buy?	How much does this snack cost?

What is the total cost of Sarah's snacks?

4) Your dad gives you $30 to go buy snacks for you and your friend. Because you want to buy snacks for your friend also, you must buy two of each snack you choose. Make sure you buy at least two different snacks, and one MUST be a combo snack!

What could you buy to spend as much of the $30 as you can?

What snacks do you buy?	How much does this snack cost?

5) Create your own question! Using your problem-solving skills, create your own addition or subtraction question using menu items and prices!

BONUS! Create your own store! What would be onyour menu? Have at least four items and include prices. Use colour, and be creative! What would be the name of your shop? What would you sign look like?

Grades 5–6 Rich Task: Candy Bar Creation

Carys Both, Madison Janzen, Tristin Bessant and Arielle Hess

Ingredient List

Use the ingredient list and prices to make your chocolate and candy choices. Record your choices on your candy sheet.

Chocolate Choices
Milk chocolate: $1205.00
White chocolate: $1025.00
Dark chocolate: $1070.00

Candy Choices
Rock candy: $312.00
Sprinkles: $250.00
Caramel: $503.00
Salted caramel: $595.00
Graham cracker: $160.00
Marshmallow: $430.00
Pecans: $604.00
Peanut butter: $191.00
Coconut: $89.00
Gummy: $341.00
Strawberry jam: $505.00
Sponge toffee: $116.00

Ingredient List - NEW Prices!

Use the ingredient list and prices to make your chocolate and candy choices. Record your choices on your candy sheet.

Chocolate Choices
Milk chocolate: $625.50
White chocolate: $610.10
Dark chocolate: $580.25

Candy Choices
Rock candy: $205.00
Sprinkles: $180.27
Caramel: $470.25
Salted caramel: $485.50
Graham cracker: $70.25
Marshmallow: $280.10
Pecans: $304.25
Peanut butter: $190.50
Coconut: $45.75
Gummy: $220.10
Strawberry jam: $250.25
Sponge toffee: $90.75

Wrapper Price
Recycled material: $500.75
Gold foil: $815.25
Solid colour: $720.10

Question 1

You have been hired by our company to make the best new chocolate bar; the GOOBER BAR! This batch of chocolate bars must be made with the finest ingredients. In your bar, you can use as many ingredients as you want, BUT you must choose at least 1 chocolate and at least 3 candies to include in your batch. How much would your chocolate bar batch cost? Show your work!

Chocolate choice(s)	Show your work
Candy choice(s)	

Total:

Question 2

You have been hired by our company to make the best new chocolate bar; the GOOBER BAR! This batch must be made with the finest ingredients. You have a budget of **$2,500.00** and must use as much money as possible without overspending, and you must use at least 3 candies and 1 chocolate. Record your choices and show your work below!

Chocolate choice(s)	Show your work
Candy choice(s)	

Total:

Extension 1

Good news! Your GOOBER BARS are a hit! The company has given you a discount on your candy but you must now pay a ‘wrapper fee’. Use the new price list to create a new Goober Bar using your previous budget of **$2,500.00**. You must use at least 1 chocolate, at least 4 candies, and the wrapper without overspending. Record your choices and show your work below!

Chocolate choice(s)	Show your work
Candy choice(s)	

Total:

Extension 2

Your GOOBER BARS are selling like crazy! You need to keep increasing your sales. Marketing your bar will double your sales. Create a drawing of your GOOBER BAR below to include on a poster. Draw your GOOBER BAR, then compare your drawing to another person who has also drawn their bar.

Carys Both, Madison Janzen, Tristin Bessant and Arielle Hess are preservice teachers at Medicine Hat College.

Hidden Mathematics

Duncan K Melville

Where does the historian of mathematics turn for sources? There is the published record, where authors typically present a definitive view of their research at that stage of their intellectual journey. Such public documents allow for tracing both individual and social developments of mathematics. But we see only what the authors wish to show the public.

Behind the published works may be an archive or *Nachlass*. Here the historian can find drafts, diaries, notebooks, scribbled calculations, correspondence: the unpublished artefacts that may enable a finer, more detailed understanding of the author's intellectual journey, the false starts, the explorations before the public unveiling of the final product. But then there is the question of curation. We see what was chosen to be retained by the principals or their executors.

What does the historian seek? If mathematics is viewed as part of intellectual history, then the historian seeks novelty. What new ideas, theorems, definitions did the subject produce? What is new must then be explained both internally and in terms of its context and significance. Those who did not contribute significantly to the development of mathematics in this narrative become marginalized.

Significance depends on what the historian seeks. If we aim to understand the place of mathematics in a culture, perhaps we should study mathematics education. Who learned mathematics, what did they learn, how did they learn it, where did they learn it and who was left out? In this case, much of the published record is in the form of textbooks, although these must be evaluated judiciously. Not every student studies, let alone retains, all the material in a text. Much of the history of mathematics education focuses on institutions, since it is the schools, universities and academies that record and preserve their records. Studies of an individual's education are more difficult to accomplish and rely on the random survival of the kinds of ephemera most people (or their executors) discard. At a deeper level is the question of what the mathematics they learned actually meant to people—how did they perceive and respond to it?

These pathways to history assume the historian has a research agenda. Armed with questions, the historian turns to particular kinds of resources to find answers, or at least a fuller understanding and maybe better questions. Often, however, archives contain hidden gems and sometimes they throw up something completely unexpected: surprise source material that provokes its own questions.

I have had this experience myself. Rummaging around in the catalogue of the National Archives at Kew one day, I stumbled across an entry for a treatise on mathematics from the early 19th century by a female author I had never heard of. I yield to no one in my ignorance, but further research revealed that she was completely unknown to the history of mathematics. No questions we had asked and no answers we had sought had ever unearthed her. Such finds are archival gold. I was intrigued. Who was this woman? Why did she care about mathematics? What had she done? How had it lain undiscovered for two hundred years?

Please allow me to introduce Rachael Frances Antonina Lee (née Dashwood) (1774–1829) and her mathematics. She was an intelligent, forceful and somewhat eccentric woman, who unfortunately attracted notoriety. RFA, as we shall call her, was the illegitimate daughter of Francis Dashwood, Baron Le Despencer, one of the most illustrious rakes in 18th-century England, a period that had no shortage of claimants to the title. At various times he was friend of the Prince of Wales, Chancellor of the Exchequer, devotee of the arts, and reviser of the Book of Common Prayer with Benjamin Franklin. He died when his daughter was six.

Dashwood left his children, RFA and her older brother Francis, well provided for. Her mother remarried, and the children were packed off to school, Francis to Eton, and RFA to an upscale convent, the Abbaye Royale de Panthemont in Paris, until its closure in 1789. Presumably, she was mostly educated in the accomplishments suitable to her class.

Figure 1. *Rachael Frances Antonina Lee, sculpted by John Bacon the Younger, c. 1820. The Metropolitan Museum of Art.*

Back in England, the teenage RFA attempted relationships with several young men, but these suitors were rejected by her mother as insufficiently eligible. Eventually, she eloped to Scotland with Matthew Lee, whose only redeeming feature was his extreme good looks. The marriage was a disaster, and the couple soon separated, although it took the lawyers two years to sort out the financial arrangements. RFA was now in a socially anomalous position and her circle contracted. Worse was to come. Ten years later, she was at the centre of a sensational abduction and rape case. The tabloid press of the era recounted every salacious detail with their customary disregard for veracity. The subsequent trial of her abductor collapsed on a technicality and RFA's reputation was in ruins. She rarely appeared in public again.

RFA thenceforth lived a secluded and peripatetic life. She studied, read and wrote incessantly. Her main interests seem to have been theology and philology, especially of ancient languages. The high spot was the publication of the first edition of her Essay on Government in 1808, shortly after her estranged husband committed suicide. At around the age of 40, she turned her pen to mathematics.

Over a period of some ten years, RFA produced three drafts of a proposed *Course of Mathematics*, an unfinished and unpublished textbook of some 300 pages that covered the standard curriculum of school mathematics. Topics included the geometry of lines and circles, select extracts from Euclid, arithmetic, algebra, fractions both common and decimal, proportions, the rule of three and the extraction of roots. RFA disclaimed much originality in the basic mathematical content. As she wrote in her preface, "In the following *Course of Mathematics*, many of the Laws and principles above defined previously discovered and established must necessarily be introduced. This is the Case with all Works of a similar kind." She did, however, argue that the arrangement and presentation of the results was "the result of profound meditation" and derived from "a deep investigation into the properties of Numbers."

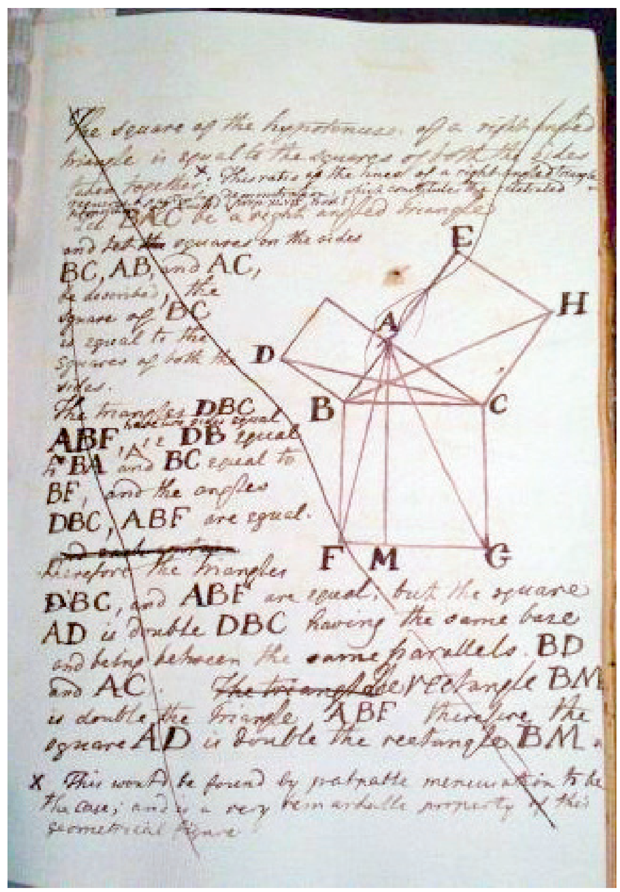


Figure 2. *A page from RFA's first draft. Provided by the author.*

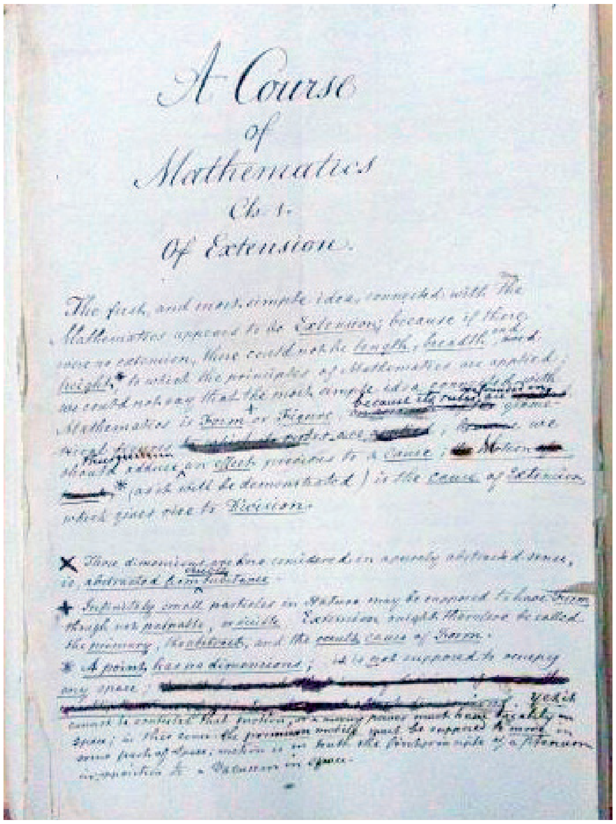


FIGURE 3. *The title page of FRA's second draft. Provided by the author.*

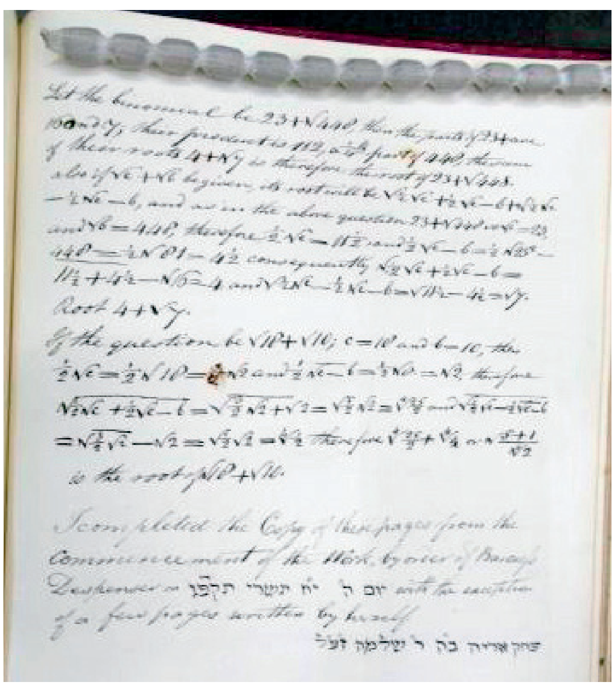


FIGURE 4. *A page from the fair copy third draft, including a note by the copyist with date and signature in Hebrew. Provided by the author.*

It is in the arrangement and presentation that RFA displays her originality in her engagement with mathematics and her mathematical philosophy or metaphysics. Declaring that mathematics begins with the notion of extension and motion of a point, she placed geometry before arithmetic, "The first and most simple idea connected with the mathematics appears to be Extension, because if there were no extension, there could not be length, breadth, and height." Numbers require division, and division presupposes the existence of something to divide; therefore, extension is the more primitive concept: "Extension may therefore rationally be denominated the primary and most simple idea connected with mathematics; Extension is produced from Motion from which lines are generated."

For RFA, mathematics was grounded in a physical, Newtonian world; therefore, "motion has a tendency to be circular" due to the actions of a projectile force and perpendicular force of gravity. Thus, from a consideration of the natural world, we have lines, right angles, diagonals and circles.

What of numbers? Taking the Biblical stance that "All the mysteries of Nature are founded on the principle that every thing was created in number, weight, and measure," RFA began with the notion of unity: "The idea of Unity is the first which arises in tracing the origin of number." Given addition, "It is evident that all numbers, from Unity, to the most complicated are in reality produced by the addition of Unity, to which they are as one, because it is the source of them." Apart from this mechanical production, numbers, at least the first few, have metaphysical properties: "The Dual or the Number z is the principle of Creation in Substance; the number 3 is produced from the Unity and Dual; this is the mystical Triune which not only the Nazarenes, but also many of the ancient Philosophers particular among the Easterns, acknowledged; no other number can, in a literal metaphysical sense exist per se." Numbers beyond three are largely arbitrary.

What were her sources? Whence came her mathematics and her sense of the overall shape and significance of it? We do not know. When she wanted to learn Hebrew, she hired a tutor. Mathematics sprang forth fully formed. There is no record of any mathematics education, either institutional or with a private tutor. Her recent biographer makes but a passing mention of mathematics, implying she studied alone.

She could have learned the mathematical content on her own from books. For instance, by 1820 she possessed a copy of Charles Hutton's *Course of Mathematics*, which was first published in 1798 and quickly went through numerous revisions, but it is not clear which edition she had nor when she acquired it. Her style and presentation are distinctive, and clearly in her own voice (she was addicted to footnotes).

The work was never finished and never published. We know of it only through a singular accident of history. She died alone, suddenly and unexpectedly, in her room in the hotel in which she was staying. She had no will and no heirs, but she was wealthy. The state swooped in and claimed everything, including every scrap of paper she had. These fill 75 boxes in the National Archives at Kew, England.[1]

How unusual was she? It certainly *appears* unusual that a woman in the early years of the19th century would have sufficient education and interest in mathematics to pursue writing a 300-page manuscript not once but through three drafts. Without the actions of the state, we would have no hint of RFA's extensive engagement with mathematics. How much other mathematics lies hidden from our usual sources?

Notes

1. The The papers of Mrs Racheal Frances Antonia Lee, the self-styled Baroness le Despenser who wrote under the nom-de-plume of Philopatria (c 1774–1829). TS 11/227-302. *Treatise on mathematics*. National Archives, Kew, UK.

Reference

Powers, A M. 2018. The Female Infidel: The Vindication of Fanny Dashwood. LuLu Press.

Duncan J Melville is a historian of mathematics and the Martha E '62 and Gregg E Peterson Professor of Mathematics at St Lawrence University. Recent publications include "Computation in Early Mesopotamia," in Computations and Computing Devices in Mathematics Education Before the Advent of Electronic Calculators, eds A Volkov and V Freiman, Springer, 2018, and "John Marsh and the Curious World of Decimal Arithmetic," in Research in History and Philosophy of Mathematics, eds M Zack and D Schlimm, CSHPM 2017 Annual Meeting in Toronto, Ontario, Birkhäuser, 2018, 23–42.

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MathemAttic: An Invitation to Participate

Kseniya Garaschuk, Shawn Godin and John Mcloughlin

Crux Mathematicorum is an internationally well-known problem-solving journal. Each issue contains original problems for readers to solve, as well as solutions from readers to past problems. It also features other notes of interest to problem solvers, including articles, regular columns, and collections of problems from Mathematical Olympiads. What many people might not know is that *Crux* started as a newsletter aimed at high school teachers in Ottawa. The level of problems in earlier volumes made the content broadly accessible as compared to those of recent years, including current volumes. Over time, the journal evolved into what it is today.

Mathematical Mayhem was a problem-solving journal for students, by students. It was started by Ravi Vakil and Patrick Surry, who had participated in the International Mathematical Olympiad, with an intention of being a journal specifically aimed at that group of people who may be prospective Olympiad participants. The journal ran for eight years before losing its funding. It was *saved* by *Crux Mathematicorum* and continued as a section of *Crux* for several years.

However, by this time, the level of problems appearing in *Mathematical Mayhem* had become increasingly difficult. Earlier scaffolding of problems or a variety of levels had seemingly been replaced by more of an Olympiad flavour. A conscious effort was made to broaden the scope of problem offerings in an effort to reach more secondary level students and teachers. Features such as *Polya's Paragon* and *Problem of the Month* were introduced so as to appeal to a wider audience. At one point in time, there was an attempt to separate *Mayhem* from *Crux* and have it continue online as a free publication. Unfortunately, this did not work out and *Mayhem* was discontinued. Eventually, *Crux* stopped being a subscription-based print journal and took its current form as a free online publication with the link below. The current issue is readily accessible as is a complete digital archive of the entire collection of the journal.

https://cms.math.ca/publications/crux

The online access offers this journal to a much wider audience than the subscribers who were regular readers. It marked a new beginning of another sort as some of us undertook an initiative to again reach a wider audience. This marked the birth of *MathemAttic*. The remainder of this article is intended to make the CMS community more aware of this part of the journal. People are encouraged to share this piece with teachers, students and others who may be interested in such a freely available publication. We welcome more problem proposers, solvers and readers, along with contributors of articles and more. The invitational spirit carries through the subsequent paragraphs. Please get in touch with us with feedback and indications of interest.

MathemAttic, like the latter versions of *Mathematical Mayhem*, is meant to appeal to a range of pre-university students and their teachers. Unlike *Mathematical Mayhem*, it doesn't have a history, so it can more easily be created from the ground up. As coeditors, we have been there from the outset.

Currently, *MathemAttic* has a problem section where problems, for the most part, are picked from a wide variety of sources with the occasional problem proposed by a reader. Two regular features have been there from the first year. *Problem Solving Vignettes* explore interesting problems and their solutions, as well as looking at techniques and ideas that would benefit high school problem solvers. *Teaching Problems* focuses attention on problems that have been used in teaching with an eye to how they can develop students' appreciation and learning around mathematical problem solving. This past year, a new feature called *Explorations in Indigenous Mathematics* was introduced with consideration of mathematics pertinent to Indigenous culture.

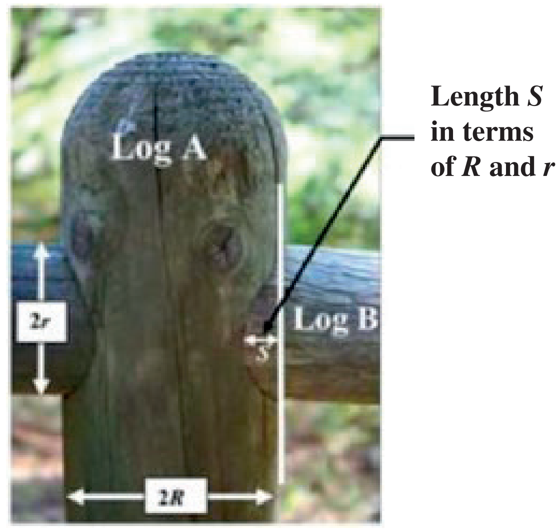
The journal continues to evolve with the newest feature appearing first in the January 2022 issue. *From the Bookshelf* will highlight books that contributors recommend for inclusion in one's own personal library. The recommendations will offer insight into the selections. In some cases, the selections will be more like mini-collections such as the work of a particular author. Longtime *Crux* readers will know that book reviews were staples in the journal for years. This feature will allow for reviews of titles that may interest pre-university students and teachers. This will open the avenue for publishers to send materials along for review. We are also developing a feature that will highlight resources on the internet, such as articles, videos, podcasts or apps of relevance to our target audience.

MathemAttic will evolve further in the coming years with the input and contributions of people interested in the spirit of mathematical problem solving, outreach and the enhancement of public appreciation of mathematics. The next step in the growth process is the development of a community of people who will act as an advisory board. We are looking to have people from across Canada engage with us in a variety of roles. The need is there for some people to edit the problem submissions, others to assist with editorial roles concerning articles, and generally a collection of people to act as a sounding board for the directions of *MathemAttic*. If you see a way that you would like to contribute, please send us a note (mathemattic@cms.math.ca) as we would welcome hearing from you. The support of the CMS community is appreciated as are efforts to circulate this notice.

Before closing, we share a couple of problems from the March issue. Solutions are welcomed from secondary level students. These can be submitted prior to the end of May through the following link: https://publications.cms.math.ca/cruxbox.

MA162. From an 8×8 chessboard, the central 2×2 block rises up to form a barrier. Queens cannot be placed on the barrier and may not attack one another across this barrier. Determine the maximal number of Queens which can be placed on the chessboard so that no two attack each other.

MA 163. In the diagram the log A has radius R. A hole of radius r is drilled through the centre of log A at right angles to the axis. Another log B of radius r passes through the hole. Find the length S in terms of R and r.



This article was originally published in CMS Notes, Volume 54, Number 2, March 2022 and is reprinted with the permission of the Canadian Mathematical Society. Minor changes have been made in accordance with ATA style.

Nerdles: The Daily Numbers Games

Nerdle, a math fact equation puzzle, began in January 2022. Since then, many teachers have challenged their students to solve this daily puzzle. The objective is to create the daily math fact equation in six guesses or less. Clues are given as to a correct number or operation in the right spot (in green) or the correct number or operation but in a wrong spot in the equation (in red).

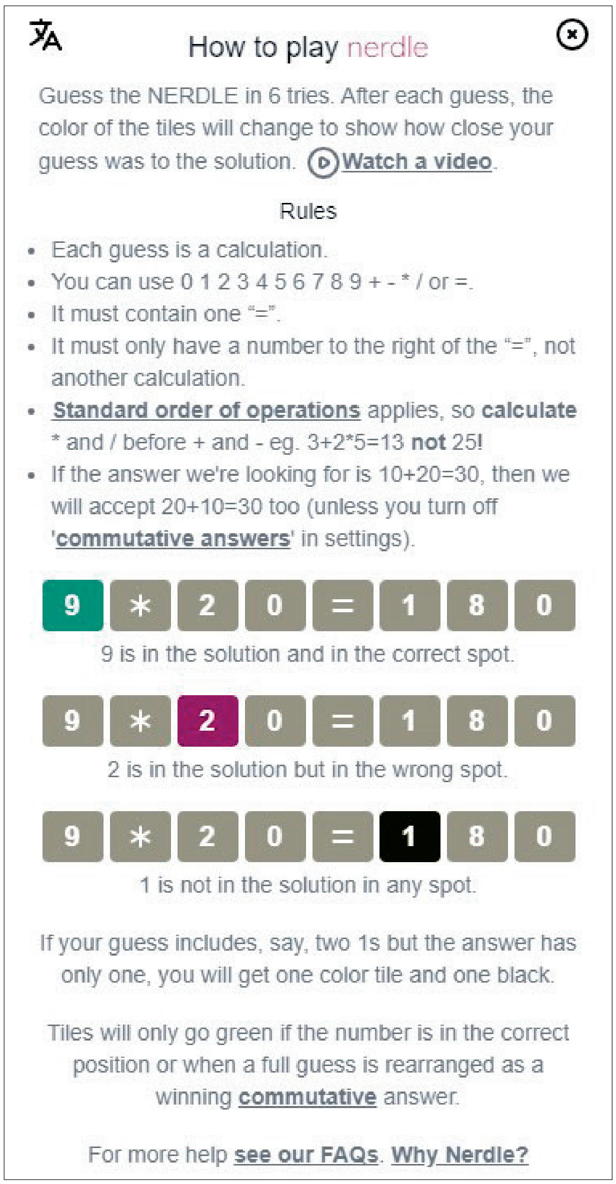


FIGURE 1. *Instructions on how to play the nerdle*

Nerdles now come in five sizes and difficulties: micro-nerdle, mini-nerdle, nerdle, maxi-nerdle and cross-nerdle. The varying levels of difficulty give students aged six and up an opportunity to use the clues to solve the math fact equation. The micro-nerdle, the mini-nerdle, and the nerdle have five, six and eight boxes to fill, respectively. The maxi-nerdle has eight boxes to fill with additional math operation buttons to use. The cross-nerdles get harder from Monday to Friday of each week. The puzzles reset and everyone gets the same puzzle each day based on the nerdle size and type, so be careful of any spoilers your students may inadvertently give out! As a bonus, the nerdles follow the correct order of operations.

For students who want an added challenge, there is also a bi-nerdle and a bi-mini-nerdle: these puzzles have two math fact equations that are solved simultaneously with clues given each time for both of them.

Students will increase their math facts as well as their reasoning abilities as they solve the nerdles. They need to take into account all the clues and use their knowledge of math operations to solve the puzzles.

Micro-nerdle (five boxes):
https://micro.nerdlegame.com

Mini-nerdle (six boxes):
https://mini.nerdlegame.com

Mini-nerdle × 2 (bi-mini-nerdle):
https://mini.bi.nerdlegame.com

Nerdle (eight boxes): https://nerdlegame.com

Nerdle × 2 (bi-nerdle):
https://bi.nerdlegame.com

Maxi-nerdle (eight boxes and more operations): https://maxi.nerdlegame.com

Cross-nerdle (nerdle in a crossword) is new as of May 2023:
https://nerdlegame.com/crossnerdle

Graham Fletcher's Three Act Tasks

Graham Fletcher has created an online *three act task file cabinet* that is free to access. The math tasks are for Grades K–7 and take only a few minutes to complete. Each requires math thinking and processing skills based around a variety of stated big ideas. Reasoning, deductive thinking, identifying information, utilizing math content knowledge, estimating, calculating and coming to a conclusion are all skills that will be developed for students through these math tasks.

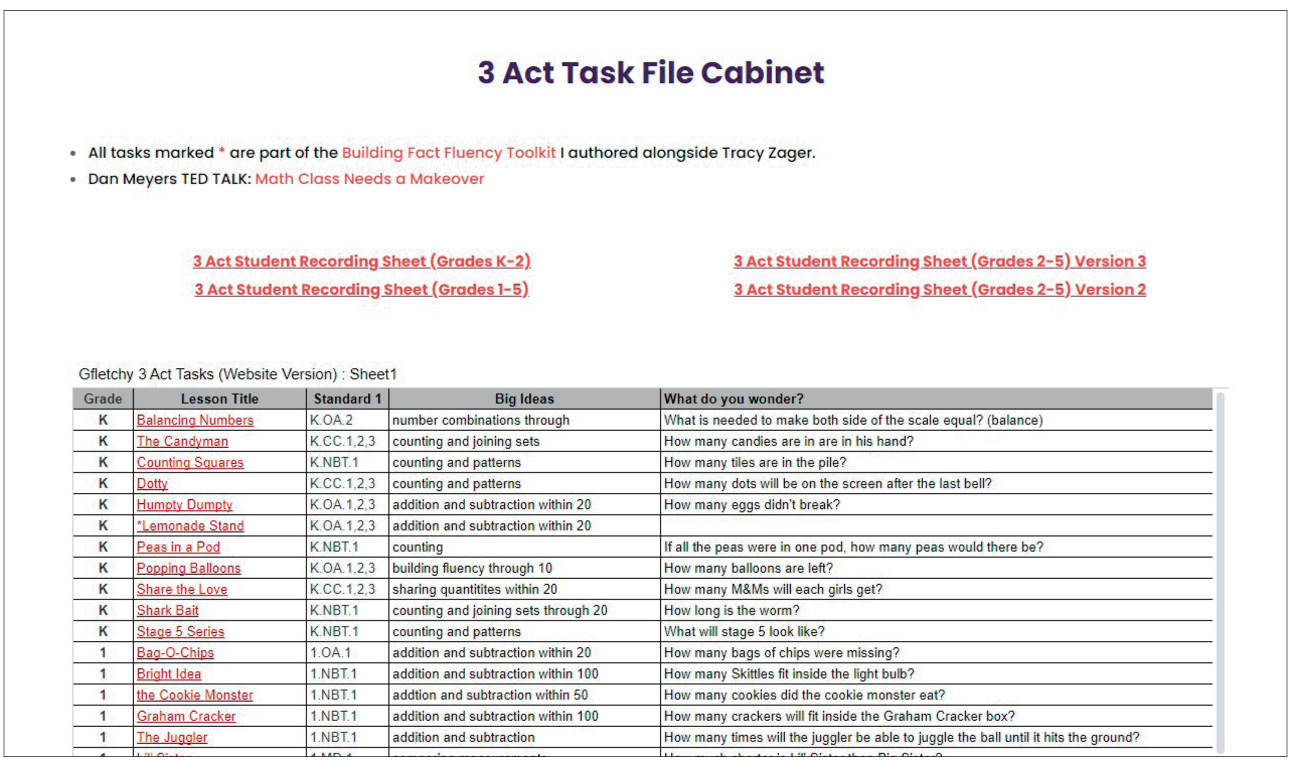
The three acts are three short videos about 20 seconds to a minute each, with math questions for students to contemplate. Ideally, the three acts/videos draw students from an initial estimate or initial thinking to a stronger conclusion as the videos provide more clarification each time.

Math talks are ideal with these tasks as the given guiding questions lend themselves well to showing and sharing students' thinking. Basic recording sheets are also provided to guide students' work with questions such as

- what did you notice?
- what did you wonder?
- identify the main question;
- make an estimate;
- what information do you need?
- show your thinking.

For older students, the guiding questions can be more complex:

- Construct a viable argument or share a reflection;
- Create a word problem that matches the task.
- The three act tasks are ready-to-use for teachers as part of a weekly math routine, math questions around a specific concept or even small group or centre use. The three act task file cabinet is available online at https://gfletchy.com/3-act-lessons



3 Act Task File Cabinet

- All tasks marked * are part of the Building Fact Fluency Toolkit I authored alongside Tracy Zager.
- Dan Meyers TED TALK: Math Class Needs a Makeover

3 Act Student Recording Sheet (Grades K-2)
3 Act Student Recording Sheet (Grades 1-5)
3 Act Student Recording Sheet (Grades 2-5) Version 3
3 Act Student Recording Sheet (Grades 2-5) Version 2

Gfletchy 3 Act Tasks (Website Version) : Sheet1

Grade	Lesson Title	Standard 1	Big Ideas	What do you wonder?
K	Balancing Numbers	K.OA.2	number combinations through	What is needed to make both side of the scale equal? (balance)
K	The Candyman	K.CC.1,2,3	counting and joining sets	How many candies are in are in his hand?
K	Counting Squares	K.NBT.1	counting and patterns	How many tiles are in the pile?
K	Dotty	K.CC.1,2,3	counting and patterns	How many dots will be on the screen after the last bell?
K	Humpty Dumpty	K.OA.1,2,3	addition and subtraction within 20	How many eggs didn't break?
K	*Lemonade Stand	K.OA.1,2,3	addition and subtraction within 20	
K	Peas in a Pod	K.NBT.1	counting	If all the peas were in one pod, how many peas would there be?
K	Popping Balloons	K.OA.1,2,3	building fluency through 10	How many balloons are left?
K	Share the Love	K.CC.1,2,3	sharing quantitites within 20	How many M&Ms will each girls get?
K	Shark Bait	K.NBT.1	counting and joining sets through 20	How long is the worm?
K	Stage 5 Series	K.NBT.1	counting and patterns	What will stage 5 look like?
1	Bag-O-Chips	1.OA.1	addition and subtraction within 20	How many bags of chips were missing?
1	Bright Idea	1.NBT.1	addition and subtraction within 100	How many Skittles fit inside the light bulb?
1	the Cookie Monster	1.NBT.1	addtion and subtraction within 50	How many cookies did the cookie monster eat?
1	Graham Cracker	1.NBT.1	addition and subtraction within 100	How many crackers will fit inside the Graham Cracker box?
1	The Juggler	1.NBT.1	addition and subtraction	How many times will the juggler be able to juggle the ball until it hits the ground?

Figure 1. *The image shows the three act task file cabinet, which includes a list of lessons.*

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