delta-k

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The Spectrum of Mathematics

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

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7. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
8. References and citations should be formatted consistently using the author-date system.
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MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-k

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CONTENTS

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From the Editor's Desk

Lorelei Boschman

I was thinking this week about the vastness of mathematics. Not only do we need to be fluent in the basics of mathematics, but we also need to recognize and use much more complicated mathematical ideas. I think of it as a spectrum—from the basics of operations to complex investigations into how mathematics works for us and applies to us and, furthermore, how we actually use mathematics and its related skills.

On the cover of this issue of *delta-K*, you will see one end of the spectrum, or perhaps the beginning, of mathematics. Throughout the issue, you will see more applications of mathematics, delving deeper into how much mathematics is woven into ideas around us, as well as some interesting and perplexing deeper mathematics and life situations that can be related to mathematics. We need to question, investigate and apply in our mathematical world today. There are so many truly interesting mathematical applications that we are only now starting to investigate. This field is growing at a very fast rate. We are seeing more connections and mathematical threads every day—if we look.

We know that there are big ideas in terms of mathematical concepts and content in mathematics. We also know that there are big process ideas that are part of the crucial goals in mathematics. When we think about the underlying procedural or process goals in mathematics that we want our students to reach, we see goals such as the following:

- Self-directed problem solving using a variety of strategies
- Effective mathematical communication
- Reasoning and generalization of connections and conclusions
- Fluency in situations that require the use of numbers

These goals take the so-called basics of mathematics and expand the scope of how we use them, throughout all content areas of mathematics.

When we think even deeper about what mathematics accomplishes, we can answer for our students the age-old question "When am I going to use this?" They will be using the process of problem solving as they investigate the best travel path and the costs for their holiday. They will be communicating, in a mathematical sense, when they sketch their kitchen with measurements for a home renovation project and set forth the materials needed and the cost. They will be reasoning, likely daily, through actions that must be taken in a multitude of situational contexts and the order of those actions, with likely consequences—all the while thinking through and reflecting on what has happened previously in a similar situation and rationalizing the worth. They will be using number sense as they examine their cellphone bill or pay stub for correctness. Wow! These are complex processes, and mathematics is the precursor to practising them in life. That is a fundamental reason *why* we practise mathematics. (Of course, as mathematics teachers, we know that there is also a certain joy in learning the mathematics involved.)

Recently, I looked at a list of skills workers will need in 2020 (Beckford 2018; Gray 2016), based on a World Economic Forum (2016) report. The first two skills on the list are complex problem solving and critical thinking. Does that sound familiar? Also, the seventh skill is judgment and decision making. When we work with our students on these big process ideas in our schools and through our content, students are practising the skills they will need for the future. Mathematics is the venue we use to develop and practise those skills.

Mathematics has a truly important place in our world today, based on the sheer frequency with which we use it, create with it, evaluate with it and solve with it. Our challenge, as mathematics teachers, is to allow our students to practise these big process ideas in meaningful situations, questions and projects and to help them develop these skills so that they can transfer them to new situations. Even if this practice does not immediately present as a mathematical situation, the skills are still in place, and we work with students on how to transfer their applicability to a novel situation. Mathematics develops skills that are necessary for our students today.

This spectrum of mathematics has far-reaching implications, far beyond simply looking at the content we teach through the K–12 curriculum. It is how we let our students practise the needed skills through the content, knowing that these skills will emerge throughout life as a necessary element of how they walk through their day.

How can we purposefully and effectively give our students practice with these big process ideas and skills? What can you do this year to further develop these skills in your students? How can we help students see the intrinsic value of mathematics beyond what they may think it is on the surface?

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An Open Letter on Standardized Testing

Alicia Burdess

Close your eyes and imagine that it's the last week of June. The weather has been beautiful for a while now, and you are excited about your summer holidays. You can't wait for camping, outdoor sports and spending time with your family. It's time for a well-deserved break from work. Just one thing is stopping you. Actually, more than one thing. You have to write final exams. The province needs to see where you are in science, social studies, reading comprehension, writing and math. To make sure, we'll double up on math and make you do a timed portion without a calculator. If you have a government job and speak French, you will have to take the reading and writing exams in two languages.

Fast-forward to the last day of exams. It's time for math. Math may be your favourite subject, or it may be the subject you struggle with the most. You may feel anxious right now just thinking about writing a math test. You have spent the past couple weeks writing exams every day, and you are tired of multiple-choice questions and bubble sheets.

Imagine that you write the first part of the math exam in the morning. It's difficult. You run out of time. You don't get to finish. You can't figure out why the test is trying to confuse you. You can't figure out why all the questions are trying to trick you. You have a break for lunch, and then you have to write the second part in the afternoon. You are supposed to answer 20 questions in 20 minutes. It takes you five minutes to finish the first question. The answer is a fraction. It takes you another couple of minutes to figure out how to put the answer into the answer boxes and fill in the bubbles. Now your heart is racing, you are sweating, and you are starting to panic. Now you think that you are stupid because you aren't going to have time to finish the test.

Imagine that you have trouble reading. Imagine that you have trouble focusing. Imagine that you are tired. Imagine that you are hungry. Imagine that you have test anxiety. Imagine that you just immigrated here a few months ago and this entire process has been done in your second or third language. Imagine that you have special needs. Imagine that you are struggling with your mental health. Imagine that your family is struggling at home. Imagine that you are a refugee. Imagine that you have all sorts of knowledge and ideas in your head that you can't explain on a multiple-choice test.

Imagine that your boss gets a letter in the mail saying that you scored 18 per cent on the math exam. Imagine that you don't get a promotion because of this mark. Imagine that this mark affects your future learning opportunities. You can't go to certain conferences and training seminars because you aren't good at math. At least, that's what the test said. Imagine that you now believe you aren't good at math and that you lack confidence to use math in your everyday life. Imagine that you tell your own children you aren't good at math and that they now think it's genetic. Imagine that you live the rest of your life trying to avoid math.

Now, imagine that you are 8 years old (Grade 3 provincial achievement test). Or 11 years old (Grade 6 provincial achievement test). Or 14 years old (Grade 9 provincial achievement test). Or 17 years old (Grade 12 diploma exam).

As an adult, would you like to be assessed, ranked and labelled? With a multiple-choice exam built to fit a bell curve? How about if you had to write exams in four subject areas? How about if you had to write exams in a language you are just learning? Just because it happened to us as kids doesn't make it right. Just because it's how we've always done things doesn't make it right.

Government, if you need data, why not use math and trust statistics? Why not use a sample population? Why not make it anonymous so that kids don't pay the price?

We are preparing our children for jobs that don't yet exist. Why are we still using standardized tests when we don't want standardized learners and workers? We need flexibility, creativity, problem solving and perseverance—none of which can be assessed through multiple-choice tests.

We know better, so let's do better. It's time to end standardized tests in Alberta.

Alicia Burdess grew up in St Albert, Alberta, and completed her bachelor of physical education degree at the University of Alberta, followed by an after degree in education at the U of A's Campus Saint-Jean and a master of education degree through Simon Fraser University. She has been a teacher with Grande Prairie and District Catholic Schools for 14 years, serving in a variety of capacities. Her passion is teaching math and working with students and teachers to increase their understanding of and confidence in learning math. She spent four years as her district's numeracy lead teacher and instructional coach. She is past president of MCATA and enjoys supporting math teachers through conferences and professional development. She is also a wife and a mother of two and enjoys spending time with family, coaching sports and reading.

How Often Do I Do Math?

David Martin

I have seen English teachers sit around and discuss the books they are reading and social studies teachers debate current issues and their impact on society. I have seen career and technology studies (CTS) teachers talk to their students about the projects they are working on—a woodworking project, an automotive problem or even an attempt to code an Arduino board to allow for more functionality in their home. As I meet more teachers, I am constantly hearing about how they are students of their own subject areas outside the walls of the classroom.

This has caused me to reflect—which I ask you to do as well—on the question, *How often do I sit down and work on mathematical problems outside my own classroom?*

When I first asked myself this question, I, sadly, had to respond with rarely or never. At the time, I would ask my students to try multiple questions daily, learn new ideas, consolidate older information and, ultimately, be problem solvers when faced with questions they had never seen before; regretfully, I modelled none of this outside the classroom.

Perseverance, resilience, creativity and critical thinking are what I expected of my students daily in mathematics, but until I embraced these practices in my own life, I didn't truly know how it feels to be stuck in a problem and not know what to do.

"What do you do when you don't know what to do in a math problem?" I asked this question to 800 Grades 4–12 students, and the number one answer (from over 80 per cent of the respondents) was "Ask the teacher." This was startling! I couldn't arm my students with authentic problem-solving strategies until I put myself in their shoes. I tried working on problems that caused me to stop and ask myself, *What should I do now?* Only then could I understand that global problem-solving strategies were missing in my own math classes.

Originally, I would teach students that when they were working on a problem from unit X, they should try certain strategies, and in unit Y, try other strategies. I wasn't teaching true problem solving; instead, I was teaching strategies specific to certain domains. When I tried solving math problems on my own time, and at my own level, I quickly learned that the following are some of the best strategies:

- *Visualize the problem.* Draw it out.
- *Guess-and-check.* Change your guess slightly and see how it changes the result.
- *Approach the problem logically.* Use if-then statements to simplify information.
- *Identify a pattern.* Change a number, a sign or something critical, and see how that changes the problem.
- *Work backward.* If we can hypothesize the result, what else would have to be true?
- *Solve an easier problem.* Simplify the problem into one that is easier to work with, and see if you can identify anything new.

My challenge for myself now—and I extend this challenge to you—is to try a math problem once a week. Ensure that the problem isn't one you can solve in seconds, or even minutes. Try to find a problem that makes you reflect on the question, *What do I do when I don't know what to do in a math problem?*

David Martin has a master's degree in mathematics, a bachelor's degree in education and, most important, a love of learning. Throughout his career, he has challenged many traditional educational practices, such as homework, tests and even grading. As a division math/science lead teacher, he has the opportunity to learn with teachers and students from pre-K to Grade 12. He is also president of MCATA. You will often find him tinkering with code, playing with mathematics or counting by prime numbers.

Problems from Open Middle

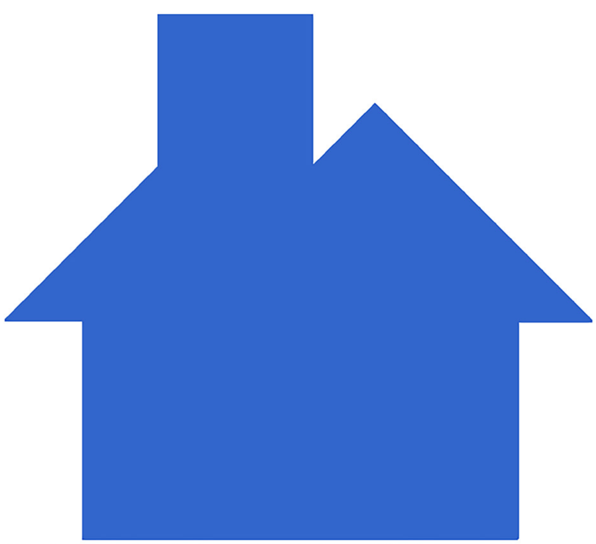
Lorelei Boschman

The website Open Middle (www.openmiddle.com) contains some great problem-solving moments for all grades. The website's tagline is "Challenging math problems worth solving," and that is a great description. What an amazing collection of math problems to give to your students today! The problems are categorized by grade level (from kindergarten to high school), as well as by mathematical area of focus. Read on for examples from Grades 1, 3, 6 and 8 and high school.

For Grade 1 (geometry), we have this problem on composite two-dimensional shapes.

COMPOSITE 2D SHAPES

Directions: What shapes could be used to create this picture?



Make a list of the shapes needed, and how many of each you would need.

Hint

Answer

Source: Bryan Anderson, Open Middle, www.openmiddle.com/composite-2d-shapes/. Licensed under CC BY-NC-SA 4.0 (https://creativecommons.org/licenses/by-nc-sa/4.0/).

Here is a problem for Grade 3 (number and operations in base 10).

MISSING DIGITS

Directions: Fill in the blanks with digits to make the answer closer to 200 than 300.

$$4\square\square - 1\square\square$$

Hint

Answer

Source: Marilyn Burns and Graham Fletcher, Open Middle, www.openmiddle.com/missing-digits/. Licensed under CC BY-NC-SA 4.0 (https://creativecommons.org/licenses/by-nc-sa/4.0/).

Here's one from Grade 6 (expressions and equations).

SOLVING ONE-STEP EQUATIONS (GREATEST SOLUTION)

Directions: Use the digits 1 to 9, at most one time each, to create an equation where x has the greatest possible value.

$$\square\square + x = \square\square$$

Hint

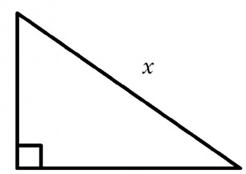
Answer

Source: Robert Kaplinsky, Open Middle, www.openmiddle.com/solving-one-step-equations-greatest-solution/. Licensed under CC BY-NC-SA 4.0 (https://creativecommons.org/licenses/by-nc-sa/4.0/).

Here is one from Grade 8 (the number system).

PYTHAGOREAN THEOREM

Directions: What could the lengths of the legs be such that the lengths of the legs are integers and x is an irrational number between 5 and 7?



Hint

Answer

Source: Daniel Luevanos, Open Middle, www.openmiddle.com/pythagorean-theorem-prob/. Licensed under CC BY-NC-SA 4.0 (https://creativecommons.org/licenses/by-nc-sa/4.0/).

The last problem is for high school (functions).

DISCRIMINANT

Directions: Using the digits 0 to 9 at most one time each, fill in the boxes to make one function have no real roots, another function have one real root, and the last function have two real roots.

$$y = \square x^2 + \square x + \square$$

$$y = \square x^2 + \square x + \square$$

$$y = \square x^2 + \square x + \square$$

Hint

Answer

Source: Lynda Chung, Open Middle, www.openmiddle.com/discriminant/. Licensed under CC BY-NC-SA 4.0 (https://creativecommons.org/licenses/by-nc-sa/4.0/).

Open Middle is worth browsing through. Most of the problems can be used immediately and as is with your students. Hints and answers are provided, as is a worksheet that students can use to think through their attempts at solving a problem. Make sure to look at the problems for other grades, as some problems are applicable for many grades.

These problems would make great cooperative learning explorations. How could you incorporate at least one or more of these per week in your classroom?

Lorelei Boschman received her bachelor of education and master of education degrees from the University of Lethbridge. She is the education coordinator at Medicine Hat College, facilitating the four-year bachelor of education program (a collaborative degree program with Mount Royal University) and instructing a variety of postsecondary courses with a mathematics focus. Previously, she taught K–8 at a rural school and spent 21 years teaching high school mathematics. Mathematics education is her passion and life work, and she has been involved in many local and provincial initiatives.

Discerning a Critical Aspect of Computational Thinking and Developing a Computational Disposition with a Logic Puzzle Game

Sally Rudakoff and Steven Khan

> Teachers are still struggling with understanding computational thinking vs. coding and how it relates to their curriculum. As well, some see this as an "add-on" that they don't have time for. I hope if we emphasize computational thinking skills in [the] elementary [grades that] students will begin to develop skills and concepts that will follow them as they advance through our school system.
>
> Sally

> Proto-Computational Thinking (PCT) might be a more realistic and achievable systemic goal for naming what we do at the earliest grade levels in a way that is intellectually honest and respectful of the multiple responsibilities around learner competencies that teachers are already charged with (read *responsible* or *accountable for*) developing.
>
> Steven[1]

We, the authors of this article, have responsibilities related to supporting teachers' growth at various stages in their careers. In both of our contexts—preservice teacher education and school district numeracy and technology support—we have noticed K–6 teachers' interest in and struggles with the relationships between computational thinking, coding and the existing curriculum. At the same time, we are attuned to concerns in the literature about levels of enactment (Namukasa 2018) and the challenges posed by the multiple origins, definitions and frameworks for the practice and study of computational thinking in multiple environments (Khan et al 2017).

We see our discussion here as propaedeutic—a preliminary excursion and exploration that invites readers to attend to the possibilities for mathematics and computational thinking in children's life-worlds. *Computational thinking* can be defined as "solving problems, designing systems, and understanding human behavior, by drawing on the concepts fundamental to computer science" (Wing 2006, 33). In a more recent review, Shute, Sun and Asbell-Clarke (2017, 142) define it as "the conceptual foundation required to solve problems effectively and efficiently . . . with solutions that are reusable in different contexts." Indeed, what separates computational thinking from other forms of mathematical problem solving is the ability to extract or abstract the answer into other domains and apply the solution to other cases. Khan et al (2017, 5) adopt a sociocultural approach to computational thinking, taking it as "an enculturated set of human practices to see, hear, encounter, and ultimately read and write the world, in a Freirian sense, in *particular* ways that are valued/rewarded in specific computational cultures." Ultimately, computational thinking is a literacy practice (Bers 2017) that is essential for understanding and participating meaningfully in the transformations of all areas of life in the 21st century (Kafai and Burke 2014).

Figure 1 shows how we conceptualize the place and importance of computational thinking from a complex systems (or transdisciplinary) perspective. In our framework, foundational computational experiences are necessary (but not sufficient) for developing computational identities (including disposition), which contributes to developing computational thinking in diverse computational communities. Such communities require members

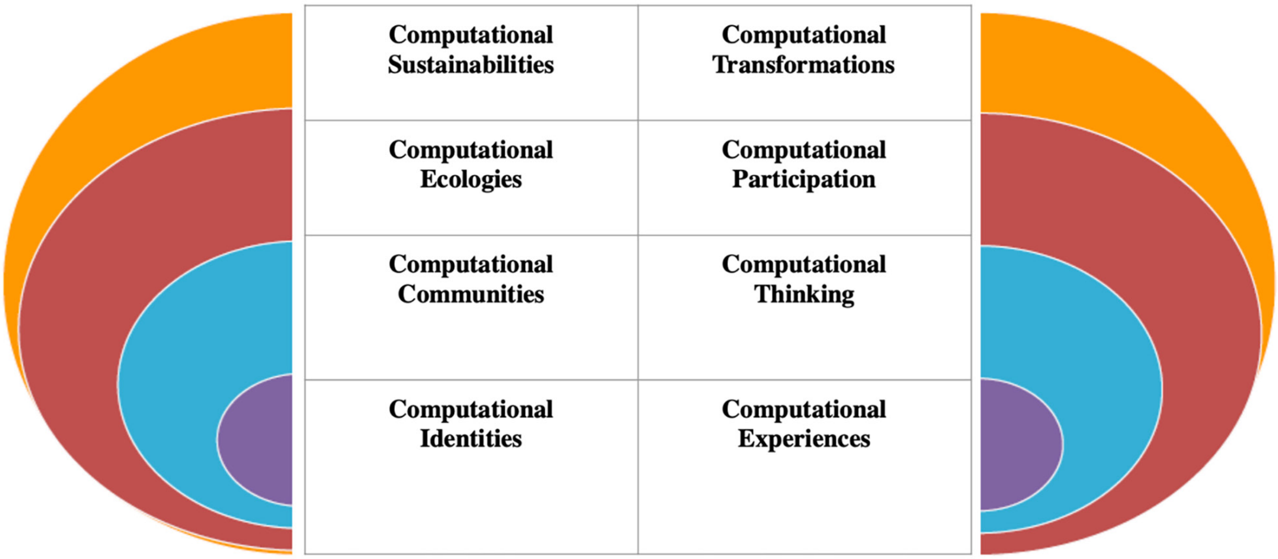


FIGURE 1. *A framework for thinking about the significance of computational thinking in schools and society.*

to participate meaningfully in diverse ways (for example, as coders, legislators or critical consumers) and give rise to webs of connected computational communities or computational ecologies. These computational ecologies contribute to the ongoing transformation of social, political, economic and ecological life-worlds and necessitate thinking in terms of their ethics and sustainability with regard to human and planetary flourishing. Our work here and that of others in early learning, such as Kotsopoulos et al (2019), is situated at the base of these nested systems—a consideration of an early experience of computational thinking and its potential for contributing to the development of some aspects of a computational identity (in particular, those aspects related to computational disposition).

Here, we think through how one might use Archelino—a commercially available (but easily generalizable) wooden logic puzzle game—to develop one aspect of computational thinking: the ability to see problems as decomposable and recomposable for learners aged four to eight years old (pre-kindergarten to Grade 3) and preservice teachers. We suggest ways that the critical aspect (Marton 2014) of decomposing and recomposing problems, while working toward a clearly defined goal, is a necessary component of computational thinking that can be developed through attending to the design of the puzzles offered by commercial logic games. We also draw attention to how the game might contribute to the development of a healthy computational disposition (Pérez 2018).

The activity we describe can be used with a whole class, with small groups or partners (station work), or at home (with parental support). Archelino presents opportunities for problem solving (especially logical reasoning) at various levels of difficulty, allows for differentiation with students who are in the early stages of developing understanding of the concept, and highlights the importance of sequencing/ordering (algorithmic thinking) and decomposition. It also affords opportunities for developing understanding of positional language related to spatial reasoning—a common learning outcome across Canadian and international curricula.

To be clear, we are *not* arguing that this should be the only experience learners have with sequencing/ordering and decomposition, or the only place to develop a computational disposition. As Resnick (2016) writes,

> For a technology to be effective, . . . it should provide easy ways for novices to get started (low floor) but also ways for them to work on increasingly sophisticated projects over time (high ceiling). . . .
>
> . . . For a more complete picture, we need to add an extra dimension: wide walls. It's not enough to provide a single path from low floor to high ceiling; we need to provide wide walls so that kids can explore multiple pathways from floor to ceiling.

We believe that Archelino, along with focused, pedagogically informed teacher- or parent-led questioning or discussion, provides excellent low-floor entry points for educators and students to begin discussing computational thinking concepts, with good potential for high ceilings and wide walls. We will not here discuss using this activity as a form of assessment, though the formative assessment aspects are evident to us as teachers, especially in student–teacher direct engagement and questioning.

We hope that teachers and other readers see how whole-class or partner activities like this can contribute to developing critical foundational aspects of both mathematical thinking and computational thinking simultaneously in young learners, as well as a productive disposition toward learning. We also hope readers will seek out and share additional opportunities with each other and the wider community, and that they will continue to develop the activity in order to draw attention to other aspects of value and to opportunities for learning that we have not yet considered.

About Archelino

Archelino is a commercial logic puzzle game created by Inon Kohn and distributed by the German game company HUCH! The game is recommended for ages four and up; it requires one or more players, and the typical playing time is approximately 10 minutes.

The game set consists of one grooved wooden ark; seven wooden figures (Noah and six animals), each about two inches tall; a puzzle book containing 60 puzzles in the categories of starter, advanced, expert and master; and one multilingual instruction booklet (see Figure 2). The game is based on the story of Noah's ark, with the premise that each animal wants to sit next to or converse with another specific animal. The essence of the game, however, can be recreated with other story scenarios not tied to this particular narrative.

Archelino allows players to solve puzzles that present a visual description of how the animals should be ordered from left to right (first, second and so on), how they should be oriented (facing left or right), and whether they want to converse (facing each other). The goal is to use the visual clues to solve the puzzle by correctly ordering, orienting and sequencing the animals on the ark. The instruction booklet and box art explicitly state that learning to think strategically is the main affordance of engaging with the puzzle.

Archelino retails for approximately C$29 on Amazon.ca and C$25 on FoxMind (www.foxmind.com). Considering the wider social content and

FIGURE 2. *The Archelino wooden figures and ark, puzzle book, and solved puzzle.*

taking a critical view of our work, we acknowledge that, as with many durable wooden resources for early learners (for example, traditional Montessori and Waldorf materials), price can be an access barrier for individuals, schools and districts. At the same time, the organizations we work for have developed and maintain libraries that lend materials such as games, puzzles and technological devices, and the public library system enables teachers and parents to access similar resources through requests and delivery to community branches. Also, at thrift stores we often find wooden animal toys that could be used in place of the Archelino materials.

In the next section, we elaborate on the specific affordances we see for developing a computational thinking skill—decomposition and recomposition—in the context of Archelino.

Choosing Archelino

We would like to say that we did an exhaustive exploration of a number of commercial games and picked the one best suited to our purpose, or that we reviewed lists of award-winning toys. However, the truth is more mundane—and, we believe, more typical of how games like Archelino get considered and incorporated into teaching and learning at home, in school settings, and in extracurricular or cocurricular groups.

A departmental colleague of Steven's, who does research on commercial games and who is also a parent, suggested the game to him for use with his three-and-a-half-year-old daughter. This was around the same time we were having conversations about the difficulties teachers have in thinking about computational thinking in elementary schools. Alberta's current curriculum does not explicitly name anything as computational thinking, but the draft K–4 revised mathematics curriculum—which we both independently reviewed in our respective professional capacities—foregrounds aspects related to algorithmic thinking.[2] We were interested in finding ways to support preservice and inservice teachers as they try to integrate and incorporate computational thinking ideas in that context. We saw an opportunity for learning, and we took it.

We had another moment of insight when we were working with Archelino while thinking about the learning of elementary mathematics alongside computational thinking. We had several moments of recognition in making connections by seeing—and naming—the familiar (decomposition) in the unfamiliar (a puzzle game context). We feel that it is also important to draw attention to such occasions that provide opportunities for building learners' capacity for transfer (Salomon and Perkins 1989), especially through noticing (Lobato, Rhodehamel and Hohensee 2012), from singular embedded learning experiences to more general applied learning in different contexts across time.

Our modest hope is that by encouraging teachers (and parents) to find the familiar in unfamiliar places, by choosing to bring opportunities for distal connections across multiple types of mathematical texts, and by guiding students to make those connections explicit (notice, name and nurture) through meaningful questioning or discussion, we can increase the probability that learners will see the relevance structure (Marton 2014) and potential of learning in general in multiple contexts. We note that educating for far transfer of learning remains an elusive goal in many areas but especially in the area of computational thinking, where the evidence for such claims is weak (Denning 2017). With a pedagogical program of selectively experimenting in deliberately combinatorial ways to prevent ideas from becoming inert (Whitehead 1967), we see increased probability of far transfer and meaningful, joyful learning.

When we were revising this article, one reviewer asked if we could offer examples of what far transfer might look like in the context of the particular learnings from this game and how one might look for far transfer. We acknowledge the value of this request and state that part of the difficulty of recognizing far transfer is that it often becomes evident long after the initial foundational experiences and cannot be explicitly drawn upon as easily as a causal phenomenon. Another difficulty with far transfer is that in real life multiple confounds occur as experiences leach into each other over time. In the case of the affordance of discerning decomposition/recomposition in this game, one might see near transfer in the short term of decomposing problems in other areas not explicitly mathematical or computational (for example, sports, cooking, science or the fine arts). Making a strong claim about far transfer is not easy, and it is precisely the overblown claims in much of the literature related to computational thinking in education that Denning (2017) explicitly critiques. Our position is to proceed cautiously, with modest claims about what might be discerned in this game while acknowledging that learning involves multiple opportunities for recursive elaboration over time and contexts.

Discernment in Variation Theory

Discernment—or the coming into conscious awareness of an object, relation, concept or phenomenon of which one was not previously consciously aware—is a focus of the variation theory of learning (Marton 2014). The aspects of the object of learning (what one is trying to learn or to have a learner notice and become aware of) are referred to as the critical aspects. So, for example, one could say that Hoyles and Noss's (2015) framework for computational thinking has four critical aspects—pattern recognition, decomposition, abstraction and algorithm design—that are to be discerned and developed by the learner, with the help of the teacher and the learning materials.

Low-floor puzzle games (such as Archelino) offer an opportunity for what Trninic (2018, 150) describes as "explorative practice": "a pedagogical approach with a *high degree of guidance* but a *minimal degree of explaining*." Trninic draws on Freudenthal's (1971) suggestion that learning mathematics (specifically, geometry) should be like learning to swim. In swimming lessons, a *qualified* teacher helps the learner come to sensory-motor and conscious awareness without initially offering explanations. This keeps the learner's limited attentional (cognitive-emotional) resources focused on a small set of critical aspects that are important in that moment.

Discerning Decomposition in Archelino

The ability to decompose a problem into smaller components is seen as an important aspect of both computational thinking and mathematical problem solving (Hoyles and Noss 2015; Polya 2014). It is also a core component of other computational thinking frameworks, such as Computing at School's concepts of computational thinking (Csizmadia et al 2015, 8) and the International Society for Technology in Education (ISTE) Standards for Students (under Computational Thinker).[3]

Figures 3–7 show several of the Archelino puzzle clue cards and the possible discernments. Players are presented with these cards one at a time, and are instructed to place the animals in the same sequence as depicted on the card or to determine the sequence of all the animals in the ark. We have arranged these clue cards in a sequence to show some of the discernments they make possible.

FIGURE 3. *Discernments: animals can all face in one direction, can face each other or can face away from each other; facing left or facing right; positional/ordinal language (such as* next to, to the left of, to the right of, first, second, third).

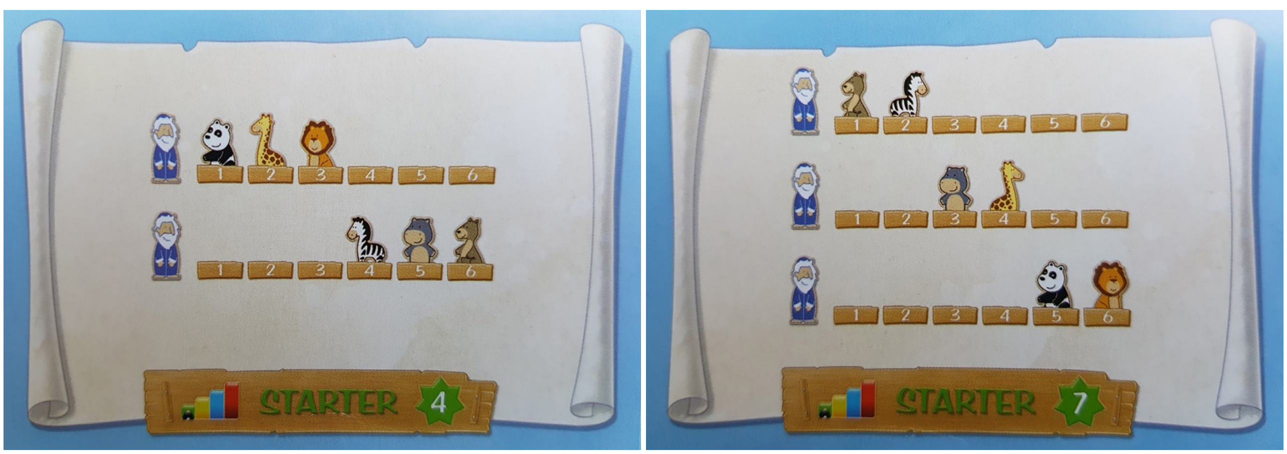


FIGURE 4. *Discernments: puzzle clues can be decomposed into simpler parts (decomposition); puzzle clues can be recomposed into one whole part (recomposition); multiple strategies can be used to construct the puzzle.*

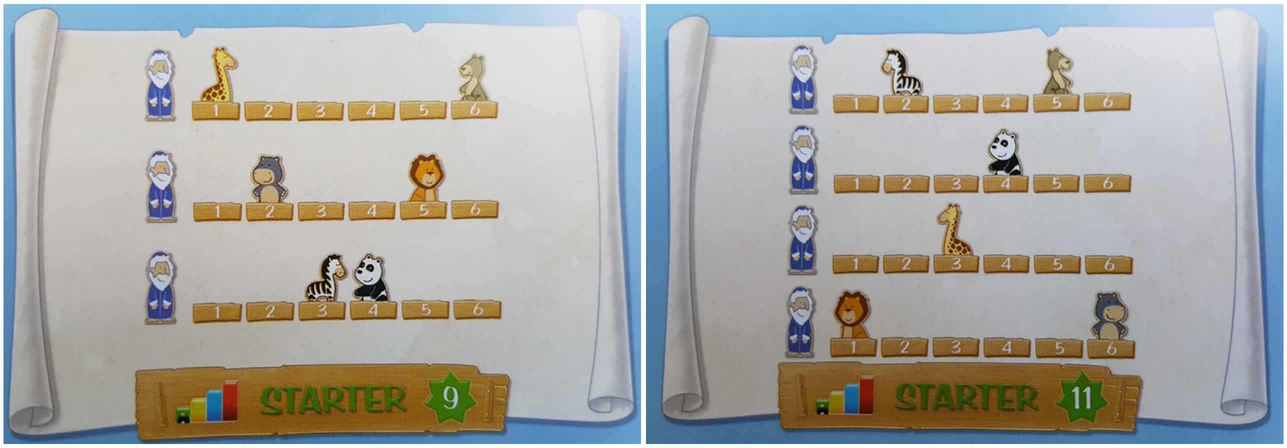


FIGURE 5. *Discernments: multiple strategies can be used to solve the puzzle; clues do not have to be followed in the same sequence presented.*

FIGURE 6. *Discernments: animals that are talking to each other are next to each other; showing who is talking to whom is a new type of clue.*

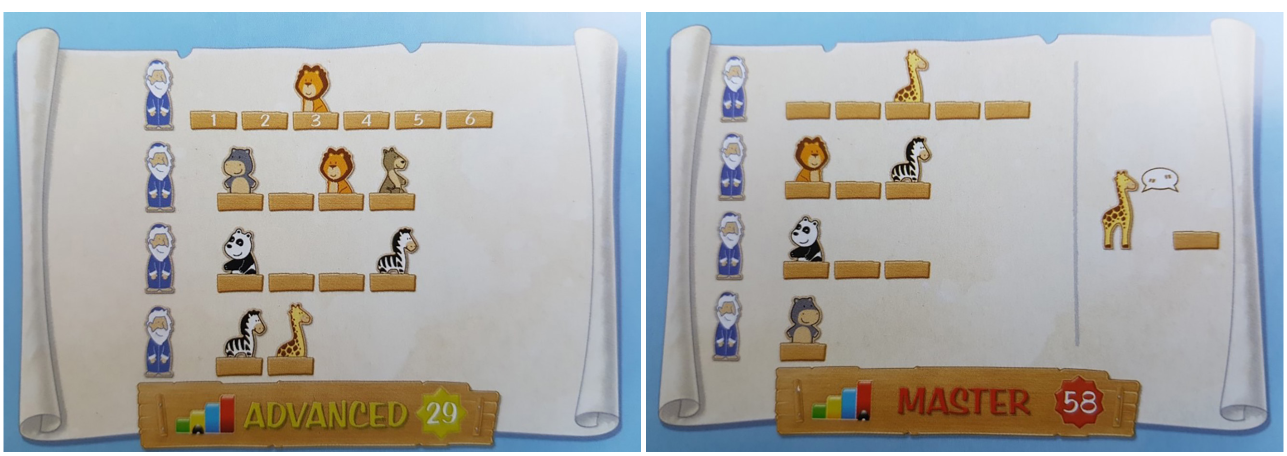


FIGURE 7. *Discernments: not all positions have to be labelled; order can be determined even if no information is given about position number.*

After exploring the puzzles, we explicitly labelled the concept we saw with the sequencing of puzzle clues as decomposition—that is, breaking down the general problem of placing the animals in the correct order (sequence) into a set of two or more discrete, non-overlapping puzzles. We also noticed that none of the clues contained redundant information (such as placing the same animal in its correct position in more than one clue).

We observed that when learners attempted to do puzzle 4 (see Figure 4), it was critical for them to first understand what the image represented. A young child's interpretation and the questions asked by several preservice teachers who played the game suggest that they saw the two images as discrete situations—asking, "Do I need two arks?"—and not as a decomposed clue to the general goal of placing the animals in the ark in the right order and orientation (facing left or right). This, we believe, offers the opportunity to draw learners' awareness to the representation in the puzzle clues as a decomposition of the final goal (placing all the animals in the correct order). Puzzles 7 (Figure 4) and 11 (Figure 5) continue this elaborative explorative practice—first, with a puzzle that is decomposed in a continuous piecewise manner, and then with a puzzle in which the animals are unevenly distributed.

We recognize that naming this process decomposition was a result of priming ourselves beforehand by considering what aspects of computational thinking might be present in this game. We had an intuitive sense that the actions the game guides players to explore could be framed within the educational discourse of decomposition. Having thus recognized the naming as a function of priming, we note that decomposition in computational thinking (in this case, with Archelino) is an illustrative example of Polya's (2014) more general heuristic regarding decomposing and recombining in problem solving, which involves analyzing a problem and breaking it into smaller, discrete problems, the sum total of which solves the original problem.

We also want to emphasize that while the sequence of puzzles in Archelino teaches the idea of decomposition, nowhere does the game explicitly name it as such, nor would we expect teachers (or parents) to do so without prompting or consideration of the particular affordances for learning. We also note that in developing explicit conscious awareness of this critical aspect of computational thinking, it is important for teachers to place a semiotic linguistic marker on the necessary aspect of the experience—that is, to label the puzzle image as "a decomposition into two parts," "a decomposition into three parts" and so on. If teachers explore and name decomposition with students in an explicit teaching opportunity, students will, ideally, be able to make connections to the importance of decomposing in other situations.

In the next section, we discuss our approaches to and strategies for playing Archelino (including our differing strategies) and move into how teachers might use the game in the classroom in various formats.

Affordances for Developing Computational Thinking

Initially, we each played the game individually. Steven also played it with his three-and-a-half-year-old daughter. We then discussed how our strategies differed for some puzzles.

For example, in puzzle 9 (Figure 8), Sally's strategy involved following each line of the visual instructions and enacting them sequentially: first, placing the giraffe and the kangaroo; then, placing the hippo and the lion; and, finally, placing the zebra and the panda. Steven's strategy involved working across the three clues to place the animals serially, following the ordinal numbers: placing the giraffe and then the hippo, the zebra and the panda, the lion, and, finally, the kangaroo.

FIGURE 8. *Puzzle 9.*

Sally's Strategy

- The giraffe is in the first position, facing to the right.
- The kangaroo is in the last/sixth position, facing left.
- The hippo is next to the giraffe, facing the giraffe.
- The lion is next to the kangaroo, facing right.
- The zebra is to the right of the hippo, facing right.
- The panda is facing the zebra.

Steven's Strategy

- The giraffe is in the first position, facing right.
- The hippo is in the second position, facing left.
- The zebra is in the third position, facing right.
- The panda is in the fourth position, facing left.
- The lion is in the fifth position, facing right.
- The kangaroo is in the sixth position, facing left.

As we talked about our different approaches, we came to appreciate not only that we approach puzzles differently but also that there is value in someone else's approach. These are valid alternative strategies, and they both will lead to successfully solving the puzzle. In a classroom, a small group or a teacher interview setting, allowing students to see and make sense of alternative strategies is important. This increases both the personal example space of strategies of which students are aware and the potential for students' flexible thinking when working on future puzzles.

Other ways of giving the instructions can draw attention to the structure of the visual clues in a mathematical way. For example, a preservice teacher in a workshop setting stated, "The animals are arranged in pairs facing each other; there is an AB repeating pattern." With a similar puzzle, a preservice teacher noticed that "animals placed on odd-numbered spaces are facing left, and animals placed on even-numbered spaces are facing right." The existence of a number of valid ways to approach the problem (depending on what is noticed and attended to) and the opportunity to bring in a variety of mathematical ideas are characteristic of a good low-floor task for elementary learners. Working one-on-one with students allows teachers to use deliberate questioning to further investigate their chosen strategies and the mathematical thinking behind their responses.

The observation that there are multiple ways to provide instructions that result in the same goal using different mathematical concepts is meaningful in the context of Rich et al (2017), who argue for an "offline-before-online heuristic" in learning computational thinking. They begin their sequencing trajectory with two fundamental ideas that relate to the Archelino puzzles. The first idea focuses on the importance of specificity when giving instructions, and the second explores the importance of the order of the instructions, as changing the order can lead to different results.

Adapting for a Partner Activity

When we were discussing our strategies, the idea emerged to play Archelino as a two-player game, with one player translating the visual clues into oral instructions and the other player building the sequence. This approach is related to our familiarity with barrier games in elementary education. Playing in this way, with the two players alternating roles, also allows for immediate feedback and a sort of debugging of instructions, as the translator can see immediately whether the builder has carried out the instructions correctly or whether steps are missing or unclear.

Displaying a poster with question prompts alongside the game would be beneficial for students when playing with a teacher or a partner. These questions could be used during the game or after solving the puzzle (as an opportunity for reflection). Questions could include the following:

- Why did you choose _________ as your first move?
- What did you first notice when looking at the visual clue?
- What strategy did you use to solve the puzzle?
- Did you notice a pattern when looking at the visual clue?
- How did you know to place the _________ in that spot?
- How did you know to place the _________ facing left?[4]

Demonstrating this type of questioning with parents and preservice teachers helps them to notice important aspects of the game play and to nurture students' thinking, problem-solving and oral communication skills. We also note the increased challenge and cognitive demand involved in attempting to verbalize visual instructions and how this verbalization helped us attend differently and effortfully to critical aspects such as the orientation of the figures, their relative positions and the sequence in which we gave the instructions, especially with the more challenging puzzles.

Adapting for a Whole-Class Activity

Exploring the activity as a class before having students play Archelino as a two-player game has benefits. It enables students to build an understanding of the goals and purpose of the activity and prepares them for playing the game with a partner. Doing a whole-class activity also allows the teacher to provide students with the appropriate language to use when completing the activity and to offer suggestions for problem solving. Here is an example of how such an activity can be structured:

- To begin, invite six students to the front of the class to represent the six animal figures on the ark. You may want to use hats or coloured fabric to identify them, as they will be moving around.
- Tape an ark (a number line) to the floor so that students know where to stand. You can identify one student as Noah to stand at the front of the ark.
- Display a visual clue on the board, using a document camera or a slideshow projector.
- Give students quiet thinking time to explore how they would place the animals on the ark.
- When students are ready, select one student to give oral instructions to the students who are representing the animals so that they appear on the ark exactly as they do in the visual clue.
- In addition to giving oral instructions, students can be responsible for moving the animals into the correct order on the ark.
- Invite students to use various strategies to place the animals on the ark, and take the time to explore their differing strategies.
- Once the animals are in the correct order on the ark, students can discuss their strategies and how they solved the visual clue.[5]

This approach resonates with Sung, Ahn and Black's (2017) discussion of the benefits of teaching computational thinking through embodied experiences, which allows for hands-on learning before the introduction of digital tools. Their study, which used the introductory programming language ScratchJr for coding number lines and doing arithmetic on the number line, found that "children who were asked to provide commands to a surrogate by decomposing steps to solutions developed robust learning and improved their number line estimation" (p 459). In particular, the ScratchJr interface afforded opportunities for discerning the equidistance of moves on a number line and enacting the composition of moves in addition.

Abstracting to Other Games and Representations

The Archelino game was a convenient choice for us. However, other commercially available games could be used to explore the same ideas. The animal figures in the Archelino set could be replaced with more readily available and affordable items (such as connecting cubes, recycled plastic toys or tangram animals) (see Figure 9), though we do not underestimate the value of investing in high-quality wooden toy puzzles and manipulatives that will last for many years and classes. The most important aspect of the Archelino set is the puzzle cards, and parents and preservice and inservice teachers should spend time thinking about how the puzzles are sequenced if they plan to remix this game using different materials or scenarios.

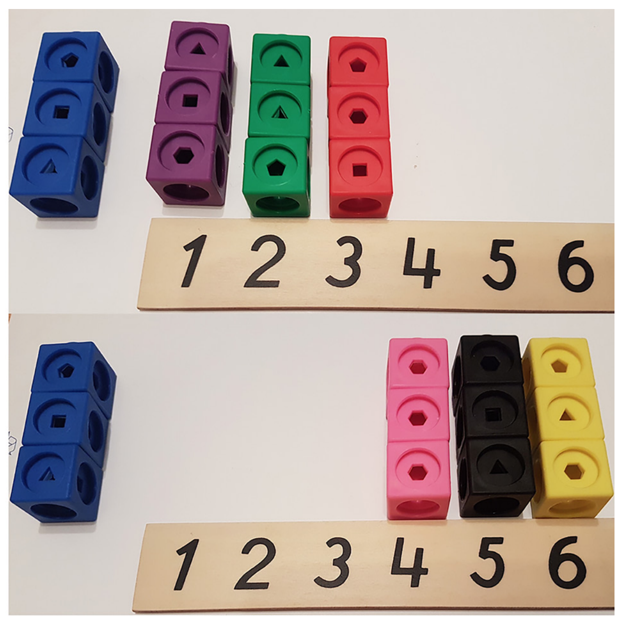


FIGURE 9. *A variant of puzzle 4, using connecting cubes.*

In the next section, we discuss an affordance for developing computational thinking dispositions with Archelino.

Developing Computational Thinking Dispositions

Computational thinking disposition refers to "the willingness and opportunity to make use of [student] knowledge and ability" in the area of computational thinking (Pérez 2018, 427). According to Pérez, students must develop computational thinking dispositions if they are to derive benefit from meaningful computational thinking learning opportunities presented in classrooms in mathematical contexts. In addition, if students can recognize computational thinking opportunities on their own, they will then be able to "recognize, respond to, and appropriately act" (p 428) on these opportunities.

Pérez's framework for the three necessary dispositions for computational thinking—tolerance for ambiguity, persistence and collaboration—is relevant to the Archelino game and can also be used when working with students on developing their computational thinking dispositions. Next, we elaborate on these dispositions and how they can be developed with Archelino.

Tolerance for Ambiguity

Pérez (2018, 444) describes the first disposition—tolerance for ambiguity—as "a tendency to experience ambiguous situations or stimuli as enriching and engaging."

As noted before, Archelino puzzle 4 (Figure 4) is an ambiguous visual stimulus for some learners and requires interpretation and teacher intervention to explain the connection between the puzzle clues and the task goal. This is an easily resolved ambiguity. When working in partners, for example, one learner must listen attentively and carefully to the oral instructions given by the partner and try to place the animals in the correct sequence and the correct orientation. Depending on the degree of specificity of the oral instructions (for example, the mathematical language used), in addition to various possible solution pathways, both learners may experience what we will call productive ambiguity (as an analogue to productive struggle). This requires them to find ways to reduce the ambiguity (for example, by asking clarifying questions) and move toward successfully completing the puzzle.

Teachers have a tendency to too quickly reduce such ambiguity for students in the same way that they sometimes too quickly reduce the cognitive demands of a task, which takes away learners' experience of productively struggling with the mathematics—the work of learning. We see a need (and an opportunity through playing Archelino) to give students more opportunities to experience productive ambiguity and to work to resolve—communicatively, mathematically and computationally—ambiguous situations that are nonthreatening. This helps build their confidence and questioning skills for when they face challenging cognitive tasks involving decomposition.

Persistence

Persistence is a necessary and valued habit for learners of all ages. It is related to developing resilience in challenging contexts, such as problem solving and computational thinking (with or without coding).

In Archelino, students display task persistence as they work to successfully complete the puzzles. Pérez (2018) argues that the classroom environment and learning opportunities should encourage persistence among students in order to develop this disposition.

To develop persistence, students need challenging tasks at the appropriate cognitive level that offer positive reinforcement upon completion. They may need numerous attempts to complete some puzzles, especially as the difficulty level increases and the puzzles begin to incorporate more elements (such as increased ambiguity and multiple possibilities for character orientation). Productive failure in mathematics (Kapur 2016), however, is one way to guide instruction. Learners' failed attempts provide teachers with rich information about their thinking and problem-solving skills that can guide further instruction.

The range of difficulty in the Archelino puzzles also allows for differentiation in the classroom. Additionally, to support all learners in solving the Archelino puzzles, teachers can use different scaffolding tools (such as a number line in front of the ark, labels for the positions on the ark, or left and right directional signs beside the ark).

Collaboration

Pérez (2018, 449) defines *collaboration* as a "tendency to coordinate effort and negotiate meaning with peers to accomplish a shared goal."

As a partner or a parent–child activity, playing Archelino allows for working together to complete the given task. At first, students can complete the puzzle by listening to instructions provided by the teacher or a parent. With explicit teaching, students can move to providing oral instructions to their peers.

Completing the puzzle as a whole-class activity, as explored earlier, encourages students to work together. In this setting, deliberate and intentional sequencing of the sharing of important variations in strategies allows students to experience situational ambiguity (Pérez 2018) and make connections as they discuss possible solution pathways. For example, immediate feedback from watching a partner place animals in the ark can lead to discussions about strategy and the specificity of instructions. Overall, this activity allows students to practise collaboration as part of developing computational thinking dispositions.

Archelino allows teachers to focus on developing computational thinking dispositions by explicitly teaching the concepts and by encouraging students to notice, name and nurture the computational thinking in this activity. Pérez's (2018) framework gives teachers a tool to guide instruction, explore student dispositions and improve how they approach computational thinking tasks in the classroom. Explicitly teaching young students this important construct can help them notice, name and nurture computational thinking opportunities in their later studies.

The framework also gives teachers an important tool for gauging students' computational thinking dispositions and planning further instruction. Like Pérez, we believe that these dispositions are malleable and that students can work toward improving their computational thinking dispositions over time.

Conclusion

Several good low-floor, unplugged activities for developing the skills and dispositions of computational thinking with young learners, as well as with teachers new to computational thinking, already exist. Commercially available or publicly accessible puzzle games, such as Archelino, allow teachers to bring computational thinking and mathematical ideas into the classroom in a nonthreatening way while developing a healthy disposition.

We have attempted to show readers how a game-based resource can be used to encourage discussion and engagement with foundational computational thinking ideas, such as decomposition and sequencing, together with early mathematical concepts, such as ordering and orientation. It is our hope that after reading this article, readers will be more alert to opportunities to explore foundational computational thinking ideas through resources they already have at hand.

Notes

1. This statement and the previous one are excerpts from early written reflections and conversations during team meetings.

2. See https://new.learnalberta.ca/Resources/content/cda/documents/math_en.pdf (accessed September 10, 2019).

3. See www.iste.org/standards/standards/for-students/ (accessed September 10, 2019).

4. A downloadable poster is available at https://docs.google.com/drawings/d/1TT4KSdJG_Taas9ksEK8qlc3t59kFPjLDsI7D6ihhNdk/edit (accessed September 10, 2019).

5. A downloadable lesson poster is available at https://docs.google.com/drawings/d/12lP87PzgFWwkkfp4oG96cDm5_8bNbOicAeI5FB9MXzg/edit (accessed September 10, 2019).

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Junior High Students' Perceptions of Mathematics Learning Experiences

Jesse Diachuk

Purpose, Question and Subquestions

The purpose of this basic qualitative research, operating within a constructivist theoretical frame (Guba and Lincoln 1994), was to explore junior high students' perceptions of mathematics learning experiences.

To facilitate this research, I interviewed four Grade 8 students from one school in an urban central Alberta school district to unveil the challenges associated with learning mathematics, the strategies and supports that increase the likelihood of success, and the impact of assessment in the classroom. I selected the student participants through convenience sampling (Merriam and Tisdell 2016), choosing the four participants by random selection from the group that returned the required forms. Though there were no criteria for selection beyond participation in Grade 8 mathematics, the participants happened to be diverse both ethnically and demographically and possessed a range of confidence levels and ability in mathematics. I conducted one-on-one, open-ended, semistructured interviews of 17–25 minutes, and had no prior relationship with the participants.

The data analysis involved transcribing, analyzing and interpreting qualitative data attained through the interviews. I sorted raw data from the interview transcripts into a number of open codes and narrowed those down through axial coding into the emerging themes presented in this article.

The research question guiding this inquiry was, What are the perceptions of four Albertan junior high students about their mathematics learning experiences? To support this key question, I used the following subquestions:

- What challenges exist for junior high mathematics students?
- What learning techniques increase the likelihood of success in junior high mathematics?
- How are students supported in their junior high mathematics learning experiences?
- How do students describe assessment practices that improve mathematics learning experiences?

Findings

After data collection and analysis, four main themes emerged from the central phenomenon of student perceptions of mathematics learning experiences:

- Barriers to student success in mathematics
- Learning supports that increase the likelihood of student success
- Factors affecting student engagement in mathematics
- The impact of assessment on mathematics experiences

Barriers to Student Success

For students, an inequitable mathematics experience can be as debilitating as an equitable experience can be affirming. Numerous factors lead to students' negative perceptions of mathematics and can contribute to the development of mathematics anxiety.

Several studies (Lin, Durbin and Rancer 2017; Núñez-Peña, Suárez-Pellicioni and Bono 2013; Ruff and Boes 2014) have defined *mathematics anxiety* as fear, nervousness, helplessness, anxiety and dread related to learning mathematics and solving mathematical problems, which can lead to the avoidance of mathematics activities altogether.

The participant interviews illuminated several barriers to student success in mathematics that may coincide with the development of mathematics anxiety. After I analyzed the participant data, it became evident that the barriers to student success could be sorted into four subcategories: the content, the teacher, peers and the classroom.

The Content

All four participants—participant 1 (P1), participant 2 (P2), participant 3 (P3) and participant 4 (P4)—indicated that curricular content served as an inhibitor to success in mathematics, an assertion consistent with the findings of prior research (Lin, Durbin and Rancer 2017; Ruff and Boes 2014).

Both P3 and P4 referred to the sequential aspect of mathematics as being a potential barrier. P3 pointed out that success in previous grades has an impact on success in future mathematics. P4 stated, "If you don't understand the first part, you're not going to understand the rest of the unit."

P2 lamented her inability to understand some curricular content, even with repeated explanations from the teacher, and P1 and P3 pointed out the content-related barriers associated with problem-solving question formats, which can undoubtedly be a struggle for struggling readers and English-language learners. P3 explained this in a unique manner, talking about "making questions bigger than they need to be."

Taken together, the participants' perceptions of content as a barrier highlight the important role of learning supports in increasing the likelihood of student success in mathematics.

The Teacher

The participants zeroed in on many teacher-centred barriers to student learning.

Both P3 and P4 mentioned having difficulty when the teacher provided too little or no explanation of the content, with P3 elaborating that some teachers "don't explain" and others "talk too fast."

P1, P2 and P3 reported that different teaching methods had an impact on their understanding. P1 expressed that she felt confused when the teacher was "teaching one way, but you actually know the other way." She queried, "If the teacher recommends one [method], but we get the other one, what should we do?" P3 echoed this sentiment and belaboured the problem of learning one method at home and then being expected to use a different method at school. P2 admitted to struggling when one teacher "used to write all around the classroom" and "all over the board." The first three participants also emphasized that they felt frustrated when teachers struggled to explain something properly.

Research has shown that ineffective teaching practices have a negative impact on student learning (Lin, Durbin and Rancer 2017; Ruff and Boes 2014; Whyte and Anthony 2012). The participants' views support the idea that aligned with inquiry-based, student-centred mathematical pedagogy, teachers must embrace various styles and methods of both teaching and learning in order to better accommodate all learners.

P1 highlighted the issue of poorly established teacher–student relationships, disclosing that she was "not really comfortable" with her teacher: "Even if I have questions, I won't ask." P2 noted that access to the teacher can be a barrier to student success because "there are a lot of people in our class, so the teacher can't focus on us individually" and sometimes "everyone needs help with different things, so the teacher gets kind of confused." These accounts demonstrate how instrumental it is for teachers to build relationships with their students and to make themselves readily available as a learning support.

Peers

The participants noted several peer-related barriers to mathematical success, in both collaborative work situations and project-based learning (PBL) settings.

P3 remarked that working with people who are at different levels of understanding can "bump you down and make you feel different," and P2 stated that, in group situations, "if it's a concept I don't get, it's going to be tough for me." She elaborated, "If I'm with people I'm uncomfortable with, I won't share my ideas. I would just do everything that they say."

P1 raised issues related to differing ideas about the direction of projects. She also noted the problems of having to work in confined spaces and of the noise level when everyone talked and planned at the same time in PBL environments.

Two participants conveyed their hesitance about depending on other people to complete their work effectively, with P4 claiming, "If they do something wrong and I've done all my work, it's all their fault." P3 and P4 both said that they preferred independent work to PBL, citing differing personalities, levels of ability, work speeds and quality of work, as well as problems related to meeting outside class time.

Teachers can address these concerns by allowing students to choose their own groups, by allowing flexible project timelines, and by coaching students on group dynamics and productivity.

The Classroom

The participants identified numerous classroom-based barriers to optimal mathematics learning.

P3 and P4 cited noise and distractions as factors that prevented them from focusing on their work.

P1 saw the lack of time given to respond to teacher questions as a barrier, asserting that it "doesn't give people that many chances to answer questions." P2 reported that it "brings me down [when] I'm trying to figure out the answer and they already said [it]."

P3 (who had taken three years of math in French before pivoting to classes in English) and P2 both discussed communication as a barrier to learning. P2 sat beside three students who often spoke another language, and she struggled because she "focused less on math and more on trying to figure out what they were trying to say."

All four participants discussed the downside of technology in the classroom, referring to its distractive nature as a barrier to learning. Participants explained that students "sneak onto YouTube" or other websites and "lie, saying they're doing their work." As summarized by P4, "It's hard to keep a junior high school student focused on the task at hand."

It would behoove teachers to be aware of the distractive elements in the classroom that serve as barriers to student learning and to mitigate their effects.

Learning Supports

During data analysis, I identified myriad factors that support student learning in mathematics. Within this theme, five subcategories emerged: the teacher, peers, personal strategies, external supports and technology.

The Teacher

All four participants spoke of the importance of having a strong teacher who is willing to help students, who explains material effectively and who allows multiple methods of finding answers. P3 spoke about how her teacher "would explain different [ways] to certain kids" and "write it down if it was a reading question to make it easier for me."

Three participants insisted on the importance of communicating directly and sharing work progress with the teacher. Teacher–student relationships directly affect student success, and the participants saw teachers who were "encouraging," "nice" and "understanding" as more approachable and supportive. P3 explained that students should "talk to [teachers] about the way [they] understand," which is far easier when the student and teacher feel comfortable with each other.

P2 and P3 noted the importance of accommodating students' individual needs and differentiating learning in the classroom. This is consistent with prior research, which found that it is vital for teachers to have the ability to nurture trusting, caring relationships with and between students, in order to create an atmosphere of safety where everyone feels involved, appreciated and able to communicate openly

(Dunleavy 2015; Griggs et al 2013; Tait-McCutcheon and Loveridge 2016).

Peers

One of the most discussed topics across the interviews was peer support. All four participants mentioned often their strong belief that peer support and collaboration were essential to their success in the classroom, a finding supported by the research (Brenner, Bianchini and Dwyer 2016; Griggs et al 2013; Tait-McCutcheon and Loveridge 2016).

P1 stated that working with peers made learning math "more fun and easy," a sentiment echoed by the other three participants, who explained that through working together, students could share methods and ideas.

All the participants made it known that they enjoyed supporting their classmates as much as seeking help themselves. According to P4, the collaborative nature of the math classroom created an environment where students were "all very supportive of each other and help each other learn what we need." P3 specified that sometimes she would ask what the right answer was and work backward to learn how to find it, while at other times she would ask other students, "Which way are you doing it?"

Three participants pointed out that group work in PBL situations could involve idea sharing and helped fill in knowledge gaps for each group member. P2 stated, "Sitting next to people I'm comfortable with is really good for me. They try to explain . . . and go over and over it until I understand."

The participants unanimously agreed that peer support was extremely important to finding success in math. Thus, teachers should be ever cognizant of the benefits of peer support and should continually promote healthy collaboration in the classroom.

Personal Strategies

Three participants disclosed the personal strategies they employed to help them achieve success in mathematics, noting the importance of practice and completing assigned work.

P1 and P3 both referred to their personal learning style as being visual-spatial. P1 stated, "If I do hands-on stuff, I understand it better." P3 seconded this sentiment and added, "I have to visually see the pictures." P1 talked about how supportive it was to her learning to explore different methods before selecting the one that worked best for her.

External Supports

Both P1 and P2 disclosed that they spent time with a tutor outside school, which helped support their mathematical understanding. P3 noted that regular time spent working on math with parents was helpful, expressing that her mom could "explain it in a smaller way." Additionally, P3 spoke of private classes she had attended in prior years to augment her learning.

Technology

All the participants confirmed that technology was present in their classrooms and that it supported their learning in mathematics, despite the barriers it presented.

P1 and P2 pointed to the academic benefits of YouTube. Many mathematics help videos are available to explain concepts when other support is not available. P1 and P4 explained that websites such as Mathletics and Math Antics were strong learning supports for them, for both extra practice and video tutorials. P2 shared that her math class used Google Classroom and that the teacher posted notes and support materials for the students. In terms of working collaboratively, P3 said that computers could be helpful "if we're working on a project."

All participants indicated that it would be helpful if teachers could figure out a way to ensure that students were responsible when using technology, but they did not seem confident that this would happen.

Student Engagement

The participants were eager to detail the factors that affected their engagement in mathematics learning. Because of the nature of the responses, I have arranged this emerging theme into two sections: engagement and disengagement.

Engagement

The participants mentioned an array of factors that increased their engagement in mathematics.

P1 shared that her teacher told students that "there are no right and wrong answers," and she noted that this process-based approach made it "nice to participate."

All four participants agreed that mathematical content could be fun and engaging. P1 and P2 both said that they enjoyed "learning new things," and P2 spoke of the sequential aspect of math, claiming that students got to "build on what [they] learned about last year." P4 indicated that he "enjoys the challenge and the subject matter."

An interesting revelation by P3 was that she would demonstrate her engagement to her teacher not by raising her hand but, rather, by "mouthing the answer to [herself] instead of [saying] it out loud." Another indicator of student engagement, suggested by both P1 and P2, was attention-seeking behaviour, such as engaging in class talks and going in front of the class to explain concepts to peers. P2 explained that doing this "shows the teacher that I care."

Three participants stated that they engaged in and were motivated by math when they experienced success. This is substantiated by other studies, which have found that self-efficacy positively correlates with effort and achievement and inversely correlates with the presence of mathematical anxiety and the perceived difficulty of the content (Çiftçi 2015; Gafoor and Karukkan 2015; Martin and Rimm-Kaufman 2015; Mata, Monteiro and Peixoto 2012). P1 stated, "When people get it, they're really noisy because they're confident."

Based on these participant perceptions, teachers would be wise to facilitate student exploration of curricular content, provide opportunities for meaningful class discussion, and celebrate students' learning breakthroughs and successes.

Disengagement

Three participants cited the mood in the classroom as contributing to their disengagement from math. P1 clarified that "if it's too quiet, it gives you a vibe that no one gets it," which "brings you down."

For all four participants, content emerged as a source of disengagement, for a range of reasons. P4 stated that the content was sometimes "too easy" and "boring," whereas P1 and P2 said that they disengaged when faced with an utter lack of understanding.

For P1, P2 and P3, feelings associated with math anxiety were also an impetus for disengagement. This is pertinent for mathematics teachers to consider, alongside several studies that have found that students who find mathematics easier and who have higher self-efficacy are more willing to seek help, whereas those who struggle and have lower self-efficacy tend to quit when the perceived difficulty is too high (Gafoor and Kurukkan 2015; Martin and Rimm-Kaufman 2015; Newman and Schwager 1993). These three participants conveyed a sense of fear associated with giving wrong answers in front of others. Two of them added that they disengaged when other students came up with the answers too fast and were not supportive.

Classroom distractions served as a source of disengagement for all four participants. P4, who claimed to have a high success rate in mathematics, asserted that he found it "very hard . . . to concentrate" when the classroom pacing was too slow. He stated, "I usually end up talking because I get too bored."

It is essential that teachers become attuned to the factors contributing to student disengagement and that they work to ensure that classroom practices afford all students the unimpeded opportunity to learn.

Assessment

The participants shared the assessment practices that enhanced their mathematics learning and helped them experience success in the mathematics classroom.

P1 and P4 both expressed the importance of teachers providing review packages for students. P1, P2 and P3 claimed to have no aversion to testing, but they felt somewhat apprehensive about concepts that they did not understand well. P3 said that she preferred assignments and book work to projects and exams, because she preferred "not to work with other people" and feared "not understanding on a test" because one misunderstood concept can "hurt my whole . . . test." P4 said that he favoured tests over other assessments because he wanted to "just get down to the material rather than spending forever on a whole bunch of tiny projects." P1 suggested that, with regard to group work, teachers should consider "marking just one person, not the whole group" because "if the group does badly, you get the mark, too."

The participants did not seem overly concerned about the type of assessments teachers provided; instead, they focused on how they could support themselves in day-to-day mathematics learning to prepare for those assessments, a notion reflected by the significantly lower number of codes in this theme.

Suggestions for Teacher Action

As evidenced by these findings, student perspectives are valuable for informing teaching practices and should be central in conversations of pedagogy. Based on my research findings, I offer the following six suggestions for teacher action, along with other supporting research:

- Be aware of content-, teacher-, peer- and classroom-related barriers that prevent students from reaching their full learning potential and that contribute to the development of debilitating mathematics anxiety (Griggs et al 2013; Lin, Durbin and Rancer 2017; Maloney and Beilock 2012; Núñez-Peña, Suárez-Pellicioni and Bono 2013; Ruff and Boes 2014; Whyte and Anthony 2012).
- Maximize the availability of learning supports in the classroom so that students are best equipped to overcome barriers to learning (Brenner, Bianchini and Dwyer 2016; Dunleavy 2015; Griggs et al 2013; Tait-McCutcheon and Loveridge 2016).

- Build healthy, supportive relationships with students to nurture the development of trust and care, which serve as the foundation of optimal learning environments (Dunleavy 2015; Griggs et al 2013; Tait-McCutcheon and Loveridge 2016).
- Encourage regular, meaningful collaboration between students, including PBL, and promote the benefits of peer support (Brenner, Bianchini and Dwyer 2016; Griggs et al 2013; Tait-McCutcheon and Loveridge 2016).
- Be aware of classroom dynamics (including the notion of productive noise, helpful and harmful student interactions, and the positive and negative impacts of technology) to ensure that students are placed in positions that will enhance their learning (Lin, Durbin and Rancer 2017; Maloney and Beilock 2012; Ruff and Boes 2014; Tait-McCutcheon and Loveridge 2016; Whyte and Anthony 2012).
- Adopt a process-based, rather than a product-based, approach to mathematics that celebrates mistakes as part of the learning process and embraces the idea of productive struggle. Using such an approach will reduce the effects of math anxiety and prevent student disengagement (Brenner, Bianchini and Dwyer 2016; Dunleavy 2015; Núñez-Peña, Suárez-Pellicioni and Bono 2013; Ruff and Boes 2014; Tait-McCutcheon and Loveridge 2016; Whyte and Anthony 2012).

Conclusion

This article has reviewed four themes that emerged surrounding the central phenomenon of junior high students' perceptions of mathematics learning experiences:

- Barriers to student success in mathematics
- Learning supports that increase the likelihood of student success
- Factors that affect student engagement in mathematics
- The impact of assessment on mathematics experiences

The findings suggest that students are attuned to the intricacies of their mathematics learning experiences. Thus, it is evident that the voices of students warrant regular consideration in discussions about pedagogical advances in and beyond the mathematics classroom. Although students seem to be less inclined to weigh in on assessment practices, they are acutely aware of the barriers that prevent them from achieving success, well versed in seeking ways to support and augment their own learning, and cognizant of the factors affecting their engagement in mathematics.

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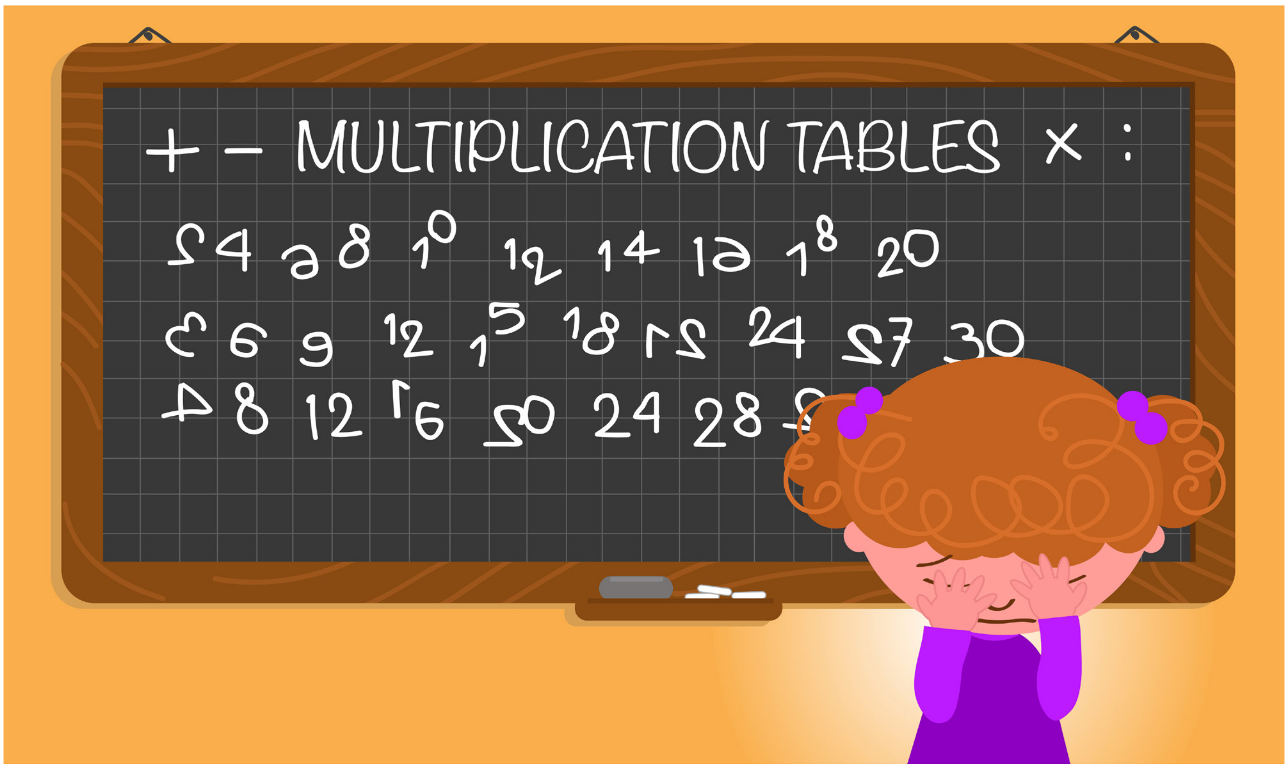
Learning from Math Class Misadventures: The Experiences of Students with Dyslexia and Considerations for Educators

Lauren D Goegan, Amy Domenique Gadsden, Wyatt Schiefelbein and Lia M Daniels



Most people have heard the term *dyslexia*. Popular culture often suggests that dyslexia involves the reversal of letters in words, leading to jokes such as "Dyslexics of the world, untie" or jokes about children writing letters to Satan instead of Santa. Others see dyslexia as a difficulty with understanding the sounds in words (phonological knowledge of the language). These perspectives lead people to interpret dyslexia as having an impact on young students when it comes to language (more specifically, English language arts class) but perhaps not mathematics. Indeed, dyslexia is predominantly characterized as a reading-based learning disorder that can also affect writing. Challenges in mathematics are often labelled as math-based learning disorders (such as dyscalculia), which is a separate category.

Although dyslexia is often diagnosed and treated as separate from mathematical skills, one can be diagnosed with both dyslexia and dyscalculia. Moreover, difficulties in reading and math can have similar underlying challenges, including effects on working memory, processing speed and oral language comprehension (Willcutt et al 2013). The experiences in math class of students with dyslexia can provide insight into how dyslexia affects learning and performance in mathematics and how teachers can support these students.

Toward that end, this narrative article has two main sections. First, we present the retrospective first-person stories of three students with dyslexia, who offer various viewpoints on what it is like to be a student with dyslexia in an elementary math class. Second, through these stories, we explore the opportunity to identify and adapt practices in the classroom.

These accounts are meant to help teachers think about their own teaching practices and how best to support student diversity in the classroom, building on the idea of universal design for learning (UDL) and motivation design principles. This examination is both timely and important. As the Alberta Education website states,

> Future curriculum focuses on foundational elements, such as reading, writing, and arithmetic, while also incorporating competencies like problem solving and critical thinking.
>
> Literacy and numeracy foundational elements are in every subject and at every grade level, and with greater emphasis on the development of competencies.[1]

Therefore, supporting students with dyslexia when it comes to mathematics instruction is vital.

Stories of Three Students with Dyslexia

Story 1: Misreading the Problems

I always hated math class. However, as I look back on my time in elementary school, I wonder if this hatred stemmed from the math itself or from the words associated with math.

The first time I remember disliking math was in Grade 2. As part of math class, we learned how to spell all the numbers up to 100. For a student who couldn't spell in general, having to spell in math class was terrible. To make matters worse, the paper would always come back covered in red ink from all my spelling mistakes, undoing any positive attitude I may have had toward math. Those days, it felt like math class was just an extension of English language arts, and all I wanted was more numbers and fewer words. Then, once we had learned how to spell all the numbers, we were given math tests that contained no numerals, only spelled-out numbers. I was lost in the sea of words, and I felt like I was drowning. I couldn't read questions properly, making it impossible for me to ever really get to the "math part" of a question. I often sat in class frustrated, knowing that I could do the math but unable to see the math in front of me. The teacher seemed to ignore my challenges and just told me to try harder next time.

Things didn't get any easier in the years to come, as math worksheets and tests began to include word problems. To this day, I refer to word problems as a reading test. The first time I remember encountering word problems was in math class in Grade 4. We were completing a unit on medieval times, and the word problems reflected this theme. Therefore, on top of the usual words I couldn't read, this unit introduced a new, specific vocabulary, which I also could not read. The words prevented me from showing what I actually knew in math because I had no idea what the questions were asking. My strategy was to guess what math skills the words were trying to get at: a big number with a small number probably meant a subtraction question; two bigger numbers probably meant addition. Guessing is never very accurate, and the red ink once again covered my paper. The words continued to get in the way, even though I could answer all the questions when they used numerals.

After a few tests like this, my teacher began to realize that something was going on when it came to the word problems. This teacher took the time to look at my tests and recognized the pattern in my errors. She recognized that my mistakes could be attributed to the words and not the math, and she was willing to make changes to support my success.

Specifically, she began to read the math tests aloud to me. Something about this combination of auditory and visual information supported my comprehension of the questions. Once I could understand what a question was asking, I could perform the necessary mathematical operation, and the red ink began to disappear. This supported my learning and became a strategy for solving word problems in the future. Offering multiple modalities for students can support the various learning needs in the classroom.

Story 2: Misunderstandings and Misperceptions

As an elementary student with dyslexia and other learning disabilities, I had many deeply wounding experiences, especially in math. As early as kindergarten, the impact of my disabilities was visible to others, as well as to myself. I had difficulty with expressive language, including articulation of letters and numbers, which included reversal, inversion and overall production. At the time, this was attributed to a behavioural challenge rather than a developmental one. As a consequence, I was often "othered" by peers and teachers. I felt the impact of my disabilities in how others responded to me, which affected my self-esteem, self-efficacy and self-worth.

One of my most devastating math experiences occurred in Grade 4. I had experienced much underachievement and failure in math up to then. I often felt frustrated in math class because of the amount of time I needed to attend to the lesson, write down the instructions and notes, process the instructions and notes, apply new knowledge, and ask questions and problem solve. As a consequence, I would often disengage and cry. I felt humiliated and stupid.

The specific incident took place following a summative assessment on long division. The teacher walked around the classroom, returning our tests. The silence as she passed my desk was deafening. She offered praise or encouragement to other students, but she had no words for me. Her silence felt like a punishment, a reinforcement of my inadequacy and inability. I felt shame. I turned my paper over. Red ink everywhere. Circles and lines scarred my worthless work. Every answer was wrong. On the bus ride home, I cried so much that the red turned to pink, and all that remained of my work were faint pencil lines.

My turning point in mathematics happened a few years later. After continued failure and underachievement, I was incredibly anxious about an upcoming test on fractions. I couldn't keep up in class—everything was moving too fast. Thankfully, an empathetic, patient and skilled substitute teacher changed the course of my achievement in math forever.

She sat with me for an entire morning, teaching me the fundamentals of fractions and the order of operations involved in solving problems. She helped me identify my challenges and areas of need, and asked me insightful questions that forced me to construct my own strategies to mitigate the impact of my disabilities in math. She took the time to respond to my questions, soothed my anxiety and was sensitive to my frustration. She also helped me construct a code book of sorts that outlined the steps involved in solving problems in multiple modalities (diagram, text, orally) that made sense to me—a key reference that would enable me to study outside of class to reinforce concepts (see Figure 1). Following this, I was motivated and committed to showing both myself and my teacher that I could be successful, that I was capable and that I did understand. The result was my first A ever in math!

That teacher helped me develop academic self-efficacy, which improved my self-esteem. As a result, I was more willing to take risks, advocate for myself and accept the impact of my disabilities as specific rather than global.

Story 3: Miscommunication When Asking Questions

In math class, I had a lot of trouble. Certain expectations in math were perhaps a bit beyond what I was capable of at the time—for example, times tables. I still don't know my sixes, sevens and eights, and I am not keen to learn them now. My experiences with word problems in math were less than ideal. I struggled with decoding the words, especially in terms of what aspects of the question were relevant. The lack of empathetic support led me to believe that my teachers were eager to fail me. They did not seem eager to support my learning needs in their classrooms.

Aside from all this, a major factor contributing to my poor performance in math class was miscommunication. I felt like my teachers couldn't understand me and the questions I asked. Oral language difficulties, such as problems conveying my ideas, often hindered my performance in math class, especially when I was required to explain my work. While I enjoyed math, math class itself was often a trying time for me.

I imagine that most people have had an experience like this: you say something you think is intelligible and receive in return a blank stare or, worse, a hesitant answer to a question you did not ask. In elementary school math class, this happened to me regularly, and it was the height of embarrassment.

Sometimes I would press on, trying to find a new way to phrase my question. After all, maybe my wording was the problem. This strategy worked maybe twice, and when it did, I felt like the teacher appreciated what I was bringing to the classroom and recognized that I was capable of understanding the material at hand. When it didn't work—which was most of the time—I was mortified, because now my peers were frustrated with my persistence and apparent lack of understanding.

I learned that it was easier to pretend the teacher had answered my question. I spent a lot of time in elementary and middle school pretending I had asked questions I had not asked. This saved me some embarrassment, but I felt largely on my own when it came to learning math. Often, it felt like teachers assumed what I was going to ask, and that was the question they would then answer, regardless of what I had actually intended. Challenges with oral language communication had an impact on my math performance, as I did not feel that I was on the same page with my teachers.

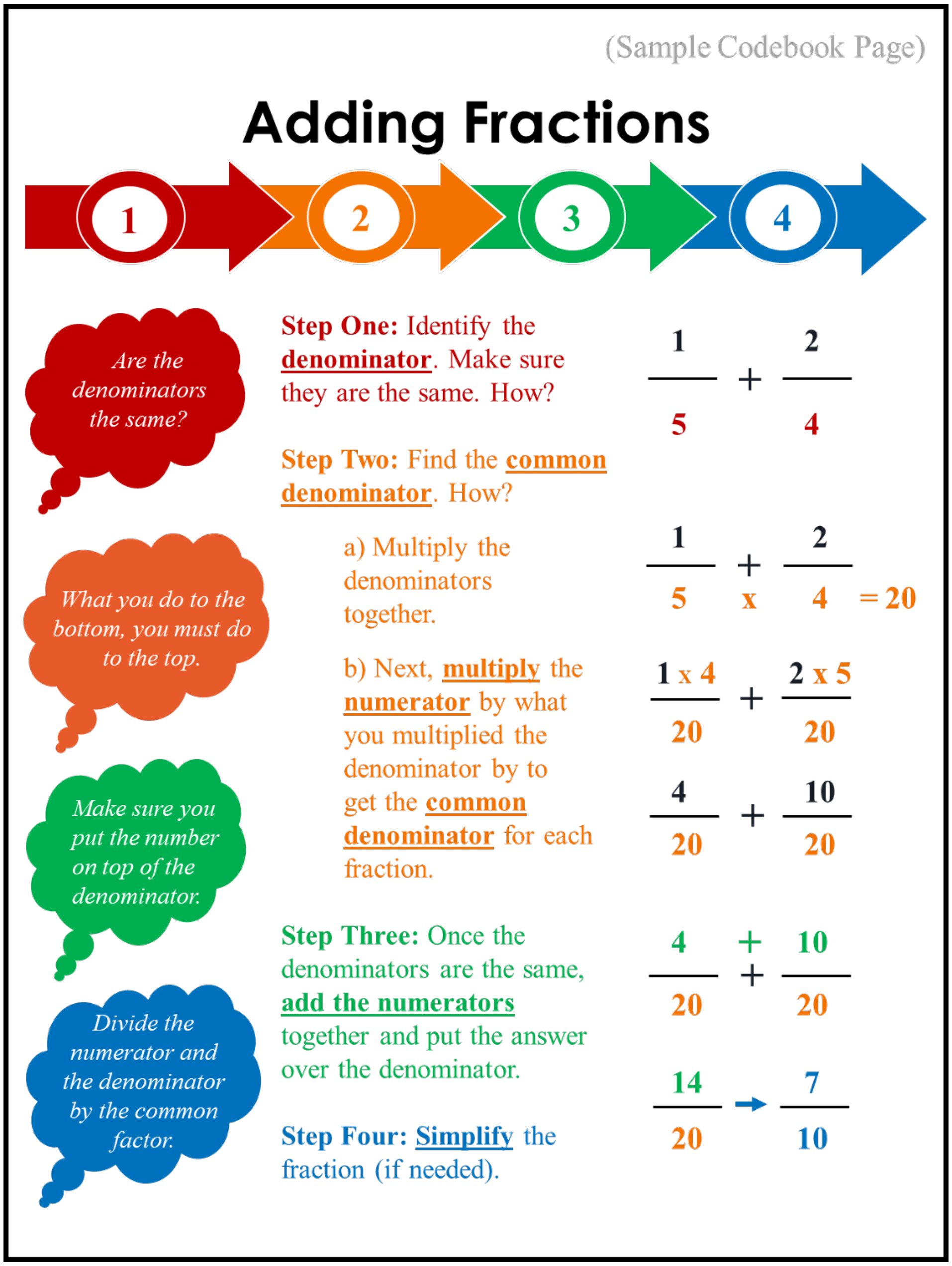


FIGURE 1. *Sample code book page for adding fractions.*

But to say that I was bad at math would be a mistake. I was rather good at math. I understood how math could be helpful in many contexts and, thus, why it was important to learn.

As I struggled to communicate in math class, I turned to a different place to learn math: art class. Through art, I could explore math in all kinds of ways. Multiplication, division, fractions, algebra, geometry—everything mathematical related to art in really fun and interesting ways. And the art teachers had fewer expectations as to what kinds of questions I would ask—miscommunication rarely happened during art class for me. My words worked there, even to sort out math. Because of this, I sincerely believe I learned more math in art class than I did in math class. Art allowed for visual exploration of concepts and ideas, an area of self-expression at which I felt adept. Art provided the freedom to explore how math was important to my goals. What is more, I was *good* at art, and thus I needed to be good at math. So I developed my math skills as they related to art.

The lesson in this story is that, as educators, we must try to really understand what a student is asking. Miscommunication is quite natural, but it takes two people to recognize that it is happening in order to then address it. Based on my own experiences with dyslexia, I would say that the root of the diagnosis is miscommunication—in my case, miscommunication that was exacerbated in math class but mitigated in art class.

Learning from These Misadventures

The stories of these three students highlight how students do not leave their dyslexia at the door of the math classroom. Dyslexia has major implications for math learning, performance and motivation. From these stories, we can extract important considerations for a responsive and inclusive math classroom that would align with Alberta's vision and advocacy for an inclusive education system.[2]

Universal Design for Learning

Universal design for learning (UDL) allows for the development of flexible learning environments that can accommodate individual learning (Edmunds and Edmunds 2018). The challenges described by the three students with dyslexia could have been mitigated with the application of UDL to math class.

The Center for Applied Special Technology (CAST) outlines three UDL guidelines: multiple means of representation, multiple means of action and expression, and multiple means of engagement.[3] Next, we describe how these guidelines could have improved the experiences of the three students.

Multiple Means of Representation

The first UDL guideline recommends providing students with multiple means of representation.

The first and second stories highlight how having access to multiple means of representation is vital for learners with disabilities. It gives them different ways to communicate and access information. For example, both stories underscore how providing alternatives to written information can contribute to better understanding. In the second story, offering ways to customize the display of information could have supported the student's understanding of the mathematical concepts to be learned.

As evidenced in all three stories, not all students learn best from reading, particularly those who have difficulty with breaking down the sounds in words, or those who have below-grade-level reading abilities. Furthermore, clarifying new vocabulary or words needed to complete the task can help reduce students' challenges with comprehension of word problems. Students must have equitable access to the content of questions so that they can meaningfully contribute to in-class activities in a timely manner. That is, time should be spent on comprehension and the development of requisite skills (such as problem solving) rather than on decoding. Knowing the words in a question is central to being able to answer that question.

Also, highlighting critical features and big ideas in a lesson can be instrumental in supporting those who have challenges with working memory or processing speed. A big idea is a place where all students can start, and they can build their knowledge from there. Using big ideas is particularly helpful for those with dyslexia because it gives them structure on which to build their learning.

Multiple Means of Action and Expression

The second guideline for UDL is multiple means of action and expression. *Action* and *expression* refer to the how of learning.

Indeed, in the second story, the teacher probed the student's learning by using questioning to help the student develop strategies for future challenges in math. This can serve as a think-aloud approach to better understand the gaps in a student's learning and how the student is making sense of the content. This approach allows the teacher to understand where the student is and to build from there, and it allows the student to be active in the learning process.

Furthermore, developing a code book with a checklist or template gives students an excellent reference for when they encounter similar problems in the future. This allows them to manage information and resources in a format that is accessible and that supports their learning needs. All students can develop a code book related to how they learn best. One student might write out the concepts, one might draw, and one might use different colours. Students can use a format consistent with how they learn.

This guideline could also be a consideration for word problems. Is there another way that word problems can be presented to show learning of math concepts? Indeed, in the first story, using a medieval setting for word problems required that students understood more concepts and terms than just those presented in math. This situation can be particularly challenging for students with dyslexia, who may struggle with vocabulary as a result of below-grade-level reading skills. While teaching across curricula can have benefits, using real-life scenarios in word problems might be more helpful for students who struggle with words. Building from that, acting out a word problem to ensure that students understand the content before they attempt to answer the math component can assist students with dyslexia.

Multiple Means of Engagement

The third guideline for UDL—and an important component of learning—is multiple means of engagement.

All three stories describe students who sat in class disengaged, uninspired, frustrated or even in tears. A teacher's use of a red pen was mentioned in two of the stories as a practice that the students experienced as punitive and unsupportive of their learning, as well as one that undermined their sense of self-efficacy and self-worth. Teachers must ensure that the learning environment is supportive and that it is a place where mistakes can happen and are, in fact, encouraged.

Teachers can support student engagement by creating opportunities for cooperation and collaboration in class and by facilitating a community environment. For example, in the first story, the student who struggled with the reading aspects of the word problem could have been paired with a student who had challenges with the numbers. Using their strengths to offset their challenges and to support each other can help students see that everyone has both strengths and areas of need. Furthermore, this can allow for meaningful peer interaction and the development of friendships, which is particularly important for those with learning disabilities, as they often have difficulties with social skills (Wiener and Schneider 2002).

Additionally, when it comes to engagement, helping students develop personal coping skills and strategies is imperative. As seen in the second story, students who have challenges in the classroom can experience high levels of anxiety and frustration. Having resources to manage that frustration and to develop coping strategies when challenges arise is important for students' engagement, achievement and continued motivation. In the third story, miscommunication was discussed as presenting obstacles to the learner in math class, but it was not an issue in art class. Math teachers must be as willing as teachers in other subjects to be patient, to understand the learner and to engage in meaningful dialogue.

Motivation Matters Most

In a video about UDL, David Rose, cofounder of CAST, notes that engagement is perhaps the most important component of UDL: "If we don't engage students in learning, don't make it important to them, make them motivated for it, then none of the other forms of representation or expression will be that important" (National Center on Universal Design for Learning at CAST 2010, 4:30). Therefore, we want to highlight the importance of supporting student motivation in the classroom.

Researchers at the Alberta Consortium for Motivation and Emotion (ACME), a research group that supports graduate students from programs across the Department of Educational Psychology at the University of Alberta, have been examining motivation design principles (Linnenbrink-Garcia, Patall and Pekrun 2016; Radil 2017). The five principles are as follows:

- Supporting student competence
- Enhancing autonomy
- Designing personally relevant and active tasks
- Modelling learning
- Encouraging relatedness

Next, we discuss how these five principles are relevant to students with dyslexia in math classes.

Supporting Student Competence

The first motivation design principle involves supporting student competence through well-designed instruction or encouraging feedback (Linnenbrink-Garcia, Patall and Pekrun 2016).

The three stories from students with dyslexia describe how hard it is to maintain motivation in math class when the words in word problems prevent students from accessing the numbers; when their efforts are met with red ink, large Xs and silence from teachers; and when their questions go unanswered. In contrast, well-designed instruction that embraces multiple modalities of presenting information or that provides multiple avenues for working with the information helps students learn and apply new knowledge and skills. When teachers ensure that students can get through the words to reach the math and then experience success, students' motivation increases.

Enhancing Autonomy

The second motivation design principle is autonomy. This involves allowing students to make decisions about or exercise control over aspects of their learning (Linnenbrink-Garcia, Patall and Pekrun 2016).

The students in the three stories experienced almost no control over their own learning; rather, they passively tried to meet the requirements set out by their teachers. However, the example of creating a code book that the student could access outside the classroom shows how empowering students not only sustains their motivation but can also bring about success.

Designing Personally Relevant and Active Tasks

The third motivation design principle is task design, which involves selecting relevant and interesting activities in order to facilitate active involvement (Linnenbrink-Garcia, Patall and Pekrun 2016).

Again, the three stories of students with dyslexia do not mention class activities or assignments linked to the students' interests. Rather, the stories describe the students' challenges with the tasks teachers presented to them and the challenge of completing those tasks with interest largely absent.

Student interest is an important consideration—not only with students with dyslexia but with all students in the classroom.

Modelling Learning

The fourth motivation design principle is modelling learning, which includes demonstrating a good attitude toward students and teaching with enthusiasm and energy (Radil 2017).

The importance of this principle can be seen most clearly in the second story. The first teacher mentioned by the student was not seen as having a positive attitude toward the student's learning challenges, and the teacher's silence when returning tests is a salient memory for the student decades later. However, the second teacher, who sat and worked with the student, had a positive effect on how the student thought and felt about him- or herself, transforming the student's frustration and feelings of inferiority into a sense of possibility.

Teachers can make all the difference through their own attitudes and enacted forms of those attitudes. Positivity and possibility are contagious (Frenzel et al 2018). Teachers set the tone through their modelled and enacted attitudes and behaviours.

The value of modelling learning is also discussed in the third story. The lack of enthusiasm and energy in the math classroom, which arose from a sense of generalized miscommunication, inhibited the student's active participation in classroom activities. Conversely, exciting and creatively engaging activities in art class fostered the student's enthusiasm for math as a means to achieve personal goals and ambitions, which in turn improved the student's math skills markedly.

Encouraging Relatedness

The fifth motivation design principle is relatedness, which means that students feel supported and have a sense of belonging among other students and their teachers (Linnenbrink-Garcia, Patall and Pekrun 2016). This can include developing genuine and caring relationships and a sense of community, which is critical to facilitating an environment where diversity is celebrated.

This sense of community was markedly absent in the third story. The student was uncomfortable with rephrasing a question that the teacher had misunderstood the first time, and the student felt a sense of impatience and pressure to remain quiet from peers. Developing a sense of relatedness helps students feel comfortable with asking questions and making mistakes.

On the other hand, a sense of relatedness was present in the second story, in the form of the teacher's enacted empathy and genuine investment. The student was able to articulate and identify this relatedness based on the behaviour of the teacher. Teachers who take the time to show genuine investment and caring are able and willing to develop positive relationships with students (Noddings 1991).

Starting a New Adventure

The math class misadventures experienced by the three students with dyslexia have allowed us to explore UDL guidelines and motivation design principles for students with diverse learning needs in the classroom. We hope that the students' accounts generate reflection and meaningful discourse about current pedagogy and teacher practice. If nothing else, these stories highlight that dyslexia is not a concern only in English language arts class. It also affects students' ability to access math concepts, as well as their confidence in expressing themselves more generally.

Notes

1. See www.alberta.ca/curriculum-development.aspx (accessed August 8, 2019).

2. See www.alberta.ca/inclusive-education.aspx (accessed August 8, 2019).

3. See http://udlguidelines.cast.org (accessed August 8, 2019).

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Inscribing Squares in Triangles

Timothy Sibbald

Through considering the problem of finding four points on the sides of a triangle that form a square, teachers can explore a variety of teaching possibilities. This article begins with an approach for intermediate math students and develops further methods suited to senior math students.

I conjecture that every triangle has four points on its sides that form a square. Whether this conjecture is true or not, the instructional value lies in having students determine where the square is for various triangles and whether the conjecture is true. The approach taken here is tiered in order to allow teachers to assess the suitability of the task for different grade levels.

Teachers can make this question accessible to intermediate students by looking at special triangles. To begin, consider an equilateral triangle. The geometry is shown in Figure 1 and has been augmented with various properties explained below. For students, constructing diagrams using dynamic geometry software provides a starting point for thinking about the location of the square.

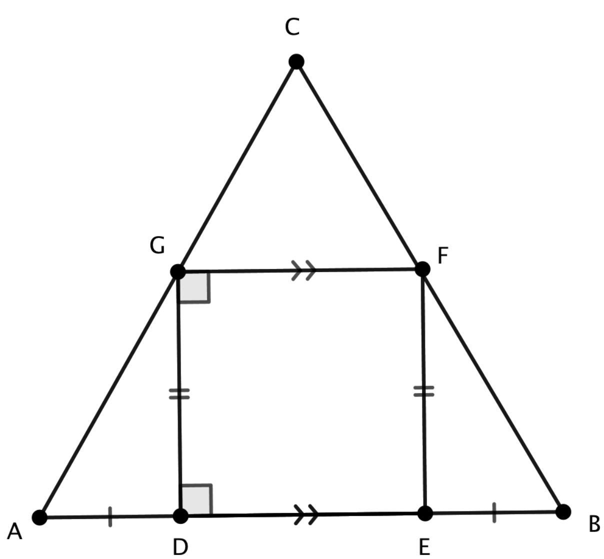


FIGURE 1. *A square with vertices on an equilateral triangle.*

For this diagram, the equilateral triangle, △ABC, was constructed in GeoGebra as a regular polygon, and then two points, D and E, were placed in arbitrary positions on side AB. Line segments perpendicular to AB were extended up from D and E to G and F, respectively, and a line segment, GF, was added to make the quadrilateral DEFG. The points D and E were moved along AB until the quadrilateral DEFG appeared, by visual inspection, to be a square. This process is not exact, but it is sufficient to provide a circumstance suitable for talking about properties in intermediate grades. The conjecture that an equilateral triangle can have an inscribed square now seems quite plausible, albeit unproven.

Suppose DEFG is a square in the equilateral triangle △ABC and that they share an axis of symmetry. This means that AD has an equal length to BE. Since DEFG is a square, DE = GF = DG = EF. Since DEFG is a square, DE is parallel to GF. The geometry is quite good for students to prepare a formal explanation that △ADG is congruent to △BEF. In addition, although it is more challenging, students can determine that △CFG is an equilateral triangle (hint: add the line of symmetry to make two congruent triangles).

Last, notice that △ABC has rotational symmetry, and any of the three sides could be selected as the side with two vertices. This implies that there are three possible squares, and adding the other two squares generates Figure 2. This facilitates a variety of classroom questions: Can you find a rhombus? How many kite shapes can you find? What is the relationship between the areas of a hexagon, three kites and an equilateral triangle? And, of course, there is the classic question of how many triangles appear in Figure 2.

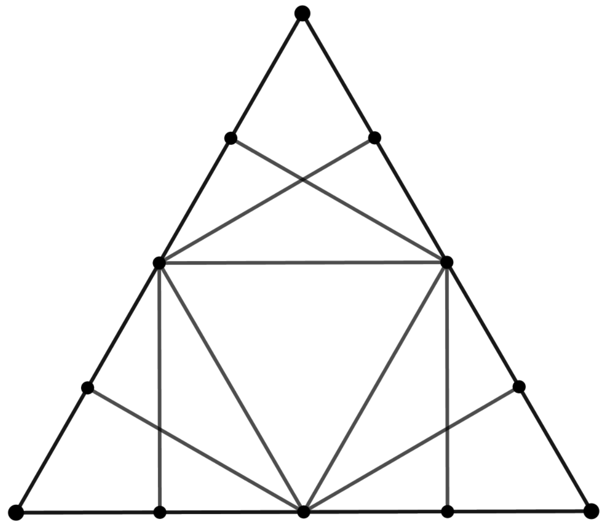


FIGURE 2. *The three squares in an equilateral triangle.*

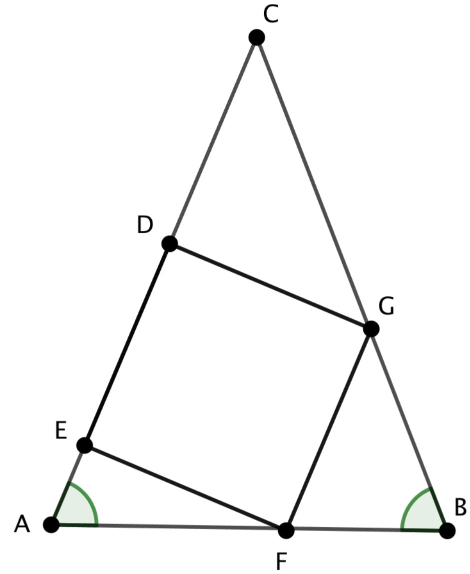


FIGURE 4. *Isosceles triangle with inscribed square with only one vertex on the side of unique length.*

An isosceles triangle has an inscribed square when there are two vertices on the side with the unique length (Figure 3). This case is similar to the equilateral case because the square shares its axes of symmetry with the triangle. The isosceles case allows for differentiation of instruction where students can explore whether there are any squares with only one vertex on the side of unique length. This scenario is shown in Figure 4.

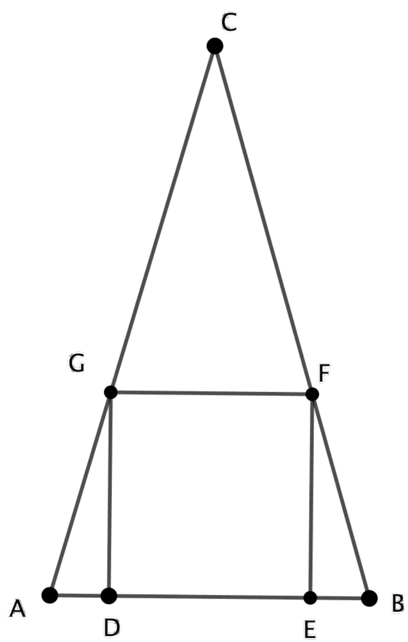


FIGURE 3. *Inscribed square in an isosceles triangle.*

In Figure 4, ∠A and ∠B are equal, and ∠AEF is a right angle, so ∠AFE is the complement of ∠A. However, ∠AFE is also the complement of ∠BFG (since ∠EFG is a right angle). That means that ∠BFG is equal to ∠B, and therefore △FBG is an isosceles triangle. This can be a teachable moment in which similar triangles are introduced.

The discovery that △FBG is isosceles leads to looking for other characteristics and making conjectures. Students may suggest that △CDG is isosceles, but it isn't generally—only when ∠C is 45°. Exploring such ideas is an important activity in the development of geometrical thinking in the intermediate grades.

High school, particularly analytic geometry, is well suited to addressing this problem more generally. Suppose two vertices of the triangle are A(0, 0) and B(1, 0). Let the third vertex be C(p, q). There is a need to discuss the domain for this point. All triangles can be formed with $0 \leq p < 1$. In particular, when q is small, ∠C will be obtuse.

This set-up also permits one to choose side AB as the side with two vertices of the square—without loss of generality. Considering these choices as an approach to simplify the interpretation is an important instructional moment in analytic geometry teaching. The set-up is shown in Figure 5, with a rectangle that is explained below.

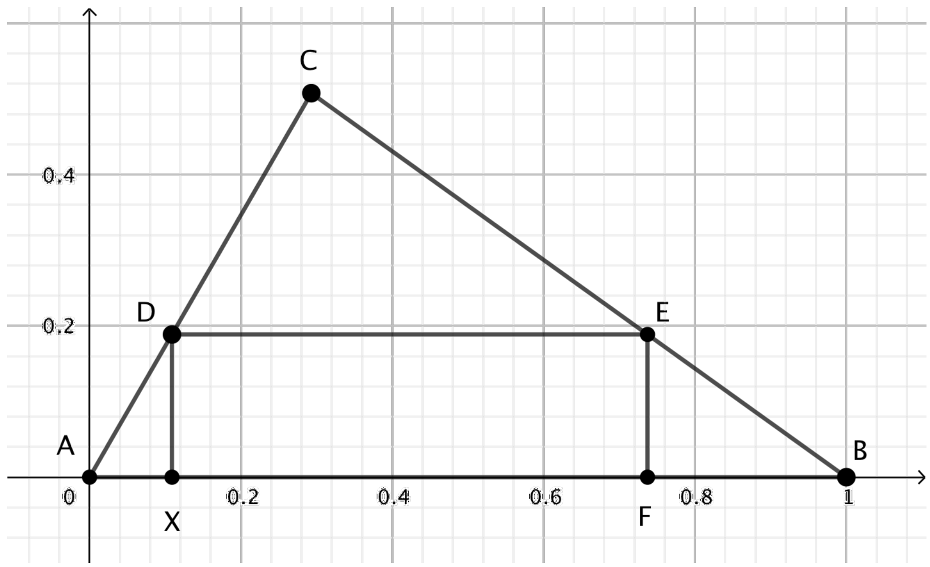


FIGURE 5. *Analytic geometry set-up.*

In Figure 5, the approach to finding the square is to add a point X at (x, 0) that generates a rectangle. First, XD is perpendicular to the x-axis; then, DE is perpendicular to XD; and, finally, EF is perpendicular to DE. The intersection points can be found, and a rectangle must arise. This explanation is well suited to instruction because it simply says that a rectangle can be made without providing the instructional details. Having students write down a recipe of calculation steps, with no actual calculations, helps organize their thinking and serves to put the organizational steps into place before focusing on the details. For example,

> A method for finding the square is to determine the equation of line AC so that both components of coordinate D can be written algebraically in terms of x. Then, by determining the equation of line BC, we can write the components of point E algebraically in terms of x. The distance XF can then be determined algebraically, as well as the distance XD. Setting these two distances equal to each other will provide an equation that is solved for x. That value of x will define the square inside the triangle (for the general values of p and q).

The specific steps are as follows:

Step 1. Find the equation of line AC.

This is direct variation with a y-intercept of 0. The equation is $y = (q/p)x$.

Step 2. Write out both coordinates of point D.

$$\mathrm{D}(x, y) = \left(x, \frac{qx}{p}\right)$$

Step 3. Find the equation of line BC.

This can be done in various ways. I will use slope-point form. Also, x is already in use, so I need to use a different letter. In the equation, I use t, which will ultimately depend on x:

$$y = \left(\frac{q}{p-1}\right)(t-1).$$

Step 4. Write out both coordinates of point E.

$$\mathrm{E}(t, y) = \left(t, \frac{q(t-1)}{p-1}\right)$$

Step 5. Determine distance XF.

Since x is the left x-coordinate of the rectangle and t is the right x-coordinate, the distance is $t - x$.

Step 6. Determine distance XD.

This is the y-coordinate of point D, which is qx/p.

Step 7. Set distance XF equal to XD, and solve for t (in terms of p and q).

$$t - x = \frac{qx}{p} \Rightarrow t = \left(\frac{p+q}{p}\right)x$$

This is major progress, but the use of two variables, *x* and *t*, lacks clarity. Hidden information is lurking, and students can be encouraged to problem solve the detail.

The resolution is that the *y*-coordinates of points D and E have to be equal. This fact was lost when *t* was introduced in order to avoid confusion over using *x* with two different meanings. Setting the *y*-coordinates of D and E equal and substituting for *t* using step 7 goes like this:

$$\frac{qx}{p} = \frac{q(t-1)}{p-1} \Rightarrow \frac{(p-1)x}{p} + 1 = t$$

$$= \frac{(p+q)x}{p} \Rightarrow (p-1)x + p = (p+q)x$$

$$\Rightarrow p = (p+q-p+1)x \Rightarrow x = \frac{p}{q+1}.$$

This can then be substituted into D in step 2:

$$\text{D} = \left(\frac{p}{q+1}, \frac{q}{q+1}\right).$$

This determines *t* in step 7:

$$t = \frac{p+q}{q+1}.$$

Then, it determines E in step 4:

$$t - 1 = \frac{p+q}{q+1} - 1 = \frac{p-1}{q+1} \Rightarrow E = \left(\frac{p+q}{q+1}, \frac{q}{q+1}\right).$$

Clearly, this will take far longer than one class. However, keep in mind that I have provided this solution in full generality. Students can choose point C and problem solve with specific values. I have provided the general form here to facilitate using a spreadsheet to verify student work. It can also be used for students to problem solve their own work. To do this, the teacher can provide the formulas, and students can substitute their own values of *p* and *q*.

An alternative approach, and the one I prefer, is to have students graph the points they determine so that they can verify graphically if their points make sense. This is shown in Figure 6 for the specific triangle shown in Figure 5.

Students who need enrichment can be asked to determine the area of the square. The simplest way to do so is to use the side length that is the *y*-coordinate of D and E. The area is

$$\left(\frac{q}{q+1}\right)^2.$$

Students can be challenged to explain why this doesn't depend on *p*. Given that analytic geometry is relatively new to students, this will help them place a familiar result in a new context. Hints can be given, such as determining whether the base or the height of the square depends on *p*.

Over the years, I have found that activities like this are beneficial for students. The activity is tiered and allows for different entry points for different students. At the same time, students find that the strategy of mapping out strategic steps separately from completing the calculations in each step is quite effective and

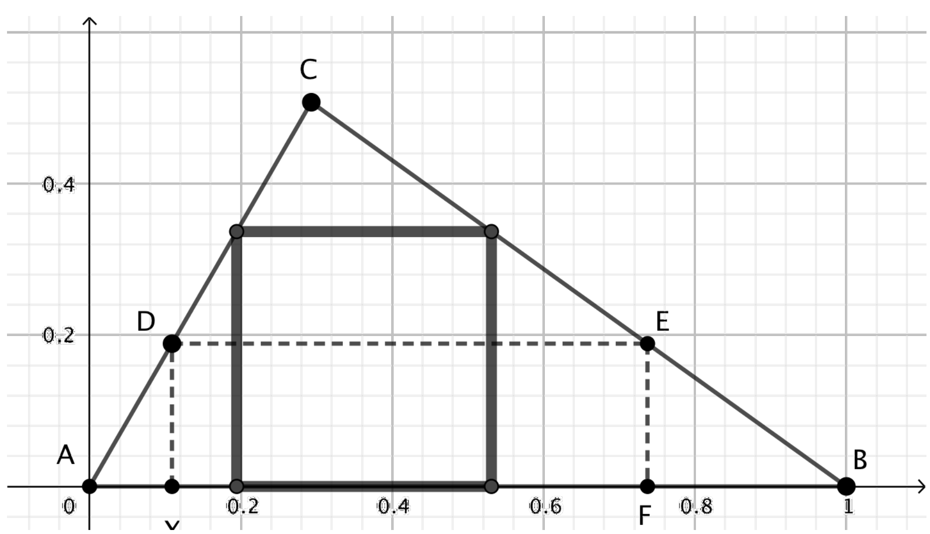


FIGURE 6. *Validating the formulas by plotting the algebraic versions of the points.*

useful. When the process of calculating is combined with the use of dynamic software for checking, students are less apprehensive about the number of steps. As they progress and see themselves achieving agreement between the algebra and the dynamic software, they become increasingly invested in achieving all the steps.

It may interest readers that the inspiration for this particular activity was an unsolved problem in geometry. The inscribed square problem asks whether every closed loop that does not intersect itself has four points that are the vertices of a square. Clearly, this is true for triangles and, in fact, true for many curves, but it has not been proven for allowable curves.

Timothy Sibbald is an Ontario certified teacher (OCT) and an associate professor in the Schulich School of Education at Nipissing University, in North Bay, Ontario. He teaches in the preservice program and the graduate programs, with a focus on mathematics instruction. He is also the editor of the Ontario Mathematics Gazette*, a publication of the Ontario Association for Mathematics Education.*

Wacky Quadrilaterals

David Martin

What happens when you draw a quadrilateral (a four-sided figure) and connect the midpoints of each side? What if you did this over and over again?

Here is an activity that investigates that!

Activity

Have students draw any quadrilateral that takes up most of the page. They will then measure the interior angles, add them all together and record the measurement. Next, they will measure and record the perimeter. (A recording sheet has been included on pages 43–44.)

Then, have students measure each side and determine the midpoint. They will then connect the midpoints and make a new quadrilateral (iteration 1). They will then measure and record the interior angles and the perimeter of this new quadrilateral. Have them repeat this step for two more iterations.

Ask students what they notice or wonder about. What would be the sum of the interior angles of iteration 10? What about the perimeter? Have students estimate the sum of the interior angles and the perimeter for the 10th iteration. (Do not have them actually create this 10th iteration. Simply have them estimate the measurements based on the pattern.)

Students then colour in the shape to make a creative design.

Figure 1 shows what the final drawing might look like.

Extensions

- Show students the animation on exterior angles of a polygon at www.mathwarehouse.com/animated-gifs/#exterior-angle-polygon. Ask, "What is this visually proving?"
- Show students Paul Lockhart's video "The World of Mathematical Reality" (https://youtu.be/V1gT2f3Fe44).

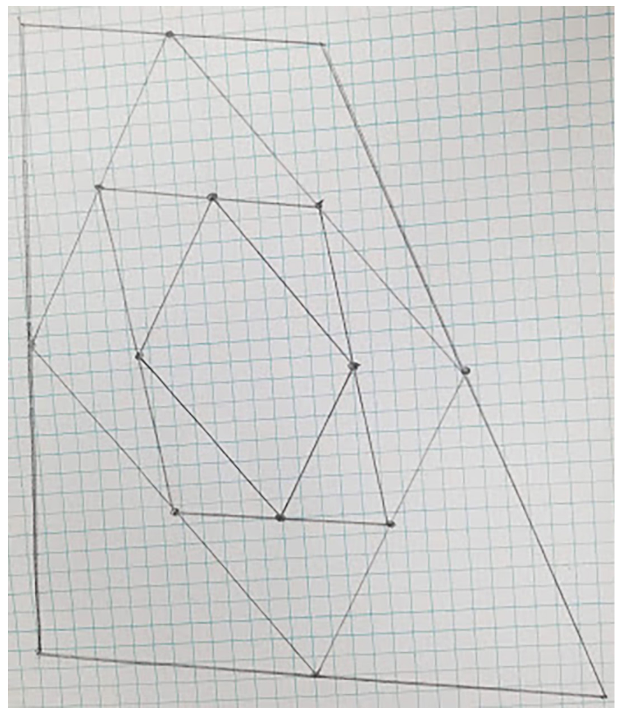


FIGURE 1. *Wacky quadrilaterals drawing.*

David Martin has a master's degree in mathematics, a bachelor's degree in education and, most important, a love of learning. Throughout his career, he has challenged many traditional educational practices, such as homework, tests and even grading. As a division math/science lead teacher, he has the opportunity to learn with teachers and students from pre-K to Grade 12. He is also president of MCATA. You will often find him tinkering with code, playing with mathematics or counting by prime numbers.

Wacky Quadrilaterals Recording Sheet

1. Draw any quadrilateral (a four-sided figure) that takes up most of the page. Measure the interior angles and add them all together and record this sum. Measure and record the perimeter.

2. Measure each side and determine the midpoint. Connect the midpoints and make a new quadrilateral (iteration 1). Measure the interior angles and the perimeter of this new quadrilateral.

3. Repeat step 2 for two more iterations. Record the sum of the interior angles and the perimeter.

4. What do you notice? Wonder about? Estimate what the sum of the angles and the perimeter will be for the 10th iteration. (Do not draw this 10th iteration. Simply estimate based on the pattern.)

5. What do you notice about the sum of the interior angles? Did you notice anything else about the angles?

6. What do you notice about the perimeter of each iteration? Did you notice anything else about the perimeter?

7. Colour in your shape to make a creative design.

Iteration	**Sum of interior angles**	**Perimeter**
0		
Determine the midpoint of each side and connect all four midpoints to make another quadrilateral.		
1		

Iteration	**Sum of interior angles**	**Perimeter**
Determine the midpoint of each side and connect all four midpoints to make another quadrilateral.		
2		
Determine the midpoint of each side and connect all four midpoints to make another quadrilateral.		
3		
What if you did this seven more times? Can you guess what the measurements might be?		
10 (estimate the measurements—do not create the quadrilateral)		

Do We Have Time for This? Analog Clocks

Sandi Berg

You may have seen articles claiming that schools are taking analog clocks off the walls because students can't read them. This has caused some controversy.

Let's look at some of the arguments from both sides:

- *Students always have their phones with them, and the phones display digital clocks.* This is often true—but not always. Some students (and adults) don't have cellphones. Also, though most phones display a digital clock, you can set up your phone to display an analog clock. Therefore, we should say that many (not all) students have a phone that displays a digital clock.
- *Learning how to read an analog clock teaches students how to skip count.* Yes, reading an analog clock certainly helps build the skill of skip counting. Even better would be if teachers focused on multiplication rather than adding—one five, two fives, three fives—as that aligns better with the numerals on the clock.
- *Learning how to read an analog clock teaches students about base 60.* True—if a teacher has talked about that concept. It also teaches students about base 13—think about a 24-hour clock.
- *If students can read an analog clock, they will be able to read airline tickets.* Yes, students should know how to read a 24-hour clock and be able to convert between the two systems. However, neither a digital clock nor an analog clock necessarily teaches that skill (unless the clock displays 13:00, 14:00 and so on).

These are just a few of the arguments for and against teaching students how to read an analog clock. I don't know how much of a role they play in analog clocks being removed. Many of the clocks have been replaced simply because of old age. However, it is possible that some have been taken down because students can't read them. I haven't had an opportunity to delve deeper into the stories, and they don't influence me at all. My goal here is not to convince you either way.

The bottom line is that I will keep teaching students how to read both digital clocks and analog clocks. Why? Because it's in the curriculum. In Alberta, reading clocks is addressed specifically in Grade 4. I am required to teach the topic. What type of clock students choose to display on their phone or to wear on their wrist is up to them.

So how do I teach students how to read clocks? The same way I teach everything—by starting slowly and using visuals. This is not a separate unit that you dedicate a week to in January. Start right away, in September, but go slow—very slow.

Activity 1

You can start this activity with students at the beginning of the school year.

First, find a clock from which you can remove the inside circle (the part the numbers are written on) (Figure 1).

FIGURE 1

Remove the inside circle. You can either do the next step on that same circle or trace the circle to create a new circle and write only the numbers 1–12 on it. I prefer to use a fresh circle so that the other numbers, which are not needed yet, are not displayed.

Draw a line from the centre of the circle to each of the numbers (1–12). This will create pie shapes. Shade the sections in three different colours. Write *-ish* next to each number (Figure 2).

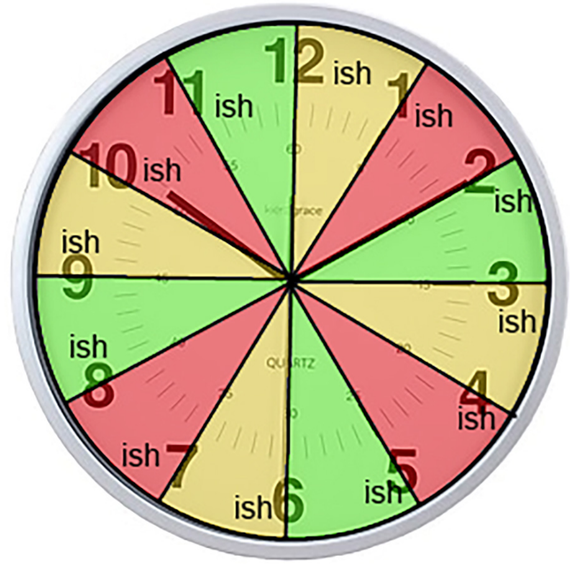


FIGURE 2

Now, if possible, remove the minute hand and the second hand from the clock. If you can't do that or don't want to, colour the hour hand so that it stands out (Figure 3).

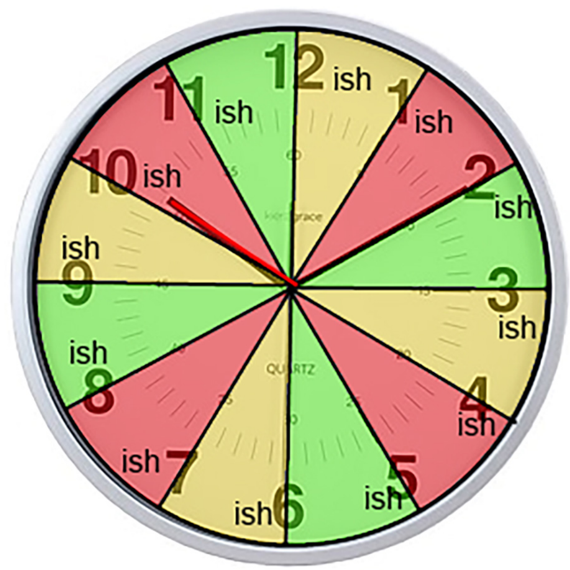


FIGURE 3

You may even want to draw arrows between the numbers (Figure 4).

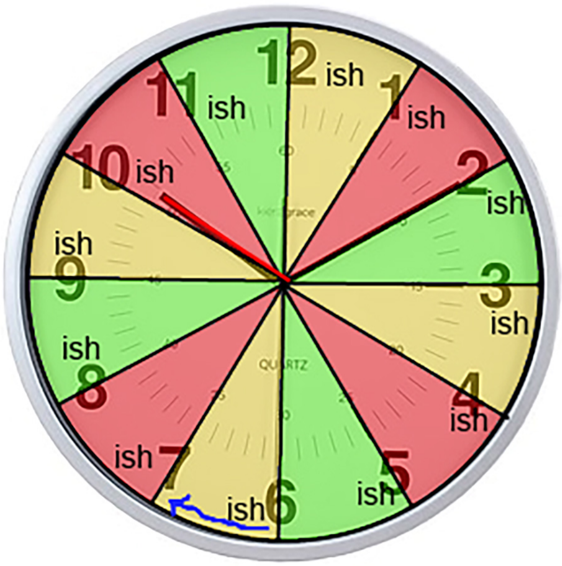


FIGURE 4

Give students a mini-lesson on how to read this clock. The clock shows that it's 10ish. Go through a few different times. What would 4ish look like?

Yes, it's not very specific. After a while, you can say something like, "It's 2ish—almost 3."

Do this for two weeks, three weeks, four weeks—however long it takes. Do not rush through. You have all year to teach students how to read a clock. Once they are comfortable, you can move on to the next activity.

Activity 2

Did you know that you can tell what time it is within about five minutes even without the minute hand? Want to try it out? Go ahead and order the clocks in the image below (Figure 5).

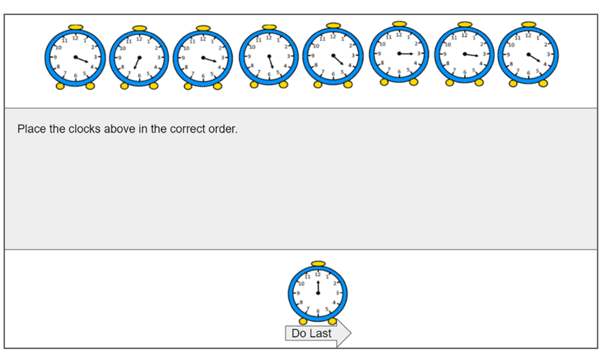


FIGURE 5

You'll notice that one clock is at the bottom, by itself. During the activity with students, this clock is sorted with the others. While you are doing the activity, feel free to consider this clock either while sorting the rest of the clocks or afterward. Later, I'll talk about how I address the clock with students.

Now, match the following times with the clocks:

A little after 5	4 o'clock	A little before 7
Noon/midnight	3 o'clock	About 3:30
About 4:30	A little after 3	A little before 4

Now, I'll explain the process I use with students. The materials for this activity are in Appendix A.

I hand out pictures of nine clocks on which only the hour hand and hour numbers (1–12) are displayed (see page 50). These are the same clocks as in Figure 5. They are separated so that students can move them around.

In pairs, students sort the clocks into the correct order by time.

Once they are done, the pairs meet with other pairs and compare their answers, share their reasoning and move the clocks around some more (if they choose).

I then open up my Google Slides version and display slide 2 (not in presentation mode).[1] This slide shows eight clocks at the top and one at the bottom (as in Figure 5). I have one student come up and move a clock into the grey box. Then, another student comes up and places a clock either before or after the first clock. The students then explain how they know where each clock goes.

We repeat this process for all the clocks until we finally get to the clock at the bottom. Where does this clock go? I ask a student to come up and place it. I then ask, "Who else placed that clock here?" and "Who placed it in a different spot?" If a student chose a different spot, I have him or her come up and show us where. Then I say, "Interesting. We have two different answers. Talk to your partner. Which do you agree with and why?" The discussion then focuses on the question of where the clock displaying 12 o'clock goes. Both spots!

Now, I distribute the time statements (see page 51). Students then match these with the clocks. They compare their answers with those of a different group, and then we discuss the answers as a class.

The handout on page 52 is for an activity that students will complete individually. Use this activity as a formative assessment to inform your teaching practices.

When you are talking about time with students, use the type of phrasing that is used in the time statements provided here. Again, don't rush this. Don't worry about using precise times yet (for example, you'll notice that none of the examples refer to times like 3:15).

Activity 3

Moving on to minutes? Don't refer to minutes at first. Instead, give students a picture of a clock with the hour numbers displayed and tick marks for minutes. Have them count the tick marks around the clock. What patterns do they notice?

Do they notice that the tick marks are in groups of five? Have students re-count around the circle, and talk them through the process: "One, two, three, four, five. That's one group of five. Notice that we're now at the 1 on the clock. One, two, three, four, five. That's two groups of five. Notice that we're at the 2. How many minutes is that? Ten minutes. Interesting. Does that always work? Keep checking. One, two, three, four, five. That's three fives. Fifteen minutes." Students then figure out the rest. Now you're making a connection to multiplication.

Spend time working with students on naming regular times (for example, *3:15*, not *quarter after*).

Activity 4

Have students draw a circle and break it into quarters. What do they notice about how this relates to the clock? If they imagine the lines going from 12 to 6 and from 9 to 3, they can make the connection to *quarter after*.

Make sure to talk about the concept of 15 minutes out of 60 minutes (15/60) and so on.

Activity 5

Ready to add seconds? Have students figure out how high they can count in one minute. Time it according to just the minute hand. (If possible, use a clock that doesn't have a second hand and that has a really obvious shift from minute to minute.) Students count in their head and then share their results. Some will count really fast, and others, really slow. That's OK. Talk about how we have decided to set a standard counting speed for time to help us be consistent. (It was humans who decided that there are 60 seconds in a minute.) Show students a clock with a second hand and tick marks indicating minutes. Then have them count again. As they count, have students focus on the second hand as it passes each tick mark. When they share their results, they will probably be more consistent now.

Now that the students have done this activity, give them their own clock with tick marks to look at. What relationships can they find between the second hand, the minute hand and the hour hand? They might say that the hour hand is shorter and the second hand is longer. The longer the hand, the faster it moves—60 seconds in a minute and 60 minutes in an hour.

Give students a puzzle: "If I count to 60, that's one minute. How would you figure out how many minutes it would take me to count to 180?" Some students may argue that it will take longer than three minutes. Why? Because it takes longer to say *one hundred seventy-nine* than it takes to say *one*. Don't discount those arguments—they are absolutely correct!

I cannot stress this enough—when teaching students how to read a clock, go slow. Build understanding. Don't move on until students understand what's going on. You have all year.

Question for Self-Reflection

How do you help students understand the concept of time? That is, how do you help them with not just reading a clock but with actually understanding what time is?

Appendix A: Reading Clocks Activity

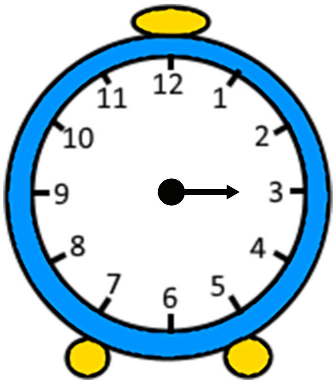
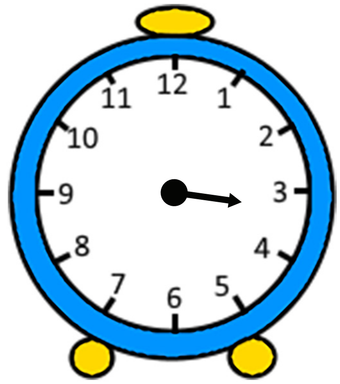
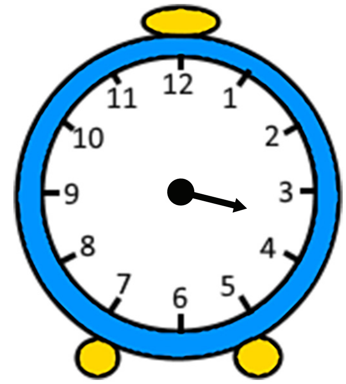
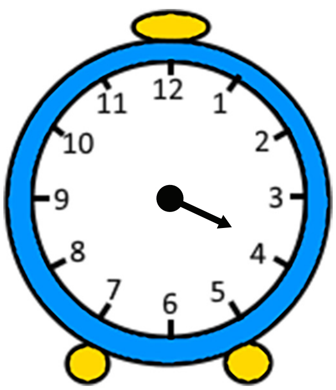
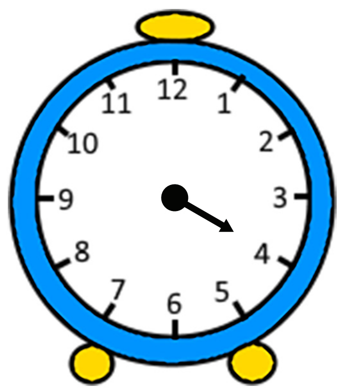
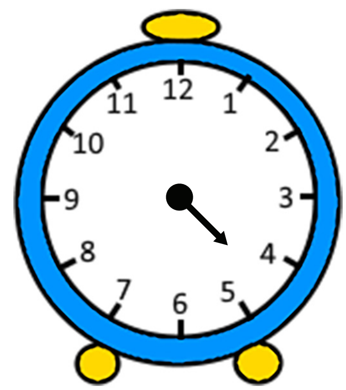
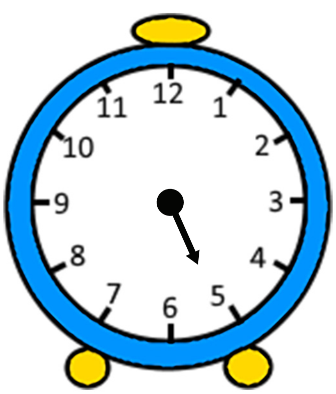
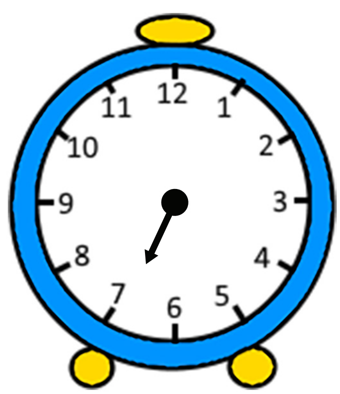
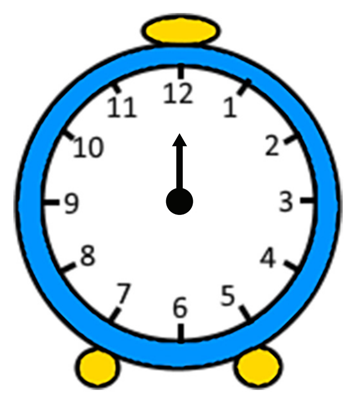
Materials

- Clock printouts (one page per group of two students)
- Time statements (one set per group of two students—each page has three sets)
- Blank clocks worksheet (one worksheet per student—each page has two worksheets)
- Smart Notebook file (available at https://educationaljourneyofatechnobabe.blogspot.com/2019/03/do-we-have-time-for-this-analog-clocks.html)

Instructions

- Print out the page of clocks and cut out the clocks so that each group of two students has nine individual clocks.

- Print out the time statements and cut the statements apart so that each group of two students has nine individual time statements. (The page contains three sets of time statements. Each group needs only one set.)
- Print off the worksheet and cut it so that each student gets half a page.
- Hand out the clock cut-outs.
- Ask students to order the clocks based on the times shown. Give them time to work. This is an opportunity for you to see what they understand about time. Do they understand that the placement of the hour hand can help them make a decent guess about the time? Where did they put the clock displaying 12 o'clock—at the start or at the end? Do not guide students. Watch them work, but do not interfere.
- Ask students to share how they decided to order the clocks. What was their process? Students can use page 1 of the Smart Notebook file to help.
- After discussion, hand out the time statements. Have students match the statements with the clocks.
- Ask students to share how they did their sorting. They can use the Smart Notebook file to help them explain.
- After discussion, hand out the worksheets. Ask students to choose four activities they do at different times throughout the day. They must use each of the phrases *a little before*, *a little after*, *about* and *at __ o'clock* exactly once. Have them draw the time on the clock, using only the hour hand.

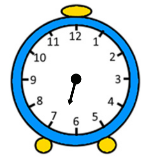










3 o'clock	A little after 3	About 3:30
A little before 4	4 o'clock	About 4:30
A little after 5	A little before 7	Noon/midnight

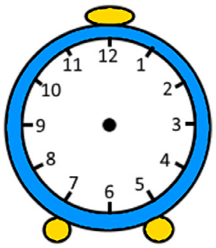
3 o'clock	A little after 3	About 3:30
A little before 4	4 o'clock	About 4:30
A little after 5	A little before 7	Noon/midnight

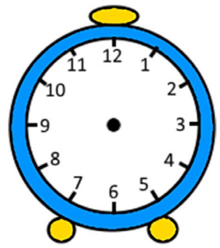
3 o'clock	A little after 3	About 3:30
A little before 4	4 o'clock	About 4:30
A little after 5	A little before 7	Noon/midnight

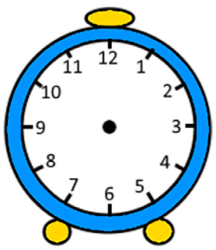
Choose four activities you do at different times throughout the day. Use each of the phrases *a little before*, *a little after*, *about* and *at __ o'clock* exactly once. Draw the time on the clock using only the hour hand. Describe the activity using the time.

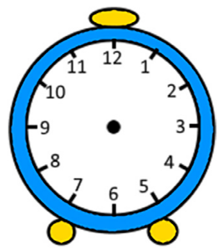


I wake up at about 6:30.

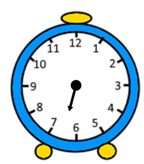




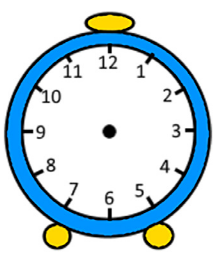


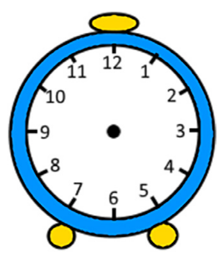


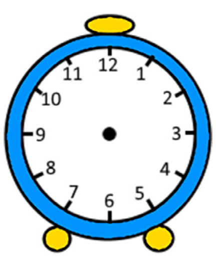
Choose four activities you do at different times throughout the day. Use each of the phrases *a little before*, *a little after*, *about* and *at __ o'clock* exactly once. Draw the time on the clock using only the hour hand. Describe the activity using the time.

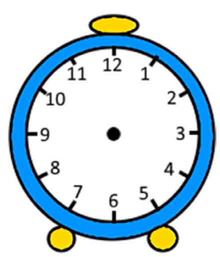


I wake up at about 6:30.









Note

1. The Google Slides presentation (as well as instructions, handouts and a Smart Notebook file) is available at https://educationaljourneyofatechnobabe.blogspot.com/2019/03/do-we-have-time-for-this-analog-clocks.html (accessed August 13, 2019).

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Alberta High School Mathematics Competition 2017/18

The Alberta High School Mathematics Competition is a two-part competition that takes place in November and February of each school year. Book prizes are awarded for Part I, and cash prizes and scholarships for Part II. Presented here are the problems and solutions from the 2017/18 competition.

Part I

November 21, 2017

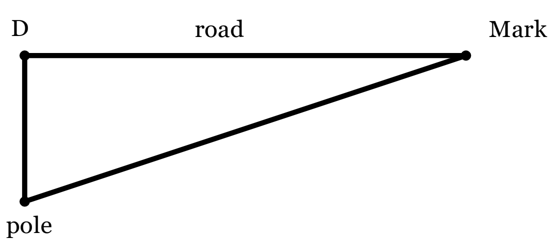
1. Peppa has twice as many apples as pears. After she eats 50 pears, she now has four times as many apples as pears. How many apples does Peppa have?

   (a) 0 (b) 100 (c) 200 (d) 400
   (e) none of these

2. A math teacher is given a big bag of 300 candies by a grateful parent. The teacher distributes as many of the candies as possible to her students so that each student gets the same number of candies. There are 14 candies left over. How many students are there in her class, if each class in the school has at most 30 students?

   (a) 20 (b) 22 (c) 24 (d) 26
   (e) not uniquely determined

3. A vertical pole is 12 m from a straight road. Mark is currently on the road, 37 m from the pole.



   How many metres does Mark have to walk on the road before he is 20 m from the pole?

   (a) 17 (b) 19 (c) 20 (d) 21 (e) 22

4. Some $n \geq 3$ different positive real numbers are arranged on a circle such that each number is equal to the product of its two neighbours. The value of n is

   (a) 4 (b) 5 (c) 6 (d) 8 (e) none of these

5. How many pairs (n, p) are such that n is a positive integer, p is a prime number and $n + p/n$ is a square of a positive integer?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) none of these

6. Ben and Cleo can paint a fence in four days. Anna and Ben can do it in two days, and Anna and Cleo can do it in three days. How many days, as a fraction, does it take all of them working if Cleo gets injured at the end of the first day and can't come back to work?

   (a) 11/6 (b) 45/24 (c) 23/12 (d) 47/24
   (e) none of these

7. The number of ways to walk from (0, 0) to (20, 2) by using only up and right unit steps and such that the walk never visits the lines $y = x$ and $y = x - 18$, except at the beginning and end, is

   (a) 151 (b) 153 (c) 189 (d) 190 (e) 195

8. Lan likes red or blue chopsticks, while Dan likes orange or green chopsticks. A drawer in a dark room contains n chopsticks of each colour, where $n > 1$ is an integer. Lan goes into the room to get chopsticks for both her and Dan. What is the smallest number of these $4n$ chopsticks that Lan must pick in order to be sure that both people

will get a matched (that is, the same colour) pair that they like?

(a) $2n + 1$ (b) $2n + 3$ (c) $3n - 3$
(d) $3n - 1$ (e) none of these

9. Let x, y and z be any real numbers such that $3x + y + 2z \geq 3$ and $2y - x + 4z \geq 5$. The minimum possible value of $7x + 5y + 10z$ is

(a) 96/7 (b) 97/7 (c) 14 (d) 99/7 (e) 100/7

10. In the trapezoid ABCD, with AD parallel to BC, the diagonals intersect at O. If the area of △AOD is 9/16 of the area of the trapezoid, then the ratio of the area of △AOD to the area of △BOC is equal to

(a) 4 (b) 6 (c) 9 (d) 12 (e) none of these

11. The area of the quadrilateral bounded by the graphs of the functions $y = |x - a|$, with $0 < a < 4$ and $y = 2 - |x - 2|$, is 15/8. The smallest value of a is

(a) 2/5 (b) 2/3 (c) 1 (d) 3/2 (e) 5/2

12. Let a and b be positive integers. The number of quadratic equations $x^2 - ax - b = 0$ having the positive root less than 10 is

(a) 391 (b) 392 (c) 441 (d) 450
(e) none of these

13. Let S be a subset of non-negative integers that contains 0, and such that for any number x in S, $3x$ and $3x + 1$ are also in S. The least possible number of elements of S less than 2,017 is

(a) 64 (b) 128 (c) 256 (d) 300
(e) none of these

14. Let a be a real number so that the equation $x^4 - 2ax^2 - x - a + a^2 = 0$ has four different real solutions. Then which of the following must be true?

(a) $a < 1/4$ (b) $1/4 < a < 3/4$ (c) $3/4 < a < 1$
(d) $1 < a$ (e) none of these

15. A positive integer is called a palindrome if it remains unchanged when written backward. Find the number of five-digit palindromes that are divisible by 55.

(a) 5 (b) 8 (c) 10 (d) 12 (e) none of these

16. How many positive integers n can be found such that the product of all divisors of n, including n, is 24^{240}?

(a) 0 (b) 1 (c) 2 (d) 3 (e) none of these

Solutions

1. Peppa initially has x apples and $x/2$ pears. Then $4(x/2 - 50) = x$ and, thus, $x = 200$. The answer is **(c)**.

2. If x represents the number of students in the class and c the number of candies received by each student, then $14 < x \leq 30$ and $x \cdot c = 300 - 14 = 286 = 2 \cdot 11 \cdot 13$. There are two possible solutions, $x = 22$ or $x = 26$. The answer is **(e)**.

3. Let D be the point on the road closest to the pole. Then the distance between the pole and D is 12 m. At first, Mark is

$$\sqrt{37^2 - 12^2} = \sqrt{1{,}369 - 144} = \sqrt{1{,}225} = 35 \text{ m}$$

from D. When he is 20 m from the pole, Mark is

$$\sqrt{20^2 - 12^2} = \sqrt{400 - 144} = \sqrt{256} = 16 \text{ m}$$

from D. Hence, Mark walked 35 – 16 = 19 m. The answer is **(b)**. (Notice that Mark can pass point D when he is 20 m from the pole. In this case, Mark has to walk 35 + 16 = 51 m.)

4. If a and b are the first two numbers in the sequence, then the third number in the sequence must be b/a (because $a \times b/a = b$). Similarly, we can find that the sequence of numbers around the circle must be a, b, b/a, $1/a$, $1/b$, a/b, a, b, ... Thus, the sequence repeats after six terms. If there were a shorter sequence of at least three different terms starting with a, b, its length must divide into 6, so it must have length 3. This would mean that the fourth term, $1/a$, would equal the first term, a, which means that $a = 1$, but then the second and the third terms (b and b/a) of the sequence are equal —a contradiction. So the only possible length is 6. The answer is **(c)**.

5. Since p/n is an integer, $n \mid p$. Hence, $n = 1$ or $n = p$. It follows that $p + 1$ is a perfect square. Thus, there is a positive integer m such that $p + 1 = m^2$ or $p = (m - 1)(m + 1)$. Hence, $1 = m - 1$ and $p = m + 1$ (that is, $m = 2$, $p = 3$). Therefore, $p = 3$ and $n = 1$ or $n = 3$. The answer is **(c)**.

6. If a is the portion of the work done per day by Anna, b by Ben and c by Cleo, then $b + c = 1/4$, $a + b = 1/2$, and $a + c = 1/3$. Hence, $a + b + c = 13/24$. Thus, 11/24 of the work has to be completed by Anna and Ben. They need for this

11/24 ÷ 1/2 = 11/12 days, for a total of 23/12 days. The answer is **(c)**.

7. The first two steps must be to the right from (0, 0) to (2, 0), and the last two steps must be to the right from (18, 2) to (20, 2). The number of ways of walking from (2, 0) to (18, 2) is

$$\binom{18}{2} = 153.$$

We have to subtract one way that contains (2, 2) and the one that contains (18, 0). Hence, the number of admissible ways is 151. The answer is **(a)**.

8. If Lan picks $2n + 2$ chopsticks consisting of n orange, n green, one red and one blue, she does not get a pair she likes. Hence, she needs to pick at least $2n + 3$ chopsticks to be sure that she gets a pair she likes. On the other hand, any choice of $2n + 3$ chopsticks will contain at least three red/blue chopsticks and at least three orange/green chopsticks, so both people will get a pair they like. The answer is **(b)**.

9. Let A and B be real positive numbers. Then

$$A(3x + y + 2z) + B(2y - x + 4z)$$
$$\geq 3A + 5B \Leftrightarrow (3A - B)x + (A + 2B)y + (2A + 4B)z$$
$$\geq 3A + 5B.$$

If we take $3A - B = 7$, $A + 2B = 5$, $2A + 4B = 10$, which is equivalent to $A = 19/7$, $B = 8/7$, the above inequality can be written as

$$7x + 5y + 10z \geq 97/7.$$

Hence, the minimum value of $7x + 5y + 10z$ is 97/7. We may take $x = 1/7$, $y = 18/7$ and $z = 0$ to justify that the minimum value is attainable. The answer is **(b)**.

10. Let area(AOD) = a, area(BOC) = b and area(AOB) = area(COD) = c. If $a/b = x$, then a/c = AO/OC = c/b, so $a/c = c/b = \sqrt{x}$. Hence,

$$\frac{\text{area(AOD)}}{\text{area(ABCD)}} = \frac{a}{a+b+2c} = \frac{\frac{a}{b}}{2\frac{c}{b}+\frac{a}{b}+1} = \frac{x}{(\sqrt{x}+1)^2} = \frac{9}{16}.$$

Solving for x, one obtains $x = 9$. The answer is **(c)**.

11. Graphing the functions, we conclude that two of the vertices of the quadrilateral are A = (2, 2) and C = (a, 0). The other two vertices,

$$\text{B} = \left(\frac{a}{2} \cdot \frac{a}{2}\right) \text{ and } \text{D} = \left(\frac{4+a}{2}, \frac{4-a}{2}\right),$$

are at the intersection of the lines $y = x$, $y = a - x$ and, respectively, $y = 4 - x$, $y = x - a$. The area of the quadrilateral ABCD is

$$4 - \frac{a^2}{4} - \frac{(4-a)^2}{4} = \frac{4a - a^2}{2}.$$

Solving the equation

$$\frac{4a - a^2}{2} = \frac{15}{8},$$

one obtains $a = 3/2$, $a = 5/2$. The answer is **(d)**.

12. The positive root of the equation is

$$\frac{a + \sqrt{a^2 + 4b}}{2}.$$

The condition

$$\frac{a + \sqrt{a^2 + 4b}}{2} < 10$$

is equivalent to $10a + b < 100$ and $a \leq 20$. There are 9 + 19 + 29 + 39 + 49 + 59 + 69 + 79 + 89 = 4 × 98 + 49 = 441 ordered pairs (a, b) such that $10a + b < 100$. The answer is **(c)**.

13. If the elements of S are written in base 3, then the conditions of the problem translate to if

$$\overline{a_1 a_2 \cdots a_n}_{(3)} \in S,$$

then

$$\overline{a_1 a_2 \cdots a_n 0}_{(3)} \in S$$

and

$$\overline{a_1 a_2 \cdots a_n 1}_{(3)} \in S.$$

Hence, S should contain all the numbers that can be written in base 3 only by using 0 or 1. Since $2 \cdot 3^6 < 2{,}017 < 3^7$, we conclude that the largest element in S that is less than 2,017 is $1111111_{(3)}$. There will be a total of $2^7 = 128$ numbers in S having the requested property. The answer is **(b)**.

14. The given equation can be written as

$$a^2 - (2x^2 + 1)a - x + x^4 = 0,$$

which has the solutions $a = x^2 - x$ and $a = x^2 + x + 1$. The solutions of the equation $x^2 - x - a = 0$ are different real numbers if $1 + 4a > 0$ or, equivalently, $a > -1/4$. The equation $x^2 + x + 1 - a = 0$ has different real roots if $1 - 4(1 - a) > 0$ or, equivalently, $a > 3/4$. The above two equations cannot have common roots. Indeed, if for $x_0 \in \mathbb{R}$,

$$x_0^2 - x_0 - a = 0$$

and

$$x_0^2 + x_0 + 1 - a = 0,$$

then $x_0 = -1/2$ and $a = 3/4$, which is not convenient. Therefore, the solutions of the given equation are all real and different if $a > 3/4$. The answer is **(e)**.

15. A five-digit palindrome has the form $abcba = 10{,}001a + 1{,}010b + 100c$, with $a \neq 0$. In order for the number to be divisible by 5, we must have $a = 5$. Therefore, the number is $50{,}005 + 1{,}010b + 100c$.

Since $50{,}005 + 1{,}010b + 100c = 11(4{,}546 + 92b + 9c) - 1 - 2b + c$, we then need to have $c - 2b - 1$ divisible by 11. Since $-19 \leq c - 2b - 1 \leq 8$, we must have $c - 2b - 1 = 0$ or $c - 2b - 1 = -11$.

If $c - 2b - 1 = 0$, then $c = 2b + 1$, which leads to five pairs $(b, c) \in \{(0, 1), (1, 3), (2, 5), (3, 7), (4, 9)\}$.

If $c - 2b - 1 = -11$, then $2b = c + 10$ and, hence, $b \geq 5$. Then we get $(b, c) \in \{(5,0), (6, 2), (7, 4), (8, 6), (9, 8)\}$.

Therefore, we have 10 possibilities. The answer is **(c)**.

16. We have $n \mid 24^{240}$. Then $n = 2^a 3^b$, with a and b integers. Thus, n has $(a + 1)(b + 1)$ divisors. Pairing them in pairs of the form $(d, n/d)$, we obtain that their product is

$$n^{\frac{(a+1)(b+1)}{2}}.$$

Therefore,

$$24^{240} = n^{\frac{(a+1)(b+1)}{2}}.$$

Hence,

$$2^{3\cdot 240}3^{240} = 2^{a\frac{(a+1)(b+1)}{2}}3^{b\frac{(a+1)(b+1)}{2}}.$$

From this equation, one obtains

$$240 = b\frac{(a+1)(b+1)}{2}$$

$$3 \cdot 240 = a\frac{(a+1)(b+1)}{2}$$

from which, taking the ratio, we get $a = 3b$ and, hence,

$$480 = b(b + 1)(3b + 1).$$

Since the right-hand side is increasing, this equation has at most one positive integer solution. Moreover, we have

$$480 = b(b + 1)(3b + 1) > 3b^3 \Rightarrow 160 > b^3$$

and, hence, $b \leq 5$. By testing, $b = 5$ gives

$$b(b + 1)(3b + 1) = 5 \cdot 6 \cdot 16 = 10 \cdot 6 \cdot 8 = 480.$$

Hence, $b = 5$, $a = 15$, and $n = 2^5 3^{15}$. The answer is **(b)**.

Part II

February 7, 2018

1. The difference between two positive integers is 18. When we divide the larger of the two positive integers by the smaller, the quotient and the remainder are equal. Find all the possible pairs of positive integers.

2. Let $\mathbb{Z}$ be the set of integers and $f : \mathbb{Z} \to \mathbb{Z}$ a function such that

$$f(f(x) + y) = x + f(y)$$

for any integers x and y.

Show that $f(x + y) = f(x) + f(y)$ for any integers x and y.

3. Prove that the numbers 26^n and $26^n + 2^n$ have the same number of digits, for any non-negative integer n.

4. A collection of items weighing 3, 4 or 5 kg has a total weight of 120 kg. Prove that there is a subcollection of the items weighing exactly 60 kg.

5. The △ABC has ∠BAC = 80° and ∠ACB = 40°. D is a point on the ray BC beyond C so that CD = AB + BC + CA. Find ∠ADB.

Solutions

1. Let x be the smaller of the two numbers. The larger number is $x + 18$. If q denotes the quotient of the division, then the remainder is also q. The long division of $x + 18$ by x gives

$$x + 18 = qx + q \Rightarrow qx + q - x - 18 = 0.$$

Then

$$qx + q - x - 1 = 17 \Rightarrow (q - 1)(x + 1) = 17.$$

Since 17 is prime and $q < x$ (as the remainder is smaller than the quotient), we have

$$q - 1 = 1, x + 1 = 17.$$

Thus, $x = 16$. Hence, there is only one pair of positive integers having the requested property, 16 and 34.

2. Setting $y = 0$ in the given relation, we get that $f(f(x)) = x + f(0)$ for any $x \in \mathbb{Z}$. Therefore, if $f(0) = a$, we have $f(f(x)) = x + a$. Then,

$$f(x + f(y)) = f(f(f(x) + y)) = f(x) + y + a.$$

Interchanging x, y in the given relation, we also have

$$f(x + f(y)) = f(x) + y.$$

The last two relations give $a = 0$, and therefore $f(f(x)) = x$. Next, replacing x by $f(x)$ in the given relation and using $f(f(x)) = x$, we get

$$f(x + y) = f(x) + f(y).$$

3. Assume by contradiction that 26^n and $26^n + 2^n$ do not have the same number of digits. If 26^n has m digits, then $26^n < 10^m \le 26^n + 2^n$, with $m > n$ and, hence, $13^n < 2^{m-n}5^m \le 13^n + 1$. Hence, $2^{m-n}5^m = 13^n + 1$, and since $13^n + 1 \equiv 2 \pmod 4$, one obtains $m - n = 1$. Therefore, n should be a solution of the equation $2 \cdot 5^{n+1} = 13^n + 1 \Leftrightarrow 10 = (1/5)^n + (13/5)^n$. It is clear that $n = 0, 1, 2$ are not solutions of this equation and also any $n \ge 3$ either, since $(13/5)^n \ge (13/5)^3 > 10$ for $n \ge 3$.

Notice: We may justify that the remainder obtained when $13^n + 1$ is divided by 4 is 2, without using congruences. Indeed, since 13, 13^2, 13^3 and in general 13^n are all of the form $4k + 1$, where k is a positive integer, $13^n + 1$ is of the form $4k + 2$.

4. We have non-negative integers a, b and c so that $3a + 4b + 5c = 120$. We wish to find a triple (a_1, b_1, c_1) of non-negative integers satisfying $a_1 \le a$, $b_1 \le b$, $c_1 \le c$ so that $3a_1 + 4b_1 + 5c_1 = 60$.

First note: If $a \ge 20$, then we can use $a_1 = 20$, $b_1 = c_1 = 0$. Thus, we can assume from now on that $a \le 19$. Similarly, we can assume that $b \le 14$ and $c \le 11$. Thus, we get

$$a = \frac{120 - 4b - 5c}{3} \ge \frac{120 - 56 - 55}{3} = 3,$$

$$b = \frac{120 - 3a - 5c}{4} \ge \frac{120 - 57 - 55}{4} = 2 \text{ and}$$

$$c = \frac{120 - 3a - 4b}{5} \ge \frac{120 - 57 - 56}{5} = \frac{7}{5},$$

and since c is an integer, we get $c \ge 2$.

From $3a + 4b + 5c = 120$, we know that a and c have the same parity. Thus, we have four cases.

Case 1. If a, b and c are all even, then we choose $a_1 = a/2$, $b_1 = b/2$, $c_1 = c/2$.

Case 2. If a and c are even while b is odd, then we claim that

$$(a_1, b_1, c_1) = \left(\frac{a+4}{2}, \frac{b-3}{2}, \frac{c}{2}\right)$$

works. These choices are all integers, and

$$3a_1 + 4b_1 + 5c_1 = \frac{3(a+4)}{2} + \frac{4(b-3)}{2} + \frac{5c}{2} = 60,$$

so we need only observe that (i) since a is even and $a \ge 3$, we know that $a \ge 4$, and thus $(a + 4)/2 \le a$; (ii) since b is odd and $b \ge 2$, we know that $b \ge 3$ and thus $(b - 3)/2 \ge 0$.

Case 3. If a and c are odd while b is even, we use

$$(a_1, b_1, c_1) = \left(\frac{a-1}{2}, \frac{b+2}{2}, \frac{c-1}{2}\right).$$

Again, these are non-negative integers, and $3a_1 + 4b_1 + 5c_1 = 60$, and $b \geq 2$ implies that $b_1 \leq b$.

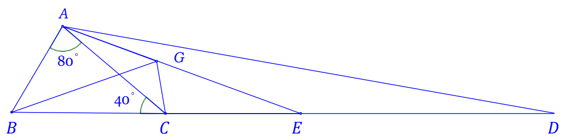
CASE 4. If a, b and c are all odd, we use

$$(a_1, b_1, c_1) = \left(\frac{a+3}{2}, \frac{b-1}{2}, \frac{c-1}{2}\right).$$

Again, these are non-negative integers, and $3a_1 + 4b_1 + 5c_1 = 60$, and $a \geq 3$ implies that $a_1 \leq a$.

This finishes the proof.

5.



Let E be on CD such that AC = CE. Since △ACE is isosceles, $\widehat{CAE} = \widehat{CEA} = 20°$. Take G on AE such that AG = AB. Since △ABG is isosceles,

$$\widehat{AGB} = \frac{180° - (80° + 20°)}{2} = 40°.$$

Now, the quadrilateral ABCG is cyclic ($\widehat{AGB} = \widehat{ACB} = 40°$) and, hence, $\widehat{BGC} = \widehat{BAC} = 80°$ and $\widehat{ACG} = \widehat{ABG} = 40°$, so $\widehat{BCG} = 40° + 40° = 80°$. Thus, △GBC is isosceles. Also, since $\widehat{GBC} = \widehat{GAC} = 20°$, △BGE is isosceles. Therefore, BC = BG = GE, and AE = AB + BC = ED. Thus, △AED is isosceles, with $\widehat{ADB} = 1/2\widehat{CEA} = 10°$.

Edmonton Junior High Math Contest 2018

Part A: Multiple Choice

1. In a triangle, all angles are integer values. The measure of the smallest angle is 20°. What is the measure of the largest possible angle?

   (a) 80° (b) 89° (c) 139° (d) 140° (e) 160°

2. The original price of an item is reduced by 20%, and the new price is further reduced by 20%. The final sale price is the same as a single reduction of what percentage of the original price?

   (a) 30% (b) 36% (c) 40% (d) 60% (e) 64%

3. The first three terms of a *geometric* sequence are 16, 24, 36, ... With the exception of the first term, each term thereafter is the product of the previous term and a constant.

   The first three terms of an *arithmetic* sequence are 30, 45, 60, ... With the exception of the first term, each term thereafter is the sum of the previous term and a constant.

   If both sequences are to continue, for which n does the nth term of the geometric sequence {16, 24, 36, . . .} first become larger than the nth term of the arithmetic sequence {30, 45, 60, . . .}?

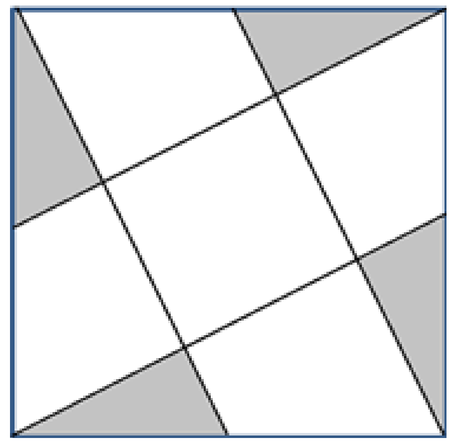
   (a) 4 (b) 5 (c) 6 (d) 7 (e) 8

4. Solve for x.

$$\frac{x}{2{,}018}+\frac{2{,}017(2{,}019)}{2{,}018}-2{,}019=-1$$

   (a) −2,018 (b) 2,018 (c) 0 (d) −1 (e) 1

5. Midpoints of each side are connected to one of the vertices of a square. What percentage of the square is shaded?



   (a) 20% (b) 25% (c) 30% (d) 35% (e) 15%

6. In the sequence of numbers 1, 3, 2, −1, . . . , each term after the first two is equal to the term preceding it minus the term preceding that: $t_n = t_{n-1} - t_{n-2}$. What is the sum of the first 100 terms of the sequence?

   (a) 0 (b) −1 (c) 21 (d) 16 (e) 5

7. A woman, her brother, her son and her daughter (all relations by birth) are chess players. The worst player's twin (who is one of the four players) and the best player are of opposite sex. The worst player and the best player are the same age. Who is the worst player?

   (a) woman (b) son (c) brother
   (d) daughter (e) insufficient information to determine

Solutions

1. The two remaining angles have a sum of 160°. Since the measurements of the angles are integers, the next smallest angle should be 21°. Thus, the largest angle would be 180 – 20 – 21, or 139°. The answer is **(c)**.

2. If the item is priced at \$100, a 20% reduction results in a reduced price of \$100 × 0.8 = \$80. A 20% reduction of the reduced price is \$80 × 0.8 = \$64. The twice-reduced price is 64% of the original, or 100 – 64 = 36, a single reduction of 36% of the original. The answer is **(b)**.

3. Each term of the geometric sequence is multiplied by the constant 1.5. Each term of the arithmetic sequence is added to by the constant 15. We can simply list out the next few terms of each sequence.

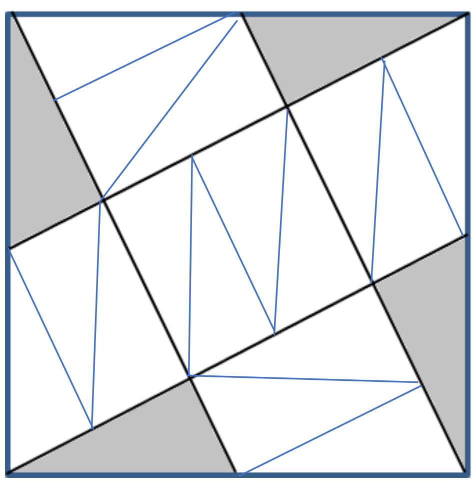
Geometric sequence	16	24	36	54	81	121.5
Arithmetic sequence	30	45	60	75	90	105

We can see that the geometric sequence is larger starting from the sixth term. The answer is **(c)**.

4. $$x = (2{,}018)^2 - (2{,}017)(2{,}019) = 1$$

The answer is **(e)**.

5. The diagram can be subdivided into congruent triangles, as follows.



The entire square is made up of 20 congruent triangles. The shaded part has four pieces out of 20. This means that the shaded part covers 20% of the square. The answer is **(a)**.

6. Writing out the sequence, we have 1, 3, 2, −1, −3, 2, 5, 3, −2, −5, −3, 2, 5, 3, −2, −5 , −3, 2, 5, 3, −2, −5 . . . . After the first four numbers in the sequence, there is a repetition of these six terms: −3, 2, 5, 3, −2, −5. The remaining 96 terms are grouped six terms at a time, and each group has a sum of 0. The sum is

$$1 + 3 + 2 + -1 + (96/6)(0) = 5.$$

The sum of the first 100 terms is 5. The answer is **(e)**.

7. If the son is the worst player, the daughter must be his twin. The best player must then be the brother. This is consistent with the given information, since the brother and the son could be the same age. The assumption that any of the other players is the worst leads to a contradiction.

If the woman is the worst player, her brother must be her twin and her daughter must be the best player. But the woman and her daughter cannot be the same age.

If the brother is the worst player, the woman must be his twin. The best player is then the son. But the woman and her son cannot be the same age, and the woman's twin, her brother, cannot be the same age as the son.

If the daughter is the worst player, the son must be the daughter's twin. The best player must then be the woman. But the woman and her daughter cannot be the same age.

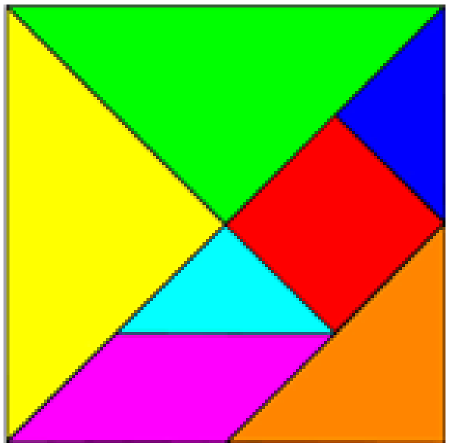
The answer is **(b)**.

Part B: Short Answer

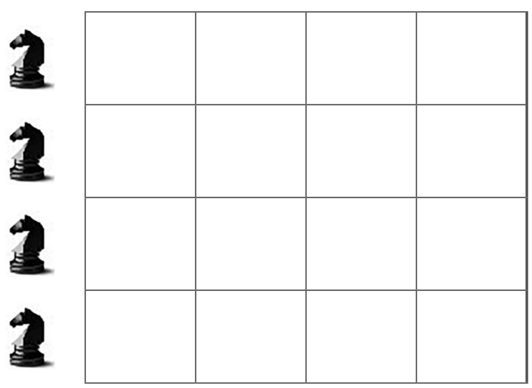
8. A large cube is assembled together with 125 smaller identical cubes. This large cube is placed on a tabletop and spray-painted on five of the faces. After drying, the large cube is disassembled, with all the smaller cubes placed in a bag. How many cubes have exactly two faces painted?

9. Juice cans are sold in a pack of six or twelve. There are 12,448 cans in the warehouse waiting to be put into different packs. If the cans are placed in any combination of six- and twelve-packs, how many cans will be left over?

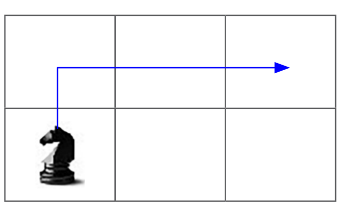
10. Below is a picture of an ancient Chinese puzzle, called a tangram. The seven pieces are arranged to form a large square. If the edge of the large square is one unit, find the perimeter of the smaller square in the picture. Leave the answer in simplest radical form.



11. Sally places four knights on a 4 × 4 board.

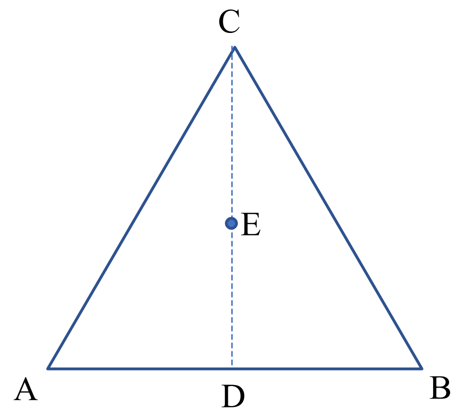


Knights can only attack a square that is 2 × 3 away.



If any square is attacked by at least two different knights, Sally colours the square black. What is the maximum number of black squares possible?

12. The △ABC is an equilateral triangle with a height of $2\sqrt{3}$. E is the midpoint of the altitude CD. Find distance $\overline{AE}$. Leave the answer in radical form.



13. In △ABC, ∡CAB = 108° and AB = AC. The bisector of ∡ABC meets CA at E, and the perpendicular to BE at E meets BC at D. Determine ∡ADE.

Solutions

8. Of the twelve edges, only eight edges have cubes that have exactly two faces painted. Each edge has three such cubes, giving a total of 8 × 3 = 24 cubes. The answer is **24**.

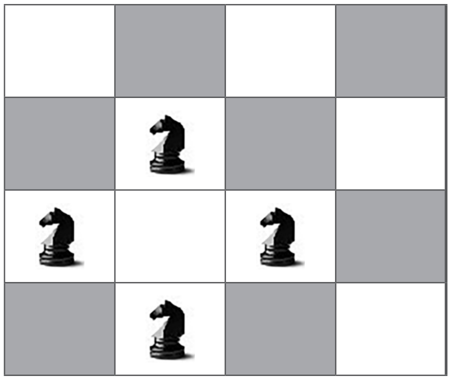
9. Let n represent the number of six-packs, and let m represent the number of twelve-packs. Then

$$6n + 12m = 12{,}448$$
$$6(n + 2m) = 12{,}448$$

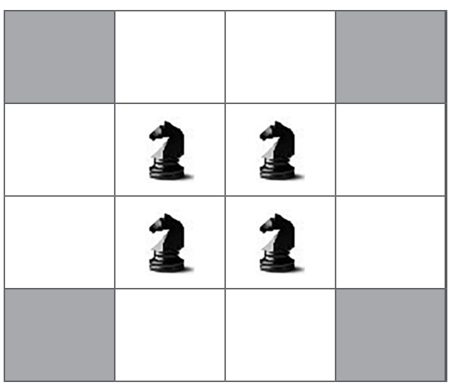
12,448 ÷ 6 = 2,074, with a remainder of 4.

Four cans would not fit into six- or twelve-packs. The answer is **four**.

10. All the triangular pieces (small, medium and large sizes) are isosceles right triangles. This means that they are all similar. The largest triangle measures $\sqrt{2}/2 \times \sqrt{2}/2 \times 1$. The smallest triangle is a reduction with scale factor of 1/2; it measures $\sqrt{2}/4 \times \sqrt{2}/4 \times 1/2$. The side of the small square is the same as one of the legs of the small isosceles triangle, $\sqrt{2}/4$. The perimeter of the small square is $4(\sqrt{2}/4) = \sqrt{2}$. The answer is **$\sqrt{2}$ or 1.414**.

11. 

Where knights will doubly attack all the blackened squares. In total, there are seven black squares. To prove that this is the maximum, we note that any knight in one of the centre squares will attack four squares, and any knight on an edge will attack three squares. Note that the above configuration maximizes the number of black squares, given that two knights are on an edge and two knights are in the centre. We only have four knights, so the only way for Sally to colour eight squares is if all knights are attacking four squares or, in other words, if they are all in the centre.



Clearly, this does not result in eight blackened squares. The answer is **seven**.

12. The $\triangle$ABC is an equilateral triangle with interior angles at 60° each. The $\triangle$CBD is a 30° – 60° – 90° triangle with side lengths in the ratio of $1 - \sqrt{3} - 2$. Since $\overline{\mathrm{CD}}$ is $2\sqrt{3}$, we must have $\overline{\mathrm{CB}} = \overline{\mathrm{CA}} = \overline{\mathrm{AB}} = 4$. It follows that

$$(\overline{\mathrm{AE}})^2 = (\overline{\mathrm{AD}})^2 + (\overline{\mathrm{DE}})^2$$
$$\overline{\mathrm{AE}} = \sqrt{(2)^2 + (\sqrt{3})^2} = \sqrt{7}.$$

The answer is $\sqrt{7}$ **or 2.646**.

13. Extend BA and DE to meet at F. By symmetry, BED and BEF are congruent right triangles. Now ∡EFB = ∡EDB = 90° – ∡EBC = 72° while ∡EAF = 180° – ∡CAB = 72° also. Hence, EA = EF = ED. Note that we have ∡BEA = 180° – ∡CAB – ∡ABC = 54° and ∡DEA = ∡BED + ∡BEA = 144°. It follows that ∡ADE = 1/2(180° – ∡DEA) = 18°. The answer is **18**°.

Part C: Short Answer

14. Each of the digits 0–9 is written on a card. You can select any number of cards from the group. The digits can then be rearranged in any order to form a number. What is the minimum number of cards from which a multiple of 3 with up to three digits can always be formed?

15. A class of 31 students invites some students from another school to the Valentine's Day dinner. There are 19 tables at which one or two students may sit. If each boy has exchanged valentine cards with exactly three girls and each girl with exactly two boys, how many students have attended the dinner?

16. How many different five-digit numbers have the property that if one digit is removed, reading from left to right, the number 2,018 will be obtained?

17. In the school badminton club, there are 18 boys and 18 girls, with 12 in each of Grades 7, 8 and 9. The school wants to enter as many mixed doubles teams as possible in the city championship. The partners in each pair must be from the same grade. What is the minimum number of mixed doubles pairs the school can enter?

18. Consider a non-negative number boring if it is made of only the same digits and cool if it is made of only distinct digits. Single digits (1, 4, 3 and so on) are only cool. Note that digits with repeats (such as 21,330) are neither cool nor boring. What is the smallest positive integer greater than or equal to 11 that cannot be represented as the sum of a boring and a cool number?

19. In $\triangle$BAD, BD = 3, AD = 4, and AB = 5. C is the point on the extension of BD such that DC = 1. PQRS is a rectangle with P and S on BC, Q on AB, and R on AC. Determine the maximum perimeter of PQRS.

Solutions

14. If we have only two cards, the numbers on them may be 1 and 4, and none of 1, 4, 11, 14 or 41 is a multiple of 3. Hence, two cards are not enough. We claim that three cards are enough. If any of 0, 3, 6 or 9 is on one of them, we have a one-digit multiple of 3. Suppose this is not the case. If we have all of 1, 4 and 7 or all of 2, 5 and 8, we have a three-digit multiple of 3, since $1 + 4 + 7 = 12$ and $2 + 5 + 8 = 15$ are multiples of 3. If this is also not the case, we take one number from each triple. Their sum will be a multiple of 3, and they will form a two-digit multiple of 3. The answer is **three**.

15. The ratio of boys to girls is 2:3. Hence, the total number of students is a multiple of 5. It is greater than 31 and less than $19 \times 2 = 38$. Hence, it must be 35. The answer is **35**.

16. If the deleted digit is in the first place, before 2, we have exactly nine choices for adding a digit, because we cannot add 0. It may appear that we have ten choices if the deleted digit is in the second place, between 2 and 0. However, 22,018 has already been counted. So we have only nine choices. Suppose the deleted digit is in the third place, between 0 and 1. We still have only nine choices, since we cannot choose 20,018. Hence, the total number of choices is $9 \times 5 = 45$. The answer is **45**.

17. Suppose that all Grade 7 students are boys and all Grade 8 students are girls. Then the school can enter exactly six Grade 9 teams. Suppose the number of teams that can be entered is less than six. Then there are more than 24 students with no partners. Such students from each grade must be of the same gender. By symmetry, we may assume that all students without partners from Grade 7 or 8 are boys, and all students without partners from Grade 9 are girls. Since the total number of boys is equal to the total number of girls, we have more than 12 girls with no partners, and they are all in Grade 9. This is a contradiction since there are only 12 students in Grade 9. The answer is **six**.

18. We claim that the smallest positive integer is 110. Note that the difference between 110 and any two-digit boring number will also be a boring number, so there are no cool numbers that can sum with a boring number to obtain 110.

    We now prove that this is the minimum. All two-digit boring numbers are the sum of the given boring number and 0. We know that all two-digit boring numbers are separated 10 numbers, which correspond to adding the cool numbers 1, 2, . . . , 10. Thus, any two-digit number is the sum of a boring number and a cool number less than or equal to 10. For three-digit numbers, note that we only need to consider numbers greater than $99 + 10 = 109$, of which there are none that are also less than 110. Thus, the minimum is proved.

    The answer is **110**.

19. We have

$$\frac{BD}{DA} = \frac{BP}{PQ} = \frac{3}{4}$$

and

$$\frac{CD}{DA} = \frac{CS}{SR} = \frac{1}{4}.$$

Note that PQ = SR. Hence,

$$BP + CS = PQ\left(\frac{3}{4} + \frac{1}{4}\right) = PQ.$$

It follows that the perimeter of PQRS is equal to

$$\begin{aligned} 2(PQ + PS) &= 2(BP + CS + PS) \\ &= 2BC = 2(3 + 1) = 8. \end{aligned}$$

Since the value is constant, the maximum value is also 8. The answer is **8**.

Quanta Magazine

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Lorelei Boschman

Go to www.wired.com/tag/quanta-magazine/ and consider some truly interesting scenarios:

- "The Math of How Crickets, Starlings, and Neurons Sync Up"
- "Is the Universe a Hologram? Maybe! This Math Trick Shows How"
- "Colorado Tried a New Way to Vote: Make People Pay—Quadratically"
- "The Sensible Math of Knocking Over Absurdly Large Dominoes"
- "The Best Way for a Mouse to Escape a Cat, According to Math"
- "A Huge Achievement in Math Shows the Limits of Symmetries"

The article that brought me to this website was "Mathematicians Discover the Perfect Way to Multiply," by Kevin Hartnett (www.wired.com/story/mathematicians-discover-the-perfect-way-to-multiply/). This article intrigued me. After all our multiplying through the ages, there is now a best way to multiply for very large numbers. The article demonstrates how this works and explains the mathematics behind it. Imagine only $2n$ steps instead of n^2 steps, and think of the neat classroom applications! The article has been reprinted in this issue of *delta-K*.

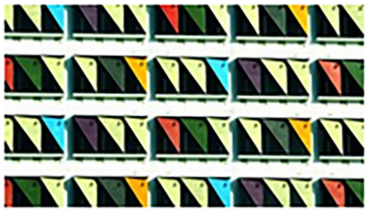
A search of "math" at www.wired.com/tag/quanta-magazine/ brought up 3,673 interesting and curious math articles. Adding more descriptors will focus your search on one of your particular topics of interest. Many of the articles are linked to science, which could further promote STEM (science, technology, engineering and mathematics) topics.

"Mathematicians Discover the Perfect Way to Multiply" was originally printed in *Quanta Magazine* (www.quantamagazine.org). What a wealth of mathematical ideas! Again, put "math" into the search bar and see what shows up. I can see many of you mathematicians out there enjoying these diverse, interesting and mind-stimulating topics and articles.

Lorelei Boschman received her bachelor of education and master of education degrees from the University of Lethbridge. She is the education coordinator at Medicine Hat College, facilitating the four-year bachelor of education program (a collaborative degree program with Mount Royal University) and instructing a variety of postsecondary courses with a mathematics focus. Previously, she taught K–8 at a rural school and spent 21 years teaching high school mathematics. Mathematics education is her passion and life work, and she has been involved in many local and provincial initiatives.

MATH

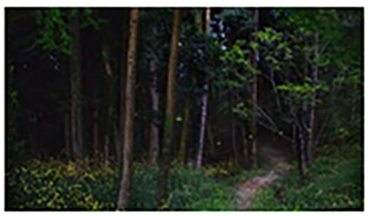
3680 RESULTS SORT BY: OLDEST NEWEST **RELEVANCE**



DECEMBER 20, 2018 | RHETT ALLAIN

Physics Owes a Lot to a Little-Loved Math Class

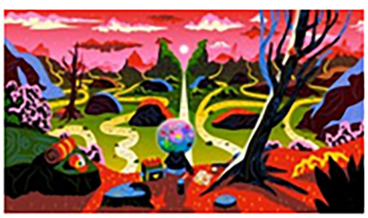
I'm talking about trigonometry, of course: that bastion of angles and triangles that is essential to calculating velocity, momentum, and much more.



APRIL 7, 2019 | NATALIE WOLCHOVER

The Math of How Crickets, Starlings, and Neurons Sync Up

Scientists have discovered new patterns that help explain the synchronized behaviors of pendulum clocks, fireflies, and even the power grid.



MARCH 24, 2019 | JORDANA CEPELEWICZ

The Mysterious Math of How Cells Determine Their Own Fate

During development, cells seem to use statistics to figure out what identities they should take on.

JANUARY 23, 2019 | RHETT ALLAIN

The Sensible Math of Knocking Over Absurdly Large Dominoes

Using a small domino to topple a taller one, you could in theory take down a domino the size of a skyscraper. Here's how it would work.

Mathematicians Discover the Perfect Way to Multiply

Kevin Hartnett

Four thousand years ago, the Babylonians invented multiplication. Last month, mathematicians perfected it.

On March 18, two researchers described the fastest method ever discovered for multiplying two very large numbers (Harvey and van der Hoeven 2019). The paper marks the culmination of a long-running search to find the most efficient procedure for performing one of the most basic operations in math.

"Everybody thinks basically that the method you learn in school is the best one, but in fact it's an active area of research," said Joris van der Hoeven, a mathematician at the French National Center for Scientific Research and one of the coauthors.

The complexity of many computational problems, from calculating new digits of pi to finding large prime numbers, boils down to the speed of multiplication. Van der Hoeven describes their result as setting a kind of mathematical speed limit for how fast many other kinds of problems can be solved.

"In physics you have important constants like the speed of light which allow you to describe all kinds of phenomena," van der Hoeven said. "If you want to know how fast computers can solve certain mathematical problems, then integer multiplication pops up as some kind of basic building brick with respect to which you can express those kinds of speeds."

Most everyone learns to multiply the same way. We stack two numbers, multiply every digit in the bottom number by every digit in the top number and do addition at the end. If you're multiplying two two-digit numbers, you end up performing four smaller multiplications to produce a final product.

The grade school or "carrying" method requires about n^2 steps, where n is the number of digits of each of the numbers you're multiplying. So three-digit numbers require nine multiplications, while 100-digit numbers require 10,000 multiplications.

The carrying method works well for numbers with just a few digits, but it bogs down when we're multiplying numbers with millions or billions of digits (which is what computers do to accurately calculate pi or as part of the worldwide search for large primes).[1] To multiply two numbers with one billion digits requires one billion squared, or 10^{18}, multiplications, which would take a modern computer roughly 30 years.

For millennia it was widely assumed that there was no faster way to multiply. Then, in 1960, the 23-year-old Russian mathematician Anatoly Karatsuba took a seminar led by Andrey Kolmogorov, one of the great mathematicians of the 20th century. Kolmogorov asserted that there was no general procedure for doing multiplication that required fewer than n^2 steps. Karatsuba thought there was—and after a week of searching, he found it.

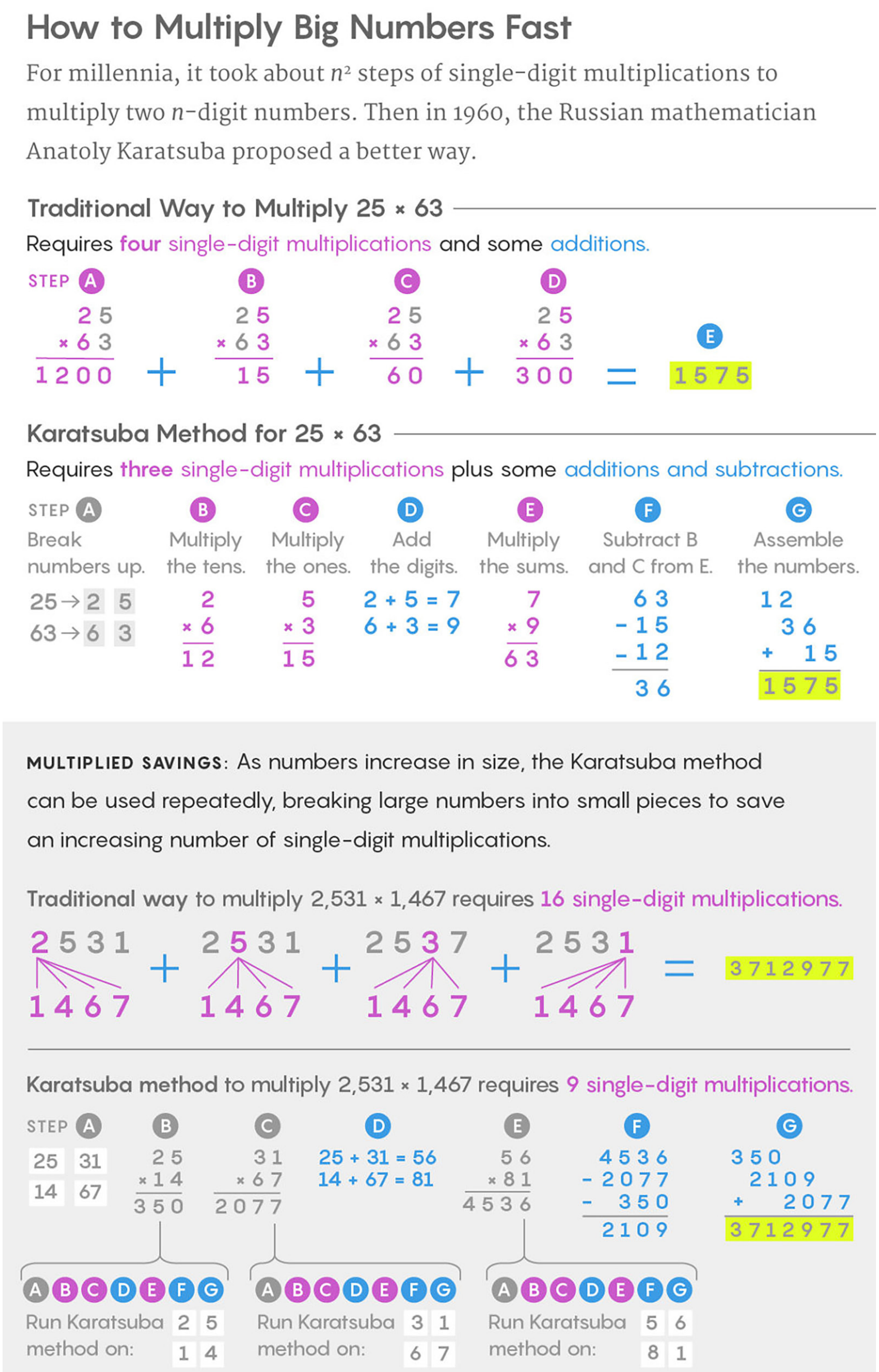
Karatsuba's method involves breaking up the digits of a number and recombining them in a novel way that allows you to substitute a small number of additions and subtractions for a large number of multiplications. The method saves time because addition takes only $2n$ steps, as opposed to n^2 steps.

"With addition, you do it a year earlier in school because it's much easier, you can do it in linear time, almost as fast as reading the numbers from right to left," said Martin Fürer, a mathematician at Pennsylvania State University who in 2007 created what was at the time the fastest multiplication algorithm.

When dealing with large numbers, you can repeat the Karatsuba procedure, splitting the original number into almost as many parts as it has digits. And with each splitting, you replace multiplications that require many steps to compute with additions and subtractions that require far fewer.

"You can turn some of the multiplications into additions, and the idea is additions will be faster for computers," said David Harvey, a mathematician at the University of New South Wales and a coauthor on the new paper.

Karatsuba's method made it possible to multiply numbers using only $n^{1.58}$ single-digit multiplications. Then, in 1971, Arnold Schönhage and Volker Strassen published a method capable of multiplying large numbers in $n \times \log n \times \log(\log n)$ multiplicative steps, where $\log n$ is the logarithm of n. For two



Lucy Reading-Ikkanda/Quanta Magazine

one-billion-digit numbers, Karatsuba's method would require about 165 trillion additional steps.

Schönhage and Strassen's method, which is how computers multiply huge numbers, had two other important long-term consequences. First, it introduced the use of a technique from the field of signal processing called a fast Fourier transform. The technique has been the basis for every fast multiplication algorithm since.

Second, in that same paper, Schönhage and Strassen conjectured that there should be an even faster algorithm than the one they found—a method that needs only $n \times \log n$ single-digit operations—and that such an algorithm would be the fastest possible. Their conjecture was based on a hunch that an operation as fundamental as multiplication must have a limit more elegant than $n \times \log n \times \log(\log n)$.

"It was kind of a general consensus that multiplication is such an important basic operation that, just from an aesthetic point of view, such an important operation requires a nice complexity bound," Fürer said. "From general experience the mathematics of basic things at the end always turns out to be elegant."

Schönhage and Strassen's ungainly $n \times \log n \times \log(\log n)$ method held on for 36 years. In 2007, Fürer beat it, and the floodgates opened. Over the past decade, mathematicians have found successively faster multiplication algorithms, each of which has inched closer to $n \times \log n$, without quite reaching it. Then, last month, Harvey and van der Hoeven got there.

Their method is a refinement of the major work that came before them. It splits up digits, uses an improved version of the fast Fourier transform and takes advantage of other advances made over the past 40 years. "We use [the fast Fourier transform] in a much more violent way, use it several times instead of a single time, and replace even more multiplications with additions and subtractions," van der Hoeven said.

Harvey and van der Hoeven's algorithm proves that multiplication can be done in $n \times \log n$ steps. However, it doesn't prove that there's no faster way to do it. Establishing that this is the best possible approach is much more difficult. At the end of February, a team of computer scientists at Aarhus University posted a paper arguing that if another unproven conjecture is also true, this is indeed the fastest way multiplication can be done (Afshani et al 2019).

And while the new algorithm is important theoretically, in practice it won't change much, since it's only marginally better than the algorithms already being used. "The best we can hope for is we're three times faster," van der Hoeven said. "It won't be spectacular."

In addition, the design of computer hardware has changed. Two decades ago, computers performed addition much faster than multiplication. The speed gap between multiplication and addition has narrowed considerably over the past 20 years to the point where multiplication can be even faster than addition in some chip architectures. With some hardware, "you could actually do addition faster by telling the computer to do a multiplication problem, which is just insane," Harvey said.

Hardware changes with the times, but best-in-class algorithms are eternal. Regardless of what computers look like in the future, Harvey and van der Hoeven's algorithm will still be the most efficient way to multiply.

Note

1. See www.mersenne.org (accessed August 19, 2019).

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Kevin Hartnett is a senior writer at Quanta Magazine *covering mathematics and computer science. His work has been included in Princeton University Press's Best Writing on Mathematics series in 2013, 2016 and 2017. From 2013 to 2016 he wrote Brainiac, a weekly column for the* Boston Globe*'s Ideas section.*

Reprinted with permission from QuantaMagazine.org, an editorially independent publication of the Simons Foundation, whose mission is to enhance public understanding of science by covering research developments and trends in mathematics and the physical and life sciences, April 11, 2019, www.quantamagazine.org/mathematicians-discover-the-perfect-way-to-multiply-20190411/. Minor changes have been made to fit ATA style.

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