delta-k

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Volume 57, Number 1 December 2021

Appreciating Perspectives

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom or a scholarly focus. They may include

- letters to the editor;
- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the National Council of Teachers of Mathematics (NCTM).

Suggestions for Writers

- *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
- If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
- Peer-reviewed articles are normally 8–10 pages in length.
- All manuscripts should be submitted electronically (in Microsoft Word format) and double-spaced. All pages should be numbered.
- The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identity to the reviewers.
- Pictures or illustrations should be submitted as separate files (such as JPEG or GIF) and clearly labelled. Their placement should be indicated in the text of the article. A caption and photo credit should accompany each.
- All sources should be properly referenced. Entries in the reference list and in-text citations should be formatted consistently, using the author–date system.
- If any student work is included, please provide a signed consent form from the student's parent/guardian allowing publication in the journal. The editor will provide this form on request.
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MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-k

Volume 57, Number 1 December 2021

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From the Acting Editor's Desk

Barbara O'Connor

Appreciating Perspectives

Each piece in this issue of *delta-K* invites readers to reflect on their own mathematical experiences with curriculum and pedagogy. Each author's unique perspective draws us into seeing more and viewing ideas from different angles, and invites us to engage in metacognitive practices for ourselves and with our students.

Each piece is grounded in a particular aspect of curriculum, but that is not the focus. Rather, what is most enticing is the subtle attention to language, gesture and experience; the invitation to attend to underlying metaphors; and the emotional involvement with mathematics. The authors share opportunities that support collaboration, exploration and frustration in doing mathematics. Most important, they explore building understanding and connections to mathematics, both for teachers' own connections to the subject and for teachers' confidence in their pedagogical choices. We are invited to create mathematical conceptualizations as genetic, which, according to Piaget, is to continually construct and reorganize our understanding and connections, as Nicole Langevin, Miwa A Takeuchi, Jenny Yuen and Shayla Jaques discuss in "Enriching Geometric Understanding Through Early STEM Pedagogy."

Taking a historical perspective, four articles by preservice teachers feature the enduring impact of female mathematicians past and present—Sophie Germain, Eugenia Cheng, Emmy Noether and Mary Everest Boole. The authors document the journeys and legacies of these gifted and tenacious women, as well as their influence even today.

In a similar pedagogical vein, in "Supporting Preservice Elementary Teachers' Growth Through Studying the Historical Development of a Mathematics Topic," Giang-Nguyen T Nguyen and Tiffany Marlow share how students' research on the historical development of a mathematics topic contextualized their understanding of the topic's importance.

Two articles delve into practices that use emotional connections to build students' confidence and interest in mathematics. In "How Do You Feel? Using Scribblers in the Math Classroom to Elicit Mathematical and Personal Connections," Josh Markle and Raeesa Shivji explore metacognitive reflection in the mathematics classroom through the use of scribblers. In "Culture—a Vital Part of Mathematics," Yvette d'Entremont and Michelle Voillot highlight how a cultural practice can lend focus to multiple mathematical concepts. These articles add to teachers' knowledge of the impact of using personal experiences to build multiple ways of knowing and using mathematics.

Tapping into metacognitive thinking is also evident in Shimeng Liu's exploration of the metaphors underlying Alberta's elementary mathematics curriculum, as well as in the cyclical phases to support building understanding of using a problem-solving approach in instruction that was used in the professional development series described by Richelle Marynowski, Shelly Wismath, Verna Mabin, Michelle Campmans, Lynn Suttie and Alana Millard. Reflective practice also plays a role in Jérôme Proulx's piece about coming to understand students' perspectives in producing equivalent fractions and in the article by Langevin et al about how teachers can support the development of a more robust understanding of geometric properties by highlighting language, gesture and technology in challenging collaborative projects.

All these pieces remind me how valuable it is for teachers to share their perspectives on teaching and learning with each other. This is especially true during these times, as we acknowledge how important personal connections are in both our relationships with each other and in mathematics, and how we can use this journal as a forum to embrace others' perspectives. The pieces invite us to continue returning to construct and reorganize our understanding of ourselves as learners and of the mathematical topics we explore.

As the acting editor of this issue of *delta-K*, I have had the immense privilege of engaging with these authors. Thank you for bringing me into discussions on how your pieces came to fruition and the story you wanted to articulate for the journal's readership. I, too, benefited from this exercise in editing, and I thank you all for sharing your experiences. Let us encourage each other to continue telling our stories of engaging daily in mathematics, both inside and outside the classroom.

One Teacher's Perspective on the Proposed K–6 Mathematics Curriculum

Darcy House

The following letter was sent to Minister of Education Adriana LaGrange on March 30, 2021. Minor changes have been made to fit ATA style.

Dear Minister LaGrange,

I am a secondary mathematics teacher, concerned parent to a young child, volunteer on the Mathematics Council of the Alberta Teachers' Association (MCATA), graduate student at the University of Alberta and member of the Métis Nation of Alberta.

I could not have been more disappointed upon reading through the proposed elementary mathematics curriculum. This new curriculum disservices Alberta. It disservices my child and those children of my fellow Albertans.

There were 14 references to First Nations, Métis and Inuit knowledges within these draft documents, 11 of which appeared in the K–3 section of the documents. None of the references appeared in Grade 6, nor did any of these references provide valuable insight into the relational nature of Indigenous world views. This mathematics curriculum quite literally phases out Indigenous Peoples by the time students reach Grade 6.

Much more concerning was the complete lack of educational form within the documents. There is little to suggest that the authors have spent any time thoughtfully addressing the valuable insights that Indigenous knowledge might provide educators as we strive to improve educational outcomes not only for Indigenous students but for all students in Alberta.

While these are huge shortcomings, my greatest concern is the framing of Indigenous knowledges as those of otherness throughout the draft document. This process, which I have come to recognize by its similarity to hydrographic art, disguises colonial knowledge as Indigenous knowledge and displaces Knowledge Holders and Elders from places of respect, history, story and relation within the classroom. I have a couple examples to share with you.

In Grade 1, the authors of this document suggest that "First Nations, Métis, and Inuit experience time through sequences and cycles in nature, including cycles of seasons and stars." But there is no mention of our responsibilities to care for one another as we each experience the (often violent) cycles of earth we are familiar with in Alberta, nor our responsibility to care for those around us when they are experiencing more difficult cycles than us.

In Grade 2, the authors reduce a valuable cultural learning opportunity to "relate First Nations' winter counts to duration." Whereas *duration* refers to a specific length of time and is often used to erase humanity in between, the cultural practice of winter counts embeds memory inside place and symbol and reminds us of the stories that are lived.

In Grade 4, students are required to "recognize the rearrangement of area in First Nations, Métis, or Inuit design." This example makes my hydrographic concerns more clear. Your curriculum developers have disguised critical mathematical discernments as those of the concerns of Indigenous design. While the curricular intention is clearly to inquire into the rearrangement of objects to cover equal area—a discernment that I use regularly in my Math 20-3 program to prove the Pythagorean theorem—the implication of this skills and procedures outcome is that this discernment is of primary concern to Indigenous artists as they produce artwork.

In Grade 5, a knowledge outcome is stated as "symmetry can be found in First Nations, Métis, and Inuit design." This demonstrates the otherness of Indigenous knowledge within this document; it isn't even hidden away. A concept from elsewhere, symmetry, is painted onto the design of Indigenous artists and coerces students to understand the art through a particular lens, with no attempt by your curricular authors to address our relations through the artist, nor through the story they tell. They assume the artist's intention is symmetry. The concept of symmetry is not enough for these authors to have fully articulated what has been created.

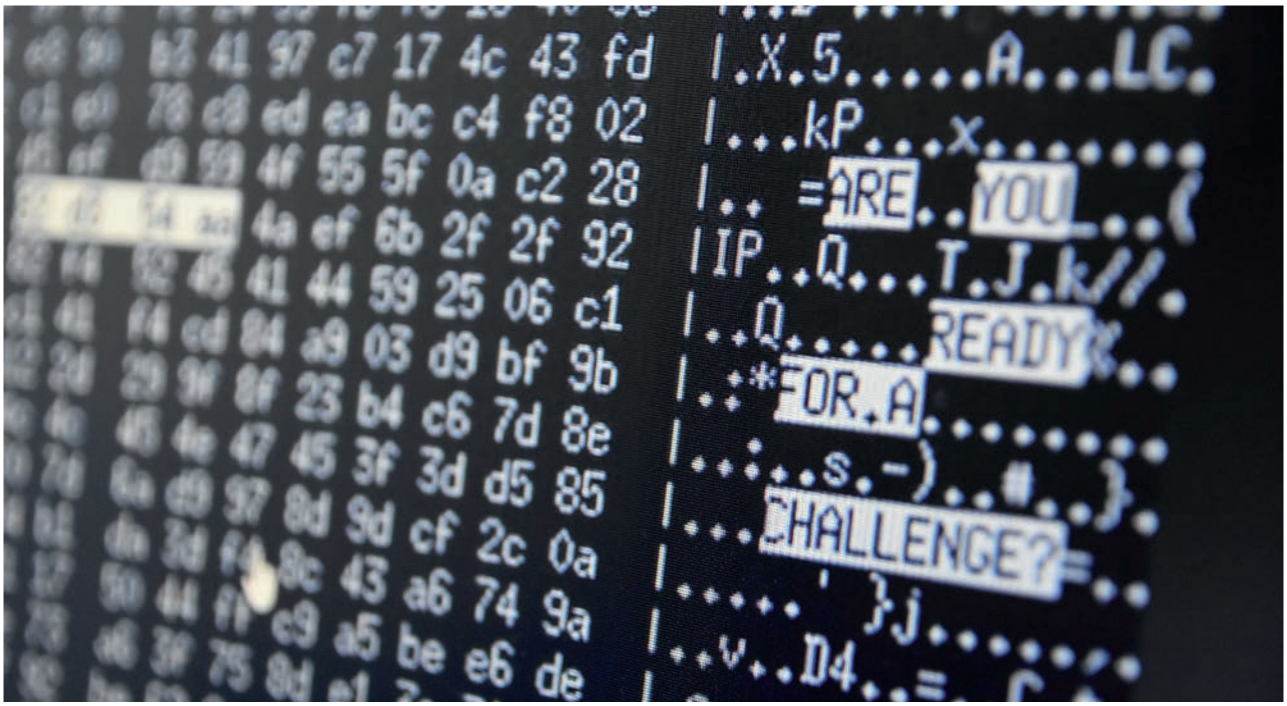
I am concerned that your curriculum might further widen the educational debt owed to Indigenous Peoples. It is clear to me that little attention has been paid to the relations this curriculum will disrupt. I am afraid that, in its current form, this curriculum will continue to miseducate young Albertans. I do hope that your finalized version is not quite so disappointing.

Thank you for your time and consideration.

Sincerely,
Darcy House

Darcy House, BSc, BEd, is a member of the Métis Nation of Alberta, a mathematics and science teacher with the Inner City Youth Development Association in Edmonton, and MCATA's treasurer. Visit his website at https://darcyhouse.ca.

Celebrating Mathematics 2020 Learning Challenge



Renert School, a private school in Calgary, has shared its Celebrating Mathematics 2020 learning challenge online. The challenge was so successful that the school made a version for anyone to play.

The challenge is available at https://renertmath.github.io/RenertMath-CelebrateMath/. See how far you can go in this increasingly difficult 42-part math maze. The challenge has been designed to cater to everyone from Grade 1 students to research-level mathematicians. Although it has been shared among many mathematics professionals, including professors at top universities, the highest level completed thus far is 39.

Special recognition goes to Renert School's math department:

- Editors and challenge engineers—Vincent Chan and Merrick Fanning
- Problem setters—Marina Barreto, Vincent Chan, Doina Corobea, Merrick Fanning, Shahin Jabbari, Vlad Rekhson and Aaron Renert

Producing Equivalent Fractions "Additively": The Case of the 1/2

Jérôme Proulx

In this article, I explore a relationship raised by Grade 5 students between equivalent halves, which can lead to a method for producing equivalent fractions "additively." I share an account of the classroom events that led to this "adding" method and outline some mathematical explorations to understand the method.[1]

During the Lesson: A Relationship Between Halves

I was once in a Grade 5 classroom, working with students on fractions. At one point in the lesson, an equivalence was traced between the fractions 3/6 and 1/2. When I asked the students if they knew of other fractions equivalent to those two, some mentioned having a method for generating halves: "You take a number, double it and then place it below in the fraction."

Right away, almost all their small hands were raised to offer other halves—20/40, 2/4, 6/12, 12/24, 4/8. Not anticipating the enthusiasm that would follow, I began writing these fractions on the whiteboard, close to where 3/6 and 1/2 were written, but scattered around. And it went on—25/50, 7/14, 24/48, 5/10, 250/500, 9/18, 83/166, 39/78, 84/168, 8/16. At this point, the whiteboard looked like Figure 1.

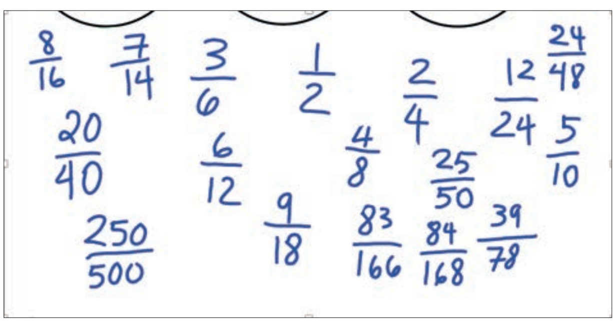


Figure 1. *Whiteboard filled with halves suggested by students.*

As I was about to ask them how these fractions were all equivalent and how they represented halves, a student at the front raised his hand. He had noticed something: "If you look at both 83/166 and 84/168 that Marco gave,[2] they are the same. He only added 1 above and then 2 below."

I thought that this was very interesting, although surprising. I also thought that it probably worked only with that specific pair of fractions. So, out of curiosity, I asked the students if there were other pairs of fractions that might be conceived that way. Enthusiasm once again filled the room—like "Oh, yes! There are plenty of others!"—and many hands went up.

"From 1/2 to 2/4," said one student, which I repeated slowly to make sure that it really was the same pattern of +1/+2. I was still a little puzzled by this. Another student said, "From 7/14 to 8/16." I began to circle these pairs in green to set them apart. However, I was still thinking that only a few would follow this pattern.

Then, another student pointed out that in moving from 12/24 to 24/48, 12 and 12 were added at the numerator and 24 and 24 were added at the denominator. I circled this pair in red, since it was produced using a different "addition."

Another student asserted, "24/48 and 25/50," as I linked them in green. A student in the back added, "6/12 and 7/14," and another continued, "3/6 and 4/8, as well." Now it was becoming increasingly obvious that the pattern worked all the time and with all halves—but I could not clearly understand why.

One student returned to the doubling–adding relationship between 12/24 and 24/48 to link 2/4 and 4/8, with 2 and 2 added at the numerator and 4 and 4 added at the denominator. I drew a red arrow between those fractions. By then the whiteboard was a little messy. We continued, with 5/10 and 6/12, 3/6 and 6/12 (3 + 3 and 6 + 6), and 8/16 and 9/18. The board then looked like Figure 2, with green and red links all over, representing well my unpreparedness for the discovery of this kind of relationship between all the equivalent halves.

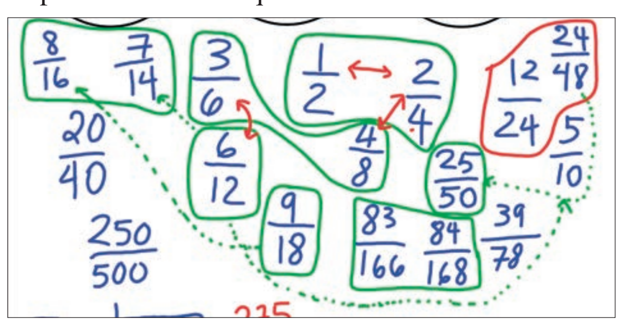


Figure 2. *Whiteboard linking halves by "additive" relationships.*

Somehow, in addition to the method that the students had initially mentioned (taking a number and doubling it), this +1/+2 relationship appeared to be another method for producing equivalent halves. I was well aware that this relationship would stand for ratios in proportional relationships (as I will show). However, in the context of equivalent fractions, as a relationship between or a process for producing equivalent fractions, these "additions" somehow troubled me.

First, on the spur of the moment, with my face close to the whiteboard, I could not relate this method well to the method we usually use to establish multiplicatively equivalent fractions. For example, for 1/2, we take a unit, split it in two and then take one of those two parts. Thus, 2/4 is said to be equivalent to 1/2, because for the same unit now split in four, our parts are twice as small, so we need to take twice as much to maintain the equivalence between fractions. Or if we take twice as many of these twice-as-small parts, we obtain an equivalent fraction.

Second, adding the numerators and the denominators of fractions is a minefield. A common misconception students have is that one can "add tops and bottoms together"—for example, 1/2 + 1/3 = 2/5 and 2/7 + 3/4 = 5/11 (which are incorrect).

However, now this relationship, or "adding" method, appears to fall right in the middle of these issues, lying between ratios, multiplicative equivalence and the common misconception about how to add fractions. Thus, I can say that I was as shocked by my students' ability to find this relationship as I was by my own inability to see how it seemed to work.

Therefore, I decided to raise the question, as much for myself as for my students, of why this method worked, how it worked and whether it worked all the time. The students answered by asserting that all these halves were produced as equivalent fractions from the outset as a variety of halves, so that was simply why it worked, and that was that! Addressing my question like that led them somehow to readdress my initial question about how and why we knew that all those fractions were halves. We explored this question for the rest of the lesson, until the bell rang.

After the Lesson: Exploring This Relationship as an "Additive" Method

Although the students seemed satisfied with their explanation of halves, I still wanted to get to the bottom of it. I was eager to know more about it mathematically, even just for myself. For this, I needed to take a step back and make sense of all of this in terms of equivalent fractions. I offer here some of my mathematical reflections.

First things first. As mentioned, I am well aware that for proportional relationships, this +1/+2 relationship stands well. To illustrate, if I have a glass of orange juice made by diluting one part concentrated orange juice with two parts water, I have a ratio of 1:2 in terms of orange juice to water. If I add another part orange juice and another two parts water to the mix, that gives two parts orange juice for four parts water, a ratio of 2:4, which is equivalently sweet in terms of orange juice to water. Put otherwise, if I add another one part orange juice to the initial mix, I must add another two parts water to maintain the sweetness of the juice. Figure 3 represents this numerically and pictorially.

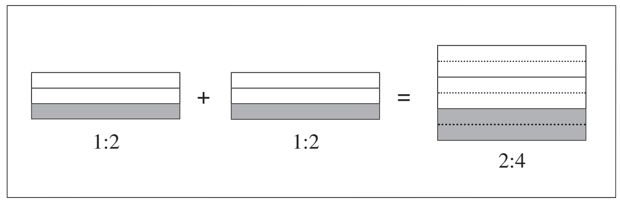


FIGURE 3. *Orange juice as an example of adding ratios.*

However, the various halves on the whiteboard were not about ratios; they were about equivalent fractions. And I wanted to make sense of this +1/+2 relationship precisely in the domain of equivalent fractions. I wanted to understand how this "adding" worked and why. Hence, the challenge!

To get my head around it, I attempted some algebra. I knew well that my Grade 5 students would not be familiar with this math, but this part was only for me, as a first attempt at the task.

Using the way the students explained it, we can represent halves as any fraction of the form $a/2a$ (where the denominator is double the numerator). Then, if we add 1 to the numerator and 2 to the denominator, we get another half. This can be represented as

$$\frac{a^{\,+1}}{2a^{\,+2}} = \frac{a+1}{2a+2} = \frac{a+1}{2(a+1)}.$$

This gives another half, where $(a + 1)$ is the number to be doubled in order to maintain the students' structure of "take a number, double it and then place it below in the fraction." Adding other numbers in the same "half" relationship, such as +3/+6, also works, giving us

$$\frac{a^{+3}}{2a^{+6}} = \frac{a+3}{2a+6} = \frac{a+3}{2(a+3)},$$

where $(a + 3)$ is the number to be doubled in the structure of halves.

This is great and seems to produce other halves by "adding" them together, because the +1/+2 and +3/+6 act as halves as well in this "addition." Thus, any half could be "added" to another half, adding numerators together and denominators together. With $a/2a$ as a half and $3a/6a$ as another one, we get

$$\frac{a^{+3a}}{2a^{+6a}} = \frac{a+3a}{2a+6a} = \frac{4a}{8a},$$

which is still in the same form. Even more generally, with $a/2a$ as a half and $b/2b$ as another half, we get

$$\frac{a^{+b}}{2a^{+2b}} = \frac{a+b}{2a+2b} = \frac{a+b}{2(a+b)}.$$

I then wondered whether this relationship, or "adding" method, would work with other kinds of fractions, and it did. For example, it works with or between thirds:

$$\frac{a^{+1}}{3a^{+3}} = \frac{a+1}{3a+3} = \frac{a+1}{3(a+1)}.$$

The same works for two-thirds (of the form $2a/3a$):

$$\frac{2a^{+2}}{3a^{+3}} = \frac{2a+2}{3a+3} = \frac{2(a+1)}{3(a+1)}.$$

This appears quite robust. It also works for any kind of fraction in which the numbers added between them would be the numerators and the denominators. For example, with two-fifths, in the form $2a/5a$ "added" with 2/5, we get

$$\frac{2a^{+2}}{5a^{+5}} = \frac{2a+2}{5a+5} = \frac{2(a+1)}{5(a+1)}.$$

It simply works all the time!

Then, in a completely generalized form to end the algebra quest, it is possible to work with the simplified fraction in an a/b form, to which another fraction a/b is "added." This gives us

$$\frac{a^{+a}}{b^{+b}} = \frac{2a}{2b},$$

which is another equivalent fraction to a/b. If this is done with ka/kb, an equivalent fraction of a/b to which $+a/+b$ is "added," we get

$$\frac{ka^{+a}}{kb^{+b}} = \frac{ka+a}{2k+b} = \frac{a(k+1)}{b(k+1)},$$

which is again an equivalent fraction to a/b and ka/kb.

Finally, with two equivalent fractions to a/b (namely, ka/kb and ha/hb), we get

$$\frac{ka^{+ha}}{kb^{+hb}} = \frac{ka+ha}{2k+hb} = \frac{a(k+h)}{b(k+h)},$$

another equivalent fraction to a/b. Somehow, algebraically speaking, this method seems to work all the time.

However, this is with algebra. Even if it seems to work, it might not be totally convincing (algebra sometimes seems magical!). Also, it is very far from the tools Grade 5 students have at their disposal. Therefore, I decided to explore the meaning of this relationship, or "adding" method, in other ways, with tools that Grade 5 students would be familiar with—words, numbers and drawings.

After the Algebra: Understanding Why This Method Produces Equivalent Fractions

Although this started as a relationship between halves, it directs one to consider it as a method for producing equivalent fractions. It does so, to some extent, by "adding" to the numerator of a fraction the numerator of an equivalent fraction, and "adding" to the denominator of a fraction the denominator of the equivalent fraction. The result is a third equivalent fraction.

Let's look at what the method looks like numerically. With 3/6 and 1/2, we get

$$\frac{3^{+1}}{6^{+2}} = \frac{3+1}{6+2} = \frac{4}{8}.$$

Using thirds, with 2/6 and 1/3, we get

$$\frac{2^{+1}}{6^{+3}} = \frac{2+1}{6+3} = \frac{3}{9}.$$

And this works with any type of fraction, as shown in the following examples.

With 1/2 = 3/6, the method gives us

$$\frac{1^{+3}}{2^{+6}} = \frac{1+3}{2+6} = \frac{4}{8},$$

which is equivalent to 1/2.

With 1/2 = 6/12, the method gives us

$$\frac{1^{+6}}{2^{+12}} = \frac{1+6}{2+12} = \frac{7}{14},$$

which is equivalent to 1/2.

With 4/8 = 3/6, the method gives us

$$\frac{4^{+3}}{8^{+6}} = \frac{4+3}{8+6} = \frac{7}{14},$$

which is equivalent to 1/2.

With 1/3 = 12/36, the method gives us

$$\frac{1^{+12}}{3^{+36}} = \frac{1+12}{3+36} = \frac{13}{39},$$

which is equivalent to 1/3.

With 2/7 = 6/21, the method gives us

$$\frac{2^{+6}}{7^{+21}} = \frac{2+6}{7+21} = \frac{8}{28},$$

which is equivalent to 2/7.

This is really interesting! As obvious as it seems now, it is important to remember that writing out the following two equalities could appear quite puzzling at first for equivalent fractions:

$$\frac{1^{+6}}{2^{+12}} = \frac{1+6}{2+12} = \frac{7}{14}$$

and

$$\frac{1^{+12}}{3^{+36}} = \frac{1+12}{3+36} = \frac{13}{39}.$$

But it does work![3]

That said, difficulties emerge when we attempt to verbalize what happens—to put this relationship, or "adding" method, into words. Usually, as mentioned, we explain the equivalence of fractions in multiplicative terms.

Let's look at the example of 1/2 and 4/8. For 1/2, we take a unit, split it in two and then take one of those parts. Then, 4/8 is equivalent to 1/2 because for the same unit now split into eight (hence, four times more), our parts are four times as small. Thus, we need to take four times as many parts in order to maintain the equivalence in fractions, as depicted in Figure 4.

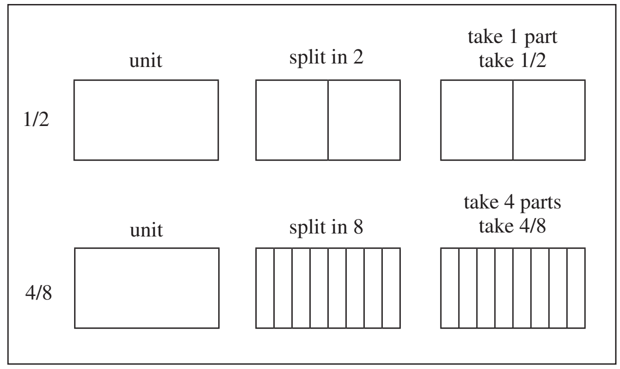


FIGURE 4. *Drawing of the equivalence of 1/2 and 4/8.*

However, putting into words this "adding" method to produce 4/8 from 1/2, as with

$$\frac{1^{+3}}{2^{+6}} = \frac{1+3}{2+6} = \frac{4}{8},$$

is a different and quite challenging matter. The multiplicative explanation of smaller/bigger parts and how much more/less must be taken to maintain the equivalence between 1/2 and 4/8 is hardly possible here. However, it is possible to explain it in relation to the size and the number of parts, as in the following:

> I have 1/2. Thus, my unit is split into two parts, of which I take one part. I then add six more parts to the unit. I now have a new unit of eight parts. If I take an additional three parts out of all those parts of the new unit, I now have four parts out of eight, giving the equivalent fraction 4/8. Or, because I already took one part of the unit, I need to take an additional three parts out of all those parts of the new unit, which represents half of the six parts that I have just added, to maintain the equivalence. I now have taken a total of four parts out of the eight parts of the new unit—hence, 4/8.

Here, the equivalency is not talked about in multiplicative terms of how much smaller or bigger the parts become but, rather, more in terms of the number of added parts and their size. Whereas one is a multiplicative method for producing equivalent fractions, this one is "additive."

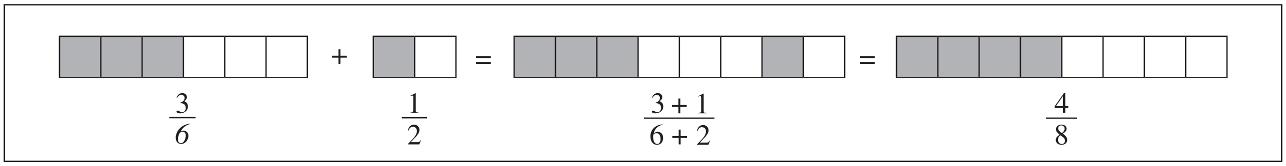


FIGURE 5. *Drawing of the equivalence between 1/2 and 4/8 "additively."*

Contrasting Units and Parts: From Equivalent Fractions to Equivalent Partitioning

The above verbalization of this relationship, or "adding" method, raises an issue concerning the unit, which is transformed in the process of adding parts to it. This transformation of the unit does not happen in the usual multiplicative explanation of equivalency, where the unit stays the same and is cut into more or fewer parts, as we saw in Figure 4.

In terms of drawing, Figure 4 also does not help much for representing the relationship, or "adding" method, as the insertion of the +3/+6 in the explanation does not explicitly appear in the drawing. In the usual multiplicative equivalency, as Figure 4 illustrates, the unit stays the same, so it is the parts that change in size. In the "adding" method, it is to some extent the opposite: the parts stay the same size, but the unit is changed. This "adding" method requires, in that sense, that we let go of the maintenance of the same unit and focus instead on the maintenance of the same size of parts, as illustrated in Figure 5.

The main difference lies in the fact that the parts are here kept equal, whereas the unit changes. By changing the size of the unit while maintaining the size of the parts, the partitioning is also maintained, as it is always in terms of halves. In that sense, this "adding" method can be seen to be more about equivalent partitioning than about equivalent fractions. In other words, it is about the conservation of the partitioning, not the conservation of the unit.

Going way back to the student's example of going from 83/166 to 84/168, this is explicable numerically as follows:

$$\frac{83^{+1}}{166^{+2}} = \frac{83+1}{166+2} = \frac{84}{168}\ .$$

Figure 6 offers a way of illustrating this, where the +1/+2 is represented by parts of the same size as 83/166, so it maintains the partitioning but produces a new unit.

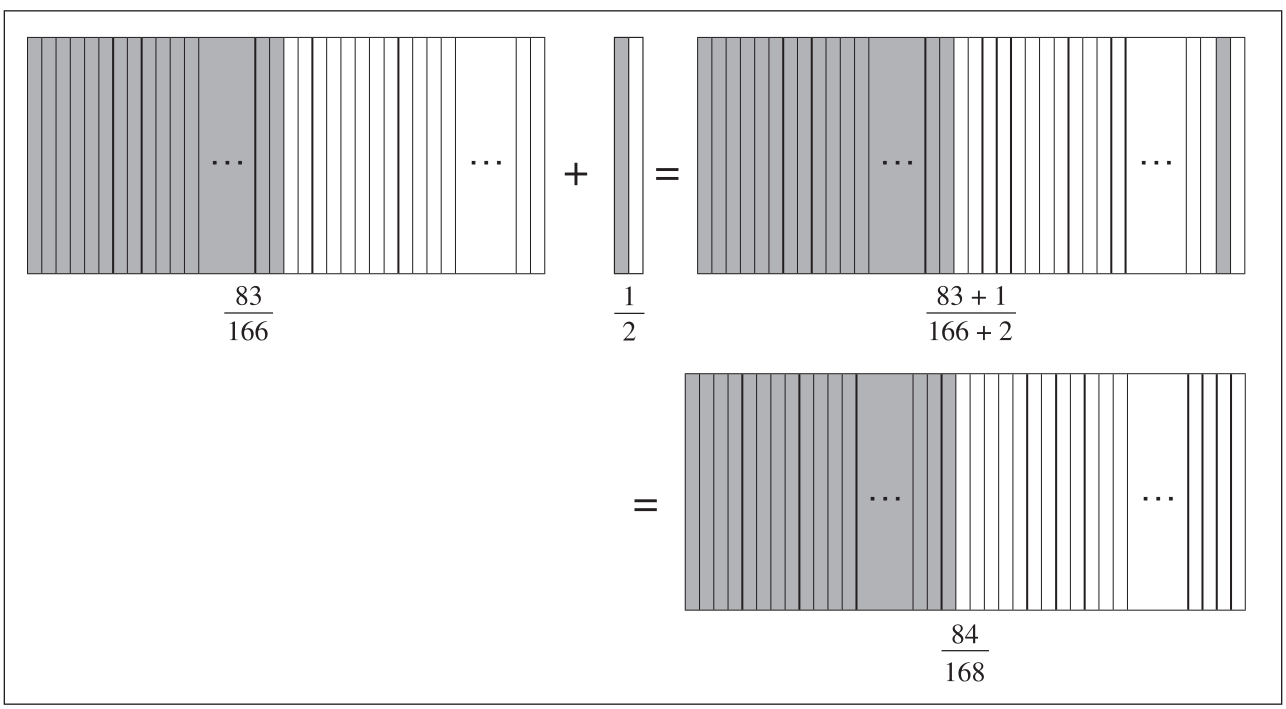


FIGURE 6. *Drawing of the equivalence between 83/166 and 84/168 "additively."*

The fractions 83/166, 1/2 and 84/168 are all equivalent and all share parts of the same size, but all relate to a different unit. It is the partitioning that is maintained in each of them. The partitioning is of central importance here; it is kept for all the actions undertaken, from the first fraction, the fraction that is "added" and the resulting fraction. The partitioning of a "half" is always maintained.

Figure 7 depicts the previously discussed equivalent fractions 2/5 and 4/10 in a similar fashion. Through this partitioning, maintaining the size of the parts makes it easier to explain (verbalize and illustrate). However, it is mostly a matter of facility of explanation, because to some extent the size of the parts of each "added" fraction does not matter. If the parts are not of the same size, then "adding" them looks a little different but works all the same. Figure 8 illustrates this for

$$\frac{3^{+1}}{6^{+2}} = \frac{3+1}{6+2} = \frac{4}{8},$$

where it is clear that it is the partitioning that is central and maintained.

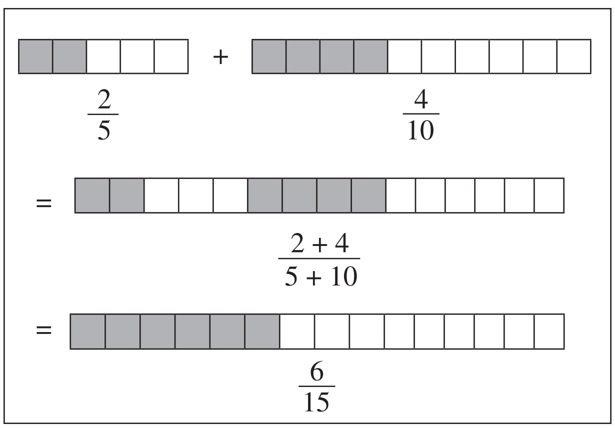


Figure 7. *Drawing of the equivalence between 2/5 and 6/15 "additively."*

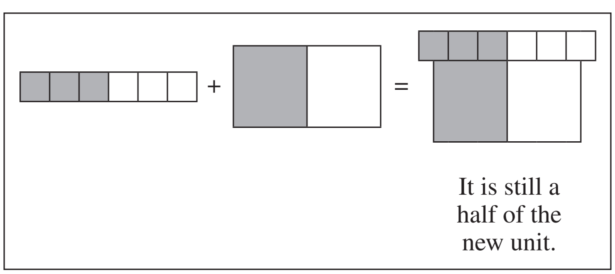


Figure 8. *Drawing of the equivalence between 3/6 and 4/8 "additively," with parts of different sizes.*

Concluding the Exploration

In sum, this relationship is fascinating, and this "adding" method works well. Not only does it work well, but it is also possible to explain and make sense of it using numbers, words and diagrams—all means that are available in Grade 5. Although it at first troubled me mathematically, this "adding" method has proven to be quite efficient for producing equivalences between fractions.

For sure, this way of reasoning about equivalent fractions is different from the usual multiplicative ways. As mentioned above, it is about the conservation of equal partitioning, which produces a new unit that maintains its partitioning. In this sense, this "adding" method appears to be less about equivalent fractions, which usually relates to the same unit, and more about equivalent partitioning of units. This equivalent partitioning places this relationship, or "adding" method, between fractions and ratios.

With all that said, I am not implying in any way that these ideas should be taught to Grade 5 students. I hope you'll agree that the discovery and exploration are pleasure enough—mathematically speaking!

Notes

1. Throughout the article, I put words such as *additively*, *adding* and *addition* in quotation marks, because what is discussed here is a different kind of addition, to say the least.

2. Names have been changed to protect anonymity.

3. These sequences could even be extended. For example, from $(1^{+1})/(2^{+2}) = 2/4$ to $(1^{+1+1})/(2^{+2+2}) = 3/6$ to $(1^{+1+1+1})/(2^{+2+2+2}) = 4/8$, and so on. This is as much surprising in terms of writing as it is in terms of a process.

Jérôme Proulx is a professor of mathematics education and runs the Laboratoire Épistémologie et Activité Mathématique (www.leam.uqam.ca) at the Université du Québec à Montréal. His research focuses on epistemological issues related to cognition and/in mathematics. His current research program is dedicated to mental mathematics and school mathematics content development. Visit his website at www.profmath.uqam.ca/~jproulx.

A Response to Proulx's "Producing Equivalent Fractions 'Additively': The Case of the 1/2"

Lynn McGarvey

In "Producing Equivalent Fractions 'Additively': The Case of the 1/2," Jérôme Proulx shares an experience familiar to teachers of mathematics, from kindergarten to postsecondary.

In this experience, we are engaged in mathematical discussion with students, and they generate a new idea, solve a problem in a novel way or ask a question we had not considered before. Even though we tell our students that mathematics is about problem solving, curiosity and persistence, we may experience a moment of panic if we do not immediately understand their question or know the answer.

How we respond in those moments says a lot about our own mathematical disposition. Do we gloss over the problem, downplay its importance or simply ignore it? Or are we willing to acknowledge that we don't know and then try to figure it out with the class, in the staff room at lunch or on our own? In his article, Proulx demonstrates all of the positive dispositions we want to see in students and teachers, including inquisitiveness, perseverance, confidence and sensemaking.

Reinterpreting the Problem: Consecutive Equivalent Fractions

The problem posed in Proulx's article wasn't one I had considered before. But mathematics isn't something you can read. It is something you have to do. So even though I read through the draft of the article, my margin notes showed that I, too, needed to work through the mathematics—usually in a different way from Proulx.

The general problem, as I interpreted it, was that of consecutive equivalent fractions, in which the numerators of two equivalent fractions had a difference of one (for example, 4/8 and 5/10). Although Proulx focuses on producing equivalent fractions "additively," I continued to forefront the notion that equivalent fractions would always have a multiplicative relationship.

Proulx presents the following representation of adding 1 to the numerator and 2 to the denominator to get another equivalent fraction in the form of one-half:

$$\frac{a^{+1}}{2a^{+2}} = \frac{a+1}{2a+2} = \frac{a+1}{2(a+1)} .$$

While I understood the equation presented, I rewrote it in the margin as

$$\frac{1}{2}\left(\frac{a+1}{a+1}\right) .$$

This revision to the equation provided a new insight for me. We typically emphasize the multiplicative relationship between equivalent fractions and use it to generate new fractions. For example, if we examined the relationship between 4/8 and 5/10, we would usually show it as follows:

$$\frac{4}{8} \times \left(\frac{1.25}{1.25}\right) = \frac{5}{10} .$$

This approach emphasizes the multiplicative relationship, but the fact that the numerators differ by 1 simply gets lost.

I thought more carefully about the fact that all fractions that are equivalent to one-half are derived from the unit fraction of 1/2. Figure 1 shows my margin notes.

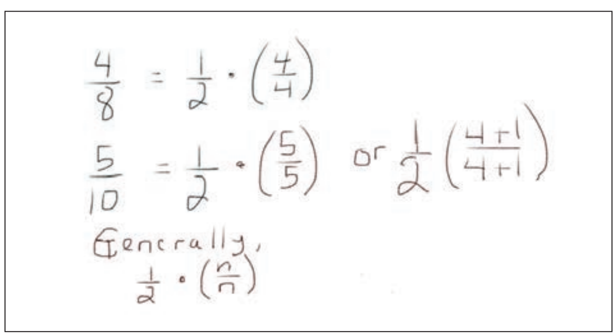


Figure 1

We can generate “consecutive” equivalent fractions by starting with the unit fraction, using

$$\frac{1}{2}\left(\frac{a}{a}\right),$$

$$\frac{1}{2}\left(\frac{a+1}{a+1}\right),$$

$$\frac{1}{2}\left(\frac{a+2}{a+2}\right)$$

and so on.

Going back to the example, where $a = 4$, we would have the following three consecutive equivalent fractions:

$$\frac{1}{2}\left(\frac{4}{4}\right) = \left(\frac{4}{8}\right)$$

$$\frac{1}{2}\left(\frac{4+1}{4+1}\right) = \left(\frac{5}{10}\right)$$

$$\frac{1}{2}\left(\frac{4+2}{4+2}\right) = \left(\frac{6}{12}\right)$$

So equivalent fractions of one-half, one-third or any unit fraction can be multiplied by any whole—a/a or $(a + 1)/(a + 1)$—to generate or analyze consecutive equivalent fractions.

Visualizing Consecutive Equivalent Fractions

The other challenge Proulx presents is how to visualize these consecutive equivalent fractions, particularly since our methods usually involve partitioning an area or measurement model into smaller but equivalent parts. We could, however, employ a sets model to illustrate the “plus 1” nature of equivalent fractions.

For example, in Figure 2, 4/8 of the set is black.

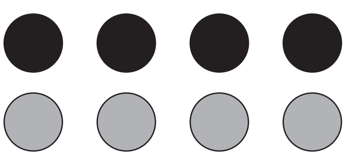


Figure 2

We can use the equation

$$\frac{1}{2}\left(\frac{4+1}{4+1}\right)$$

to illustrate “plus 1” to the numerator and denominator to generate the equivalent fraction, 5/10, as shown in Figure 3.

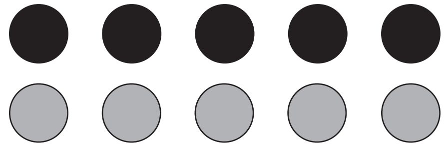


Figure 3

In both cases, we can say that 4/8 and 5/10 are generated from the unit fraction 1/2 (Figure 4), where both numerator and denominator are multiplied by the same whole—either 4/4 or (4 + 1)/(4 + 1).

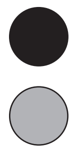


Figure 4

Closing Remarks

I recognize that in making sense of the problem posed by Proulx, I have reinterpreted it and examined it from a different perspective. Such is the beauty of mathematics. I expect that every article with a mathematical curiosity can be examined and explained from different perspectives.

It is interesting that even as I write this response, I wonder how it will be critiqued. That underlying feeling of uncertainty when presented with a novel problem doesn’t necessarily dissipate after you have worked through a problem on your own. However, as teachers, we need to be curious and courageous and to put our mathematical thinking out there, so that others can engage in the problem and offer their own ideas and interpretations.

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Alberta's K–9 Mathematics Program of Studies: Metaphor and Implied Meaning

Shimeng Liu

In this study, I read behind the text of Alberta's K–9 mathematics program of studies (Alberta Education 2007) by scrutinizing the metaphors it uses or implies in conceptualizing images of knowledge, learning and teaching.

Understanding metaphors and related associations embedded in curriculum sheds light on how concepts of mathematics learning and teaching are constructed. Metaphor fundamentally structures human conceptual systems and decisively affects our perceptions, thoughts and actions (Fauconnier and Turner 2008; Lakoff and Johnson 1980).

In the late 1990s, Sfard (1998) outlined two predominant metaphors in learning mathematics—learning as acquisition and learning as participation—and their implications for educational practice. Since then, the realm of educational research has witnessed a more dynamic conglomeration of figurative associations acting to expand the interpretations of what knowledge, learning and teaching are. In addition to learning as acquisition and learning as participation, the metaphors of learning as construing and learning as viability-maintaining, along with their supportive ideas, are formulating into the grander networks of education paradigms. The four paradigms are standardized education, authentic education, democratic citizenship education and systemic sustainability education (Davis 2018).

The Discourses on Learning in Education website maps the predominant discourses around these education paradigms and the associated metaphors commonly applied to the school setting.[1] It draws from a wide range of sources, including scholarly publications, research reports and other professional resources. My findings are drawn from metaphors and definitions documented on this website. Revealing fragments of the underlying education paradigms, metaphors in a curriculum framework disclose the beliefs that go into daily pedagogical practices without profound reflection.

Returning to Alberta's K–9 mathematics program of studies (Alberta Education 2007), I will first examine the preamble and then move to the conceptual framework for K–9 mathematics and student outcomes. A careful textual analysis, with a focus on the early years (K–4), reveals a multi-metaphorical structure rooted in the discursive and conceptual system. This indicates the palimpsest-like composition of the curriculum. A palimpsest, which is a widely used trope in studying cultural discourses, is a parchment on which layered traces from past writing can be seen, and the term has come to refer to the idea of superimposed temporalities (Stam 1997).

The program of studies presents its palimpsestic discursive multiplicity in using metaphors to convey the struggle between correspondence and coherence discourses in conceptualizing mathematics learning and teaching. More specifically, assumptions from mentalism permeate the text, ideas of constructivism are foregrounded in depicting learning and learner, and awareness of complexity discourses is emerging but not confirmed.

Correspondence and Coherence Discourses

Table 1 is an overview of the metaphors for knowledge, knowing, learner, learning and teaching used in the preamble to Alberta's K–9 mathematics program of studies. It also lists the learning theories associated with those located metaphors and the learning discourses (correspondence or coherence) that the learning theories belong to.

The *correspondence discourses* are a set of theories about learning that assume the separation of mind and body and that perceive learning as setting up a correspondence between the outer, objective world and the learner's inner, subjective world.[2] For example, *cognitivism*—which is nested in the correspondence discourses and originates from the metaphor of the brain as a digital computer (Searle 1990)—understands knowledge as information and learning as acquiring and retrieving ideas or concepts from the outside. A more commonly known example of a correspondence discourse is *rote learning*, a learning theory that emphasizes knowing as recalling and learning as storing memorized information.

In contrast, the *coherence discourses* do not perceive learning as being generated from the correspondences between the external and the internal. Rather, they understand knowledge as an interconnected system and the learner as "an evolving coherence" that continuously adapts to the web of meaning and interpretations through experiential being. For instance, *inquiry-based learning* focuses on learning as problem solving in authentic contexts and knowing as the learner's intellectual *and* physical participation in generating knowledge. Likewise, *embodied cogni-*

Table 1. *Overview of Metaphors in the Preamble to Alberta's K–9 Mathematics Program of Studies*

	Metaphors	**Learning theory**	**Learning discourse**
Knowledge	Materials, tools and contexts	Mentalism	Correspondence
	Ideas and concepts	Cognitivism	Correspondence
	Problem-solving situations	Inquiry-based learning	Coherence
Knowing	Applying	Inquiry-based learning	Coherence
	Observing and interacting	Social constructivism	Coherence
	Setting goals and working toward them	Attainment metaphor	Correspondence
	Recalling	Rote learning	Correspondence
Learner	Active agent with individual context	Embodied cognition	Coherence
Learning	Attaching meaning to actions and construing meaning individually	Non-trivial constructivism	Coherence
	Having meaningful discussions	Commognition	Coherence
	Being embedded in everyday activities	Embeddedness discourses	Coherence
	Gaining understanding	Acquisition metaphor	Correspondence
Teaching	Addressing individual needs	Learning styles theories	Correspondence
	Enhancing the formation of mathematical understandings	Facilitation theory	Coherence

tion views the learner as a social-cultural-situated bodied agent and learning as being initiated with bodily movements and perceptions.

In the context of the program of studies, the object-related metaphors align with correspondence discourses, and the ecological-based metaphors respond to coherence discourses.

The program of studies depicts knowledge of mathematics as materials, ideas and situations. It states that working with "materials, tools and contexts" (Alberta Education 2007, 1) helps students construct meaning about mathematics. This aligns with the *mentalism* learning theory, which posits that knowledge is constituted by facts of the objective world and that subjective interpretation needs to be created by internalizing those external truths. For example, the program of studies states that through communication, students build "links among concrete, pictorial and symbolic representations of mathematical concepts" (p 1).

Existing as the objectified reality, mathematics knowledge can be gleaned from its various forms (materials, visuals and symbols), but the contents are ideas and concepts. *Cognitivism* also assumes that knowledge exists as information but uses the metaphor of the brain processing knowledge like a computer.

In the program of studies, reasoning and thinking in "problem-solving situations" are the foundation for students "to develop personal strategies and become mathematically literate" (p 1). Thus, mathematics knowledge can also be seen as inquiry. In *inquiry-based learning*, students become an indispensable part of knowledge production by exploring problems and applying their individual epistemological resources.

The idea of knowing as applying is consistent with the idea that mathematics knowledge is produced in situations of inquiry, indicating the context-dependent nature of understanding mathematics.

As noted in the program of studies, the sense-making of young children is stimulated by their observation and interaction "embedded in everyday activities" (p 2), both in and out of school. This stance extends mathematics knowledge from classroom-based situated inquiry to the broader social constructs. In *social constructivism*, knowing means to observe and interact with others as social practice.

However, as indicated in the program of studies, knowing also necessitates that students "set achievable goals and assess themselves as they work toward these goals" (p 2). As one of the most powerful folk theories in formal education, the *attainment metaphor* sees mathematics as a specific field wherein students learn by following a path and reaching milestones.

Occasionally, the program of studies indicates that knowing also involves recalling concepts (p 10), which is *rote learning*. This goes back to the *mentalism* that mathematics knowledge is context-free external truths and that knowing requires memorizing and retaining information.

As the program of studies emphasizes, learners are active agents with individual contexts, bringing their own prior knowledge, experiences and backgrounds to the learning community (p 1). This implies an association with *embodied cognition*, which assumes that learning starts with physical movements and perceptions and then is refined and blended into higher-order concepts within one's sociocultural context.

However, metaphors for learning go beyond embodied cognition to embrace a group of analogues with conflicting connotations.

First, the program of studies suggests that learning could be seen as attaching meaning to actions and constructing (or construing) the meaning of mathematics individually (p 1). Learning as construing implies the perspective of *non-trivial constructivism* because of its focus on the individuality of meaning production derived from experiences.

Second, learning could be having meaningful discussions about mathematics (p 1). Learning as interaction through communication, or *commognition*, "foregrounds the role of verbal language" in generating knowledge.

Third, learning mathematics could involve ordinary activities, such as playing or baking (p 2). Seeing learning as embedded in everyday activities refuses the division between self and other, body and mind, and individual and collective, suggesting an affinity with *embeddedness discourses*, which "understand[s] social and cultural collectives as dynamic, learning phenomena."

Last, learning is to "gain understanding" (p 3) and to attain specific skills and knowledge (p 9), which signifies *acquisition metaphor*.

As with the other four constructs, teaching is presented in the program of studies with polarized metaphors and implied meanings.

As to the role of the teacher, the program of studies states that "through the use of manipulatives and a variety of pedagogical approaches, teachers can address the diverse learning styles, cultural backgrounds and developmental stages of students" (p 1), while promoting robust mathematical understanding. *Learning styles theories* see the learner as "an information processor," with individual personal characteristics and preferences, and teaching thus means arranging information according to the peculiarities of each learner and making the inputting of that information smoother.

Another metaphor used in the program of studies—teaching as enhancing the formation of mathematical understandings (p 1)—speaks to *facilitation theory*, in which there is a "positive learner–teacher relationship" and teaching is facilitating mathematical interpretation in the learner.

Thus, the program of studies interweaves learning styles theories (from the correspondence discourses) with facilitation theory (from the coherence discourses), creating paradigmatic tensions.

Mentalism, Constructivism and Complexity Perspectives

I also investigated the stated learning outcomes and beliefs about mathematics in the program of studies to determine whether they are in accordance with assertions made in the preamble.

The concept of performance outcomes itself is an inference made on the basis of standardized education, which was conceived as the foundation for industrial societies by taking knowledge as a standardizable object and the learner as measurable worker (Davis 2018, 188). Hence, the *acquisition metaphor* is intrinsic to the specific outcomes in the program of studies, which serve to "identity the specific skills, understanding and knowledge that students are required to attain by the end of a given grade" (Alberta Education 2007, 9).

For instance, a Grade 1 student is expected to "estimate quantities to 20 by using referents" (p 13), in order to meet the standard of number sense development at this stage. This reveals a limited consideration of the "individual interests, abilities and needs" and "varying knowledge, life experiences and backgrounds" (p 1) mentioned in the preamble.

Nevertheless, the learning outcomes are not entirely confined within the perceptions of *mentalism*. While specifying outcomes, the program of studies does underscore the importance of daily activities in fostering mathematics understanding. This indicates agreement with *non-trivial constructivism* in the belief that knowledge is the "sum of already-established construals/constructs" and the learner is "a meaning-maker" who derives meaning from individual experience.

For instance, the number strand in the program of studies implicitly integrates Lakoff and Núñez's (2000, 53) four "grounding metaphors" as strategies to develop higher number sense by correlating the innate capabilities of subitizing and counting with everyday activities. For example, the ability to "demonstrate an understanding of counting by . . . using parts or equal groups to count sets" (Alberta Education 2007, 13) involves the metaphor of "arithmetic as object collection" (Lakoff and Núñez 2000, 54). "Demonstrat[ing] an understanding of fractions by . . . explaining that a fraction represents a part of a whole" (Alberta Education 2007, 21) suggests the metaphor of "arithmetic as object construction" (Lakoff and Núñez 2000, 65). Moreover, the "measuring stick metaphor" (p 68) and the "motion along a path" (p 71) metaphor could also be identified among the specific outcomes. In sum, knowledge of arithmetic as count, size, distance and position is not external truth to be transferred into the learner's mind but, rather, the interpretations produced and refined by interconnections of the learner's intuitive understanding of numerosity with the experience of using objects.

Since mathematics is framed by "our bodies, our brains, and our everyday functioning in the world" (Lakoff and Núñez 2000, 5), the program of studies points out the mathematical processes of significance in achieving the outcomes: communication (C), connections (CN), mental mathematics and estimation (ME), problem solving (PS), reasoning (R), technology (T), and visualization (V) (Alberta Education 2007, 4).

Table 2 shows that in the program of studies the most frequently mentioned mathematical processes for each grade are connections in kindergarten; visualization in Grade 1; and communication in Grades 2, 3 and 4. Taking all the grades together, the top three processes are communication, connections and visualization.

Table 2. *Frequency of Mathematical Processes Mentioned in the K–4 Learning Outcomes*

	K	Gr 1	Gr 2	Gr 3	Gr 4	Total
C	8	18	26	24	22	98
CN	10	17	22	22	21	92
ME	2	8	8	12	6	36
PS	5	8	9	18	12	52
R	5	13	24	23	18	83
T	0	0	0	0	1	1
V	6	19	20	21	21	87

Communication refers to various modes of expression students can use to help them "make connections among concrete, pictorial, symbolic, oral, written and mental representations of mathematical ideas" (Alberta Education 2007, 5), and *connections* refers to "contextualization and making connections to the experiences of learners" (p 5), to each other and to the world beyond the classroom, for the sake of integrating mathematics into micro- and macro-coherence. *Visualization* refers to visual images, visual reasoning and spatial perception in making sense of number, geometric shapes and measurement (p 6). All three processes are interrelated and nested within each other. They represent the necessary steps before reaching "formalising understanding" (Pirie and Kieren 1994, 39) in mathematics development, in which the learner is able to move beyond using concrete objects in certain contexts into working with symbolic representations in diverse contexts.

The incongruity appears when communication, connections and visualization are not identified as pedagogical focuses in the conceptual framework for mathematics. Instead, the program of studies deems problem solving to be the mathematical process that is "the focus of mathematics at all grade levels" (Alberta Education 2007, 6), because it asks students to apply their prior knowledge to new contexts and "empowers students to explore alternatives and develops confident, cognitive mathematical risk takers" (p 6). Underlying this statement is the idea that mathematics knowledge represents a range of possibilities and that learning means expanding and transcending the given possibilities. Going back to Discourses on Learning in Education, this alludes to *eco-complexity discourses*, which is a subcategory of the coherence discourses that relies on ecological metaphors in perceiving learning as an evolving ecosystem of complexity.

A complex phenomenon has two characteristics: it is adaptive because it evolves its own structure, and it is emergent because of the co-evolving interactions of individual agents and the ensemble (Davis and Simmt 2003). The emergence of new knowledge and creativity goes hand in hand with transcendental moments in the collective of bumping ideas and bodies. For example, in a Grade 7 class learning multiplication, stimulating opportunities for mathematical engagement arise after intentional efforts have been invested in creating an adaptive and self-organizing collective of learning, rather than just a collection of individuals (Davis and Simmt 2003). Shifting dichotomies of person and nature, material and transcendent, and one and many into nested constructs, the complex system of learning does not view the learner as a discrete agent in their own existence but, rather, as an agent in the intersectionality of bodily, social, cultural and biological subsystems. This perspective synthesizes the constructivist and sociocultural perspectives on mathematics development, because of its emphasis on the inseparability between individual-in-construing and individual-in-social-action (Cobb 1994).

If we can say that the curricular focus on problem solving gestures toward the complexity perspective of mathematics education, then the implied text of the learning outcomes in the program of studies is calling for the return of constructivism, in which learning is individual doing and thinking.

Conclusion

In this study, I have striven to understand the metaphors attached to knowledge, knowing, learner, learning and teaching in Alberta's K–9 mathematics program of studies. The findings reveal that underlying the curriculum is a pervasive wrestling between the correspondence and coherence discourses of learning, with a particularly strong voice from mentalism, a growing recognition of constructivism and an emerging awareness of complexity discourses.

Notes

1. "Map," Discourses on Learning in Education website, B Davis and K Francis, https://learningdiscourses.com/learning-discourses/ (accessed October 13, 2021).

2. Information on this term and subsequent italicized terms comes from the alphabetical index of the Discourses on Learning in Education website at https://learningdiscourses.com/alphabetical-index/ (accessed October 13, 2021).

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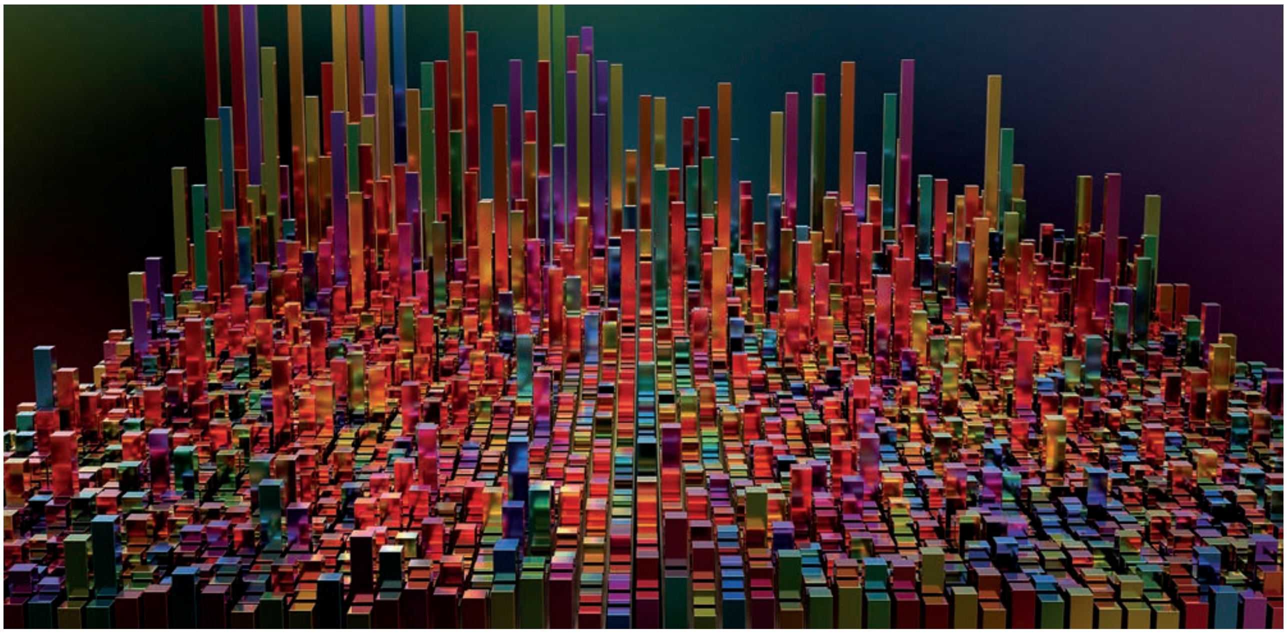
Engaging K–12 Teachers in Mathematical Puzzles and Problems

Richelle Marynowski, Shelly Wismath, Verna Mabin, Michelle Campmans, Lynn Suttie and Alana Millard



Engaging students in solving mathematical problems and puzzles is an excellent way to get them thinking mathematically and communicating their ideas (Karp and Wasserman 2015). Often, the challenge for teachers is finding not only problems and puzzles but also the time to experiment with them before giving them to students.

This article describes the experiences of a group of K–12 mathematics teachers who participated in a professional development series on puzzles and problems. The PD series introduced the teachers to mathematical problems and theories related to problem solving in the mathematics classroom, as well as giving them follow-up opportunities to visit each other's classrooms.

The intent of this PD series was twofold:

- To provide teachers with experience in engaging with mathematical puzzles and problems themselves in a safe environment in order to increase their confidence in implementing those puzzles and problems in their classrooms
- To provide opportunities for teachers to observe each other as they engaged in mathematical activities with their students and then reflect on the outcomes with each other

Puzzles and Problems Sessions

The PD series was aimed at increasing the teachers' confidence in themselves as problem solvers and their confidence in engaging their students as problem solvers.

Elements of the PD sessions were based on Marynowski's (2013, 2014, 2015) previous work with secondary mathematics teachers. The essential components of the sessions were

- content provided by an expert in the field,
- an expectation that the teachers would integrate the ideas into their practice,
- a commitment to observing each other's practice, and
- time provided for teachers to engage in the sessions and peer observations (Alberta Teachers' Associa-

tion 2014; Darling-Hammond and Richardson 2009; Desimone and Pak 2017).

The content of the sessions was drawn from the Liberal Education 2200: Problems and Puzzles course offered at the University of Lethbridge.[1] The format of the sessions incorporated elements of the school division's peer mentorship program. About half of the usual course content was covered, in six three-hour sessions, and the content was modified to integrate instructional strategies and peer mentorship. The participants had the opportunity to obtain credit for Education 4850: Problem Solving in the K–12 Mathematics Classroom, a new undergraduate course that integrated mathematical content and professional development.

For the peer mentorship portion, the school division provided release time and substitute teachers so that the participants could both engage in the half-day sessions and have time to observe each other's practice.

Each session integrated literature on and support for the teachers in peer mentorship, offered opportunities for the teachers to share what they had tried in their classrooms between sessions and what they had seen and experienced during their peer observations, and engaged the teachers as active problem solvers. The teachers were invited to observe at least one other teacher's class between sessions. The series of sessions took place from September to December 2017.

As already noted, the teachers could participate as part of a course for university credit or participate in the PD series only. Twenty teachers engaged in the PD series only, and six teachers opted to take the course. Those who took the course were able to apply for a mathematics bursary provided by Alberta Education.

Since the participants taught at various levels from kindergarten to Grade 12, the focus of the PD series was not on specific learning outcomes that could be addressed through the problems but, rather, on general problem-solving strategies, competence and confidence. Thus, the teachers were asked to adapt their learning about the puzzles and problems and integrate that learning into their individual contexts.

Research Questions and Processes

Several themes have previously been identified in work with university students in regular offerings of the Problems and Puzzles course, including patience, persistence and other mental attributes in problem solving; solo versus collaborative work; and confidence building (Wismath and Orr 2015; Wismath, Orr and Zhong 2014).

One goal of the research project described here was to examine those themes in this new context, with an abridged version of the course and in work with practising K–12 teachers. Another goal was to explore the impact of engaging in a four-month PD series on problem solving in the teachers' own mathematics classrooms, as well as the impact of their visits to each other's classrooms.

The participants completed a 15-question pre- and post-survey (Appendix A) on how they envisioned themselves as problem solvers to see whether engaging as problem solvers influenced their perceptions of themselves (Wismath, Orr and Zhong 2014). Additionally, the post-survey invited participants to share their experiences and what they had noticed about themselves and their students.

The quantitative results and qualitative comments from the survey are not the focus of this article; however, they are used to illustrate that a change in the participants' perceptions occurred through engaging in problem solving as active learners. The quantitative data illustrates the change in the participants' ratings of their attitudes toward problem solving, and the qualitative data provides illustrative comments from the participants. The qualitative data was analyzed for specific themes and illustrative examples following a thematic analysis approach (Braun and Clarke 2006).

What follows are comments from the lead facilitator of the sessions, the survey results and detailed reflections from four participants.

Lead Facilitator's Experiences

As a professor of mathematics who has taught a variety of undergraduate mathematics courses over many years, I (Shelly Wismath) have long been aware of the tension between content and process and have been frustrated at how little we talk about the process of thinking about and creating mathematics when we focus on content. This concern has become stronger in recent years, as I have turned to teaching general math and quantitative skills to students majoring in subjects other than math and science.

Such concerns were at the forefront when I had the wonderful opportunity to develop a course on problem solving called Problems and Puzzles, which I have taught regularly since 2012. In an attempt to keep the focus on content to a minimum and to make the course accessible to students of all majors, I have used puzzles as the vehicle for problem solving, including math word puzzles, counting problems, logic puzzles, and historical examples of puzzles and riddles.

The math level required for the course is generally not more than beginning high school level. Although we spend one week on using pairs of linear equations, the methods needed are taught in class, as is the often more challenging process of translating sentences into equations (for example, "Mary is five years older now than twice John's age four years ago"). Minimal class time is spent on lecturing to identify themes or strategies; rather, most of the time is allotted for students to work on new puzzles and discuss a variety of approaches and solutions afterward.

A number of key themes have emerged from my teaching of this course, as well as from an associated research project carried out over several course offerings (Wismath and Orr 2015; Wismath, Orr and MacKay 2015; Wismath, Orr and Zhong 2014; Wismath and Zhong 2014).

First, a primary goal of the course is to allow students to develop metalevel and metacognitive skills. Rather than focusing on specific content, students gradually focus on abstracting ideas from a complex context, observing and testing for patterns, using careful reasoning, and communicating their thinking process to others. They learn strategies such as working backward; identifying subgoals; using charts, tables and diagrams to represent and organize information; and making small-scale models. In weekly reflection assignments, they are encouraged to assess their own skills and growth and to think about how problem solving is used in their particular areas of study. This metacognitive reflection produces increased awareness of transferable metalevel skills, as students realize that they can use these strategies in other academic work, such as writing essays, analyzing textual arguments, setting subgoals and studying for tests (Wismath, Orr and Zhong 2014).

Mathematics textbooks that include problem solving usually start with the four-step method of George Pólya (1973):

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

In class, students can choose to work alone or in informal small groups and can shift back and forth as they wish. Our research team has found distinct cycles in collaborative versus solo work that correlate with Pólya's four steps (Wismath and Orr 2015). Students usually start working on the puzzles alone, to get a full understanding of the information and the goals of the problem. If they are able to, they work alone until they reach a successful conclusion. However, if they get stuck, they then turn to collaborative work, comparing notes and brainstorming new approaches, as in Pólya's second step. They then go back to working alone for the third step (carrying out the plan). They cycle through this consult-and-carry-out phase as many times as needed. Finally, the metacognitive "looking back" step is a collaborative one, as students increasingly benefit from sharing their approaches and hearing the many ways other students thought about the same problem.

Researchers on learning in a variety of disciplines have tried to identify plateaus and thresholds in learning—stages in which student growth seems to level off for a while and stages in which sudden growth leading to a new plateau can occur as students grasp some "threshold concept" that allows for a breakthrough in understanding (Cousin 2006). Our research team found three such thresholds in student growth over the course (Wismath, Orr and Mackay 2015).

The first threshold involves getting started. Instead of waiting for guidance, students gradually became more willing to plunge in and try one of their strategies. We think that the tendency to wait for guidance stems from a pedagogical approach that inadvertently teaches students that the worst thing they can do is make a mistake, or "fail." In fact, in problem solving, making mistakes is where the learning occurs, as it gives students a chance to figure out what went wrong and why and how to fix it. Solutions usually don't come immediately; rather, they are achieved through an iterative process of trying and fixing.

The second threshold is related to patience and persistence. As students cross this threshold, they no longer give up as soon as they get stuck on a problem but, rather, persist in trying various methods.

Finally, many students move over a third threshold, involving increased attention to Pólya's first step, as they realize that acquiring a deeper understanding of a problem upfront and spending time making a careful mental model mean less of a guess-and-check approach and less work overall.

These thresholds also offer a rich metacognitive learning experience, as students become more aware of how to be successful problem solvers.

A final component of growth that we studied was confidence. We measured students' confidence as problem solvers through a pre- and post-course survey, using items on a five-point Likert scale, and observed a statistically significant increase in their confidence over the duration of the course. A gender breakdown of this data revealed, however, that despite decades of effort to improve both the confidence and the success of girls and women in math and science,

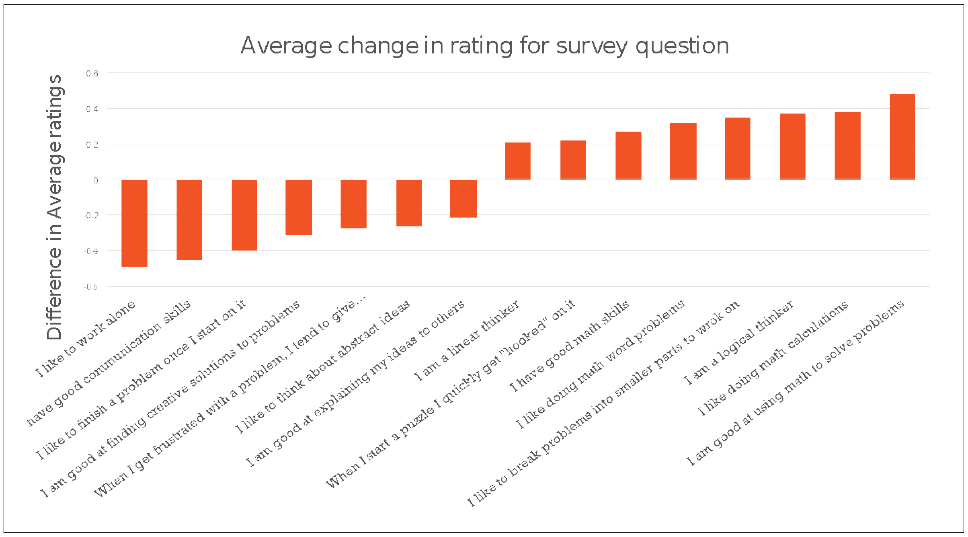


FIGURE 1. *Average change in participant ratings from the pre-survey to the post-survey.*

there was still a strong gender difference in confidence (Wismath and Zhong 2014). At the start of the course, the female students had a much lower average confidence score than the male students (3.14 versus 3.94). By the end of the course, the female students' average confidence score had increased more than that of the male students (3.86 versus 4.44). Again, this shows both a metacognitive skill gain and the importance of addressing gender in teaching mathematics and problem solving.

Survey Results

Before and after the series of PD sessions for K–12 teachers, the participants responded to a questionnaire about their perceptions of themselves as problem solvers (see Appendix A). Figure 1 illustrates the average change in rating for each survey question, from the largest decrease to the largest increase.

The two most striking changes were that, after the sessions, the participants liked working alone much less and saw themselves as being good at using math to solve problems. They were also less likely to give up on a problem when they were frustrated, an improvement in persistence. In addition to providing teachers with experiences to improve their problem-solving skills, these sessions promoted resilience in persisting with difficult problems and built confidence.

In an open-ended question, the participants were asked what makes a good problem solver. Of the 17 comments, 8 focused on persistence, openness and flexibility. The following is a sampling of the responses:

- "Resilience, the drive to continue in the face of failure; Persistence; Tenacity."
- "Open-minded and the willingness to consider all possible solutions and weigh out which has the most benefits."
- "Flexibility—ability to look at different angles/viewpoints."
- "Being open to try."
- "Zoom out to see the forest. Get lay of the land. Then incrementally zoom in. . . . When you get frustrated, walk away to engage in 'diffuse mode' thinking or switch to a different aspect or perspective."

Five other comments focused on specific approaches to problem solving, such as the "ability to break the problem down to pieces that are manageable" and "us[ing] a variety of strategies to attempt the problem at hand [and] tak[ing] time to look at the whole problem and work through it piece by piece." None of the responses included proficiency in mathematics or in calculations as an important characteristic of a good problem solver. The participant comments echoed the idea that metacognitive and critical thinking are what are important in mathematics, beyond competence with numbers or arithmetic (Karp and Wasserman 2015).

The participants were also asked what had changed in their teaching practice after engaging in the sessions. Five participants mentioned incorporating more time for problem solving in their classes: "I am taking more time to introduce problem solving opportunities for the students. We take one math period a week to simply work on math problems such as frogs on a log or play spatial games." Four of the comments were more general, describing the overall feeling of the class: "I have introduced techniques, methods, ideas into my class. It has brought some fun back into the sometimes dry environment." Three participants commented on the peer mentorship process itself: "[I] gained courage to go outside of my walls to borrow ideas and thoughts from others—at all grade levels. Important to remember we as teachers are not an island." All 15 comments indicated that there had been some sort of change to the participants' practice.

Participant Experiences

The following reflections highlight the experiences and learnings of four participants in the PD series. Their stories illustrate how engaging in problem solving as learners translated into their teaching practice.

Verna's Experience

It was an honour to be part of this project. I have actively been involved in the school division's peer mentorship program for five years, and every year I evolve so much as a teacher and a lifelong learner. I became involved in peer mentorship because there were not a lot of PD opportunities I could participate in outside of the classroom. Having three children at home limited when I could be away and how much I could afford to spend engaging in PD opportunities. This year was an exceptional year, because in addition to being part of a wonderful program, I was able to take a university math course at the same time and earn credits toward my total years of education.

It was challenging for me, because as a social studies major, math scared me. I have never felt confident in my ability to think or teach mathematically, and I knew this class would make me do just that. It was an eye-opening experience for me, and I learned strategies that could help students who felt just like I did about math. The series of sessions was very condensed, which made it a little difficult to engage in classroom visits with colleagues. It meant a great deal of time outside of my own classroom, which was challenging, but it was well worth it.

I gained a lot of confidence throughout this collaborative project, and that confidence is now evident in my students, as well. From participating in this project, I have learned that math is more than calculations and worksheets, and math in my classroom is now fun!

Michelle's Experience

I joined the project because I was looking for new ways to make math fun and new ways to encourage problem solving in young students. I teach kindergarten, where we naturally do a lot of real-life problem solving every day, but I wanted an opportunity to introduce some critical thinking. I was also interested in seeing what teachers do in older grades and whether I could tweak anything to make it more age-appropriate. Sometimes we don't give young students the benefit of the doubt, and we assume that they can't do things, even though they may be up for the challenge.

Many activities in the course required reading or advanced knowledge that kindergarten students haven't acquired yet, so it was unrealistic to take everything from the course back to the classroom. But the idea of thinking about the next step and trying to use reason can be used at any age, so I focused on that.

I introduced logic puzzles to my students. While they did enjoy the puzzles, we had to do them as a whole group, because my students cannot read. Wanting to stay in the area of logic, I moved to a couple of board games that require thinking about the next steps before making a move. I introduced Connect Four and Rush Hour to my students, and both have become popular choices during free play time. To win at Connect Four, a player must create a line (vertical, horizontal or diagonal) of four disks of the same colour before their opponent does. This is a difficult idea for some students, especially since they also have to closely monitor their opponent's lines. In Rush Hour, the player must figure out how to get an ice cream truck out of a traffic jam. The player is given a card that shows exactly how to place the vehicles on a grid and then must move the vehicles forward and backward until they have created an empty path for the ice cream truck to travel on. The game has various levels of difficulty, so everyone in my class can be successful. The only downfall is that it is a single-player game, so we don't have the bonus of learning how to take turns or what it means to win or lose.

These are just two examples of problem-solving games that can be used in the kindergarten classroom. If my budget allowed for it, I would stock my classroom with as many of these games as possible, as I clearly see the benefits every time my students play them. I will continue to use logic puzzles, as well, but I strongly recommend board games, because children can play them without adult guidance and are always happy to teach the games to their peers.

Lynn's Experience

In my teaching, I focus on developing understanding in math, not memorizing questions. My aim is to create problem solvers and critical thinkers who can use skills in a variety of situations, and this program seemed like a great opportunity to develop strategies for fostering this level of learning. This program invited me to see my classroom and my teaching through a unique lens of discovery and pursuit. Allowing the problem-solving process to occur can be frightening for a teacher. We do not want any wasted time or the stress of being "behind."

Through participating in the program, I started to view lessons and ideas through a lens of problem solving and processes, and I used puzzles to promote and reinforce the qualities that make a proficient critical thinker and a resilient problem solver. The answer to a problem became less and less important, as dead ends were rewarded and productive time did not always move in a straightforward, linear way.

Looking at my class through this lens has enabled me to let go of the constraint of time, and it has helped me promote the problem-solving process in my classroom. Rewarding resilience more than a correct outcome has helped my students feel more confident in math. This opportunity has rejuvenated my spirit in the classroom and has motivated me to continue challenging my students through trying different styles of learning and use of time at school. I, too, have found the puzzles engaging, and they have motivated me in my own learning and intellectual wellness. My students value effort and resilience more than they did before; they still get a great deal of satisfaction from finding the correct answer, but they no longer shy away from the sometimes-frustrating process of getting there.

Alana's Experience

Problem solving is an engaging and all-encompassing activity to do with children. I wanted to take part in this project because I had done research on teaching problem solving in an elementary classroom and wanted to further extend my knowledge and to see these practices used in everyday classrooms. How lucky I feel to have been part of this program and experience! From assisting in teaching this workshop series, I learned a great deal about how to help students learn how to struggle. Finding that perfect zone of proximal development—where the problems are challenging enough to stretch students' understanding and thinking to make new connections but are not so difficult that students get frustrated and give up—is essential. It was interesting to assist practising teachers through this process and offer hints or guiding questions. I found that the more I predicted where the participants would struggle, the better I was able to prepare questions that could guide them without giving the answer away or stopping their thinking.

This work has extended into my final teaching internship in Grade 2. I have been able to modify many of the problems to work for younger students. Currently, my students love the game Polar Bears Around the Drinking Hole (a version of Petals Around the Rose).[2] They ask to play during snack time and whenever we have a few free minutes. We have talked about making a prediction or a hypothesis and then testing it. About one-quarter of my students know the rules and can clearly explain them; we call them our drinking hole masters. My students have also played Frogs on a Log, after reading the book *Frog on a Log?* by Kes Gray (2015)—a great literacy tie-in. Students broke into pairs to work on the puzzle, and it was exciting to observe them as they worked together to move the frogs the quickest way possible. They have also started to work on simple logic puzzles, mostly about shapes and colours. These have required some direct instruction initially to help move students into independent work. The one concern I have with these puzzles is that they are often very text heavy, so students who struggle to read also struggle to understand the clues. During our measurement unit, we used toothpicks as a manipulative to measure; this was a perfect segue to toothpick puzzles (also known as matchstick puzzles). The students started with a shape formed by the toothpicks, and they had to change it to a different shape, using a limited number of moves. They spent time thinking about the definition of various shapes and enjoyed the kinesthetic learning.

The two biggest challenges I faced were time and students' attitudes. We have only a certain number of hours in the day to address all of the curriculum content, and many of these problems do not have a direct connection to the curriculum. However, I find that the problems help students develop the skills of mathematical thinkers. They learn how to break down a problem and better understand what is being asked, how to articulate their ideas and how to have meaningful mathematics conversations, and they also start to see the mathematical connections to everyday life. When we first started to work on problems, many students became frustrated and upset, so we had to take a step back and talk about persistence. This ended up being our focus in health, and it has been amazing to see the students use strategies we have talked about in order to persevere through a problem.

Conclusion

As already described, this PD series involved six sessions, offered by the school district, as a vehicle to offer approximately half the content of an undergraduate course in problem-solving skill development. The associated research project looked at the previously identified themes of confidence, threshold (or transition) concepts, persistence and patience, and cooperative learning in this new context. The data collected, both quantitative and qualitative, shows that the teachers built both metacognitive appreciation for and skill and confidence in problem solving. As measured by a post-survey, two of the three largest increases in average data scores occurred on the indices measuring persistence and confidence.

The participants' qualitative reflections also demonstrate their increased confidence in their own abilities, both in solving problems themselves and in guiding their students' development as problem solvers. Verna's reflection highlights this growth: "I gained a lot of confidence throughout this collaborative project, and that confidence is now evident in my students, as well."

Michelle and Alana commented on how the university-level math content had to be modified for lower grade levels, but they identified metacognitive skills that they were able to use and build with their students, such as "thinking about the next step and trying to use reason." Alana also identified how she had helped her students pass the metacognitive threshold of persistence, along with other skills appropriate at the Grade 2 level: "They learn how to break down a problem and better understand what is being asked, how to articulate their ideas and how to have meaningful mathematics conversations." This idea of mathematical conversation also appeared widely as a benefit of both the sessions and the in-class peer mentorship. The teachers appreciated the value of working together and continued to do so outside of the formal sessions. Again, our quantitative data bears out this significant increase in positive teacher attitudes toward mathematical conversation.

Finally, the participants showed awareness of the significant thresholds in the development of problem-solving skills. All participants developed a tool kit of strategies to try themselves when working on new problems and to pass on to their students, and Lynn and Alana both noted the increase in students' effort and resilience, which mark the threshold of patience and persistence.

Overall, we argue that our survey data, as well as feedback from the participants, shows that this project was successful in meeting its goals. The puzzle-based approach to problem solving, sessions that allowed participants to practise this approach among a supportive group, and the opportunity to extend the learning to their own classrooms with peer support gave the participants a multifaceted way to develop their own skills in a fun and safe way, which they could then extend to their own students.

A useful follow-up to this project would be doing an assessment with the participants after one year, to measure the long-term impact on their teaching and on their students' progress as problem solvers.

Appendix A: Attitudes and Attributes Survey

The following survey was administered to participants both at the start of and after completion of the series of sessions.

Level of agreement with each statement: strongly disagree, disagree, neutral, agree, strongly agree

I like to work alone.

I have good communication skills.

I like to finish a problem once I start on it.

I am good at finding creative solutions to problems.

When I get frustrated with a problem, I tend to give up.

I like to think about abstract issues.

I am good at explaining my ideas to others.

I am a linear thinker.

When I start a puzzle I quickly get "hooked" on it.

I have good math skills.

I like doing word problems.

I like to break problems into smaller parts to work on.

I am a logical thinker.

I like doing math calculations.

I am good at using math to solve problems.

Notes

1. Liberal Education 2200: Problems and Puzzles, University of Lethbridge, list of topics covered, 2012, www.cs.uleth.ca/~wismaths/pandppage/topicslist.pdf (accessed October 18, 2021).

2. Petals Around the Rose game, Illuminations website, National Council of Teachers of Mathematics (NCTM), https://illuminations.nctm.org/lessons/petals/petals.htm (accessed May 6, 2019).

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Enriching Geometric Understanding Through Early STEM Pedagogy

Nicole Langevin, Miwa A Takeuchi, Jenny Yuen and Shayla Jaques

Attention to science, technology, engineering and mathematics (STEM) education has been rapidly growing nationally and internationally (Sengupta, Shanahan and Kim 2019; Takeuchi et al 2020). In Alberta, STEM is not yet a curriculum subject in itself, although related pedagogical efforts—such as makerspace and design-based learning—are being taken up in Alberta schools.

STEM is a relatively recent domain of studies in the education field. In the EBSCO Academic Search Complete database, the term *STEM* first appears in education journals from the early 2000s. Since then, the number of publications on STEM education research and teacher development resources for STEM education has significantly grown.[1]

Despite this growth, there is a scarcity of research on STEM education in the early years (kindergarten to Grade 3), compared with the research on secondary and postsecondary STEM education, which could be partially a result of the implicit research agenda of preparing students as future workers for STEM industries (Takeuchi et al 2020). An aesthetic vision for STEM education that centralizes *seeing more* and *seeing the complex* (Farris and Sengupta 2016) stands apart from those instrumental visions focusing mainly on workforce preparation. Aesthetic expansion through STEM education enables learners to make familiar phenomena unfamiliar and to notice previously unnoticed elements and meanings in those phenomena. What does STEM education in the early years look like when the teacher aims for aesthetic expansion?

This article sketches a scene of students *seeing more* in geometry through an early STEM class in Alberta, designed by the mathematics and STEM learning leader at the school. All students in the school experienced a one-hour STEM class twice per week. The teacher arranged the classroom space (by removing individual desks and seats) so that students could move their bodies freely. The classroom housed commercial resources (such as Lego Mindstorms robotics kits, Sphero robots and Ozobot robots), and the teacher also created a free makerspace with a collection of recycled materials (Figure 1). The teacher actively used the classroom wall and the hallway to document and demonstrate the process of the students' projects, highlighting the design cycle (Figure 2). The final iterations of the students' creations, as well as their initial plans and photos of the process, were included.

Figure 1. *Recycled materials for the free makerspace.*

Figure 2. *Classroom wall displaying students' work.*

The school was located in a neighbourhood with many apartment buildings, where many recently arrived immigrant and refugee families had settled. Approximately 90 per cent of the students were identified as English-language learners, with various home languages.

In this article, we demonstrate how students in a Grades 1/2 combined class learned key mathematics concepts in this STEM classroom by describing the students' learning during the final project of the year. The students received two STEM classes per week, and this project took place over six weeks. We will focus on two students, Alex and Hakim, who were both newly arrived students and whose first languages were not English.[2] We videotaped classroom interactions over six instruction days and repeatedly reviewed the videos. We identified the trajectory of engagement by Alex and Hakim as illustrative of the learning that was happening in this STEM class.

In the final project, the students created an art piece with geometric shapes that they had been learning, using Sphero robots as paintbrushes. Each phase of this project presented unique mathematical learning opportunities, as demonstrated in the following sections.

Understanding Polygons Through Coding

In many early mathematics classrooms, students explore polygons by describing, sorting and comparing provided shapes. In this interactive STEM classroom, the teacher started by challenging the students to draw various polygons based on their definitions.

Figure 3 is the teacher's sketch of the shapes students drew (through gestures). Not only did the students recognize the regular hexagon shape, but they also drew two other shapes that, by definition, are both valid hexagons (six-sided polygons). Many of the students' existing understanding was based only on exploring regular polygons, such as those found in pattern blocks. As a class, they explored the idea that any closed shape with six sides is a hexagon, even if the sides are not equal.

Figure 3. *Teacher's sketch of the shapes students drew (through gestures).*

The students' understanding of regular and irregular polygons was continually facilitated as they thought about how to code a polygon. After free exploration with Sphero robots and graphing software that tracked the movement of the Sphero, students engaged in discussion about how to code a polygon, as seen in the following interaction:

> Teacher. Some triangles have all equal sides, and some triangles are really pointy and long and skinny. Some triangles have some big corners and some small corners. Some triangles point this way or this way. *(Gestures her hands toward two different sides of the room.)*
>
> Student. Triangles can go in any way!

Again, Figure 3 shows that the students were aware that even triangles can have different angles and directions. The teacher then showed them an example of code with three blocks, as shown in Figure 4.

Figure 4. *Example of code for a triangle.*

The following interaction shows the student's understanding that there must be three blocks of code for a triangle, because a triangle has three sides:

> Teacher. If I'm making a triangle, why did I start with putting three blocks of code?
>
> Student. Because the triangle has three blocks.
>
> Teacher. Has three . . . ? Do you remember what we call those? (*Makes a triangle with her hands and uses her right index finger to draw along the side of the triangle on her other hand.*)
>
> Student. Has three sides.

In this interaction, the teacher brought the student's attention to a mathematical term—*side*. However, instead of correcting the language the student used (*blocks*), the teacher amplified the linguistic cues with gestures to facilitate fuller student participation in the linguistically diverse classroom, as discussed by Takeuchi and Dadkhahfard (2019).

After this class discussion, the students worked in pairs to code the Sphero robots to roll in the shape of polygons of their choice. By the next class, Alex and Hakim had successfully made two quadrilaterals and a triangle. After this practice with the programming language, they challenged themselves by attempting to create a more difficult shape—a pentagon.

They started with five blocks of code for the five sides of a pentagon and programmed angles, speeds and time (all seemingly guesses) into the code. After their first test, they noticed that the Sphero had made an L shape, not a closed shape like they wanted. They decided to increase the numbers on the speed and then test the code again. They then saw that the Sphero had traced the same L shape but on a larger scale.

They concluded that they needed to change something in the code other than the speed. Alex took the tablet and stared at the wall, as if trying to picture a pentagon. He then changed the angles in the code. They tested the code again, with the new angles, and found that the Sphero was making a different shape and kept running into walls.

They decided to test their code in a bigger space (the hallway), and Alex changed the angles again. This time, he drew the shape of a pentagon in the air with his right index finger, moving the angles on the code with his left hand as he came to each point of the shape he was drawing with his right hand. He then turned to Hakim excitedly and shouted, "I fixed the program!"

By the end of this class, Alex and Hakim had successfully adjusted the angles in their code so that the Sphero almost made the shape they wanted; however, the lengths of the sides in the code were off, so the Sphero did not yet make a closed shape.

During the next class, Hakim worked to ensure that the angles in the code would make a pentagon and that he had the right number of seconds coded so that the Sphero would return to where it had started, making a closed shape. To do this, he drew the pentagon he wanted on a small whiteboard, as pictured in Figure 5. He drew lines as markers along the edges of the shape to estimate how far the Sphero should be moving along each side. He eventually got the code to a point where, as he said, "it [the Sphero] came back," making a closed pentagon shape, as pictured in Figure 6.

Figure 5. *Hakim using visuals to correct the code.*

Figure 6. *The pentagon shape created by the completed code.*

Through Alex and Hakim's creation of the code that made the Sphero move in a closed pentagon shape, we can see the thought process of computational thinking that they engaged in while getting to the shape they wanted. They started by creating a code with the right number of blocks but estimations for the angles, speeds and time, to see what the Sphero would do. As they became more fluent in the programming language, and as they came to understand how the code translated to the Sphero's movement, they were able to make more-accurate adjustments to the code to make the shape look more and more like a pentagon. After they had created a shape with more-reasonable angles, they used a visual tool—the grid on a whiteboard—to measure how far they wanted the Sphero to roll along each side. These acts of mapping out the shape and estimating the numbers for the code based on a drawing show that they were able to use their learned knowledge of shapes and programming to help them get the pentagon they wanted.

Painting with Polygons (Scaling)

In the next phase of the project, the teacher explained to the students how they would be using the codes they had created to paint their shapes with the Sphero. To do this, they would need to adjust their programs so that the shapes would be small enough to fit in the cardboard Sphero Arena (Figure 7).

Figure 7. *The Sphero Arena.*

The teacher talked to the students about the ways they could test their shapes to see if they would fit in the arena and gave them the option of estimating whether the shapes would fit. Through an open discussion on how they could scale their shapes, the students verbally showed that they understood the relationship between the code and how the Sphero would move. For example, one student suggested, "You could change how many seconds it goes for." This student and others demonstrated that they grasped how changing the time and the speed in the code would translate to the size of their shape, showing that they were figuring out how the mathematics and the technology worked together.

As they started to use the Sphero as a paintbrush, Alex and Hakim made sure that their pentagon would fit inside the Sphero Arena by estimating its size. They then went to the arena to test their program and paint their first shape. After testing the code, they found that their pentagon fit "perfectly on the paper," which they were ecstatic about.

With that, Alex painted a pentagon on his paper, and the Sphero moved around as expected to form a pentagon. It fit almost perfectly. During Hakim's attempt, the Sphero got stuck and was not positioned the way he wanted, as pictured in Figure 8. He was upset that his pentagon did not look the way he wanted. The teacher pointed out that even though it was not quite as they had pictured it, it was still a pentagon. She said, "Do you think that making a five-sided shape deserves a high-five?" Once the teacher had acknowledged Hakim for accomplishing such a difficult task, he was satisfied with and proud of what he had created.

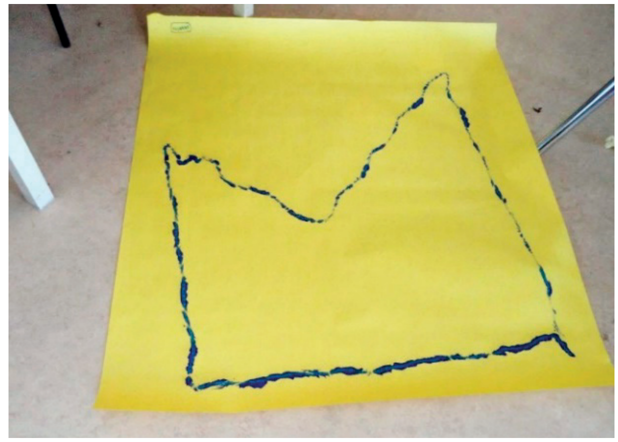

Figure 8. *Hakim's finished painting of a pentagon.*

By the last class, the students had all painted one shape on paper, using their codes. The teacher then encouraged them to paint a second shape so that it overlapped the first shape. She talked to them about thinking "like an artist" as they thought about how they could overlap their shapes, and she explained that artists overlap shapes in various ways and that often artists have shapes at different angles crossing each other. The teacher then explained that before they painted the second shape, the students would need to determine whether it would fit in the Sphero Arena by "eyeballing" (or estimating) it.

When the students went to work on their second shape, Alex and Hakim fixed one of their previously coded rectangles to make sure that it was a closed rectangle that would fit in the Sphero Arena. When they brought the Sphero over to the arena for testing, the teacher noticed that the angles were out of order and went over it with Hakim. She helped him see that the code made an L shape instead of a rectangle and then helped him reorder the code. When they tested the program and found that the rectangle fit, Alex got excited: "It's perfect! We can go in any rotation!" He explained that because of the size of their rectangle, they could put it anywhere on the paper, in any direction, and it would fit over the pentagon.

Alex and Hakim finished creating their artwork, and they put the rectangle in different spots over the pentagon, so that their paintings would be different, as pictured in Figures 9 and 10.

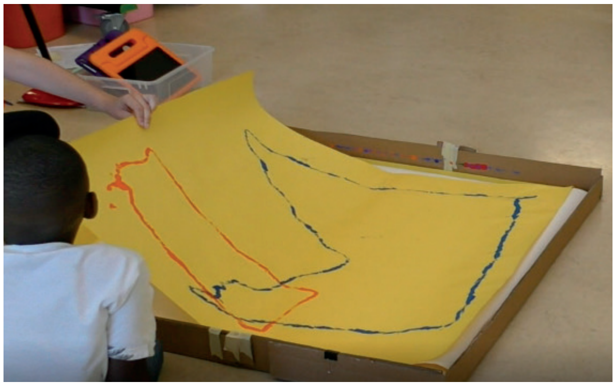

Figure 9. *Hakim's STEM final project.*

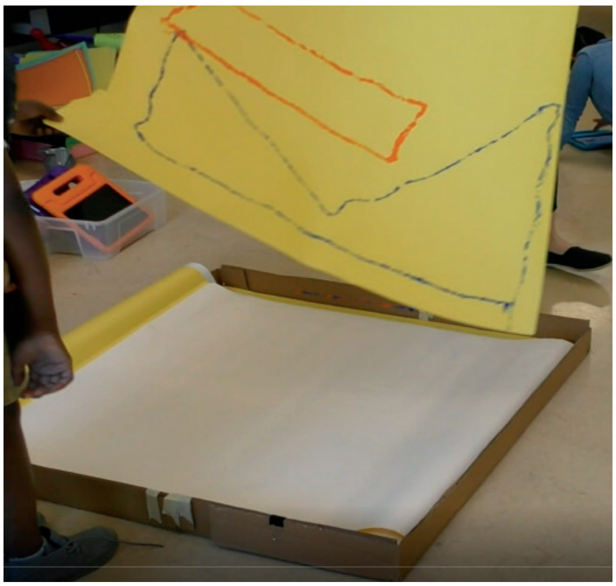

Figure 10. *Alex's STEM final project.*

By the end of this project, both Alex and Hakim showed an understanding of how the program worked and an ability to alter their code in order to fix errors in it until it ran the way they wanted. They also showed that they had developed emotional connections with the project. This was evident in their disappointment when the artwork did not look the way they had imagined it, as well as in their excitement and happiness after it finally came together and they could see the finished product.

Conclusion

If we were to ask Grade 1 students what a pentagon looks like, not many would picture what Alex and Hakim coded as a pentagon. The students in this STEM class started *seeing more* (Farris and Sengupta 2016) in geometry as they traced the movements of the Sphero robots they had coded. Coding was not the end in itself in this STEM class but, rather, a secondary skill taught as a means to explore geometry more deeply.

As it was the end of the school year, the teacher designed a project in which students could paint a take-home piece by arranging polygons of their choice. The students' pieces were all different and uniquely reflected their ownership, as seen in the case of Alex and Hakim.

Sphero robots were used in this lesson, but we are not necessarily advocating the use of Sphero (or any other commercial product that may perpetuate the instrumental agenda of STEM education). Nor are we implying that STEM class is defined by the use of technology itself. Instead, what we have shown here is how one technology can serve multiple purposes when the teacher's pedagogical hopes are clear.

To create this STEM class, the teacher removed siloed desks in the classroom to facilitate collaboration and to create an open space where students could move their bodies freely. The teacher brought in recyclable materials from home for students to transform through their designs. In this classroom, by tracing the movements of Sphero robots in the Cartesian coordinate system, students started to see the relationship between code and the elements of a polygon (angles and sides). As they painted with the Sphero, they encountered the notion of scale. In this pedagogical space, we demonstrated how students' mathematical conceptualization became genetic, which Piaget (1970, 2) characterizes as "continual construction and reorganization."

This early STEM class allowed the curriculum to be connected vertically, as well as horizontally. In terms of Alberta's mathematics curriculum (Alberta Education 2007), the lesson described here addressed key curricular expectations for each grade level—for example, "replicate composite 2-D shapes" (p 15) for Grade 1 and "describe, compare and construct 2-D shapes" for Grade 2 (p 18). However, the lesson could be adapted to stretch students' learning to deeper concepts—for example, "identifying examples of angles," "classifying angles" and "demonstrat[ing] the sum of interior angles" in geometric shapes (p 37) for Grade 6. This project could also be connected to Alberta's art curriculum (Alberta Education 1985)—for example, "shapes" and "movement of figures and objects" (p 5). In this sense, what we have documented in this article shows an expanded possibility for student learning in early STEM curriculum and pedagogy.

In these times, as we work collectively through the COVID-19 global pandemic, projects like this increasingly show their importance in early childhood education. In the midst of the pandemic, many educators are experiencing heightened difficulty in helping their young students understand abstract concepts, such as constructing and scaling shapes, with limited access to tangible learning tools that students can manipulate. Documenting pre-pandemic pedagogical innovation by teachers in public schools is, thus, important as we envision and reimagine post-pandemic education.

We asked the students, "What does STEM education mean to you?" Alex responded, "We can be scientists, and we can be anything!" As more crises arise in our world and our lives become more dependent on mathematical understanding (with and without technology), it is essential for young learners to see themselves in roles that will help advance the world they will live in.

Notes

1. See, for example, *STEM Education by Design: Opening Horizons of Possibility* (Davis, Francis and Friesen 2019), which was published in alignment with the teacher education course offered at the University of Calgary.

2. The names of the students have been changed to protect their anonymity.

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Culture—a Vital Part of Mathematics

Yvette d'Entremont and Michelle Voillot

Math was my (Yvette's) favourite subject in school. It was logical and practical, and I was good at it. I now realize that I was good at applying the rules and procedures necessary to solve the problem. However, very few of my classmates felt as I did.

Many students perceive mathematics as being difficult. Mathematics "is one of the most important subjects in schools in a modern society" (Bishop 1997), yet it remains a challenge to master. It has often been perceived as a body of facts, procedures, theorems and algorithms (Banks 1991; Bishop et al 1993). Some students grasp the meaning related to mathematical concepts, whereas others simply try to memorize what they see as a set of abstract rules (Barta, Eglash and Barkley 2014; Bishop 1997). In 1992, Teen Talk Barbie uttered the words "Math class is tough." After backlash from women's groups, that phrase was quickly removed from Barbie's vocabulary. It did, however, leave the impression that "math is tough" was a common sentiment. As a young teacher, I realized that most of my students saw mathematics as abstract, difficult and not related to anything real.

An important aspect of mathematics that should be considered in mathematics instruction is culture. *Culture* can be defined as the beliefs, values, attitudes, customs, social relationships, art and literature that define an ethnic group (Abidi 1996; Banks 2008). In the mathematics classroom, teachers should consider the cultural backgrounds of the members of the classroom, the school and the community.

Many teachers of subjects such as mathematics and the sciences are under the impression that mathematics is a noncultural subject (Banks 2008; Dalley and d'Entremont 2004; Lee 2003). Mathematics is often taught as a set of universally accepted facts (Bishop et al 1993; D'Ambrosio 1990), leading to the perception that it is a culture-free discipline (Banks 1993; Lee 2003; Peterson and Barnes 1996; Secada 1989). Nothing could be further from the truth. Mathematics is not waiting to be discovered but, rather, can be constructed through culturally symbolic representations.

Acharya (2015) suggests that mathematics seems to be influenced by learners' cultural backgrounds and that it should be connected to students' lives and cultures. Similarly, Glanfield and Sterenberg (2020) discuss culturally responsive education in mathematics.

The purpose of this article is to demonstrate how basic mathematics is rooted in culture through the example of the cultural activity of quilting.

Culture, Teaching and Learning

My teaching career began in an Acadian school in southwest Nova Scotia. The community was rich in personal resources and practical applications: fishermen, carpenters, women who made beautiful quilts and the Mi'kmaq who made sturdy woven baskets. Although integrating culture and school subject content is not a new concept, back then I did not take advantage of the resources available to me that would have made mathematics real to my students.

D'Ambrosio (1985) uses the term *ethnomathematics* to describe the relationship between culture and mathematics. It is important to understand how culture relates to students and their ways of thinking and learning mathematics (D'Ambrosio 2001). The National Council of Teachers of Mathematics (NCTM 1991) *Professional Standards for Teaching Mathematics* highlights the importance of building connections between mathematics and students' personal lives and cultures. Rosa and Orey (2009) have examined mathematics in a variety of cultural contexts and have concluded that mathematics cannot be culture-free, as learning is influenced by society. As Stevens, Sharp and Nelson (2001, 376) write, "When mathematics lessons are linked with personal experiences, typically, the result is that the student gains a stronger understanding of the content than if the lessons are isolated and unconnected." Personal experiences relate to students' multiple ways of knowing that integrate their background knowledge and home and community experiences.

The term *culturally relevant pedagogy* has been used to stress the importance of using home and community cultural experiences to cultivate learning with understanding (Gay 2002; Ladson-Billings 1994). Gay believes that "there is a place for cultural diversity in every subject taught in schools" (p 107) and notes that "research indicates that culturally relevant examples have positive effects on the academic achievement of ethnically diverse students" (p 113). Students' learning is influenced by their cultural backgrounds (Stylianides and Stylianides 2007). Using cultural knowledge and personal frames of reference can make learning encounters more relevant and strengthen connections to mathematics, thereby enhancing learning (Boaler 1993; Eglash 1997; Rosa and Orey 2010).

Harackiewicz, Smith and Priniski (2016) state that interest is essential to academic success. Although interventions that develop student interest matter in all educational contexts, the authors note that such interventions are most needed in academic domains that many students do not initially find interesting, as well as in domains in which student interest tends to decline over time, such as mathematics and other STEM (science, technology, engineering and mathematics) subjects. The relationship between interest and learning seems self-evident: the more interest students have in a topic, the more they are willing to learn about it (Schraw, Flowerday and Lehman 2001).

Various pedagogical approaches have been used to demonstrate the importance of the relationship between personal interest and meaningful mathematics. Wiseman et al (2020) discuss three learning opportunities that demonstrated how Indigenous and Western ways of knowing can work together. The projects took place in three Indigenous communities: the Mi'kmaq Territory in Nova Scotia, the Anishinaabe Territory in Ontario and the Kanien'kehá:ka Territory in Quebec. The Nova Scotia project—Show Me Your Math—demonstrates the impact of making meaningful personal connections to mathematics.[1]

Culture, Mathematics and Motivation

Integrating culture into mathematics can be challenging. However, as Rosa and Orey (2010, 24) write,

> There are evidences from research (Bishop, 1988; Boaler, 1993; Eglash, 1997; [Orey & Rosa, 2007]; Zaslavsky, 1996) that including cultural aspects in the curriculum will have long-term benefits for mathematics learners; that is, cultural aspects contribute to recognizing mathematics as part of

daily life, enhancing the ability to make meaningful connections, and deepening the understanding of mathematics.

Integrating culture and content to illustrate mathematical concepts can be done by using examples, artifacts and information from various cultures (Banks 1993). Many students are not motivated to learn mathematics. However, they become motivated to learn when teaching is related to their real world and identity (Zaslavsky 2002). Activities and concepts that are personally relevant are often more interesting to students and can motivate them to learn something about a new topic (Sansone and Thoman 2005). As students' knowledge increases, they find more value in the topic and make connections to things they already know and care about. In turn, this can further trigger their curiosity and motivate them to continue learning about the topic.

Motivation drives many behaviours, including the active process of learning. Learning mathematics in a cultural context can provide motivation for students and enhance the link between school mathematics and things students can relate to (Boaler 1993). Educators and psychologists have paid increasing attention to learning with understanding as an important goal for students (Stylianides and Stylianides 2007). Fredrickson and Branigan (2001) suggest that interest can be considered a motivational tool, as interest leads to short-term exploration and, over time, to increased personal knowledge and improved cognitive abilities.

Transforming Pedagogy

As Alberta's K–9 mathematics program of studies (Alberta Education 2007) states, students by nature are curious and want to learn. They bring their personal values and beliefs from home and community into the classroom. The references to home and community in the program of studies should not be ignored, as mathematics has more meaning when integrated with culture (Wiest 2002).

A variety of cultural artifacts and expressions (such as weaving, knitting, beading, rug making and quilting) make use of mathematics. Studying the patterns in these artifacts can increase understanding of mathematical concepts.

The quilt is one of many cultural traditions and artifacts that can be used to teach a variety of mathematical concepts in a context that is motivating and interesting and that supports a connection between subject and student interests. A quilt is a cultural symbol. Whereas traditional quilts were made for practical use, modern quilts are not only practical but also works of art. This form of artistry involves improvisation, experimentation and creativity—and mathematics. Even a basic quilt involves tessellating patterns of geometric shapes, which requires knowledge of transformational geometry. (For example, determining how tessellating shapes will fit together with no gaps requires some thought.)

I made my first quilt (Figure 1) at the beginning of my teaching career. It wasn't fancy, the pattern wasn't complicated, and the colours were quite dull, but it was proof that I could take my place among the quilters in the family. The many quilt shows and quilting stores in Alberta indicate that quilting is not a dying pioneer art. It is, in fact, an example of cultural knowledge that is passed on to the next generation, and "the acquisition of cultural knowledge is important to all peoples" (Alberta Teachers' Association 2016, 36).

Figure 1. *My first quilt.*

A simple quilting pattern (as shown in Figure 2) illustrates symmetry, rotations, reflections and translations. By analyzing simple quilting designs, students can determine the lines of symmetry and the transformations involved.

Once students are familiar with the properties of symmetry and with transformations, they can analyze a variety of quilt block designs, as shown in Roscoe and Zephyrs (2016, 25).

Designing a quilt block is an excellent way to explore symmetry, tessellations and transformational geometry. Providing students with a blank 3 × 3 or 4 × 4 grid (Figure 3) and asking them to create their own design (Figure 4) allows them to put into practice the properties they have studied. Students can then explain what properties they used to create the design. The same design can look quite different in other

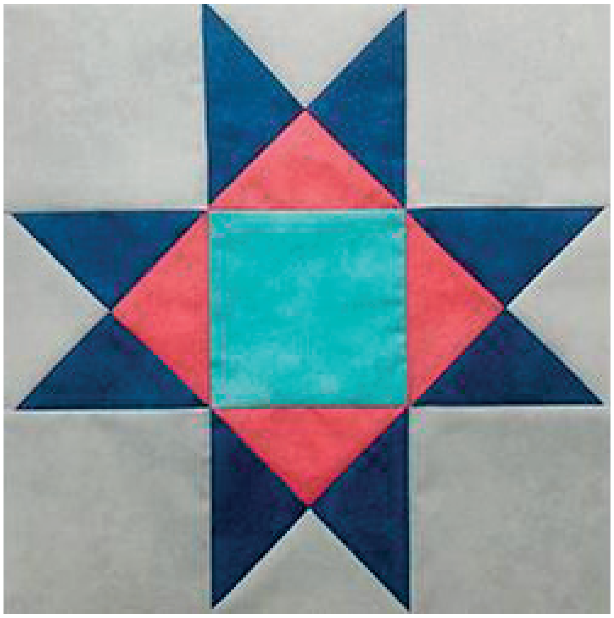

Figure 2. *A basic quilt design.*

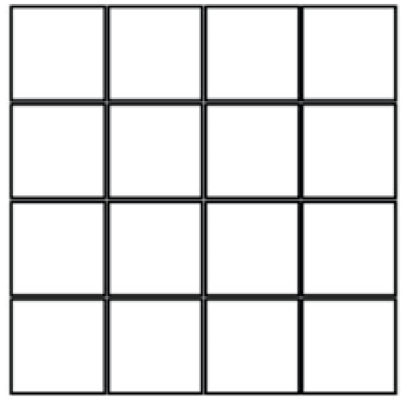

Figure 3. *A 4 × 4 grid.*

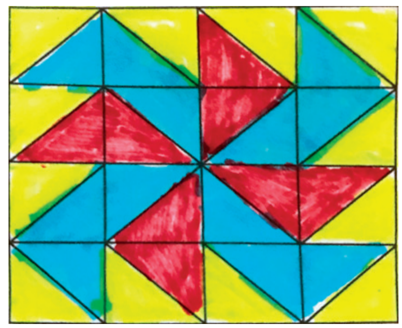

Figure 4. *Quilt design.*

colour combinations. Students' creativity and their understanding of the concepts will determine the designs. Upon completion, the blocks can be displayed together as a paper quilt. This activity allows students to practise mathematics as a quilter would.

Harackiewicz, Smith and Priniski (2016) recommend providing students with content and academic tasks that facilitate the connection between academic topics and existing student interests. Creating a quilt block design is an interesting, motivating and practical way to teach a variety of mathematical concepts, as well as encouraging creativity and mathematical investigation.

Members of the community can be an important educational resource. A community member who quilts can demonstrate quilting for students while explaining the process, the pattern, the colours and the choice of materials. The many mathematical concepts that are involved in making a quilt can be pointed out during the demonstration.

This same activity can be done using other artifacts, such as the Ukrainian pysanka, the Métis sash or the French Canadian ceinture fléchée. Teachers can incorporate these activities into the mathematics curriculum.

Conclusion

Many students and teachers believe that no connection exists between mathematics and culture, but nothing is further from the truth.

Studies have demonstrated that culture is a vital part of mathematics (Eckermann 1994; Massarwe, Verner and Bshouty 2012; Zaslavsky 1996, 1998). Mathematical activities that reflect cultural traditions can be used to teach mathematical concepts, and mathematics knowledge is inherently tied to cultural practices (Nasir, Hand and Taylor 2008). The education that students receive at home and in their communities greatly influences how they learn, and they bring those values and beliefs to school.

Integrating familiar objects and contexts in the teaching of mathematics can facilitate learning. Mathematics can best be understood when it is experienced in the context in which it is used. Gay (2002, 106) concludes that "when academic knowledge and skills are situated within the lived experience and frames of reference of students, they are more personally meaningful, have higher interest appeal, and are learned more easily and thoroughly."

Promoting interest in the classroom contributes to a more motivated and engaged learning experience, as interest is a powerful source of motivation that energizes learning (Harackiewicz, Smith and Priniski 2016). Interest has been cited as a central component of motivation in learning, and interest in activities can be developed through finding meaning and value in those activities (Hidi and Renninger 2006).

Relating concepts and culture helps students discover relationships between mathematics, the real world and the history of various cultures (not only their own). This creates a learning environment that is interesting and motivating. To create and maintain a culturally relevant pedagogy that reflects students' cultures, teachers must continually re-examine their pedagogical practices and strategies.

Note

1. Show Me Your Math, https://showmeyourmath.ca (accessed September 29, 2021).

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How Do You Feel? Using Scribblers in the Math Classroom to Elicit Mathematical and Personal Connections

Josh Markle and Raeesa Shivji

In this article, we discuss using what we call math scribblers in two very different contexts: in a mathematics curriculum and instruction course for preservice teachers and in a Grade 3 mathematics classroom.

Despite the varying contexts, we used the math scribblers to similar ends. In particular, our focus was to heighten students' awareness, of both the *doing* of mathematics and *how they felt* in doing mathematics. The scribblers, which we will describe in greater detail, functioned as both a repository of students' mathematical problem solving and a space where students could reflect on how they attended to the mathematics, how it resonated in their lives and the feelings it elicited.

In this article, we discuss artifacts of student work from both contexts and suggest implications for classroom practice.

Positioning Ourselves

Josh is an instructor in the Faculty of Education at the University of Lethbridge, where he teaches various courses, including a curriculum and instruction course for mathematics majors. Raeesa is a recent graduate of the education program, in which she

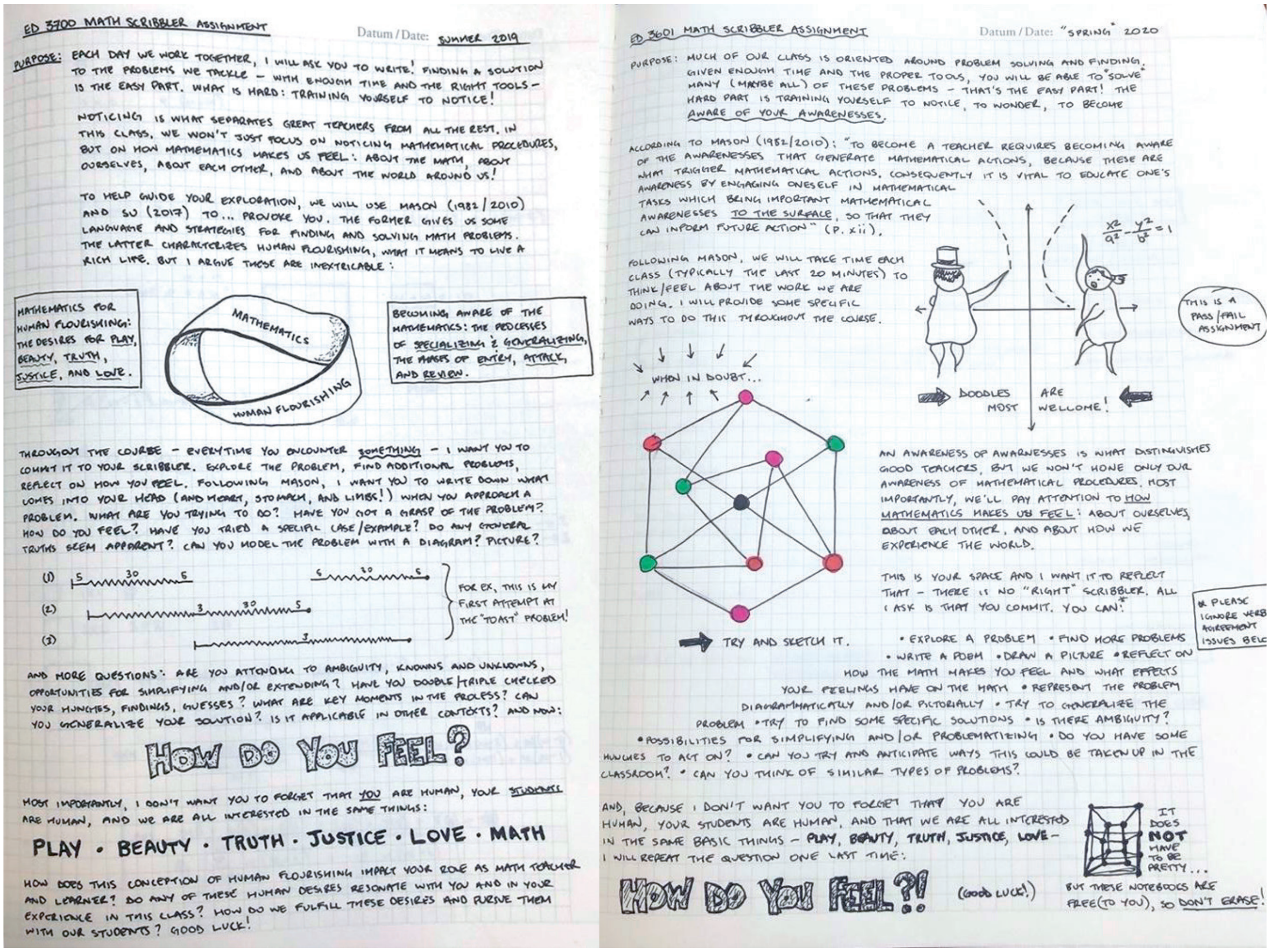

FIGURE 1. *Math scribbler assignments from a curriculum and instruction course for preservice teachers.*

majored in mathematics education. We worked together first in a general curriculum and instruction course (fall 2018) and then in a curriculum and instruction course for mathematics majors (spring 2020).

It was in the latter course that Josh assigned the math scribbler. A copy of the assignment was stapled to the inside of a blank notebook distributed to each student. Figure 1 shows the assignments from summer 2019 and spring 2020. Both emphasize heightening students' awareness of how they do mathematics and end with the question, How do you feel?

Raeesa had the unique experience of working with the math scribbler both as a student and as a teacher. In her final teaching internship, in which she held a half-time teaching assignment in a Grade 3 classroom for four months, she developed and implemented her own version of the math scribbler with her students. This idea was included in her professional inquiry project, in which she investigated, researched and implemented ideas on the importance of reflection in the mathematics classroom. Her idea was that reflection in mathematics could help students better understand fundamental math concepts, deepen their comprehension and keep them engaged. One way she brought reflection into her classroom was through the use of math scribblers.

Using the Math Scribbler with Preservice Educators

I (Josh) teach math-focused curriculum and instruction courses for both preservice teachers majoring in math and preservice teachers majoring in other subjects who want to explore math as a teachable subject.

In my experience, the relationships those two groups have with doing mathematics are complex and often significantly different.

Students in the former group tend to be comfortable with mathematics and have a history of success in the subject. This is perhaps unsurprising, given their choice to become math teachers. Yet, despite their proficiency in solving mathematics problems, I often observe that these students struggle to make explicit their problem-solving processes and personal connections to the material. Mathematics is conceived as something to be done, not lived.

A similar issue arises with the students who are not mathematics majors—but for different reasons. These students often associate mathematics with failure, anxiety and frustration, which also seems to be grounded in their history with the subject. In the same way that proficiency and familiarity might affect students who major in math, anxiety and fear hamper these students from making rich personal connections to mathematics.

I implemented the math scribbler as an assignment in my classes to cultivate a heightened awareness in students of their approach to problem solving and to give them the opportunity to realize that mathematics was a part of their lives that they had reason to value. Mason, Burton and Stacey (2010, xii; italics added) note that "to become a teacher requires becoming *aware of the awarenesses* that generate mathematical actions, because these are what trigger pedagogical actions." The scribbler was a space where students could engage in tasks oriented toward making explicit how they approached mathematics, not simply finding a solution.

However, I wanted students to heighten their awareness of more than just their problem-solving processes. I also wanted them to reflect on how mathematics manifested in their lives. Discussion prompts for each problem asked students to make connections to Su's (2017) mathematics for human flourishing and how his five human desires—play, beauty, truth, justice and love—resonated in a piece of mathematics. Above all, I wanted students to make an emotional connection to the mathematics they did. This was often provoked by returning to a simple prompt throughout the problem-solving process: How do you feel?

In their discussion of mathematical identity, Black et al (2019) identify the importance of emotional experiences in doing mathematics. "An experience of emotion," they write, "can become crystallised into a mathematical identity if one encounters reiterations of that emotional experience, allowing time for the essential reflections on the experience that may be verbalized as 'I am . . . .'" (p 381). The math scribbler became a space for those encounters.

Two Perspectives on Using the Math Scribbler

In this section, I (Raeesa) discuss my experiences with using math scribblers both as a student and as a teacher.

Using the Math Scribbler as a Student

When Josh first introduced the math scribbler assignment in our curriculum and instruction class, I immediately started to worry. A math assignment with no clear rules or guidelines was distressing to someone like me. I wanted to know what I had to do to get a good grade and what Josh expected from me as a student.

The open-endedness of the math scribbler made me uncomfortable, but as we progressed through the first week of the course, I began using my scribbler as a place to jot down questions, answers, connections, ideas, thoughts and feelings about what we were learning and the math we were interacting with. I began to really like the openness of the assignment. It was different from anything I had experienced before. As I continued to use my scribbler, I worried less about the grade attached to it and focused more on how much freedom I had with it and how much I felt I could express myself through it.

My math scribbler was my own personal place to voice anything related to mathematics. I would connect what we were learning to previous mathematics I had learned and reflect on how it made me feel. If something challenged me, I would express my anger about not being able to figure it out. I would also express my confidence and happiness when I was able to get past a math problem that had tested me.

Using the math scribbler was a unique way of approaching a mathematics class, and I found it engaging and personal. I wasn't worried about how neat my scribbler looked, the grade I would get or whether my answers were correct. I knew that Josh cared more about my feelings and ideas when I was interacting with the math: How did I approach the problem? How did I revise my thinking to accommodate new information? What did the math remind me of? How did the math make me feel?

Figure 2 shows pages from my math scribbler. The top right and bottom left panels show my thinking around the use of manipulatives in the classroom and how I felt about using them.

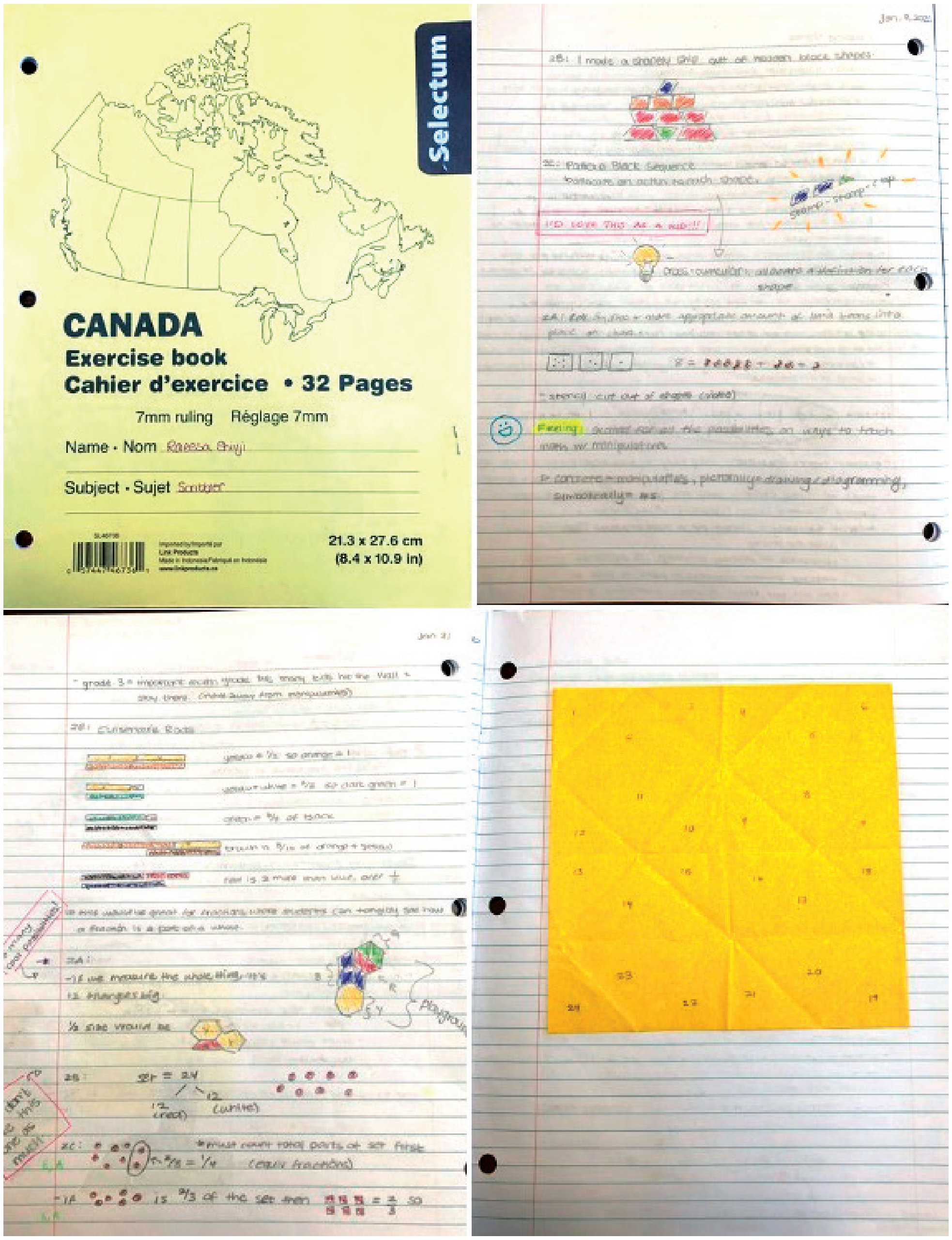

Figure 2. *Pages from Raeesa's math scribbler.*

Using the Math Scribbler with Grade 3 Students

When I began my final four-month teaching internship, I knew that I wanted my Grade 3 students to have a similar experience with math scribblers, but I was also uncertain about how to implement the scribblers and how they would be received. I was especially intimidated as an intern teacher, a situation in which I would be assessed by more-experienced colleagues based on the lessons I taught. But I had thoroughly enjoyed the assignment as a student, and I wanted to give my young students the opportunity to enjoy it too.

I introduced the scribbler assignment as a way for my Grade 3 students to express their mathematical ideas, thoughts and feelings to me. I used pictures and keywords to modify Josh's scribbler assignment, since many of my students were still learning to read (Figure 3). When I told my students that I had done the same assignment for my university class, they were excited to try it.

Here, I describe work my students did on a numberless word problem that I created to address

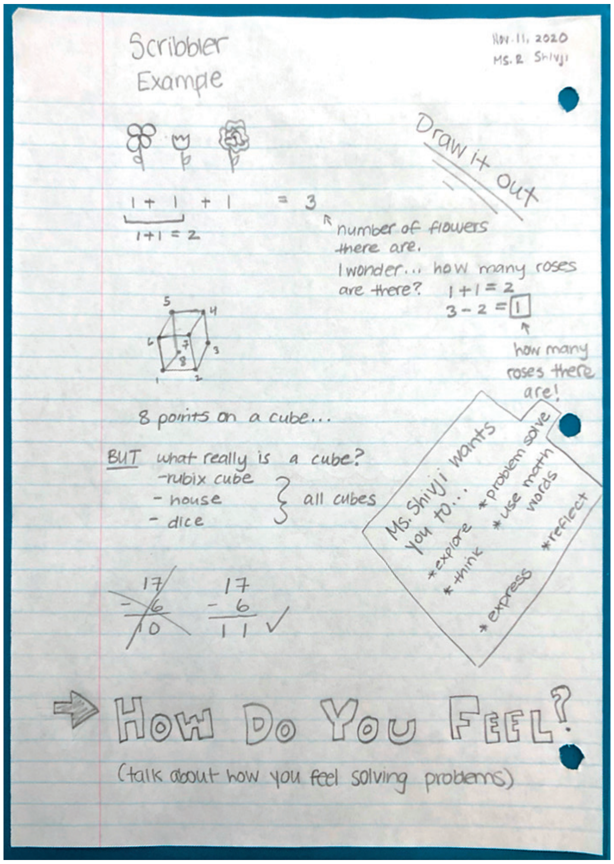

Figure 3. *Raeesa's Grade 3 scribbler assignment.*

one- and two-digit addition. I presented the following problem to my class:

> Ms Shivji had some money in the bank. She added a few more dollars yesterday.

My objective was to have students think critically about addition. Along with the numberless problem, I provided guiding questions and prompts to encourage personal and emotional connections to the problem. The students were forced to challenge the way they thought, to dig deep, and to make connections with the word problem and what they had learned previously.

The students were uncertain at first. This was a task different from anything they had seen before. Despite their uncertainty, they bravely embraced the task. Many of them responded to my written and verbal prompts and used them as starting points for further exploration, often feeding off the ideas of their classmates. Some students doodled as they thought, but as they began to mathematize the problem, their hands started shooting up and they were eager to share what they were noticing.

I first asked the students, "What math do you notice in this problem?" This was meant to get them thinking about where math exists, other than in the form of a number. For a Grade 3 student, this is a tough concept to grasp, but my students were eager to find ways to show that there was math in the problem. At first, they expressed how difficult it was for them to see the math. I encouraged them to look harder and told them that there was indeed math in the problem.

One student wrote that there was "no math at all" (Figure 4). But, after some discussion, the student wrote that I had some money in the bank and I added a few more dollars. This student, like others, was able to realize that the phrases *some*, *added* and *a few more* were related to amounts. They were mathematical words. The students immediately began to write the problem down in ways they considered to be more mathematical.

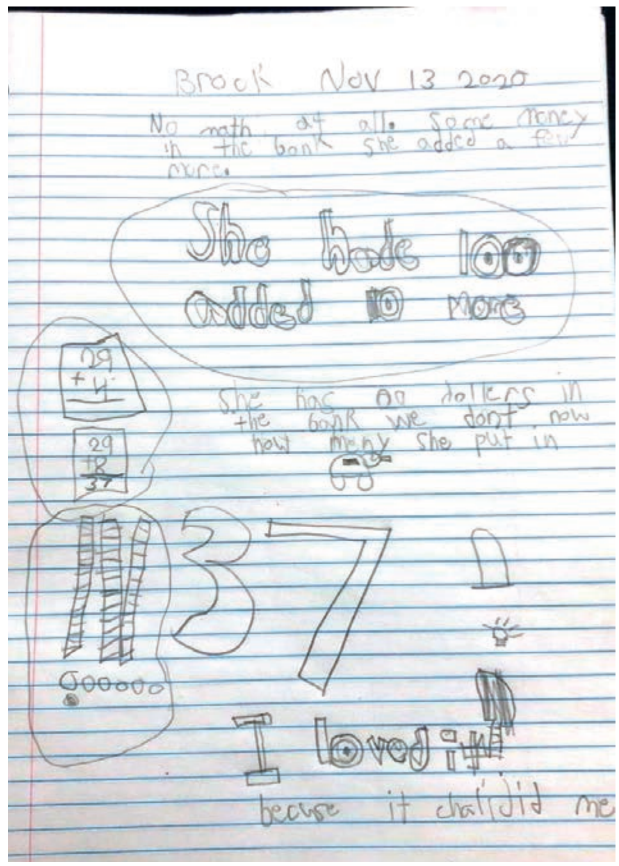

Figure 4. *A student beginning to see some math in the problem.*

Gradually, I provided additional information and a number, followed by another piece of information and another number, until the students had all the information and numbers, as well as a question to solve. After introducing a new piece of information, I would ask students questions such as the following:

- How does this change what we know?
- What else do we need to know in order to solve this problem?
- What question could I ask you about this situation?
- What type of problem is this?
- Can you estimate an answer to this problem?

I consistently encouraged students to answer these questions in their scribblers. The questions got them thinking about the problem critically, as well as mathematically, and allowed them to build on what they knew in order to expand the problem.

One student noted that we would need to know whether the problem involved addition or subtraction (Figure 5). This student was also able to recognize that we needed to know how much money I had put in the bank in order to solve the problem (which at that point didn't actually present a question to solve).

The students' math scribblers captured not only the mathematical products of their work but also their processes. The scribblers reflected their heightening *awarenesses* (Mason, Burton and Stacey 2010, xii). From the beginning, I encouraged the students to tell me how the mathematics made them feel, no matter what that feeling was. I encouraged them to write down how they felt at the beginning and at the end—to see if their feelings had changed. I told them that all feelings were valid. Whether they were confused, excited or frustrated, I wanted to know.

When the students had all the information and a question to answer, I encouraged them to show their work while solving the problem and to use strategies they had learned in class—such as drawing a picture, counting forward from the biggest number and being mindful of how they felt. I challenged students who finished early to find other ways to solve the problem. Many tried solving the problem in multiple ways and recorded their strategies in their scribblers. Some of the strategies they used were vertical addition using regrouping, base-10 blocks, drawing pictures, and using a number line or hundreds chart. They were able to see that the solution could be reached in

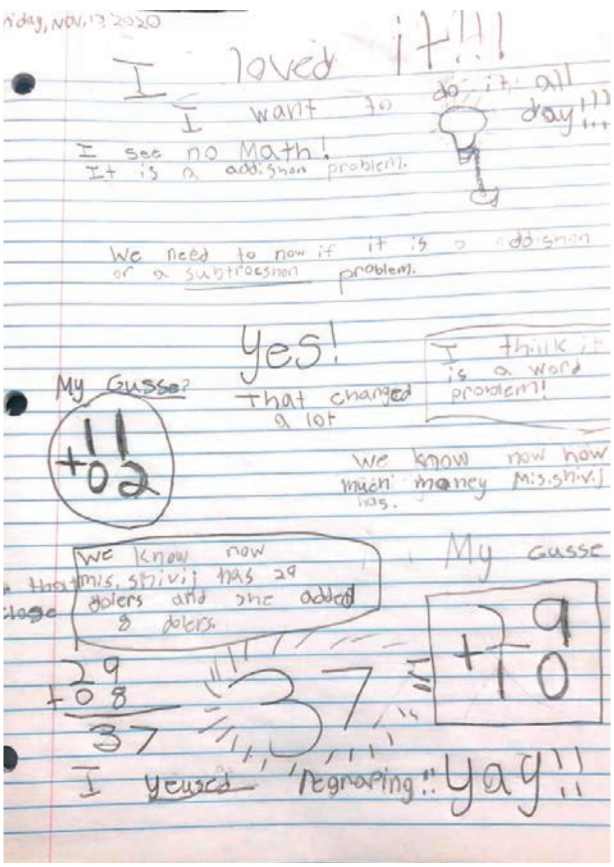

Figure 5. *Modelling a problem.*

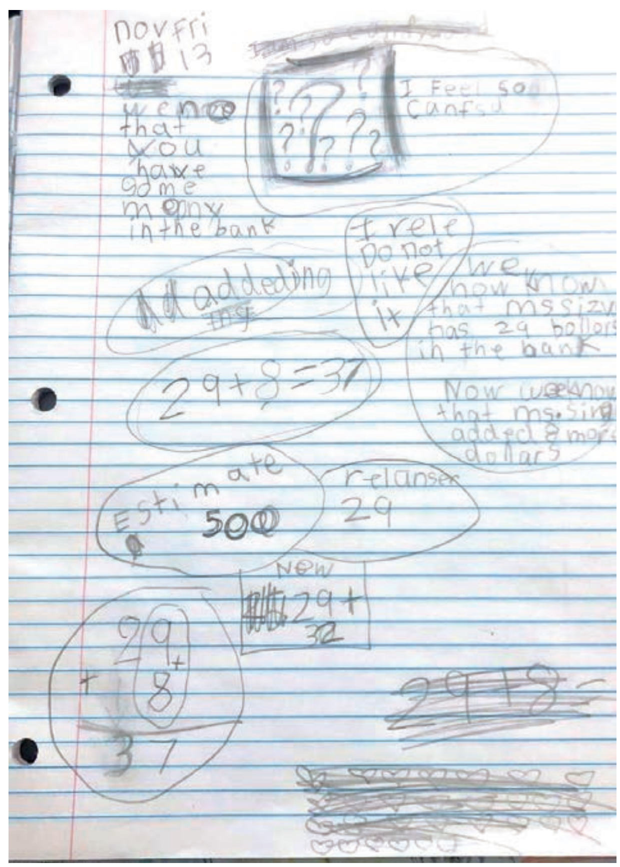

Figure 6. *A student's feelings of confusion and frustration, as expressed in the math scribbler.*

multiple ways and that they often had different ways of approaching the problem.

For me, the most valuable aspect of this process was being able to observe the emotional connections my students had with mathematics. The student who at first saw "no math at all" in the problem reported that they had "loved" the task (Figure 4). Other students wrote about how the work made them feel good, and others expressed frustration and confusion (as shown in Figure 6).

I was so glad to see my students expressing all these feelings in their scribblers, but I paid particular attention to the students who reported feelings other than simply enjoying working on the problem. Telling your teacher that you are confused is daunting, and I hoped that my students would trust me enough to tell me what they were feeling. The student in Figure 6 expressed her confusion through both question marks and words. At the top of her page, she drew a box filled with question marks and wrote, "I feel so [confused]." She even wrote, "I [really] do not like it." This allowed me to reassure her that it was perfectly fine to be confused and to not like something. Some students reported being scared. One student was so confused that all he wrote was "What are we doing what." This helped me understand which students needed more support and which students I would need to closely guide the next time we used our scribblers.

Implications for Classroom Practice

We have discussed using math scribblers in disparate contexts—a university course and a Grade 3 classroom. In both cases, math scribblers became a space for students to focus on the process of doing mathematics and to become more aware of their personal connections to doing mathematics.

Raeesa's account of her experiences as a student and the work of her Grade 3 students demonstrates some of the benefits of using a math scribbler from a student's perspective. The math scribbler becomes a space for making visible the messiness of mathematical thinking, making explicit one's mathematical awareness and voicing one's feelings about doing mathematics (whether frustration or joy).

From a teacher's perspective, the benefits are clear. In both Josh's curriculum and instruction course and Raeesa's Grade 3 classroom, the math scribbler became an invaluable formative assessment tool, yielding rich insight into not only students' mathematical problem solving but also their emotional connections with the subject.

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Supporting Preservice Elementary Teachers' Growth Through Studying the Historical Development of a Mathematics Topic

Giang-Nguyen T Nguyen and Tiffany Marlow

In this article, we review instructional methods used with preservice elementary teachers in a teacher education program. Specifically, we reflect on a pedagogical approach in a mathematics methods course that focused on researching the historical development of a mathematics topic and how that approach supported preservice elementary teachers' growth in learning and teaching mathematics. We share information about the assignment, discuss the results and suggest implications for future practice.

Researchers have suggested several methods to help teachers overcome mathematics-related anxiety, including engaging in reflection through journal writing, using manipulatives (Brown, Westenskow and Moyer-Packenham 2012), and researching the history of mathematics to foster their beliefs and attitudes about teaching and learning mathematics (Charalambous, Panaoura and Philippou 2009).

A recommended approach is promoting preservice teachers' reflective processes before they enter the classroom. According to Leijen et al (2014), promoting reflective processes fosters the development of preservice teachers' reflective professional competencies as a way to support their practical knowledge. Metacognition plays an important role in developing the link between theory and practice and the instruction of practical knowledge. *Metacognition* can be defined as "the capability to gain conscious insight into one's thinking processes" (Brycz, Brejwo and Imach 2018, 289). *Metacognition* refers to what we know about our cognitive processes and how we regulate those processes to facilitate learning and memory (Schraw and Moshman 1995). Reflective processes tend to be performed infrequently in teacher education programs; as a result, preservice teachers often lack reflection skills as compared with inservice teachers (Juklová 2015).

The instructional approach discussed here gave preservice elementary teachers opportunities for reflection to better understand how it supports their growth in terms of knowledge for teaching and learning mathematics.

Instructional Method

In a mathematics methods course for preservice elementary teachers, students completed an assignment on the historical development of a mathematics topic, including

- the origin of the topic and its development,
- how the topic is applied in the real world,
- how the topic is taught at various grade levels, and
- instructional practices that foster student learning about the topic.

After researching their selected topics, the preservice teachers responded to the following two prompts put forward by Clark (2012, 83):

- "What mathematical knowledge related to {*mathematical topic*} do you have now that you did not have before? In what ways did studying some historical aspect(s) of the topic contribute to this knowledge?"
- "In what ways has studying the historical development of {*mathematical topic*} helped you to think about teaching mathematics in general?"

Preservice Elementary Teachers' Reflections on Their Learning

The responses to the prompts were coded into themes focused on preservice elementary teachers' growth as learners and as future teachers. Selected examples of the responses are shared in this section.

Reflection as Learners

The preservice elementary teachers referred to a change in their mathematical knowledge, including

- developing a new understanding of the rules they had memorized,
- deepening their mathematical knowledge,
- seeing real-world applications of a concept, and
- identifying the connection between mathematics and other subjects.

They felt that exploring the origins, development and modern uses of a mathematical topic had enhanced their knowledge. As one preservice teacher noted,

> Before I had completed my research on decimals, I had no historical background knowledge on decimals. I had only known how to compute mathematical equations using decimals and how to convert fractions into decimals. Their origin was surprising to me. Now that I have researched how decimals developed throughout history, I have a better knowledge of where decimals came from and their original purpose as a means for currency and measurement of weight. (Fall 2015)

They also indicated a change in how they felt about the mathematical topic they had researched, such as

- a deeper understanding of the topic,
- a new appreciation of mathematics and
- a new understanding that mathematics learning takes place in community, not only as individuals.

As one preservice teacher shared,

> I also learned a lot about the history of exponents, how they all started, and how exponents were developed. I think it is interesting that multiple mathematicians had a role in the discovery and development of exponents and each one built off of the other. (Fall 2015)

Several of the preservice teachers reflected on ways to connect their topic to real-world events or to other mathematics topics. They often made connections to their own experiences, such as previously acquired historical knowledge or personal areas of interest. One preservice teacher noted,

> I learned that tessellations are determined by the measurement and sum of angles, and that some shapes will tessellate and others will not. By researching the historical aspect of the topic I was able to learn the known origin of tessellations and how they were artistic before they were recorded as a mathematical concept. (Spring 2016)

Reflection as Future Teachers: Pedagogical Considerations

The preservice elementary teachers' reflections focused on the pedagogical aspects of teaching mathematics, particularly when looking at it from their future students' perspectives, as seen in the following two responses:

> It has helped me think about ways to teach things in a perspective that the students understand the whole meaning of the concept from the beginning and not to just assume they will learn the true purpose later. If they do not learn the whole meaning at a younger age, they will have a hard time understanding why the concept has more meaning at a later age. (Fall 2015)

> In preparation of this research, I was skeptical about a paper on fractions. I thought this paper would be boring and factual in nature. I believe that I took more time researching this paper than I did writing because of the mathematical knowledge that I was learning about teaching fractions and how easy fractions should be to students. I've found that the more a teacher knows about a concept, the more the students will learn. (Spring 2016)

They reflected on how the experience of researching a mathematics topic had changed their beliefs about the nature and purpose of mathematics, the role and importance of learning mathematics, and the applicability of mathematics to the real world. One preservice teacher reflected,

> Overall, studying the historical development of [the concept of] area has helped me view the teaching of mathematics in a better way. I now see that there are many visualizations to help students truly understand what area is and can be applied to. For instance, if I can help students visualize an Egyptian agricultural plot, then they can remember what area really is. This will help me be a facilitator of knowledge for my students because they will be able to create meaning on their own by thinking about the interesting information I provide. Also, I have realized that arts and crafts can be included into a math lesson, such as area. This will help me create a fun and interactive learning environment when teaching math. In conclusion, the history of area in and of itself is interesting, but it can also be used to help students learn about area in our present day and time. (Fall 2015)

The preservice teachers reflected on pedagogical considerations. Specifically, they considered various approaches to teaching mathematics that go beyond using a textbook and described teaching mathematics in new or creative ways. For example, two preservice teachers discussed specific pedagogical practices:

> Studying fractions has made me realize that a lot goes into teaching mathematics. It takes time, effort, and creativity. It is not like reading where you are teaching the same concepts over and over. (Fall 2015)

> Though it was quite a challenge, the study of the history of teaching fractions provided me with a perspective on teaching that concept that will definitely benefit me as an educator. I learned that in America we are often too reliant on textbooks, when the real learning takes place in a constructivist environment, and can be created by the students, and merely guided by the teachers. It gives me confidence in knowing that teachers and learners share the same goals throughout time and across the continents. (Fall 2015)

They envisioned the importance of knowing the historical development of a mathematics topic for both themselves and their students. Through understanding the creation and evolution of a topic and why it remains applicable to the curriculum, the reason behind learning a concept becomes more clear, as seen in the following comments:

> When I think about mathematics in general I think about drill and practice of equations. . . . Through the research I have noticed how much math I use in my everyday life without even knowing it and how important it is to teach it correctly and to make it fun for students. (Fall 2015)

> I think that looking into history showed me how to bring number sense to life in other subjects. Incorporating mathematics into all subjects shows students the real world connections for future careers and trades. This will help students see the value in number sense and the importance in learning it. (Fall 2015)

The preservice teachers indicated that they could now see the relevance or importance of real-world experiences when teaching and learning mathematics. Additionally, they were able to see the social and cultural influences that led to the development of the topic. This realization helped them understand the relevance of the topic as they discovered how it continued to be applied over time. One preservice teacher noted,

> Learning about how differently we teach fractions today compared to when I was a young student was fascinating to me. I now have a better understanding of how today's approach to fraction instruction is rooted in ancient practices, particularly with regard to using real-world contexts and modeling. I enjoyed learning through examples and non-examples about how teachers have struggled to convey meaning using realistic scenarios even in ancient times. (Fall 2015)

Discussion and Implications

This pedagogical approach supported the development of the preservice elementary teachers' practical knowledge about their selected topics and influenced how they thought about teaching mathematics in general.

The preservice elementary teachers' responses to the reflection prompts show that they were thinking about their future teaching through the lens of their previous experiences as learners. Moreover, their reflections indicate that they had expanded their experiences with mathematics to include a new understanding of its origins and development. As Fauvel (1991) notes, the development of mathematical concepts is tied to human development, as well as to the development of societies. Allowing students the space to explore the cultural and historical significance of the development of mathematical concepts can influence their motivation to learn mathematics and feel enthusiastic about the subject.

Implementing this pedagogical approach revealed some practical reasons why studying the historical development of a mathematics topic was meaningful for preservice elementary teachers. Their responses indicated that this research

- enhanced their mathematical knowledge of the topic,
- encouraged them to reflect on their current knowledge and identify areas in need of improvement,
- influenced their future pedagogical decisions,
- initiated self-reflection, and
- helped them gain new knowledge.

We also want to note a limitation of our research. The information we discovered was based on the preservice teachers' reflections, and we did not conduct in-depth interviews with them to gain more evidence or clarify concepts they might not fully understand.

Implications

Adopting and reflecting on this pedagogical approach to teaching and learning mathematics has revealed implications to share with those who are responsible for teaching preservice teachers.

First, looking at the preservice teachers' responses to the reflection prompts, we see various levels of reflection. Metacognitive frameworks look at not only what people know about their cognitive processes but also how they regulate those processes to facilitate learning and memory (Schraw and Moshman 1995). When opportunities for reflection are embedded in the curriculum, instructors can gain insight into the extent that preservice teachers are gaining knowledge related to teaching and learning mathematics. This insight is beneficial in terms of learning how to meet the needs of preservice teachers as future teachers. Research has indicated that reflective processes are performed infrequently in teacher education programs (Juklová 2015). Through this pedagogical approach, we provided preservice teachers with an opportunity to reflect, and we found that the process was important for their growth.

Second, this assignment has the potential to influence both teacher and student motivation to explore mathematics and the applicability of mathematics beyond the classroom. This mathematics course for preservice teachers did not require any field experience hours or specific preparation in the history of mathematics. Hence, we suggest that the assignment could also be used with students in upper elementary through high school.

Through their lived experience, preservice elementary teachers could carry this activity into their own classrooms and expose upper-elementary students to this process. Depending on the grade level, they could use scaffolds to assist their students with the research process, such as providing them with research materials or a short list of mathematics topics to choose from. The final product could take on various forms, such as poster presentations, short journal entries or one-pagers to extend student learning. These methods may be more appropriate for younger students than asking them to write a full essay on the selected topic. In addition to a deeper understanding of mathematical concepts, this assignment could provide an opportunity for cross-curricular instruction in upper-elementary classrooms and higher grades.

Moreover, because of the heavy emphasis on preparing students for reading and mathematics assessments, elementary teachers have limited opportunities to incorporate social studies instruction (Heafner 2018). Casler-Failing (2018) found that cross-curricular approaches to teaching in the elementary classroom increased teachers' ability to make connections across a variety of subject areas. Through creating cross-curricular activities, such as having students uncover the history behind the development of a mathematical concept, teachers can increase time spent on social studies instruction.

Doucet and MacCabe (2016) describe a lesson in which high school students approached concepts in a history class from an interdisciplinary lens and how it led to an increase in passion. Perhaps the interdisciplinary assignment recommended here also has the potential to spark student interest in mathematics and history. We humbly suggest that this assignment has the potential not only to influence preservice teachers' motivation to better understand and appreciate mathematical concepts but also to influence their future students' motivation.

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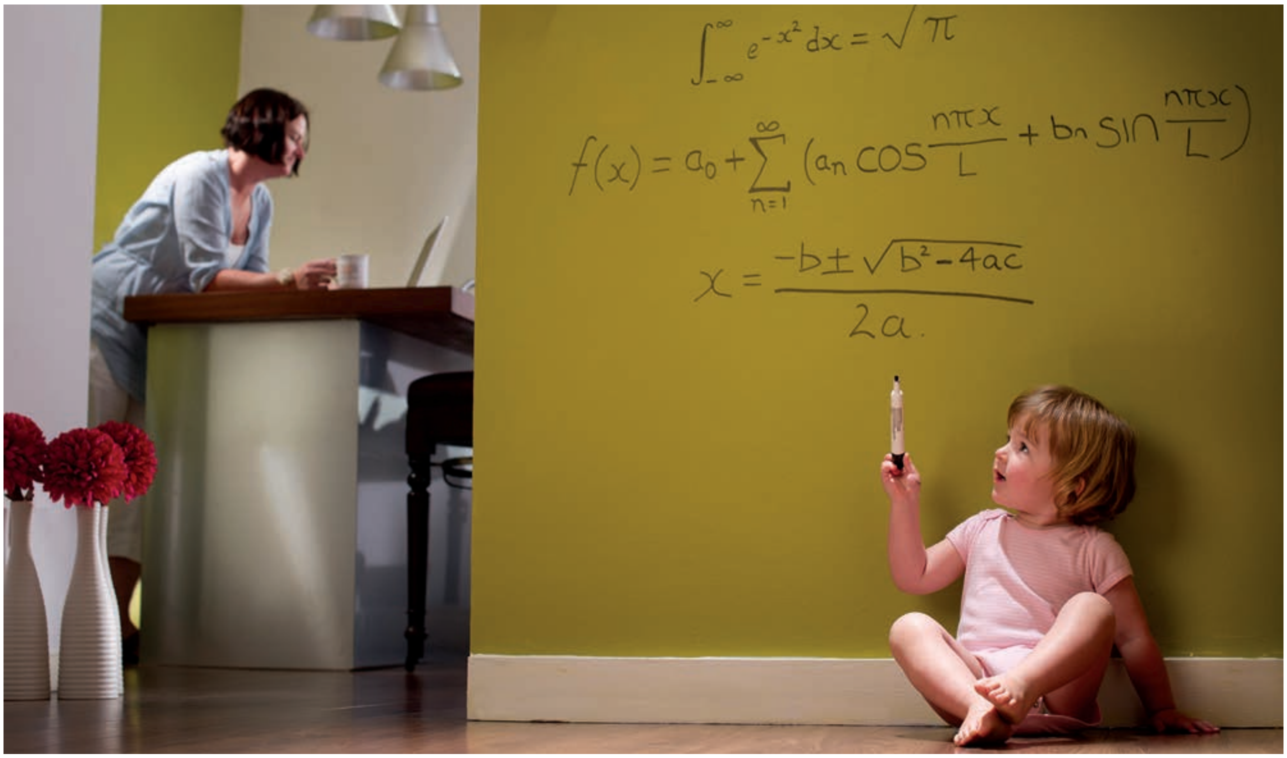
Giang-Nguyen T Nguyen, PhD, is an associate professor of mathematics education at the University of West Florida. She teaches courses in education, research methods, mathematics methods and educational technologies. Her research focuses on factors that influence preservice teachers' motivation for learning and teaching mathematics and how to support their development of mathematical knowledge for teaching.

Tiffany Marlow is a history teacher in California and a doctoral student at the University of West Florida. She has taught world history at the middle school and high school levels. Her research focuses on history curriculum—specifically, women's representation.

Inspiring Female Mathematicians

Indy Lagu



In the fall of 2020, I had the privilege of teaching an undergraduate course called Mathematics Through the Ages. This senior course is taken almost exclusively by preservice teachers. As COVID-19 restrictions forced us into Google Meet instead of the classroom, the semester could have been very difficult. It was made a lot easier by the lovely group of students I had the pleasure to work with.

My idea for this course was to look at several mathematics topics through a historical lens. Most mathematics classes spend almost zero time on the history of the discipline, and even less on the people involved.

For the final project, I asked my students to investigate any topic in mathematics that interested them, write a paper and then give a 15-minute presentation to the class.

Several students decided to write about female mathematicians. I suggested that they submit their papers to *delta-K*. The resulting articles, which are intended for a general audience, are as follows:

- "A Brief Biography of Sophie Germain," by Jennifer Kraft
- "Eugenia Cheng: An Inspiring Mathematician," by Mackenzie Ha, Ihn-Ah Jung and Joseline Ortiz Cardenas
- "The Greatest of All: Female Mathematician Emmy Noether," by Shenaé Richards
- "Mary Everest Boole: Present-Day Uses of Math from the Past," by Kaitlyn Neal

I hope that the work of these students inspires you to investigate some mathematicians and weave their remarkable stories into your classes.

Indy Lagu is an associate professor of mathematics at Mount Royal College, in Calgary, and MCATA's department of mathematics representative.

A Brief Biography of Sophie Germain

Jennifer Kraft

Her Life

Sophie Germain (1776–1831) was a lifelong reader and academic. When the French Revolution erupted in Paris, young Sophie occupied herself at home by reading classic works from her father's library, teaching herself Latin and Greek, and studying mathematics and geometry (Zielinski 2011). She is most known for her contributions to the study of acoustics, elasticity and the theory of numbers (Barrow-Green 1998).

Despite not being allowed to study at the École Polytechnique, because she was female, Germain received lecture notes and submitted papers (under a false name) to a faculty member named Joseph-Louis Lagrange (Barrow-Green 1998; Zielinski 2011). Lagrange became her mentor, which led to her corresponding with other prominent mathematicians (Zielinski 2011). Lagrange continued to support and encourage her work for many years.

Before diving into Germain's work in mathematics, it is important to consider the many obstacles she faced along the way. Her work suffered as a result of her lack of formal training, and because she lacked access to the resources a male mathematician would have had, she missed out on opportunities that would have contributed to her credibility and knowledge (Barrow-Green 1998; Zielinski 2011).

Despite this lack of opportunity, Germain became the first woman to win a prize from the French Academy of Sciences (Zielinski 2011). This in itself was by no means an easy process. It took Germain three attempts before she was finally awarded the prize in 1816, for her account of the phenomena exhibited in experiments done by a German physicist on vibrating plates (Barrow-Green 1998; Famous Scientists 2017). In addition to this, German mathematician Carl Friedrich Gauss persuaded the University of Göttingen to award Germain with an honorary degree, but unfortunately she died of cancer in 1831 before she could receive it (Del Centina 2008, 351).

Her Mathematical Contributions

Germain never published her work on Fermat's last theorem (FLT), although her work is credited in a footnote in a treatise by French mathematician Adrien-Marie Legendre (Barrow-Green 1998; Del Centina 2008, 349; Famous Scientists 2017). However, unpublished manuscripts, including letters from Germain to Gauss, have since been uncovered and transcribed by Del Centina. As he states, these manuscripts

> throw new light on her strategy toward a proof of Fermat's Last Theorem, which . . . appears more comprehensive and far-reaching. . . . Some of the results she stated, and often proved, were later rediscovered and published by other authors, without acknowledgement to her. (p 350)

We know that Fermat's claim states that

> if n is a positive integer greater than 2, the equation $x^n + y^n = z^n$ does not admit integer non-trivial solutions, i.e. no three integers x, y and z exist, such that $xyz \neq 0$, which satisfy the equation above. (Del Centina 2008, 351)

When Germain began her work on FLT, only the cases $n = 3$ and $n = 4$ had been proven (Limaye 2018). She saw that there was no real point in attempting to prove FLT for an individual number, because there are infinite numbers, so she began to consider proving it for whole classes of numbers (Famous Scientists 2017). Hence, she split FLT into two fundamental cases, and this idea is commonly referred to as the key theorem, as it constitutes the fundamental step toward her proof of the first case of FLT (Limaye 2018).

Case 1 states that "$x^p + y^p = z^p$ has no integer solution for which none of x, y, and z are divisible by p," and case 2 states that "$x^p + y^p = z^p$ has no integer solution for which one and only one of the three numbers is divisible by p" (Limaye 2018).

Germain then focused on case 1, where she stated that "if p is an odd prime and $q = 2p + 1$ is also a prime, then p must divide one of x, y, or z and therefore Case 1 of Fermat's Last Theorem is true for p" (Limaye 2018). This is known as Sophie Germain's theorem and constitutes the special case, proved by Germain, in which x, y, z and n are all relatively prime. As Limaye states, "Her idea to split the theorem into two cases—and her approach—revolutionized how mathematicians thought about this problem."

It is important to understand that Germain's unconventional approach to number theory resulted in the development of many new theories in both abstract and applied mathematics (Bailey 2006). A few applications that resulted from number theory are coding, cryptography, testing conjectures and determining primes, as well as the discovery of many other relationships between different sorts of numbers. In addition, Germain's mathematical background allowed her to make further contributions to material science and acoustics.

Despite her lack of recognition in the field of mathematics, Germain's great passion and benevolence led her to persevere in all areas of her life.

Teaching Context

Interestingly, a children's book was recently published about Germain, called *Nothing Stopped Sophie* (Bardoe 2018). The book tells the true story of her life as a mathematician who never gave up on achieving her dreams.

This book could be a powerful teaching resource in the elementary classroom. It could be used in many contexts—for instance, in studying number theory and discovering relationships between different sorts of numbers.

The book would also be a great opportunity to showcase a female mathematician who not only stood up for herself but was vulnerable and took many risks to stand up against social prejudices. Germain is a model for female academics, as well as for anyone who lacks opportunities to have their accomplishments acknowledged by society.

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As a recent graduate of Mount Royal University (2021), Jennifer Kraft, BEd, recognized that her aspirations in the teaching profession were rooted in her experiences coaching academy soccer, leading summer day camps and working as a child development mentor in her community. Growing up in a small town, she was part of a tightly knit school system where building connections and fostering a sense of belonging were of utmost value, and this feeling shaped her early learning experiences and led to her love of helping others and growing alongside them. She is pursuing further research and a master's degree in speech-language pathology.

Eugenia Cheng: An Inspiring Mathematician

Mackenzie Ha, Ihn-Ah Jung and Joseline Ortiz Cardenas

While researching Eugenia Cheng, we discovered that because of her unique approach to mathematics, she is a remarkable mathematician worth knowing. Most important, the heart of her work with mathematics and beyond lies in her passion for inspiring and helping others.

Her History

Eugenia Cheng, PhD, was born and raised in Hampshire, UK, and her family originated from Hong Kong.[1]

Currently, Cheng resides in Chicago, where she teaches mathematics to art students at the School of the Art Institute of Chicago.[2] At the institute, she is the scientist in residence, a position designed for a scientist who wants to contribute to the world and instill curiosity in young adults.

When Cheng is not busy teaching university students, she hosts professional development workshops for teachers around the world, does school visits and volunteers to help teach mathematics in elementary schools.[3] In addition, she is an engaging public speaker who has appeared at TED Conferences in London and Vienna. She also creates online videos to educate the general public about mathematics and has authored several books.

Clearly, Cheng is passionate about teaching and uses various platforms to educate people. Her mission is "to bring mathematics to a wider audience and help reduce maths phobia."[4] To do so, she explains concepts simply and entertainingly, so that they reach a broad scope of people. She especially seeks to include those who do not have a strong background in mathematics or who have developed a fear of it. One way she helps people learn math is through making comparisons to food, as well as revealing other unexpected connections to many aspects of our lives.

Mathematics Found in Baking

Cheng has written a number of books, and her best-known book is *How to Bake Pi: An Edible Exploration of the Mathematics of Mathematics* (Cheng 2015). As the publisher's description states, the book "provides an accessible introduction to the logic and beauty of mathematics," while also making creative parallels between mathematics and baking.[5] The question she presents is, What is math and how does it work? To explore this question, she draws inspiration from the kitchen and shows that mathematics and baking share several similarities in their techniques and ideas. Every chapter begins with a recipe, followed by an explanation of how that recipe connects to a specific topic in mathematics.

Another example of this is a video in which Cheng (2014) uses the dessert mille feuille to explain exponents. *Mille feuille* is French for "one thousand layers," and although this dessert has only three visible layers of pastry, each layer actually contains multiple layers. The process of making mille feuille shows how numbers combine together to become exponentially larger. The steps are as follows:

1. Place a rectangular, flattened slab of butter on a sheet of pastry. Then fold the pastry over the butter to create two layers and roll it out until it's flat.
2. Fold the pastry into three, which creates six layers in total ($2 \times 3 = 6$), and again roll it out until it's flat.
3. Repeat step 2 five more times (a total of six). This can be written as $2 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$, or 2×3^6. Then, multiply by three one more time, because of the three distinct layers of pastry. The final equation becomes $2 \times 3^7 = 4{,}374$, signifying 4,374 layers of pastry.

Connections Between Mathematics and Social Issues

Another theme that Cheng explores is how mathematics relates to social issues, and how mathematics could be the answer to social progress. She states that "mathematics is the *logical* study of how *logical* things work" (Cheng 2018a, 8), but it does not have to be restrictive. In school, math is often taught as a set of rules to follow, but Cheng believes that mathematics should instead be something that invites students to solve problems creatively by expanding their thinking. She believes that this creative quality in math is critical. Without creativity in the world, we would not have the music and art that we enjoy. Cheng believes that progress begins with creativity and dreaming, and that mathematics is a place where that progress can be shown.

To illustrate how mathematics can help us to understand our world and other people and to make social progress, Cheng uses abstract mathematics. In her TED Talk (Cheng 2018b), she explains that applied mathematics involves using math in real-world situations, such as how to build a bridge or fly an airplane. She describes mathematical thinking as thinking about the human world and applying math to real issues, not just thinking about the quadratic equation.

Cheng proceeds to demonstrate how math can be used to understand the world. She begins by showing a diagram of the factors of 30. She then rearranges the diagram so that she can list the factors of 30 using only prime numbers. Then, she replaces the prime numbers with variables, so that they become abstract—able to represent anything.

Next, Cheng labels the variables according to three types of absolute privilege—rich, white and male—so that a represents a white person, b represents a rich person, and c represents a male person. She explains that this idea of absolute privilege is a sensitive subject that can make people angry. However, she believes that understanding the root of people's anger is more productive than being angry in return.

She then looks at a diagram containing only women and adds cisgender as a variable. Although women in general feel less privileged than men, some women forget how privileged they are compared with women of different backgrounds and status. In this diagram exploring women and their comparable privilege, rich, white cis women are at the top, and non-rich, non-white trans women are at the bottom.

On the topic of gender, Cheng often shares her own experiences as a female mathematician—specifically, how she found herself having to behave a certain way to be accepted in the field. She believes that the words *feminine* and *masculine* are problematic when used to describe certain behaviours, because people have been conditioned to believe that it is wrong or undesirable for a man to be seen as feminine or a woman to be seen as masculine.

In her book *x + y: A Mathematician's Manifesto for Rethinking Gender* (Cheng 2020), she encourages the use of ungendered words, such as *congressive* and *ingressive* (pp 132–33). She describes *congressive* as bringing things (such as people) together and *ingressive* as entering into things without being stopped by other people's opinions and feelings. A person who is ingressive is focused on themself, independent, individual, competitive and adversarial, whereas a person who is congressive is focused on the community and society, considerate toward others, interdependent, connected, collaborative, and cooperative. Cheng believes that creating a congressive environment, in which we work together instead of against each other, is better for mathematicians, schools and society.

Fascination with Higher-Dimensional Category Theory

Cheng's main interest in mathematics is higher-dimensional category theory. Since category theory is far too complex for us to understand at this point in our education, we will reflect on what we have learned about context and content from our research and how we intend to apply these concepts in our current education, as well as in our future endeavours.

The dictionary definition of *context* is "the interrelated conditions in which something exists or occurs."[6] For example, the meaning and use of the word *bark* depends on the context. Is a dog barking? Or is *bark* referring to the bark of a tree? *Content* is defined as "the topics or matter treated in a written work."[7] For example, the table of contents in a book lists the chapters in that book, and those chapter titles give the reader an idea of what the book is about—or, in other words, what content the book holds.

As we were introduced to category theory, we learned that meaning should be derived from context. To take a categorical perspective in math means to consider context over content. An educational video by MathProofsable (2018) provides an analogy that helped us understand this concept. Imagine that you are in need of a spear, but all you have is a hammer. You realize that since you already have a hammer, you could sharpen its handle to make a spear. By doing this, you are deriving meaning from context by considering context over content. In one context, you could view the tool as a hammer, and in another context, you could view the same tool as a spear. This process is called recontextualizing. It is also called construction, because you are changing the context from using the tool as a hammer to using it as a spear.

We pondered how we could use these mathematical concepts in our current lives. Understanding category theory was out of our reach, but we knew that we could learn something from this simple idea of taking meaning from the context before the content. What was the point of learning about this? Figure 1, which was inspired by the MathProofsable (2018) video, illustrates some connections that showed to us the purpose of our understanding of category theory.

What we learned about category theory taught us the skills to solve any problem, whether in mathematics or in our lives. Guided by the steps of solving a categorical problem, the first thing we can do when we encounter a problem is to look at the content: What exactly is the problem? What does the problem contain? Once we have defined the problem and found the content in the problem, we can then look at the context: How does this problem relate to other aspects of our lives? How does x relate to y and z? Following this, we can reformulate the problem. We can rewrite or rethink the problem in a way that we can better understand or that is more relevant to any new information we have gleaned throughout the process. This will help us to eventually solve the problem by using the knowledge, structure and relationships that we found.

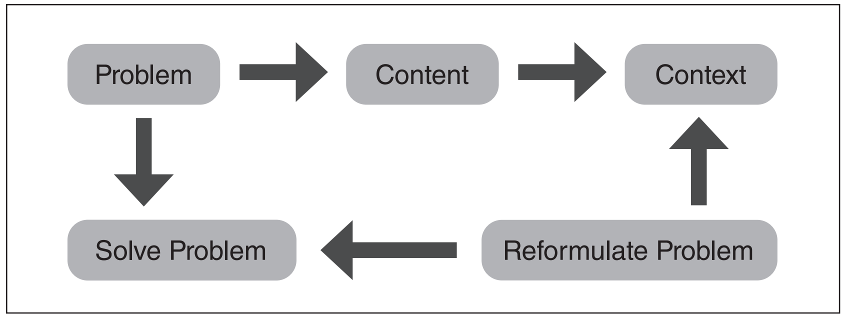

Figure 1

Because researching category theory taught us skills and strategies for solving problems through changing our perspective, we know that as we act on our new understandings, category theory will continue to show its value and help us become keen and open-minded problem solvers.

We find Eugenia Cheng incredibly inspiring because of her desire to use her talents to help those who need help and those who are interested in learning more about how math works. As aspiring educators, seeing her exercise divergent thinking to work toward her goals encourages us to do the same and inspires us to use our creativity in both our future careers and our everyday lives. There are so many ways to approach learning. We just need to be innovative and discover those ways.

Notes

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Mackenzie Ha, Ihn-Ah Jung and Joseline Ortiz Cardenas are undergraduate students in elementary education at Mount Royal University. With a keen interest in mathematics education, they hope to contribute to exciting educational advances through progressive and creative teaching methods. They are inspired daily by their professors, the children they teach, and each other's positivity and diligence. They each have a variety of interests, such as playing musical instruments, creating lifestyle montage videos and learning new languages, but together they love to unwind with some bubble tea.

The Greatest of All: Female Mathematician Emmy Noether

Shenaé Richards

Emmy Noether, circa 1900–10.
(Source: www.agnesscott.edu/lriddle/women/noether.htm.)

> If one proves the equality of two numbers a and b by showing first that "a is less than or equal to b" and then "a is greater than or equal to b," it is unfair, one should instead show that they are really equal by disclosing the inner ground for their equality.
>
> Emmy Noether

Emmy Noether was a German mathematician who was born on March 23, 1882, and passed away on April 14, 1935. Though many people know her as Emmy, her full name was Amalie Emmy Noether.

In 1900, Noether was certified to teach English and French at girls' schools; instead, she studied mathematics at the University of Erlangen. Back then, women could only audit classes (with permission from the instructor). Auditing a course means that one attends classes but is not required to submit any coursework or write exams, which means that women did not receive credits for a course. Despite this, Noether audited classes at the University of Göttingen in 1903/04. In 1904, the University of Erlangen allowed women to be full students, so she returned there to study mathematics and earned a PhD. She stayed there to conduct research (without remuneration) while assisting her father, Max Noether, also a well-known mathematician.[1]

In 1915, Noether was invited to the University of Göttingen to work with Felix Klein and David Hilbert on Einstein's general relativity theory (Kidskonnect 2018). In 1918, she published Noether's theorem, which "considered Hilbert and Einstein's problem: that General Relativity Theory seemed to break the law of conservation of energy" (Famous Scientists 2015). She discovered that there is a conservation law for every invariant (such as symmetry) in the universe (Famous Scientists 2015); every differentiable symmetry of a physical system's action has a corresponding conservation law.[2]

Einstein was impressed with Noether's work, writing to Hilbert that her theorem was a piece of "penetrating mathematical thinking."[3] At the time, her work was somewhat controversial because of its abstractness, but over the years, it has proven to be useful in many areas of theoretical physics, engineering and crystallography, not just applicable to the theory of relativity. Noether's theorem is still in use today. For example, Kleinert (2016) discusses the theorem in relation to point mechanics, continuous symmetries and conservation law, alternative derivation, displacement and energy conservation, momentum and angular momentum, and translational invariance in space.

Though the formula is not used in most K–12 mathematics classes, Noether's theorem comes up when introducing students to the real number line's symmetry or how number line points become arbitrarily close to each other. The theorem also demonstrates a clear relationship between quantities conserved in physics (such as energy and momentum) and physical symmetries.

During this time, Noether also published several papers on theoretical algebra, working with algebraist Ernst Otto Fischer, and became increasingly interested in abstract algebra. She would go on to make her most important and substantial contributions in this area of mathematics.

After the success of Noether's work on the general relativity theory, she earned a position as a lecturer at the University of Göttingen in 1919, and in 1922, she began receiving a small salary. She was considered to be one of the most formidable mathematics instructors at the university.

Throughout the 1920s, Noether continued to work on new mathematical theories, especially in abstract algebra. In 1927, she started to focus on linear transformations in algebra and how they could be applied to commutative number fields. She then "investigated the structure of noncommutative algebras and their application to commutative fields by means of cross product (a

A postcard Noether sent to Fischer in 1915 that was filled with her ideas about abstract algebra. (Source: https://germanculture.com.ua/famous-germans/emmy-noether-german-mathematical-genius/.)

form of multiplication used between two vectors)."[4]

In 1933, when the Nazis took power, Noether and many other Jewish professors at the University of Göttingen were dismissed. Noether moved to the United States to become a mathematics professor at Bryn Mawr College. Unfortunately, she passed away suddenly due to complications from surgery.[5] Though she was only 53 years old, she left her mark as a female mathematician. After World War II, the University of Erlangen paid tribute to Noether by naming a co-ed gymnasium that supported female mathematicians in her honour (Kidskonnect 2018).

Noether "is best known for her contributions to the development of the then-new field of abstract algebra" (American Physical Society 2013). Einstein referred to her as "the most significant creative mathematical genius thus far produced since the higher education of women began" (American Physical Society 2013). Throughout her career, she worked as a researcher and a teacher, published over 40 papers, and taught at several universities. Moreover, Noether's theorem has proven enormously useful in theoretical physics.

Notes

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Achieving her goal of becoming a teacher and educating young minds, Shenaé Richards graduated from Mount Royal University in 2021. She is eager to learn more about female mathematicians and the hard work and dedication it took for them to shine.

Mary Everest Boole: Present-Day Uses of Math from the Past

Kaitlyn Neal

Mary Everest Boole. (Source: www.agnesscott.edu/lriddle/women/boole.htm.)

> You may some day become a teacher. If ever you are teaching a class how to set down a sum or an equation, say "This is my way," or "This is the way which I think you will find most convenient."
>
> Mary Everest Boole, *Philosophy and Fun of Algebra* (1909)

Mary Everest Boole was born in 1832 in Wickwar, Gloucestershire, England. During her childhood, the family moved to Poissy, France, and upon their return to England, when she was 11, she was removed from school to be an assistant to her father, who was a minister (Frost 1995).

From a young age, Boole was intrigued by mathematics. Later, she met a famous mathematician, George Boole, whom she would marry after her father's passing (Frost 1995).

You may be wondering about the famous name Everest. Boole's uncle, George Everest, was the surveyor general of India. He surveyed the yet-to-be-named Mount Everest and calculated its summit height; hence, the mountain was named in his honour (Frost 1995).

The first step in Boole's career was her role as a librarian at Queen's College, which was England's first women's college. Soon after, she began to teach children and was recognized as an outstanding teacher by the head of the London Board of Education. However, she was forced to resign after a controversy over one of her books (Frost 1995).

Boole spent the remainder of her career and life regularly publishing books and articles (Frost 1995). In my research, I found mention of at least 21 publications authored by her. This seems to be one of her most significant accomplishments, as she left a paper trail for subsequent mathematicians and psychologists to study and review her work.

The University of Cambridge's Darwin Correspondence Project notes that Boole "was the author of several progressive educational texts, publishing '*The Preparation of the Child for Science'* in 1904 and *'Philosophy and Fun of Algebra'* in 1909."[1] Those are the two books most relevant to this article.

The Preparation of the Child for Science (Boole 1904) is said to have had "a great impact on progressive schools in England and the United States in the first part of the twentieth century" (Frost 1995). Most of us are familiar with progressive teaching methods, such as cooperative learning, making content relevant, practical uses and hands-on learning. However, I was amazed to learn that a female mathematician had been practising these progressive teaching methods that long ago. Although we most often learn about these methods through the teachings of John Dewey, Boole's ideas for progressive teaching are still very present in classrooms today.

The book I will discuss in detail is *Philosophy and Fun of Algebra* (Boole 1909). This book introduces readers to algebra, including word problems; solving for x, y and z; quadratic equations; square roots; and infinity. All these topics are covered throughout math education today, indicating that Boole's ideas are still very much relevant. Throughout the book's chapters, she explains her reasoning and logic in a uniquely creative manner.

Chapter 1, titled "From Arithmetic to Algebra," explains how "arithmetic means dealing logically with facts which we know." Boole emphasizes the idea of dealing logically with the known information through a word problem that involves how much a certain amount of sugar will cost. She later writes that, instead of guessing what to call a number that we don't know, "Let us agree to call it x, and let us always remember that x stands for the Unknown." In this chapter, we see that the mathematical language Boole used is the same language used today.

Chapter 2, "The Making of Algebras," includes an explanation of how we approach problems. Boole states that "we make an Algebra whenever we arrange facts that we know round a centre which is a statement of what it is that we want to know and do not know."

Boole provides examples of solving problems concurrently in chapter 3, "Simultaneous Problems." For example,

x equals half of y;
y equals twice x;
z equals x multiplied by y.

In this problem, one cannot determine the value of x or y or z, because they depend on one another. Boole provides an algebraic way of solving the question by working through the possible answers. She argues that investing the time into this process "adds to our knowledge of the structure of Algebra, and assists us in solving other problems."

Boole dives into quadratic equations in chapter 15, "The Square Root of Minus One," where she refers to the square root of minus one as "an entity (or non-entity)" that "come[s] to tell us where we are to go next; and the shortest road to get there; and where we ought not to go just at present." She suggests that the only way to solve this problem of the square root of minus one is to "treat him logically, exactly as if he were six or nine."

In chapters 16 and 17, Boole focuses on infinity, offering problems such as the following to illustrate the concept: "Suppose there is a cake on the table. How many children can go through the room without the cake being all eaten up?"

In chapter 9, Boole describes how she uses sewing cards to help students make connections to geometry—learning the relation between dimensions. By her own account, she was very young when she connected sewing cards to curve stitching (Boole 1904; Innes 2004, 37). In modern mathematics, *curve stitching* is defined as "the practice of constructing straight-line envelopes by stitching colored thread through a pattern of holes pricked in cardboard" (Ross 2015). This method was devised "to help children grasp geometric shapes, forms and structure."[2] This activity, which Boole recommended for children in kindergarten, "excites mathematical and artistic imagination but requires only the most basic skill level" (Innes 2004, 38). In classrooms today, curve stitching is mostly seen in the form of string art.

In addition to being a female mathematician and a teacher, Boole "considered herself a mathematical psychologist" (Frost 1995). She "had a lifelong interest in the unconscious mind, the psychology of learning, and how these relate to the teaching of mathematics and science" (Innes 2004, 36). She based her advice about educating children "on theories about the role of the unconscious and the psychology of learning gleaned from her own experience" (p 37).

Today, Boole is most known for her progressive teaching ideas and practices, as well as her creative methods and exercises for teaching mathematics to young children. The mathematical topics she explored in her books and her invention of curve stitching are still taught and relevant in mathematical settings today.

Notes

1. "Mary Everest Boole," Darwin Correspondence Project, University of Cambridge, www.darwinproject.ac.uk/mary-everest-boole (accessed October 20, 2021).

2. "Mary Everest Boole," Darwin Correspondence Project.

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Born and raised in the small town of Okotoks, Alberta, Kaitlyn Neal recently graduated with a BEd from Mount Royal University, with minors in teaching English as a second language (ESL) and mathematics. She now resides and teaches in Kelowna, British Columbia. After hearing a young girl say that she was not good at math because she was a girl, Kaitlyn was inspired to research and shed light on female mathematicians.

Mathemagical Black Holes 1, 153, 370, 371, 407 and Cyclical

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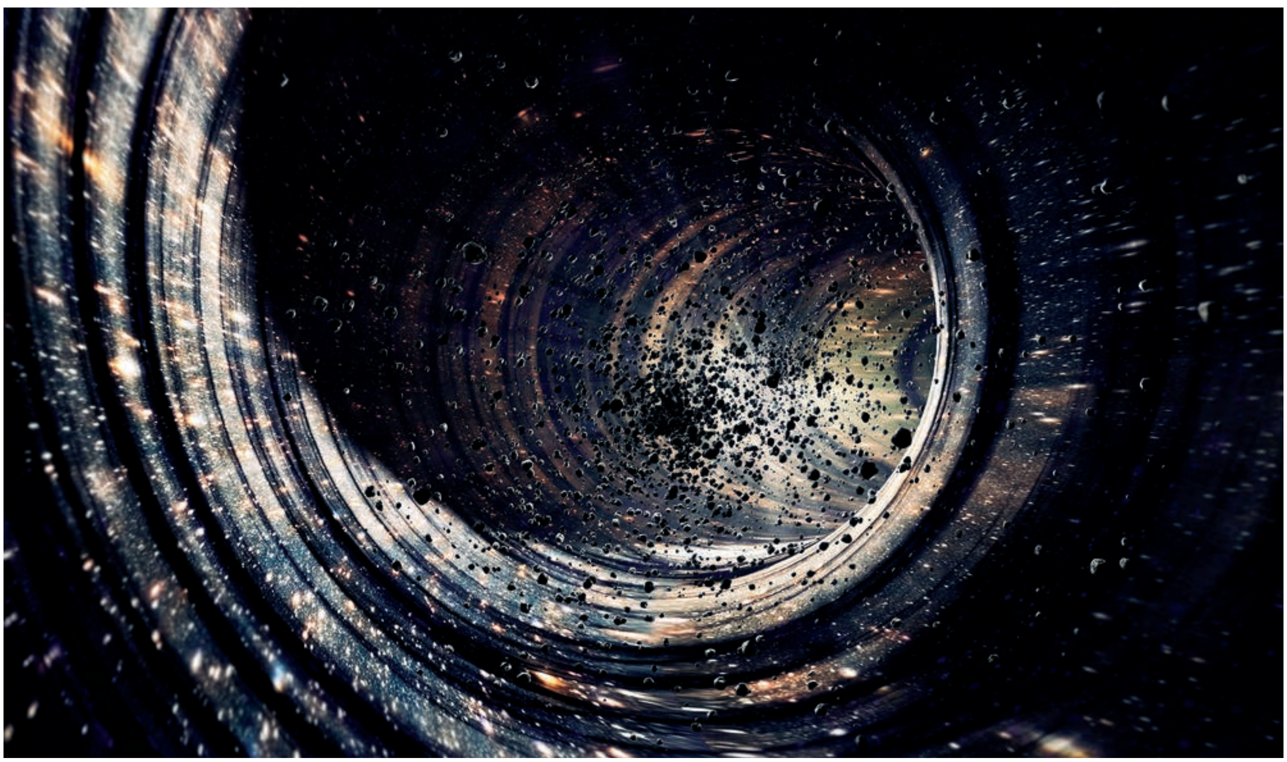

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