And Then the Imaginary Becomes Real

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The number "i" has the interesting property to become real every time its exponent is of the form 2k, k ε $\mathbb{N}^*$.

$$x = 2k,\ k \ \varepsilon\ \mathbb{N}^* \Rightarrow i^x \ \varepsilon\ \mathbb{R}.$$

What will happen when the exponent is i itself?

The demonstration given here is based upon the famous Euler's Formula:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

This formula is proved in any first course of Complex Variable or Complex Analysis. It is a key to successful work in the complex field. One of the first uses of this formula leads to a result that is always a surprise for the beginner

What happens when $\theta = \pi$?

$$e^{i\pi} = \cos \pi + i \sin \pi$$

$$e^{i\pi} = -1.$$

We have a relation between irrational numbers, imaginary and integers.

What happens when $\theta = \pi/2$?

$$e^{i\pi/2} = \cos \pi/2 + i \sin \pi/2$$

$$e^{i\pi/2} = i.$$

If we want to get the value of i^i the temptation will then be to write

$$i^i = (e^{i\frac{\pi}{2}})^i$$

$$= e^{i^2\frac{\pi}{2}} = e^{-\pi/2}$$

so $i^i \ \varepsilon\ \mathbb{R}$

Let us face the problem in another way.

We know that

a) $e^{i\theta} = \cos\theta + i \sin\theta$

b) $\ln e^{i\theta} = i\theta \ln e = i\theta$

Thus if we write down

$$\omega = i^{i}$$

$$\ln\omega = \ln i^{i} = i \ln i.$$

$$= i \ln e^{i(\pi/2 - 2k\pi)}, \quad k \in \mathbb{Z}$$

$$= i \cdot i(\pi/2 - 2k\pi)$$

$$= -\pi/2 + 2k\pi$$

then $\omega = e^{-\pi/2 + 2k\pi} = i^{i}$

$$\therefore i^{i} \in \mathbb{R}.$$

Thus i^{i} belongs to the real field, but there are many values for this expression, all of them being real. We can have a restriction and take the *principal value* only, in which case $i^{i} = e^{-\pi/2}$.

The same procedure for i^{-i} will give $i^{-i} = e^{\pi/2 + 2\lambda\pi}, \quad \lambda \in \mathbb{Z}$

Here again we are dealing with a multi-valued expression whose principal value may be taken for technical purposes.

Is i^{i} greater or smaller than i^{-i}? Are these expressions sometimes equal? The comparison is easily handled with the preceding results.

Let us first answer the second question:

$$i^{i} \stackrel{?}{=} i^{-i}$$

$$e^{-\pi/2 + 2k\pi} \stackrel{?}{=} e^{\pi/2 + 2\lambda\pi}, \qquad k, \lambda \in \mathbb{Z}$$

$$-\pi/2 + 2k\pi \stackrel{?}{=} \pi/2 + 2\lambda\pi$$

$$k - \lambda \stackrel{?}{=} 1/2$$

$k, \lambda \in \mathbb{Z}$ we cannot have $k - \lambda = 1/2$ so i^{i} is never equal to i^{-i}.

Which one is the greater ? A simple look at the principal values will give

$$e^{-\pi/2} < e^{\pi/2}$$

$$\Rightarrow \quad i^{i} < i^{-i}$$

but this relation is not the same if we are looking at the different values of i^{i} and i^{-i}. The values will vary according to the variation of k and λ.

If $k \leq \lambda$ then $i^{i} < i^{-i}$ and if $k > \lambda$ then $i^{i} > i^{-i}$.

Further studies will bring the following relations:

a) $i^{i} = (-i)^{-i}$

b) $i^{-i} = -i^{i}$.

All these results are astonishing at first sight but they remind us that we are in a land of interesting discoveries when moving in the complex field.

MATH LUNCHEON

Math Luncheon of GREATER EDMONTON TEACHERS' CONVENTION will be held in Rupert's Land Room of the Macdonald Hotel, 12:30 pm, Thursday February 24.

Speaker: Dr. Doyal Nelson, University of Alberta.

Topic: "Mathematics Education in Japan"

Price: $3.50

Contact William Gee at Afton School, 16604 - 91 Avenue, Edmonton.

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