On the Russian Peasant Multiplication Algorithm

- *William J. Bruce*

This simple algorithm for multiplication of natural numbers involves addition, multiplication only by two, and division only by two. It is commonly known as the Russian peasant method because of the limited arithmetical skills required. We consider an example in detail and then prove by mathematical induction that the algorithm holds for any two natural numbers.

Our problem is to find the product of 226 and 35 and the format given below is chosen so as to fully explain each step:

	A	B		Product
a	226	35	b	ab
$2a$	452	17	b_1	$ab = 2ab_1 + a$
$4a$	904	8	b_2	$ab = 4ab_2 + 2a + a$
$8a$	1808	4	b_3	$ab = 8ab_3 + 2a + a$
$16a$	3616	2	b_4	$ab = 16ab_4 + 2a + a$
$32a$	7232	1	b_5	$ab = 32ab_5 + 2a + a$
	7910			

The procedure illustrated above is as follows:

1. Double $a = 226$ in column A to obtain $2a = 452$. Reduce $b = 35$ to an even number, namely 34, and divide 34 by 2 to obtain $b_1 = 17$ in column B. Our new product, $2ab_1$, is short by $a = 226$, since we reduced b by 1. We now have

$$ab = 2ab_1 + a.$$

2. Double $2a = 452$ in column A to get $4a = 904$. Reduce $b_1 = 17$ to an even number again, namely 16, and divide 16 by 2 to obtain $b_2 = 8$ in column B. Our new product, $4ab_2$, is short by an <u>additional</u> amount $2a = 452$, since we reduced b_1 by 1. We now have

$$ab = 4ab_2 + 2a + a.$$

3. Double $4a = 904$ in column A. Since $b_2 = 8$ is even, it is divisible by 2 and we obtain $b_3 = 4$. Our new product, $8ab_3$, is not short by any <u>additional</u> amount,

since b_2 is even. We now have

$$ab = 8ab_3 + 2a + a.$$

4. Continue this procedure to obtain 16a and 32a in column A, and b_4 and b_5 in column B. In these steps no additional omissions have been made in obtaining $16ab_4$ and $32ab_5$. Thus

$$ab = 16ab_4 + 2a + a$$

and

$$ab = 32ab_5 + 2a + a.$$

Now $32ab_5$ is trivially obtained by multiplying by 1. Hence we have

$$P = ab = 32ab_5 + 2a + a$$
$$= 7232 + 452 + 226 = 7910.$$

We can verify that this is correct since

$$b_5 = \frac{b_2}{8} = \frac{b_1 - 1}{16} = \frac{\frac{b-1}{2} - 1}{16} = \frac{b-3}{32}$$

so that

$$P = 32a\,\frac{b-3}{32} + 2a + a$$
$$= ab - 3a + 2a + a = ab.$$

This final step in the algorithm can now be stated briefly. Every time the number in column B becomes even, there is no remainder upon division by 2, hence for even numbers in column B we omit the corresponding numbers of column A. The sum of the remaining numbers in column A gives the product ab.

The simplicity of the steps in the example indicate that, intuitively at least, we can see that this algorithm generalizes for products of any natural numbers. Our proof by induction proceeds in the following manner -

The algorithm is true for $b = 1$.

We assume it is true for $b \leq k$, $k \leq b-1$.

Case I: b even. We must prove true for $b = k+1$.

Now $a(k+1) = 2a\,\frac{k+1}{2} = 2ab_1 = 2a\,\frac{b}{2} = ab$, since b is even.

Case II: b odd. We must prove true for $b = k+1$.

Now $a(k+1) = ak + a = 2a\,\frac{k}{2} + a = 2ab_1 + a$, since b is odd.

But $2ab_1 + a = 2a\,\frac{b-1}{2} + a = ab$.

It follows by the principle of mathematical induction that the algorithm is true for every natural number b. Note that the algorithm is commutative and that the number of steps will be fewer when we choose the smaller number for division by 2.