Alberta High School Mathematics Competition 2017/18

The Alberta High School Mathematics Competition is a two-part competition that takes place in November and February of each school year. Book prizes are awarded for Part I, and cash prizes and scholarships for Part II. Presented here are the problems and solutions from the 2017/18 competition.

Part I

November 21, 2017

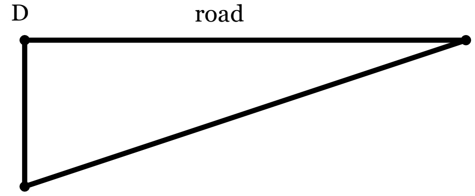
1. Peppa has twice as many apples as pears. After she eats 50 pears, she now has four times as many apples as pears. How many apples does Peppa have?

   (a) 0 (b) 100 (c) 200 (d) 400
   (e) none of these

2. A math teacher is given a big bag of 300 candies by a grateful parent. The teacher distributes as many of the candies as possible to her students so that each student gets the same number of candies. There are 14 candies left over. How many students are there in her class, if each class in the school has at most 30 students?

   (a) 20 (b) 22 (c) 24 (d) 26
   (e) not uniquely determined

3. A vertical pole is 12 m from a straight road. Mark is currently on the road, 37 m from the pole.



   How many metres does Mark have to walk on the road before he is 20 m from the pole?

   (a) 17 (b) 19 (c) 20 (d) 21 (e) 22

4. Some $n \geq 3$ different positive real numbers are arranged on a circle such that each number is equal to the product of its two neighbours. The value of n is

   (a) 4 (b) 5 (c) 6 (d) 8 (e) none of these

5. How many pairs (n, p) are such that n is a positive integer, p is a prime number and $n + p/n$ is a square of a positive integer?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) none of these

6. Ben and Cleo can paint a fence in four days. Anna and Ben can do it in two days, and Anna and Cleo can do it in three days. How many days, as a fraction, does it take all of them working if Cleo gets injured at the end of the first day and can't come back to work?

   (a) 11/6 (b) 45/24 (c) 23/12 (d) 47/24
   (e) none of these

7. The number of ways to walk from (0, 0) to (20, 2) by using only up and right unit steps and such that the walk never visits the lines $y = x$ and $y = x - 18$, except at the beginning and end, is

   (a) 151 (b) 153 (c) 189 (d) 190 (e) 195

8. Lan likes red or blue chopsticks, while Dan likes orange or green chopsticks. A drawer in a dark room contains n chopsticks of each colour, where $n > 1$ is an integer. Lan goes into the room to get chopsticks for both her and Dan. What is the smallest number of these $4n$ chopsticks that Lan must pick in order to be sure that both people

will get a matched (that is, the same colour) pair that they like?

(a) $2n + 1$ (b) $2n + 3$ (c) $3n - 3$
(d) $3n - 1$ (e) none of these

9. Let x, y and z be any real numbers such that $3x + y + 2z \geq 3$ and $2y - x + 4z \geq 5$. The minimum possible value of $7x + 5y + 10z$ is

(a) 96/7 (b) 97/7 (c) 14 (d) 99/7 (e) 100/7

10. In the trapezoid ABCD, with AD parallel to BC, the diagonals intersect at O. If the area of $\triangle$AOD is 9/16 of the area of the trapezoid, then the ratio of the area of $\triangle$AOD to the area of $\triangle$BOC is equal to

(a) 4 (b) 6 (c) 9 (d) 12 (e) none of these

11. The area of the quadrilateral bounded by the graphs of the functions $y = |x - a|$, with $0 < a < 4$ and $y = 2 - |x - 2|$, is 15/8. The smallest value of a is

(a) 2/5 (b) 2/3 (c) 1 (d) 3/2 (e) 5/2

12. Let a and b be positive integers. The number of quadratic equations $x^2 - ax - b = 0$ having the positive root less than 10 is

(a) 391 (b) 392 (c) 441 (d) 450
(e) none of these

13. Let S be a subset of non-negative integers that contains 0, and such that for any number x in S, $3x$ and $3x + 1$ are also in S. The least possible number of elements of S less than 2,017 is

(a) 64 (b) 128 (c) 256 (d) 300
(e) none of these

14. Let a be a real number so that the equation $x^4 - 2ax^2 - x - a + a^2 = 0$ has four different real solutions. Then which of the following must be true?

(a) $a < 1/4$ (b) $1/4 < a < 3/4$ (c) $3/4 < a < 1$
(d) $1 < a$ (e) none of these

15. A positive integer is called a palindrome if it remains unchanged when written backward. Find the number of five-digit palindromes that are divisible by 55.

(a) 5 (b) 8 (c) 10 (d) 12 (e) none of these

16. How many positive integers n can be found such that the product of all divisors of n, including n, is 24^{240}?

(a) 0 (b) 1 (c) 2 (d) 3 (e) none of these

Solutions

1. Peppa initially has x apples and $x/2$ pears. Then $4(x/2 - 50) = x$ and, thus, $x = 200$. The answer is **(c)**.

2. If x represents the number of students in the class and c the number of candies received by each student, then $14 < x \leq 30$ and $x \cdot c = 300 - 14 = 286 = 2 \cdot 11 \cdot 13$. There are two possible solutions, $x = 22$ or $x = 26$. The answer is **(e)**.

3. Let D be the point on the road closest to the pole. Then the distance between the pole and D is 12 m. At first, Mark is

$$\sqrt{37^2 - 12^2} = \sqrt{1{,}369 - 144} = \sqrt{1{,}225} = 35 \text{ m}$$

from D. When he is 20 m from the pole, Mark is

$$\sqrt{20^2 - 12^2} = \sqrt{400 - 144} = \sqrt{256} = 16 \text{ m}$$

from D. Hence, Mark walked $35 - 16 = 19$ m. The answer is **(b)**. (Notice that Mark can pass point D when he is 20 m from the pole. In this case, Mark has to walk $35 + 16 = 51$ m.)

4. If a and b are the first two numbers in the sequence, then the third number in the sequence must be b/a (because $a \times b/a = b$). Similarly, we can find that the sequence of numbers around the circle must be a, b, b/a, $1/a$, $1/b$, a/b, a, b, ... Thus, the sequence repeats after six terms. If there were a shorter sequence of at least three different terms starting with a, b, its length must divide into 6, so it must have length 3. This would mean that the fourth term, $1/a$, would equal the first term, a, which means that $a = 1$, but then the second and the third terms (b and b/a) of the sequence are equal —a contradiction. So the only possible length is 6. The answer is **(c)**.

5. Since p/n is an integer, $n \mid p$. Hence, $n = 1$ or $n = p$. It follows that $p + 1$ is a perfect square. Thus, there is a positive integer m such that $p + 1 = m^2$ or $p = (m - 1)(m + 1)$. Hence, $1 = m - 1$ and $p = m + 1$ (that is, $m = 2$, $p = 3$). Therefore, $p = 3$ and $n = 1$ or $n = 3$. The answer is **(c)**.

6. If a is the portion of the work done per day by Anna, b by Ben and c by Cleo, then $b + c = 1/4$, $a + b = 1/2$, and $a + c = 1/3$. Hence, $a + b + c = 13/24$. Thus, 11/24 of the work has to be completed by Anna and Ben. They need for this

$11/24 \div 1/2 = 11/12$ days, for a total of 23/12 days. The answer is **(c)**.

7. The first two steps must be to the right from (0, 0) to (2, 0), and the last two steps must be to the right from (18, 2) to (20, 2). The number of ways of walking from (2, 0) to (18, 2) is

$$\binom{18}{2} = 153.$$

We have to subtract one way that contains (2, 2) and the one that contains (18, 0). Hence, the number of admissible ways is 151. The answer is **(a)**.

8. If Lan picks $2n + 2$ chopsticks consisting of n orange, n green, one red and one blue, she does not get a pair she likes. Hence, she needs to pick at least $2n + 3$ chopsticks to be sure that she gets a pair she likes. On the other hand, any choice of $2n + 3$ chopsticks will contain at least three red/blue chopsticks and at least three orange/green chopsticks, so both people will get a pair they like. The answer is **(b)**.

9. Let A and B be real positive numbers. Then

$$A(3x + y + 2z) + B(2y - x + 4z)$$
$$\geq 3A + 5B \Leftrightarrow (3A - B)x + (A + 2B)y + (2A + 4B)z$$
$$\geq 3A + 5B.$$

If we take $3A - B = 7$, $A + 2B = 5$, $2A + 4B = 10$, which is equivalent to $A = 19/7$, $B = 8/7$, the above inequality can be written as

$$7x + 5y + 10z \geq 97/7.$$

Hence, the minimum value of $7x + 5y + 10z$ is 97/7. We may take $x = 1/7$, $y = 18/7$ and $z = 0$ to justify that the minimum value is attainable. The answer is **(b)**.

10. Let area(AOD) = a, area(BOC) = b and area(AOB) = area(COD) = c. If $a/b = x$, then a/c = AO/OC = c/b, so $a/c = c/b = \sqrt{x}$. Hence,

$$\frac{\text{area(AOD)}}{\text{area(ABCD)}} = \frac{a}{a+b+2c} = \frac{\frac{a}{b}}{2\frac{c}{b}+\frac{a}{b}+1} = \frac{x}{(\sqrt{x}+1)^2} = \frac{9}{16}.$$

Solving for x, one obtains $x = 9$. The answer is **(c)**.

11. Graphing the functions, we conclude that two of the vertices of the quadrilateral are A = (2, 2) and C = (a, 0). The other two vertices,

$$\mathrm{B} = \left(\frac{a}{2} \cdot \frac{a}{2}\right) \text{ and } \mathrm{D} = \left(\frac{4+a}{2}, \frac{4-a}{2}\right),$$

are at the intersection of the lines $y = x$, $y = a - x$ and, respectively, $y = 4 - x$, $y = x - a$. The area of the quadrilateral ABCD is

$$4 - \frac{a^2}{4} - \frac{(4-a)^2}{4} = \frac{4a - a^2}{2}.$$

Solving the equation

$$\frac{4a - a^2}{2} = \frac{15}{8},$$

one obtains $a = 3/2$, $a = 5/2$. The answer is **(d)**.

12. The positive root of the equation is

$$\frac{a + \sqrt{a^2 + 4b}}{2}.$$

The condition

$$\frac{a + \sqrt{a^2 + 4b}}{2} < 10$$

is equivalent to $10a + b < 100$ and $a \leq 20$. There are 9 + 19 + 29 + 39 + 49 + 59 + 69 + 79 + 89 = 4 × 98 + 49 = 441 ordered pairs (a, b) such that $10a + b < 100$. The answer is **(c)**.

13. If the elements of S are written in base 3, then the conditions of the problem translate to if

$$\overline{a_1 a_2 \cdots a_n}_{(3)} \in S,$$

then

$$\overline{a_1 a_2 \cdots a_n 0}_{(3)} \in S$$

and

$$\overline{a_1 a_2 \cdots a_n 1}_{(3)} \in S.$$

Hence, S should contain all the numbers that can be written in base 3 only by using 0 or 1. Since $2 \cdot 3^6 < 2{,}017 < 3^7$, we conclude that the largest element in S that is less than 2,017 is $1111111_{(3)}$. There will be a total of $2^7 = 128$ numbers in S having the requested property. The answer is **(b)**.

14. The given equation can be written as

$$a^2 - (2x^2 + 1)a - x + x^4 = 0,$$

which has the solutions $a = x^2 - x$ and $a = x^2 + x + 1$. The solutions of the equation $x^2 - x - a = 0$ are different real numbers if $1 + 4a > 0$ or, equivalently, $a > -1/4$. The equation $x^2 + x + 1 - a = 0$ has different real roots if $1 - 4(1 - a) > 0$ or, equivalently, $a > 3/4$. The above two equations cannot have common roots. Indeed, if for $x_0 \in \mathbb{R}$,

$$x_0^2 - x_0 - a = 0$$

and

$$x_0^2 + x_0 + 1 - a = 0,$$

then $x_0 = -1/2$ and $a = 3/4$, which is not convenient. Therefore, the solutions of the given equation are all real and different if $a > 3/4$. The answer is **(e)**.

15. A five-digit palindrome has the form $abcba = 10{,}001a + 1{,}010b + 100c$, with $a \neq 0$. In order for the number to be divisible by 5, we must have $a = 5$. Therefore, the number is $50{,}005 + 1{,}010b + 100c$.

Since $50{,}005 + 1{,}010b + 100c = 11(4{,}546 + 92b + 9c) - 1 - 2b + c$, we then need to have $c - 2b - 1$ divisible by 11. Since $-19 \le c - 2b - 1 \le 8$, we must have $c - 2b - 1 = 0$ or $c - 2b - 1 = -11$.

If $c - 2b - 1 = 0$, then $c = 2b + 1$, which leads to five pairs $(b, c) \in \{(0, 1), (1, 3), (2, 5), (3, 7), (4, 9)\}$.

If $c - 2b - 1 = -11$, then $2b = c + 10$ and, hence, $b \ge 5$. Then we get $(b, c) \in \{(5,0), (6, 2), (7, 4), (8, 6), (9, 8)\}$.

Therefore, we have 10 possibilities. The answer is **(c)**.

16. We have $n \mid 24^{240}$. Then $n = 2^a 3^b$, with a and b integers. Thus, n has $(a + 1)(b + 1)$ divisors. Pairing them in pairs of the form $(d, n/d)$, we obtain that their product is

$$n^{\frac{(a+1)(b+1)}{2}}.$$

Therefore,

$$24^{240} = n^{\frac{(a+1)(b+1)}{2}}.$$

Hence,

$$2^{3\cdot 240} 3^{240} = 2^{a\frac{(a+1)(b+1)}{2}} 3^{b\frac{(a+1)(b+1)}{2}}.$$

From this equation, one obtains

$$240 = b\frac{(a+1)(b+1)}{2}$$

$$3 \cdot 240 = a\frac{(a+1)(b+1)}{2}$$

from which, taking the ratio, we get $a = 3b$ and, hence,

$$480 = b(b + 1)(3b + 1).$$

Since the right-hand side is increasing, this equation has at most one positive integer solution. Moreover, we have

$$480 = b(b + 1)(3b + 1) > 3b^3 \Rightarrow 160 > b^3$$

and, hence, $b \le 5$. By testing, $b = 5$ gives

$$b(b + 1)(3b + 1) = 5 \cdot 6 \cdot 16 = 10 \cdot 6 \cdot 8 = 480.$$

Hence, $b = 5$, $a = 15$, and $n = 2^5 3^{15}$. The answer is **(b)**.

Part II

February 7, 2018

1. The difference between two positive integers is 18. When we divide the larger of the two positive integers by the smaller, the quotient and the remainder are equal. Find all the possible pairs of positive integers.

2. Let $\mathbb{Z}$ be the set of integers and $f : \mathbb{Z} \to \mathbb{Z}$ a function such that

$$f(f(x) + y) = x + f(y)$$

for any integers x and y.

Show that $f(x + y) = f(x) + f(y)$ for any integers x and y.

3. Prove that the numbers 26^n and $26^n + 2^n$ have the same number of digits, for any non-negative integer n.

4. A collection of items weighing 3, 4 or 5 kg has a total weight of 120 kg. Prove that there is a subcollection of the items weighing exactly 60 kg.

5. The $\triangle$ABC has $\angle$BAC = 80° and $\angle$ACB = 40°. D is a point on the ray BC beyond C so that CD = AB + BC + CA. Find $\angle$ADB.

Solutions

1. Let x be the smaller of the two numbers. The larger number is $x + 18$. If q denotes the quotient of the division, then the remainder is also q. The long division of $x + 18$ by x gives

$$x + 18 = qx + q \Rightarrow qx + q - x - 18 = 0.$$

Then

$$qx + q - x - 1 = 17 \Rightarrow (q - 1)(x + 1) = 17.$$

Since 17 is prime and $q < x$ (as the remainder is smaller than the quotient), we have

$$q - 1 = 1, x + 1 = 17.$$

Thus, $x = 16$. Hence, there is only one pair of positive integers having the requested property, 16 and 34.

2. Setting $y = 0$ in the given relation, we get that $f(f(x)) = x + f(0)$ for any $x \in \mathbb{Z}$. Therefore, if $f(0) = a$, we have $f(f(x)) = x + a$. Then,

$$f(x + f(y)) = f(f(f(x) + y)) = f(x) + y + a.$$

Interchanging x, y in the given relation, we also have

$$f(x + f(y)) = f(x) + y.$$

The last two relations give $a = 0$, and therefore $f(f(x)) = x$. Next, replacing x by $f(x)$ in the given relation and using $f(f(x)) = x$, we get

$$f(x + y) = f(x) + f(y).$$

3. Assume by contradiction that 26^n and $26^n + 2^n$ do not have the same number of digits. If 26^n has m digits, then $26^n < 10^m \leq 26^n + 2^n$, with $m > n$ and, hence, $13^n < 2^{m-n}5^m \leq 13^n + 1$. Hence, $2^{m-n}5^m = 13^n + 1$, and since $13^n + 1 \equiv 2 \pmod 4$, one obtains $m - n = 1$. Therefore, n should be a solution of the equation $2 \cdot 5^{n+1} = 13^n + 1 \Leftrightarrow 10 = (1/5)^n + (13/5)^n$. It is clear that $n = 0, 1, 2$ are not solutions of this equation and also any $n \geq 3$ either, since $(13/5)^n \geq (13/5)^3 > 10$ for $n \geq 3$.

Notice: We may justify that the remainder obtained when $13^n + 1$ is divided by 4 is 2, without using congruences. Indeed, since 13, 13^2, 13^3 and in general 13^n are all of the form $4k + 1$, where k is a positive integer, $13^n + 1$ is of the form $4k + 2$.

4. We have non-negative integers a, b and c so that $3a + 4b + 5c = 120$. We wish to find a triple (a_1, b_1, c_1) of non-negative integers satisfying $a_1 \leq a$, $b_1 \leq b$, $c_1 \leq c$ so that $3a_1 + 4b_1 + 5c_1 = 60$.

First note: If $a \geq 20$, then we can use $a_1 = 20$, $b_1 = c_1 = 0$. Thus, we can assume from now on that $a \leq 19$. Similarly, we can assume that $b \leq 14$ and $c \leq 11$. Thus, we get

$$a = \frac{120 - 4b - 5c}{3} \geq \frac{120 - 56 - 55}{3} = 3,$$

$$b = \frac{120 - 3a - 5c}{4} \geq \frac{120 - 57 - 55}{4} = 2 \text{ and}$$

$$c = \frac{120 - 3a - 4b}{5} \geq \frac{120 - 57 - 56}{5} = \frac{7}{5},$$

and since c is an integer, we get $c \geq 2$.

From $3a + 4b + 5c = 120$, we know that a and c have the same parity. Thus, we have four cases.

Case 1. If a, b and c are all even, then we choose $a_1 = a/2$, $b_1 = b/2$, $c_1 = c/2$.

Case 2. If a and c are even while b is odd, then we claim that

$$(a_1, b_1, c_1) = \left(\frac{a+4}{2}, \frac{b-3}{2}, \frac{c}{2}\right)$$

works. These choices are all integers, and

$$3a_1 + 4b_1 + 5c_1 = \frac{3(a+4)}{2} + \frac{4(b-3)}{2} + \frac{5c}{2} = 60,$$

so we need only observe that (i) since a is even and $a \geq 3$, we know that $a \geq 4$, and thus $(a + 4)/2 \leq a$; (ii) since b is odd and $b \geq 2$, we know that $b \geq 3$ and thus $(b - 3)/2 \geq 0$.

Case 3. If a and c are odd while b is even, we use

$$(a_1, b_1, c_1) = \left(\frac{a-1}{2}, \frac{b+2}{2}, \frac{c-1}{2}\right).$$

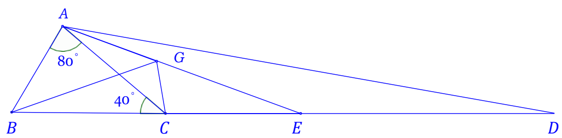
Again, these are non-negative integers, and $3a_1 + 4b_1 + 5c_1 = 60$, and $b \geq 2$ implies that $b_1 \leq b$.

Case 4. If a, b and c are all odd, we use

$$(a_1, b_1, c_1) = \left(\frac{a+3}{2}, \frac{b-1}{2}, \frac{c-1}{2}\right).$$

Again, these are non-negative integers, and $3a_1 + 4b_1 + 5c_1 = 60$, and $a \geq 3$ implies that $a_1 \leq a$.

This finishes the proof.

5. 

Let E be on CD such that AC = CE. Since $\triangle$ACE is isosceles, $\widehat{CAE} = \widehat{CEA} = 20°$. Take G on AE such that AG = AB. Since $\triangle$ABG is isosceles,

$$\widehat{AGB} = \frac{180° - (80° + 20°)}{2} = 40°.$$

Now, the quadrilateral ABCG is cyclic ($\widehat{AGB} = \widehat{ACB} = 40°$) and, hence, $\widehat{BGC} = \widehat{BAC} = 80°$ and $\widehat{ACG} = \widehat{ABG} = 40°$, so $\widehat{BCG} = 40° + 40° = 80°$. Thus, $\triangle$GBC is isosceles. Also, since $\widehat{GBC} = \widehat{GAC} = 20°$, $\triangle$BGE is isosceles. Therefore, BC = BG = GE, and AE = AB + BC = ED. Thus, $\triangle$AED is isosceles, with $\widehat{ADB} = 1/2\widehat{CEA} = 10°$.