T.H. McConaghy

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1970

Active Learning in Mathematics
A Set of Resource Materials for Teachers

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Preface

The theme of this *Annual* is ACTIVE LEARNING IN MATHEMATICS - its major purpose is to serve as a set of resource materials for teachers interested in pursuing mathematics labs, workshops, and similar activities with their own classes. Each of the contributors was invited to describe sample mathematics activities in sufficient detail that another teacher could use them in his own classroom. The resulting papers give a brief account of the authors' conception of active learning in mathematics, followed by descriptions of activities that have been used successfully with elementary or secondary school mathematics students. The papers are arranged roughly in order of the grade level for which the activites are intended, from elementary through junior high to high school.

Trivett's article suggests a "systems approach" interpretation of active learning in mathematics with illustrative examples of how creative mathematics learning situations can be devised. *Vance* describes how to establish a mathematics laboratory program and details laboratory lessons that have been used in Grades IV to VIII. *George Cathcart* discusses the lab approach and provides some sample assignment card descriptions for elementary students. In her article, *Gloria Cathcart* describes drill activities that have been enjoyed by her Grade III and IV students.

Dawson describes a "Fallibilistic" teaching strategy and illustrates the approach with a sequence of lessons based on Madison Project techniques that have been used successfully with Grades II through IX students. *Neufeld* details a sequence of pencil and paper activities on systems of numeration designed to challenge upper elementary school and junior high school students and teachers.

Sigurdson discusses how to structure a unit along discovery lines and then recounts a two-week-long "inventing" unit on finding areas of geometric figures that he has tried with a Grade VII class. *Wasylyk* and *Kieren* provide principles for designing mathematics activities for low ability students and they detail sample "area of a circle" activities successfully used with low ability 14 to 15-year-olds. *Bale* outlines some general guidelines for teachers wishing to create their own mathematical laboratory experiences and gives some sample activities suggested by topics in *Seeing Through Mathematics*. *Fisher* describes some of the 46 student activities incorporated in his active learning unit on real numbers which has been tried with Grade VIII students and which covers all the topics found in the real numbers unit in the 1969 Alberta Grade VIII mathematics curriculum.

In an article reprinted from the November 1968 issue of the *Manitoba Journal of Education*, *Sigurdson and Johnston* discuss the meaning of discovery in mathematics and give detailed descriptions of 11 classroom activities designed to lead Grade XI students to discover properties and applications of the quadratic function through exploration, forming hypotheses, testing hypotheses, summing up and practising.

Finally, the editor has included an annotated bibliography of some presently available published resource materials for promoting active learning in mathematics.

The editor wishes to thank the contributors to the *Annual* for the excellent practical papers they have written and to thank the MCATA Executive for the free hand he has been given in putting together this issue. He would also like to thank his Ed. C.I. 470 students for their timely "eleventh hour" assistance.

Hopefully, mathematics teachers will find in these pages a wellspring of ideas for pursuing active learning in mathematics with their students.

The Editor

Active Learning in Mathematics

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Everyone in Education knows that there are ebbs and flows, waves, kicks, the latest method, gimmicks, projects and innovations. There are discovery methods, rote drill, team teaching, open areas, traditional ways, my approach and yours, and one great difficulty in penetrating this maze to select what each of us does, is that the words are discrete and life is not. I mean, the mere saying or writing of the phrase "active learning in mathematics" is easily done and when it is said, it is finished. The phrase has four words, each is easily understood, it heads a piece of writing and because of such simplicities we can easily jump to the conclusion that similar clarity will automatically follow. More, it can easily be concluded that the simple ideas simply described can be as easily made active within classroom situations. Oh, if it were so easy!

Then, too, the words 'active learning' suggest that there exists something which is 'non-active learning'. I, for one, don't know what that could be. *All* learning is surely an activity, or it involves activity. It may be of the mind or it may be otherwise, but it must necessarily include a minimum of physical activity even if it be eye movements or the finger and hand movements needed to hold a pencil. However, although the word 'active' is strictly redundant, the phrase is one which is frequently used these days so there is value in pursuing some meanings for which the words might stand.

I am trying to suggest that what is important to discuss is not 'this' as opposed to 'that' in simple terms, but rather to ask *what* activities are relevant to the learning of mathematics, what *quality* of activities do we want, activities with *what?* What activities of the traditional kind of mathematics lessons are going to be continued regardless of our opinions, our preferences and current fads or styles? What traditional activities have we the power to retain or discard, yet wish to keep? What activities, under-emphasized in previous years, should we increase in density in the future? And, lastly, are there any activities never used before which need to be introduced? I suggest that careful blendings of many possible activities will lead to the best learning and to the fastest progress.

What components exist from which the blends can come? What will give us guidance in the choice of activities? Clearly it must be from the consideration of, firstly, the nature of the human beings involved in the learning situations; secondly, the physical situation of schools, classrooms and so on; thirdly, the nature of what it is we intend to communicate - that's the mathematics; and fourthly, some of the application requirements of the students' future.

Here we choose to deal only with the first and the third, for it is those that are frequently neglected in other than obvious ways. With regard to the other two, it can be said that the physical situations are usually adequate for most activities once the determined teacher is convinced of their value, while applications tend to look after themselves when deep understanding is present.

What follows, then, is an introduction to some of the important aspects of active learning in mathematics. They apply to all students at any grade, whether they be deemed by some to be slow or fast, dull or bright. The examples quoted can be multiplied almost without limit and they have to be for adequate classroom implementation. Here, however, hints only can be given.

THE DISCOVERY ASPECT

In a sense, every student has to discover everything. The mere projection of words, explanation, facial looks or activity by a teacher, does not necessarily imply or get a reaction by a student, let alone a hoped-for reception. The text may be handled and so may physical aids, but there have to be continual acts of volition by every student to trigger some kind of reception in his eyes or mind.

There are, however, varying levels of reception needed for understanding. That a fraction is conventionally written in the form $\frac{a}{b}$ needs little discovery. It is seen and almost immediately accepted by the youngest of students in school. That it represents not only a particular element as well as the whole of a particular class of ordered pairs of whole numbers, this, on the other hand, needs more and deeper discovery with a different quality and quantity of what has to go on within the person to make it as secure as the symbol seen with the eyes.

In mathematics teaching, we have traditionally acted as though simple surface-level discovery is all that is needed. We explain, we do examples, show how and get the children to practice on paper. We try to define subtleties and when we see the pupils failing we tend to the conclusion that it is because some are not capable. We call then "inattentive", "lazy", "slow" and by using more simple words, seem to suggest that it is they who are totally at fault. Seldom do we think that maybe it is we teachers who have not yet attained the highest forms of communication and activities to induce what is needed to have all students learn the mathematics.

If, therefore, discovery is a fact in learning processes, not just the name of a method, we surely have to enquire more into the ways and means by which students are compelled to take responsibility for their own learning, for no one ever did anyone else's. Teachers must minimize correction from an apparent authoritative standpoint and only insist on being an authority where the student cannot be. That can be condensed to just two functions: orchestrating the activities of the group and telling the children what are the agreed conventions of symbols and form. In all other matters - in the widening consciousness of the fundamental facts - the pupils must be helped to discover at first hand, and although little attention has been paid to this under the weight of verbal and written tables and formulae to be learned, every man and woman knows the effect of a general understanding into which many details fit, compared with the knowledge of many details with no connecting substratum of understanding.

Teachers need to be learning facilitators, creators of learning environments, flexible ones, appropriate to the tasks, to the actual students involved and capable of minute to minute changes. As a result of his wisdom and knowledge, using the art and craft of his trade, the teacher of mathematics in particular initiates activities which are such that anyone engaged in them can hardly help but be affected by the concentration of the latent mathematical ideas and concepts built in. It is arranged deliberately that the chances are very high that no student can avoid the inundation and, what's more, enjoy it!

The environment will include the use of physical aids, the writing on paper, the usual symbol work and texts, but it will also take into consideration the inner consciousness of every person in the room. No system can be ignored. Inner systems of humans must be recognized as essential concomitants of the learning environment - thoughts, wishes, day dreams, mind wanderings, feelings. It is these that constitute the stuff of existence as much, or more than, the manipulative aids, the books, the diagrams and the mathematics which we aim for.

> It is the teacher's responsibility to watch the learning process and to introduce creative conflict at appropriate moments.

THE INDIVIDUAL ASPECT

Every student has to do his own learning, with his insights, his past, and his present as he sees it. These are the filters through which pass all that occurs in our math lessons. We had better be aware that this is so.

Each student is unique and may differ much in his reception of something public from all the other receptions of the individuals in the group. It is impossible that two people receive the same thoughts, the same understanding, no matter how the teacher tries. Whereas one child happily accepts the following sequence:

$$\begin{aligned} \log\ 20 &= 1.3010 \\ \log\ 2 &= 0.3010 \\ \log\ .2 &= \bar{1} + .3010 \\ \log\ .02 &= \bar{2} + .3010 \end{aligned}$$

another's difficulty may lie in his worry that in the third and fourth lines there are + signs but in the other lines there are not. His difficulty evaporates as soon as he sees that he *could* write the equivalents:

$$\begin{aligned} \log 20 &= 1 + .3010 \\ \log 2 &= \underline{0} + .3010 \\ \log .2 &= \bar{1} + .3010 \\ \log .02 &= \bar{2} + .3010 \end{aligned}$$

Now there is a balance of the forms which satisfy him. If the teacher does not understand a possible obstacle here and cannot help, the student may well feel he is just incapable of understanding where apparently the others do.

Another example, from elementary school:

Student A.	B.	C.	D.
2.48	2.48	2.48	2.48
-1.69	-1.69	-1.69	-1.69
1.21	.81	$1.\bar{2}\bar{1}$	.79

Because child D is "right" it does not follow that we know what he did, how he thinks. Correct answers are frequently written for the "wrong" reasons. What is needed is a suspended judgment, not a stamp of approval or disapproval. If we can find out what each child really did (and a good way of doing this is to ask them and *listen* to replies!) this is what might emerge:

A says "9 from 8, can't, so 8 from 9 is 1 ("always do what you can")
6 from 4, can't, 4 from 6 is 2 . . . etc."
B says "9 from 8, can't, so 8 from 9 is 1.
16 from 24 is 8."
C says "9 from 8 is $\bar{1}$, so 6 from 4 is $\bar{2}$ . . . etc."
D says "9 from 48 is 39; put down 9 and carry 3.
3 from 20 is 17, but 1 from 17 is 7.
Put down 7."

That's the tip of the iceberg of what goes on *within* individuals. It occurs all the time, in all lessons, with all students. As the understanding that this is so grows (together with an increase in knowledge of more and more possibilities within this vast, commonly unexplored field), every teacher can open for himself new vistas in his approach to teaching. The consequences are greater self-respect and successful learning for all students.

So then individualism also is a fact, not a method, not an opinion. We cannot choose to treat humans non-individually and succeed. If an "individual approach" appears to be only a fad in which it is assumed that it is organization which is at the bottom of individualism and if this passes next year to some new urge which has all students reacting the same way, people will still go on acting individually. Even if they are overwhelmingly herded and made to jump through the same hoops of authority, they still won't oblige except for a short while. Man's eternal quest for freedom will be maintained despite temporary setbacks.

THE GAMES ASPECT

Children and adults like playing games. Games are concentrated, challenging, enjoyable activities which can be given up at will, whereas work is more unwilling and there is more accepted pressure before it can be abandoned.

Children often like to play games with physical objects and they have show this in their preschool experience. However, games are not restricted to the use of physical objects nor competition against others. Word games, written games, games with oneself and with textbooks can all be rewarding.

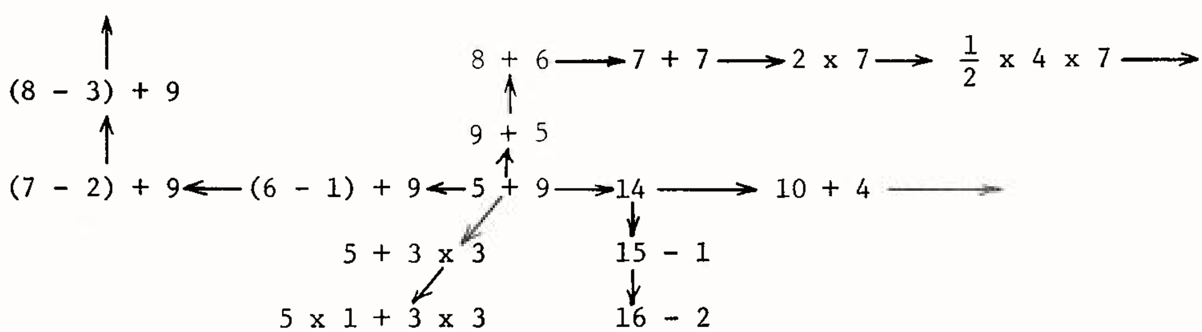
Some of the physical games are quite new in mathematics lessons. There are teachers who for years have had children apply mathematical and arithmetical principles to measuring, shopping and surveying, but today there are also many physical materials for use whose importance is not in application but in the lead they provide to the awareness of mathematical principles by the very act of using them and becoming more conscious of what one is doing. The colored rods, attribute blocks, multibase blocks, geometry models, geoboards, pebbles and counters, mirrors, cut-out figures, films, one's own body and body movements - these are some of the newer opportunities to be taken by good teachers.

As an example, consider the use of the so-called attribute blocks which are of different shapes and different colors. A red triangular block and a red square are alike because they are both red but they are different in shape. When we use blocks of other shapes, other colors, and use some with different thicknesses and even different materials, then we have pieces for many games which demand, in the very act of playing, understandings of differences, equivalences, intersections and union, elements and subsets. And although the use of such is by no means the only way, it can hardly be denied any longer that such notions are essential to a proper understanding of numbers, addition and multiplication, and more of the familiar list of what we want children to know.

Moreover, by the use of games, improved mastery of the traditional mathematics and arithmetic can be attained. Instead of laborious, non-related facts which somehow have to be memorized, the games approach involves thoroughness and intensity and subtly demands every student's on-going dedication.

Games with physical objects are only one type of game. There are many others so here we illustrate one which seems quite traditional:

CONTINUE THIS "WEB": From any name draw an arrow and write another name for the same number which comes to *your* mind. Have a reason why you went from one name to the next.



Do not ally the word "games" with something frivolous, a pastime, something we do "after work". One look around at man playing games shows clearly that the most played games are very demanding in energy, devotion, thought, and time and that they increase learning and powers. To try to meet mathematics as a series of games is not to drag it down to the wasteful, trivial, filling-time attitude. Rather, it is to elevate certain mental activities to the level of some of man's finest activities - the game of space exploration, of serving others or of life itself.

THE MATHEMATICAL ASPECT

Mathematics is concerned with the dynamic use, in the mind, of relations and relations of relations, with some applications to social and economic life.

Arithmetic is one of its branches having a main emphasis on the renaming of numbers by means of algorithms, tables, figuring, calculations and other devices. To learn that 2 + 3 = 5 *may* only be a matter of remembering that "5" is a word which is an acceptable substitute for the phrase "3 + 2", and "6" is not the accepted word. To recognize, however, that 3 + 2 = 2 + 3 implies much more. Now we get a hint of a basic principle (commutativity) and this may lead to other interests. Alternatively, it gives us a pattern to get new names for numbers we have not met before. (4576 + 687 surely must equal 687 + 4576 even if this - 4576 + 687 - has not been confronted previously).

To write or say 2 + 3 = 5 is a convention, a *convenience*. The symbols as such give no clue to what they stand for. It is just as easy to accept the writing 2 + 3 =7, but historically that is not what was developed, so we think that 2 + 3 = 5 is correct and find 2 + 3 = 7 uncomfortable. The meanings behind the symbols, however, are *not* convention. They are based on fact over which man has no control. He can only be ignorant of the fact. The fact is embedded in life and everyone of us is capable of firsthand experience and enlightenment. We therefore encourage mathematics partly to understand and use the ideas, the concepts, the relations, the facts and partly to help us maneuver the old arithmetic which is still vital and valuable.

On the whole, we do not question the kinds of activity suggested by the se of mathematics text books so long as the teacher appreciates that the one rinted form, the book order and the style of presentation is not necessarily ight for any student whatsoever! Those mass produced aids show the sophisticated utcome of years of thought and centuries of argument and symbol evolution. They ust be seen as such, used for the embodiment of desirable aims maybe, but approached along different and unsophisticated paths. They rarely exhibit the form f progress precisely needed by any child to reach many of the same conclusions.

EXAMPLE: Usually in text books the addition of fractions is dealt with before the operation of multiplication.

One student pointed out that addition is so:

$$\frac{2}{3} + \frac{5}{7} = \frac{(2 \times 7) + (3 \times 5)}{(3 \times 7)} = \frac{29}{21}$$

and multiplication is thus:

$$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

Addition, it seems, implies three multiplications and one addition. Multiplication needs only two multiplications. On what grounds, he asked, is *addition considered easier than* multiplication and taught first? Had he a point?

THE INTEGRATIVE ASPECT

If math is indeed concerned with relations and activities with relations, then it is essential that every student constantly has experience which relates things for *him*. It is *his* meaning which is vitally important, his continuous progress will be unique for *him*, and the relations which he explores will not by any means be confined to that portion of his school day called the "math lesson". Relations are abstracted by all humans in all kinds of situations. The math lesson will emphasize them, show what can be done with them, and greater power in their use will be gained. But the existence of relations must be from all subjects studied, from all life. Appreciation of integration across the curriculum is therefore important.

Integration is also needed within the math study itself so that growing experience is had in relational activities which will become second-nature and pervade all the educational scene.

Within the math study, such integrating, creative aspects can be dealt with in many ways. Here are two, briefly:

1. Reverse the usual task of getting an answer to a problem. Instead, get a problem to fit the answer: $x = 7$. Write some other equations of which $x = 7$ is the solution.

   The sum of four numbers is 2489. What are they?

2. Rather than set word problems, give an equation and from it invent an appropriate social situation.

   $3y + 4 = 19$. What situation in a grocery store might fit this?

Across the curriculum, apart from some obvious use of number work and equations, there are the rich, powerful mathematical ideas of relations, differences, attributes, sets and subsets, transformations, symbolisms, reversals and repetitions to explore and use. All of these occur and have their importance in social studies, art, science and music.

GRADE I

If a child can read and write in some form the words "step up pat" he can also read and write all six permutations of these same words.

He can be asked to do so therefore with teacher *expecting* him to do it. He can also be expected to read every one of the six sentences and know whether to him they make sense or not.

GRADE VI

In discussing "Man", students can be helped to list equivalences and differences of men and see intersections of attributes. This leads to an improved awareness of racial similarities and differences, all within the same human species.

THE HUMAN ASPECT

No mathematics lesson ever attended took place without human beings being present! Even if it is true that mathematics is devoid of feelings, opinions and other vagaries of everyday life, it is not true of mathematics learning situations, for the humans present have such attributes all the time. In the learning of the subject, therefore, there are facts of this kind to be taken into full consideration by the teacher: the need for understanding, respect for others, tolerance, frustration, tiredness, hope, lack of communication, annoyance, satisfaction. Unless teachers are aware of these as being present in every lesson, in every human, then a large part of what is really going on in the lesson will be completely missed and the teachers will be forced to resort to the traditional rationalization of "the students won't attend", or "they are slow and incapable".

It is not that we necessarily wish to change such feelings, but these are very real parts of everyone's life and we must certainly recognize their existence and the part they may be playing. Because of such awareness and allowance, we shall help the students commune with each other and with ourselves, since we shall no longer be dependent on outward signs alone which give little clues to the thinking and the perceptions going on within.

Perhaps the future progress of all learning depends solely upon the harmonizing of what has to be passed onto the new generation within the context of the "species specific mode" of every learner human in this case. It is fruitless to try to teach a dog to do what a dog is incapable of doing. The same holds for humans. We assume that telling is sufficient, whereas if we pause and think for a moment, each of us has plenty of evidence that telling is seldom sufficient for communication for anything other than trivialities. *It is not a human mode that learners automatically learn by being told.* Humans do not pay much attention to what others are saying. The mode surely suggests that we have to encourage activities in which it does not matter if the learners attend or not - except to themselves, which they cannot escape.

If a 5th grader wants to add fractions thus:

$$\frac{3}{7} + \frac{4}{9} = \frac{7}{16}$$

let him and encourage him to do more like it.

$$\frac{3}{4} + \frac{4}{12} = \frac{7}{16} \qquad \frac{2}{5} + \frac{5}{13} = \frac{7}{16}$$

$$\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{4}{13} = \frac{7}{16} \qquad \frac{7}{16} = \frac{3}{1} + \frac{2}{1} + \frac{1}{1} + \frac{1}{13} \quad (!)$$

When he has developed other names for $\frac{7}{16}$ according to the same rules, he most probably will begin to sense a contradiction with other thoughts he has about fractions. It is that contradiction within himself that will provide motivation and at least in this case the realization that that is not the way fractions can be combined.

SUMMARY

What we suggest, therefore, is a "systems approach" - one which takes into consideration *all* the systems which are active within the learning situation, outward and inward. Much cannot be avoided, some of it can be and some can be initiated by the teacher. Every learning situation is a very complex set of circumstances, whereas the history of education has tended to act more and more as though the learning is simple even if the organization is complex. A simple solution or a straight-forward outward appearance can invariably be assumed to be misleading and wrong. There has to be discovery and non-discovery; rote learning will still play a role; there will be partial understandings and superficial insights along a spectrum. There will not be artificial discussion about the group as opposed to the individual, for every group is composed of individuals, and every individual is a member of many groups. The variation of possible individual responses to every tiny situation, the variation of movements, thoughts, things to do and the way for people to do them - all contribute to a fantastic number of alternatives that in practice, whether we like it or not, make every lesson different, no matter how hard we try to repeat or homogenize.

Most of what goes on, most of the active learning, will be unseen by the members of the group. Each individual will have a concentrated view only of his own incoming sensations and inner thoughts, and on top of all this there will be the mathematics, the activity with relations, numbers, space, sets, operations, functions and the arithmetic, as a by-product of some of this. The complexity is inevitably so great that it makes one wonder how we ever got the belief that we can determine in any but rough form how some piece of learning *should* take place. Why do teachers plan lessons rather than prepare themselves to meet their students as they *will* be by the time of the next lesson?

A balance is needed, not one which attempts to remain still, but one which is nevertheless stable. Sometimes the kinds of activity will seem to be of one kind but because of the teacher's innate stability and his understanding, purposes and leadership, there will be some vaguely perceived fulcrum about which the kinds of activity will oscillate. The mathematics classroom will sometimes appear to be a hive of physical activity. Children will be measuring, weighing, or using blocks and ropes to discuss sets. Others may be constructing tetrahedra and investigating their rotations. Computers may be in use or some pupils may be outside the school with angle meters doing elementary surveying. Sets of cubes, colored or plain, may be seen, although the students using them may really be discovering the relations inherent in successive cube numbers. All the fun of the fair may seem the order of that day!

On other days, or with some children perhaps during the same time period, the activity will be with paper and pencil and the appearance will be traditional.

The facts concerning the consciousness of the different systems extant must lead to variety for each learner, for the whole class, and for the teacher, too. Underlying all, however, will be the commonality of an increasing mathematical consciousness for each child within a human community. This will result from the effects of an environment purposely engineered by the knowledgeable teacher in which the learning takes place intensely, individually, integratively and joyfully. That is the "active learning in mathematics" to aim for!

Establishing a Mathematics Laboratory Program

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WHAT IS LABORATORY LEARNING IN MATHEMATICS?

The term *laboratory method* is commonly used today to refer to an approach to teaching and learning mathematics which provides opportunities for the students to abstract mathematical ideas through their own experiences. Laboratory programs can, in general, be characterized by the following features:

1. The student is actively engaged in the doing of mathematics; he is not a passive observer in the learning process.

2. Concrete materials, structured games, or environmental tasks are used to give meaning to mathematical concepts.

3. The students work much of the time individually or in small groups from written instructions.

WHY USE A LABORATORY APPROACH?

Some of the functions and aims of the laboratory method are:

1. to permit students to learn abstract concepts through concrete experiences and thus increase their understanding of these ideas;

2. to enable students to personally experience the joy of discovering principles and relationships;

3. to arouse interest and motivate learning;

4. to cultivate favorable attitudes toward mathematics;

5. to encourage and develop creative problem solving ability;

6. to allow for individual differences in the manner and speed at which students learn;

7. to enable students to see the origin of mathematical ideas and to participate in "mathematics in the making";

8. to enrich and vary instruction.

HOW CAN LABORATORY ACTIVITIES BE USED IN CONJUNCTION WITH A REGULAR MATHEMATICS PROGRAM?

Laboratory lessons can be inserted into a school's ongoing mathematics program in several ways. One period a week might be reserved for a laboratory program consisting of a series of special activities and games selected to supplement and to enrich the regular course. Such a program would function as as adjunct to the prescribed course of studies and would be aimed at fostering the development of independent and creative thinking and at improving attitudes, as well as providing a change from the textbook-chalkboard approach to learning mathematics.

Teachers can also use laboratory lessons to introduce new concepts and to review previously taught material. Students often require concrete experiences before a concept can be meaningfully developed in class at an abstract or symbolic level. Other ideas, first introduced in class, can be further explored and investigated in a laboratory setting. Topics such as probability, numeration measurement, and geometry are particularly well suited for laboratory study.

HOW ARE LABORATORY LESSONS DEVELOPED?

Each of the activities later described in this paper is designed to lead to the development of a concept on the discovery of a relationship in mathematics.

CONCRETE MATERIALS: Each lesson is based on some type of physical material or manipulative device. Concrete materials serve several important functions:

1. They create interest and provide motivation.

2. They provide a real-world setting for the problem to be investigated or concept to be developed.

3. They provide a physical means by which the learner can begin to solve the problem or explore the concept.

4. They provide the learner with a way of verifying his hypotheses and checking his calculations independently of a teacher or textbook.

SEQUENCE OF LEARNING ACTIVITIES: In general, each lesson is structured to permit the students to arrive at the desired conclusion or generalization inductively. The students first familiarize themselves with the structure or operation of the apparatus or physical objects. They then use the material to gather information relating to the problem, recording this data in a table or on a graph, if possible. On the basis of these observations, hypotheses are formulated and tested. Finally, generalizations are stated. The newly discovered rule might then be used to answer additional questions or to do practice exercises.

WRITTEN INSTRUCTIONS: Although the lab lessons described later in this article can be presented to whole classes of students by their teachers (who would demonstrate with the concrete materials), the activities are most successful when performed by small groups of students working from written instructions. In preparing the "assignment cards", the teacher must consider the amount and kind of direction his students might require. It has been the experience of the writer that students new to a laboratory approach in mathematics usually want and need fairly specific instructions which indicate how they are to proceed and which provide feedback relating to their progress. However, after a short time, most students develop more confidence and can become independent to the point where they are willing to determine their own procedure for solving problems posed in rather broad terms. The lessons outlined in this paper are fairly detailed, but the instructions can be made more "open" or "closed" depending on the needs of the particular students who will be using them.

HOW IS A MATHEMATICS LABORATORY ORGANIZED?

GROUPING STUDENTS: Small group learning can offer several advantages over individual work or whole class instruction in mathematics. Most students (of both elementary and junior high school age) like school work which involves some physical activity and opportunity to talk to their classmates. In small groups, each pupil has an opportunity to work directly with the concrete materials and to take part in discussions relating to the activity and possible methods of solving the problem. Many laboratory activities must be performed by the students working as a team in which one person manipulates and another records. With activities of this type, if a group consists of more than two students, there is the danger that one of them will be "left out" of the activity and discussion.

Groups can be formed on the basis of friendship or ability. The writer suggests that to organize the pupils to perform the activities described in this paper, the teacher designate pairs of students who are compatible (but not "too" friendly) and who work at about the same speed and level of understanding. Placing a high ability student with one of low ability is not necessarily advantageous for either, as the two are likely to have different learning styles. The more able learner may wish to move from the concrete to the abstract more quickly than the student of lesser mathematical ability. Of course, the composition and size of the groups can be changed periodically.

ROOM: Almost any classroom can be readily transformed into a mathematics laboratory. "Stations" or "centers" for the activities are created by moving two or three desks (flattop are best) together in various locations around the room. Concrete materials and instructions for the activities can be kept on shelves or in cupboards where they will be easily accessible to the students on "lab days".

ROLE OF THE TEACHER: Although the teacher has a somewhat different role in a laboratory setting than in a more traditional classroom situation, he is still the key to a successful program. The teacher must first select or devise worthwhile activities which will be appropriate for his class. Students can often assist by bringing or making materials and by contributing task cards that they have developed.

During a lab period, the teacher acts as a guide or counselor, giving assistance when requested or needed, but encouraging the students to develop concepts through their own efforts. He must guard against giving students information that they are not ready to assimilate. On the other hand, a teacher would not allow students to become discouraged or waste time from lack of direction. Evaluation and recording of pupil progress is another important, although difficult, responsibility of the teacher.

HOW MIGHT STUDENTS BENEFIT FROM LABORATORY EXPERIENCES?

A recent study conducted by the writer to investigate the effects of implementing a mathematics laboratory at the Grade VII and VIII levels confirmed that students enjoy learning mathematics in this way, and indicated that such a program might be one way of achieving certain important objectives of teaching mathematics. Students in the laboratory group worked in pairs from written instructions, rotating through a set of 10 activity lessons based on concrete materials on a once-a-week basis.

Several measures were used to compare the lab students with students who had taken the experimental lessons in a teacher-directed class setting, and also with students who had not been exposed to the experimental materials but who had continued to study the regular program (*Seeing Through Mathematics*) full time. It was found that the use of 25 percent of class time in mathematics for informal exploration of new mathematical ideas did not adversely affect achievement in the regular program over a three-month period. In addition, tests of learning, retention, transfer, and divergent thinking indicated that students in both experimental groups had benefited mathematically from participation in the program. Although test scores were slightly higher for the students who had studied the lessons under a teacher in a class situation, the reaction of the lab group to their insturctional setting was more favorable. The lab students also rated higher than students in the other two groups (a) in feeling that learning mathematics is fun and enjoyable, and (b) in the view that mathematics is a subject which can be investigated and developed experimentally by using real objects rather than restricted to a textbook subject in which symbols are manipulated.

The most popular feature of the laboratory method as identified by the students was the opportunity which it provided for working independently of the teacher. The following are comments made by students in the laboratory group:

- I liked the privilege of working at your own speed and without a teacher always telling you what to do. It was fun and helped me a great deal. I think it is better than teaching from the book and is a lot more interesting.

- I liked where you could find out and prove things yourself so you would know for a fact that something is true.

SOME LABORATORY ACTIVITIES

The laboratory lessons now described have been used with students ranging from Grade IV to VIII and also by prospective elementary teachers. (It appeared that the latter category of students usually benefited from the concrete experiences leading to the mathematical ideas as much as the younger pupils.) It should be pointed out that these lessons deal mainly with abstract mathematical concepts. Other kinds of worthwhile laboratory tasks might be more real-life oriented or science-related.

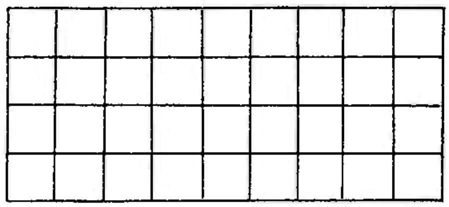
AREA AND PERIMETER

Purpose: To investigate the perimeters of the family of rectangles having a constant area.

Materials: 36 unit (one-inch) squares, graph paper.

Procedure:

1. Form a rectangle from the squares by arranging them in 4 rows and 9 columns as shown.



   Find the *area* - the number of squares needed to cover the rectangular region.
   Find the *perimeter* - the number of units around the outside edge of the figure.

2. Rearrange the squares to form another rectangle. (Make a different number of rows and columns.) Find its area and perimeter.

3. Form as many rectangles as you can using all 36 squares. Find the area and perimeter of each and record your answers in a table.

4. (a) Did you make a rectangle with 5 rows? Discuss.
   (b) Could you now find the area and perimeter of the rectangle with 3 rows and 12 columns without first making it and counting? How?
   (c) What did you notice about the area of each of the rectangles? Why is this so?
   (d) Which of the rectangles had the greatest perimeter?
   Which of the rectangles had the smallest perimeter?

5. Complete the table below. Use the information to make the graph showing the relationship of the perimeter to the number of rows.

No. of rows	No. of columns	Perimeter
1		
2		
3		
4		
6		
9		
12		
18		
36		

6. Use your graph to answer the following questions:

(a) Give the dimensions of the rectangle having the smallest and greatest perimeters.
(b) Find (approximately) the perimeter of the rectangle with a base of 15 units.
(c) Find the dimensions of the rectangle having a perimeter of 50 units.

POLYHEDRA[1]

Purpose: To discover Euler's rule for simple polyhedra through examination of models of the regular polyhedra, prisms, and pyramids.

Materials: Models of various prisms and pyramids and the regular polyhedra (or materials and instructions for constructing them).

Procedure:

1. Find the model of the tetrahedron. It has 4 triangular *faces*. The line along which two faces meet is called an *edge*. The point or corner where three faces meet is called a *vertex*. How many edges and vertices (plural of vertex) does the tetrahedron have?

2. Count the number of faces, vertices, and edges of each of the solid shapes and record your findings in the table below.

Name of Shape	Number of Faces	Number of Vertices	Number of Edges
tetrahedron	4	4	6
cube			
square pyramid			
triangular prism			
octahedron			
pentagonal pyramid			
pentagonal prism			

3. Try to find a rule relating the number of faces, edges, and vertices, which holds for each of the above solid shapes. (Hint: Add the number of faces and vertices.)

4. See if your rule holds for the other solid shapes.

5. Examine the pyramids and prisms again. What is the difference between a (pentagonal) prism and a (pentagonal) pyramid?

6. In what way are the tetrahedron, cube, octahedron, dodecahedron and icosahedron special? Why are they called the *regular* polyhedra?

[1]Suggested by F. Bassetti *et al.*, *Solid Shapes Lab.* (New York: Science Materials Center, 1961).

7. Examine all the shapes again and fill in the table below on the faces of the solid shapes.

NAME	NUMBER OF FACES			
	TRIANGLES	SQUARES OR RECTANGLES	PENTAGONS	HEXAGONS
PYRAMIDS tetrahedron				
square pyramid				
pentagonal pyramid				
hexagonal pyramid				
PRISMS triangular prism				
cube				
pentagonal prism				
hexagonal prism				
REGULAR POLYHEDRA tetrahedron				
cube				
octahedron				
dodecahedron				
icosahedron				

INTERSECTING SETS[2]

Purpose: To find the number of members in the union of two intersecting sets. $[N(A \cup B) = N(A) + N(B) - N(A \cap B)]$

Materials: Set of 24 attribute blocks (3 colors, 4 shapes, 2 sizes); 2 hula-hoops.

Procedure:

1. Arrange the blocks in piles according to (a) shape, (b) color, (c) size.
   How many shapes (colors, sizes) are there?
   How many blocks are there of each shape (color, size)?

2. (a) Place all the red blocks inside one of the hoops. Count them.
   (b) Place all the square blocks in the other hoop. Count them.
   (c) Now arrange the two hoops so that all the red blocks are in one hoop and all the square blocks are in the other hoop (at the same time).
   How many red square blocks are there?
   How many blocks are there altogether in the two hoops (Number which are either red or square or both red and square)?

3. Answer the following questions. Try to find a way of answering the questions without placing the blocks in the hoops and counting.

   Find the number of blocks which are:

   (a) circular
       large
       both circular and large
       either circular or large or both

   (b) small
       blue
       both small and blue
       either small or blue or both

   (This activity can be extended to consider 3 intersecting sets by using the complete set of 48 logic blocks, with the additional attribute of thickness.)

[2]Suggested by Z. P. Dienes and E. W. Golding, *Learning Logic, Logical Games.* (New York: Herder and Herder, 1966).

MATHEMATICAL BALANCE[3]

Purpose: To investigate certain properties of the whole numbers and to solve simple linear equations using a balance beam.

Materials: Balance beam.

Procedure:

A.

1. On the right hand side of the balance beam, place one ring on hook 10. In how many ways can you place hooks on the left hand side to balance this? Write a mathematical sentence for several ways.

2. Can you balance 10 using only a single hook other than hook 1 or hook 10? In how many ways can you do this?

3. Find which weights (from 3 to 25) can be balanced by placing rings on a single hook other than hook 1 or the hook corresponding to the weight. Record your findings in a table similar to the one below.

Weight	Yes or No	How
3	No	
4		
. . .		
10	Yes	2 rings on hook 5, 5 rings on hook 2
. . .		
25		

B.

1. 3 rings on hook 5 balance 5 rings on hook ________.
   Do some other examples like this.

2. 3 rings on hook 4 and 3 rings on hook 6 balance 3 rings on hook ________.
   Do some other examples like this.

C. Use the balance beam to find the solution set of each of the following:

1. $3 + n = 8$
2. $n + 2 < 6$
3. $3n = 6$
4. $2n > 8$
5. $2n + 7 = 15$
6. $4n + 6 = 3n + 8$

[3]Suggested by Z. P. Dienes, *Task and manual for use with the algebraic experience materials*. (Harlow: The Educational Supply Association Limited, Schools Material Division).

HOW MANY SUBSETS?

Purpose: To determine the relationship between the number of members in a set and the total number of subsets of that set.

Materials: Five Cuisenaire rods (white, red, green, yellow and black rods) in a small box.

Procedure:

Consider the 5 rods in the box as a set {W, R, G, Y, B}. A *subset* of the set would be determined by drawing from the box and number of the rods.

For example, if you drew the red and yellow rods, you would have the subset {R, Y}. How many other subsets have two members? List them. How many subsets have 1 member?
One possibility is to draw *all* five rods. This is a subset of a set. Another possibility is to draw *none* of the rods. This is also a subset, called the empty set - ϕ.

So subsets may have 0, 1, 2, 3, 4 or 5 members. How many subsets are there althogether? List all the subsets you can and then guess how many you think there are altogether.

To help answer this rather difficult question, first consider a series of related, but easier questions.
Find all the subsets of the sets {W}, {W, R}, {W, R, G} and {W, R, G, Y}.

Record your findings in the table below:

Set	No. of Members	Subsets: 0 Members	1 Member	2 Members	3 Members	4 Members	Total No. of Subsets
{W}	1	ϕ	{W}	-	-	-	
{W, R}					-	-	
{W, R, G}						-	
{W, R, G, Y}							

Can you see a pattern? How many subsets will {W, R, G, Y, B} have?

Can you determine how many of these subsets have 0, 1, 2, 3, 4 and 5 members?

How many subsets would a set with 6 members have? With 10 members?

THE CIRCUMFERENCE OF A CIRCLE[4]

Purpose: To discover the relationship between the diameter and the circumference of a circle.

Materials: Wooden discs, tin cans, and so on, of various diameters; string; ruler.

Procedure:

1. Take a disc from the box. Using only the string and ruler, what distances can you measure on the disc?

2. Measure the *diameter* (distance across the disc) and the *circumference* (distance around the outside of the disc) and record your findings in the table below. Do this for each circular object.

Diameter	Circumference

3. Examine the table. Can you write a rule which approximately relates the diameter and the circumference of a circular object?

4. Use your rule to predict the circumference of a circular object of diameter (a) 1 inch (b) 10 inches (c) 15 inches.

5. Predict the diameter of a circular object which has a circumference of (a) 19 inches (b) 44 inches (c) 1 inch.

6. The radius of a circular object is the distance from the center to the outside edge. How is the radius related to (a) the diameter (b) the circumference.

7. If a circle has a radius of 5 inches, find (a) its diameter, (b) its circumference.

8. Find the radius of a circle having a circumference of (a) 25 inches (b) 63 inches.

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[4]Adapted from The Madison Project's Independent Exploration Material, *"Discs,"* distributed by Math Media Division, H & M Associates, Danbury, Connecticut.

PROBABILITY[5]

Purpose: To introduce basic concepts in probability through experiments with coins, dice and sampling urns.

Experiment 1: Flipping a coin.

Materials: A penny.

Procedure:

Flip a coin ten times and record the number of times it lands heads and tails. Repeat this procedure five times and record your findings in the table below.

Trial	1	2	3	4	5	Total
No. of heads						
No. of tails						

What would be your best guess as to the number of times a coin would land heads out of 10 flips? 50 flips? Why?
How close were your results to what you expected?

Experiment 2: Rolling a die.

Materials: A die, paper cup.

Procedure:

Roll the die 30 times and record your findings in the table below (under "First Trial").

Outcome	First Trial	Second Trial	Total
one			
two			
three			
four			
five			
six			

[5]Adapted from E. C. Berkeley, *Probability and Statistics - An Introduction Through Experiments.* (New York: Science Materials Center, 1961).

How many one's, two's, etc. would you expect to obtain in 30 rolls of the die?
Did you get exactly 5 one's, two's, etc.? Discuss.

Repeat the experiment and record your findings under "Second-Trial". Total the results for the two trials. Compare your results with what you would have expected to obtain in 60 rolls of the die.

Experiment 3: Guessing the Urn

Materials: 4 sampling urns labelled A, B, C, and D, each with 20 beads as follows:

1. 18 black, 2 white
2. 14 black, 6 white
3. 10 black, 10 white
4. 4 black, 16 white

(Boxes or cups containing marbles, which are replaced after each draw, may also be used.)

Procedure:

Find Urn A. Can you determine which of the urns (1, 2, 3 or 4) this is? Perform the following experiment to help you make your decision. Shake a bead in the bubble ten times and count the number of times a black bead appears and the number of times a white bead appears. Now which of the four urns do you think is Urn A?

Identify B, C, and D in the same way.

Urn	Black	White	Guess
A			
B			
C			
D			

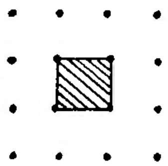
To see how well you guessed, remove the lids from the urns.

THE AREA OF A TRIANGLE[6]

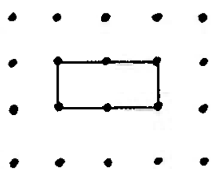
Purpose: To determine methods of finding the area of triangular regions which can be formed on a geoboard.

Materials: Geoboard and rubber bands.

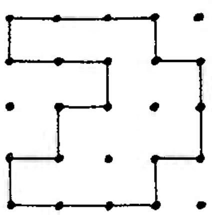
Procedure:



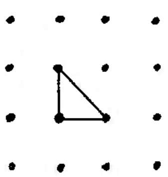
Let the shaded area represent 1 square unit on the geoboard.



How many square units are within this figure? __________

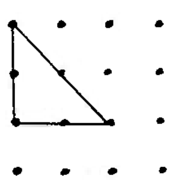


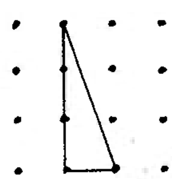
What is the area of this shape? __________ square units

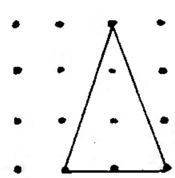


How many square units are within this triangle? __________

What is the area of the following triangles?







__________ __________ __________

Did you find an easy way to figure out the areas of triangles?

Explain how you did this.

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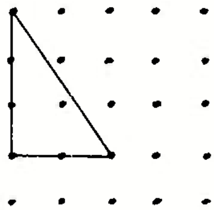
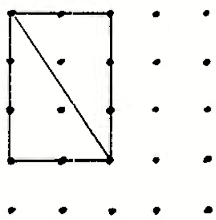
[6]Adapted from Donald Cohen, *Inquiry in Mathematics Via the Geo-Board*. Teacher's Guide. (New York: Walker, 1967).



Make this triangle on the geoboard.

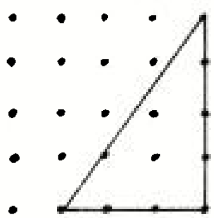


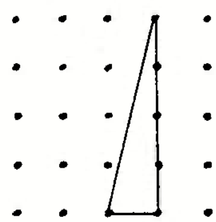
With another elastic band make a rectangle.

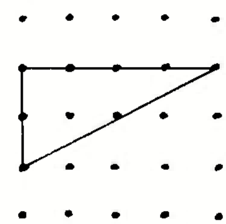
The area of the rectangle is ________ square units.

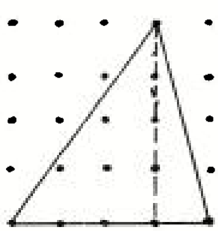
The area of the triangle is ________ square units.

Use this method to find the areas of the triangles below:

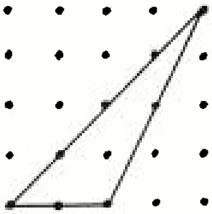




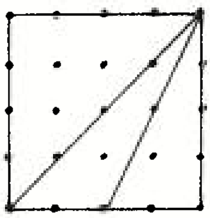




The last triangle can be thought of
as being made up of two triangles.



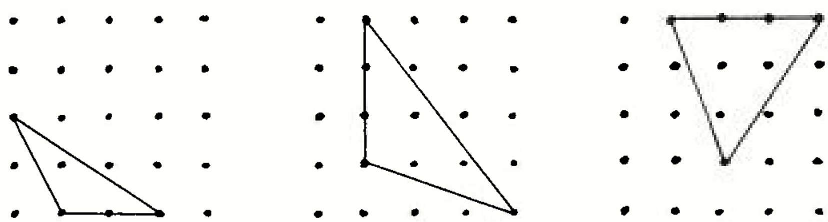
How would you find the area of this triangle?



One way would be to complete the rectangles as shown and subtract two triangular areas.

Area = 16 sq. units - 8 sq. units - 4 sq. units = 4 sq. units.

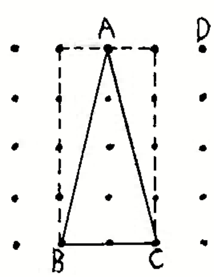
Find the area of these triangles:



Given a triangle, choose a side you can determine the length of and call this side the *base* of the triangle. The *height* of the triangle is the length of the other side of the rectangle enclosing the triangle.

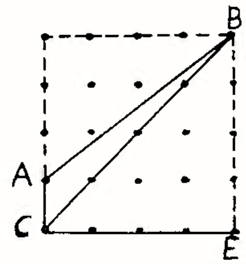
<u>Example 1</u>

In triangle ABC, if BC is the base, CD is the height = 4 units.

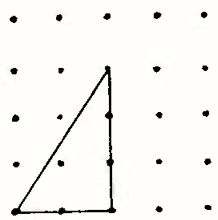


Example 2

In triangle ABC, if AC is the base, CE is the height = 4 units.



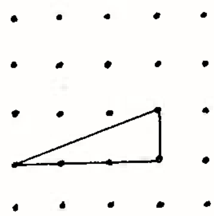
For each of the triangles below, find the lengths of the base and the heigh
Also find the area.



base ________

height ________

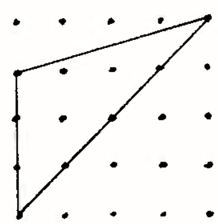
area ________



base ________

height ________

area ________



base ________

height ________

area ________

If you know the length of the base and the height of a triangle, how can you find its area?

The Laboratory Approach to the Teaching of Mathematics

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The mathematics laboratory has received much attention in recent years as an approach to the teaching of mathematics. This emphasis is obvious from even a casual reading of the professional journals in mathematics education. It is also exemplified in numerous publications by the Nuffield foundation and in the book *Freedom to Learn.*[1] What is the nature of this approach?

COMPONENTS OF A MATHEMATICS LABORATORY

Active Student Involvement

One of the distinguishing features of a mathematics laboratory is that pupils are actively involved in the learning process. This is in contrast to the conventional approach where children were expected to sit in five or six neat rows, passively absorbing (?) the material the teacher was presenting. In the laboratory setting, a small group of students may be cutting or folding paper in one corner of the room as they investigate some geometric concepts, another group may be out in the hallway involved with measurement while others may be rolling dice on the floor in an investigation of probability. In other words, the philosophy of the mathematics laboratory is "learning by doing".

Manipulative Aids

If pupils are to be actively involved in the learning process, they have to have something to be involved with. Pupils investigate ideas of numeration by manipulating counters, popsicle sticks, or commerically available apparatus such as Dienes' Multibase Arithmetic Blocks. An understanding of volume is achieved through pouring liquid or sand into containers of different sizes and from one container to another. The meaning of a fraction can be discovered by the manipulation of cardboard cut-outs. Similarly, other mathematical concepts are investigated through the use of appropriate manipulative or concrete aids.

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[1]Edith E. Biggs, and James R. MacLean, *Freedom to Learn.* (Don Mills, Ontario: Addison-Wesley (Canada), 1969).

Problem-Solving Through Open Search and/or Guided Discovery

A common connotation of a laboratory is a place for experimentation. Experimentation involves the search for a solution to a problem. In a mathematics laboratory, the pupil is confronted with a mathematical problem and he uses the manipulative aids to investigate various hypotheses and eventually arrives at a solution. The solution is the pupil's own solution - he has "discovered" a mathematical concept, relationship, principle, or process by himself or with the help of a partner.

The problem investigated in the laboratory may be a problem posed by the teacher, or pupils may work on mathematical problems they have encountered in their daily life. The latter is often a more meaningful problem for the pupil.

The foregoing discussion assumes that a problem is given and that the pupils search for a solution with a minimum of guidance from the teacher. This is good, but a more common approach is to pose a problem and then guide the pupils to a discovery through a series of intermediate steps. These steps are usually outlined on what is frequently called an "assignment card". Some examples of assignment cards will be given shortly. The latter approach is more structured than the open-ended approach discussed earlier. For this reason, it is the recommended approach for a teacher who is beginning to experiment with the laboratory approach to teaching mathematics.

VALUE OF A MATHEMATICS LABORATORY

Obviously, you can see problems in the laboratory approach to teaching mathematics. For example, there is the problem of organizing the class so that each pupil is engaged in meaningful activity. In addition, there are problems in assigning grades to pupils and an increased noise level which could be a problem in some schools.

Clearly, the above problems can be overcome with a little effort. But, you ask, are the benefits of using a laboratory approach worth the additional time and effort needed to overcome the problems encountered in setting it up? Why use the laboratory or activity approach?

Involves the Real World

First of all, the laboratory approach translates abstract mathematical concepts into concrete representations. This is important in the elementary school since, according to Piaget, most children in the elementary grades do not have the cognitive structure required to deal with formal abstract mathematical notions. Therefore, the best understanding should occur when pupils are allowed to discover the concepts themselves through the use of concrete aids.

Fosters Inquiry

In the laboratory setting, children learn to solve problems independently or in small groups. As students become more familiar with this approach, many of them will be able to pose their own problem and determine a method for solving it. They may even be able to devise their own materials if such are needed to solve the problem. When pupils are able to do this they have developed an attitude of inquiry.

Individualization

The laboratory approach allows pupils to progress at a rate suitable to their ability. A pupil may spend as long investigating a concept as he requires to understand it. On the other hand, children who understand a concept or process need not spend time doing repetitive exercises.

The laboratory approach allows for individualization not only in terms of rate of learning, but also with respect to the concepts covered. There probably are some core concepts which everyone needs to investigate, but the laboratory approach allows individual pupils the opportunity to explore concepts that are of interest to them.

HOW TO GET STARTED

Different teachers have used different techniques to start a mathematics laboratory in their classroom. Some have found that it works well to send one row of pupils each day to the back of the room where the teacher has set up a few activity stations. When this procedure is used, children always anxiously await their day to have lab.

Other teachers have started a mathematics laboratory by setting aside one mathematics period per week to work on projects and activities.

An important feature of the above two approaches is that they expose pupils (and teachers) to the activity approach slowly. It may be harmful to jump into the laboratory setting too deeply too quickly. Pupils need time to adjust to the new setting and to learn how to make use of their additional freedom and responsibility.

No matter what technique you use to get started, you probably want to structure the activities at first. Then, as pupils learn to accept more responsibility for learning, you can make the activities more open ended and even allow pupils to work on their own projects. Following are a number of assignment cards which have been successfully used in the elementary school. Their purpose in this article is to give you some ideas and to help get you started on an activity approach to teaching mathematics.

SAMPLE ASSIGNMENT CARDS

Measurement

The following assignment card could be used to introduce linear measurement at about a Grade II or III level. This assignment card is quite structured and does not give pupils much of an opportunity for independent exploration. The following assignment card may accomplish three objectives:

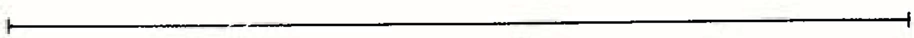
1. Improve the ability of the pupils to estimate lengths.
2. Enable pupils to discover the concept that linear measurement is the iteration of a unit, whatever the unit may be.
3. Lead pupils to the realization that standard units are needed. This objective may be accomplished by having pupils compare their answers, especially in part 3.

For parts 1 and 2 pupils will need a rod and a piece of string. A six centimeter rod (Cuisenaire) and about a 10 centimeter string work well, but the actual length of the unit is not important.

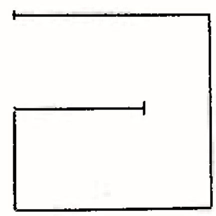
LET'S MEASURE

Use the string and stick on the table to measure.

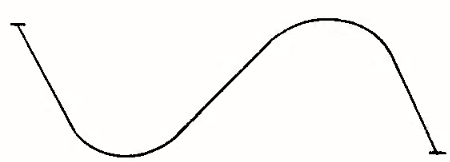
1. Look at each of these lines. First write down how many sticks or strings long you think the line is. Then measure it and write down your answer.



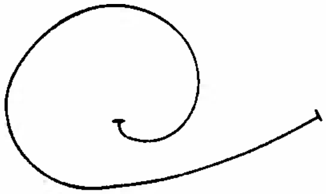
My guess: ____________________ sticks
Measured length: ______________ sticks



My guess: ____________________ sticks
Measured length: ______________ sticks



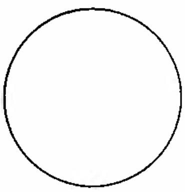
My guess: ____________________ strings
Measured length: ______________ strings



My guess: ______________________ strings
Measured length: ________________ strings

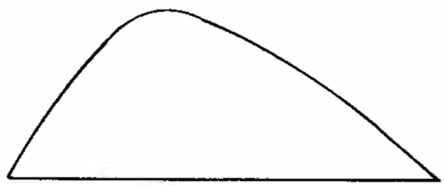
2. Decide whether the stick or the string would be the best to measure the following lines. Write down your guess and answer as before.

My guess: __________ ___________
Measured length: ________ _________

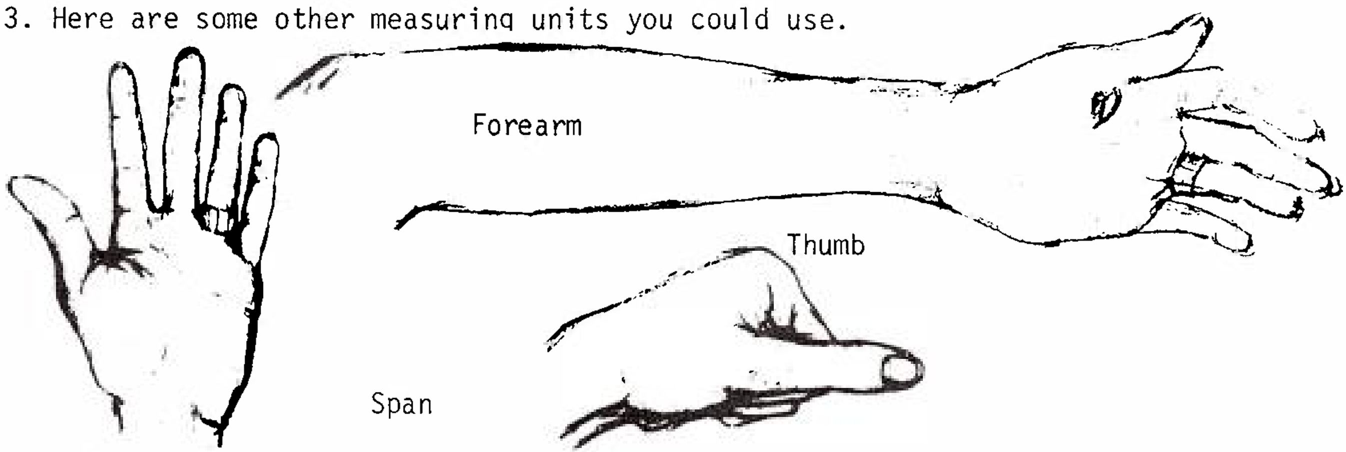


My guess: _______ _________
Measured length: _______ ________

My guess: _______ _________
Measured length: ________ _________



3. Here are some other measuring units you could use.



Use these units to measure the following. First guess and then measure.

		GUESS	MEASURE
(a)	Length of the blackboard in forearms	______	______
(b)	Width of this paper in thumbs	______	______
(c)	Width of the door in spans	______	______
(d)	Width of a book in thumbs	______	______
(e)	Length of your pencil in thumbs	______	______
(f)	Height of the door in forearms	______	______
(g)	Width of a window in spans	______	______

Geometry

The next sample assignment card is suitable for the upper elementary grades. Notice that it is much more open-ended than the first one was.

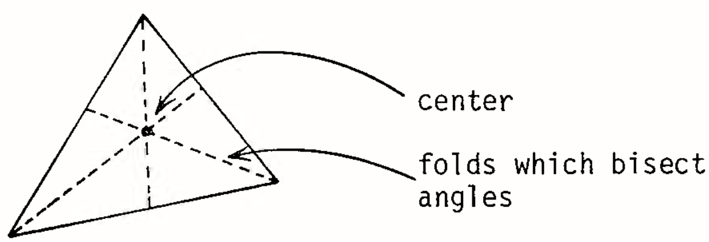
The only material the students will need for this activity is a supply of paper and a pair of scissors. This activity was taken from *MATHEX: An introduction*[2].

Finding a Center of a Triangle

1. Take a sheet of white paper. Fold it to make a triangle. Cut out the triangle.

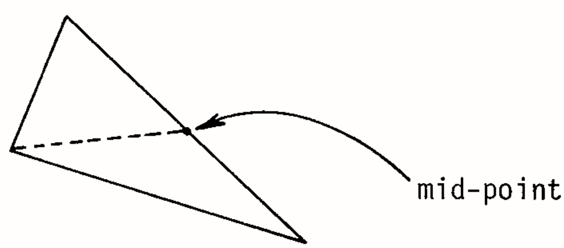
2. Use any method you like to find a center of the triangle by folding.

HINT:



3. Fold and cut out another triangle. Use a different procedure for finding a center.

HINT:



[2]L. D. Nelson and W. W. Sawyer (Editors), *MATHEX: An Introduction.* (Montreal: Encyclopedia Britannica Publications, 1970).

4. Fold and cut out a triangle whose center is the same regardless of the procedure you use to find the center.

5. Fold and cut out a third triangle. Try to find a third procedure for finding a center.

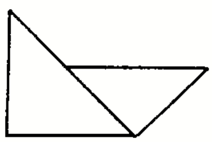
Tangram Puzzle

Puzzle

The following assignment card provides some valuable experiences with plane geometric shapes as well as being a mathematical puzzle.

1. The figure on the next page is an old mathematical puzzle called the Chinese Tangram. Trace the figure onto a piece of manila tag or cardboard and then cut out the seven pieces.

2. Experiment with the seven pieces to see how many *different* shapes you can make. For example, using two different triangles you can make the following shape:



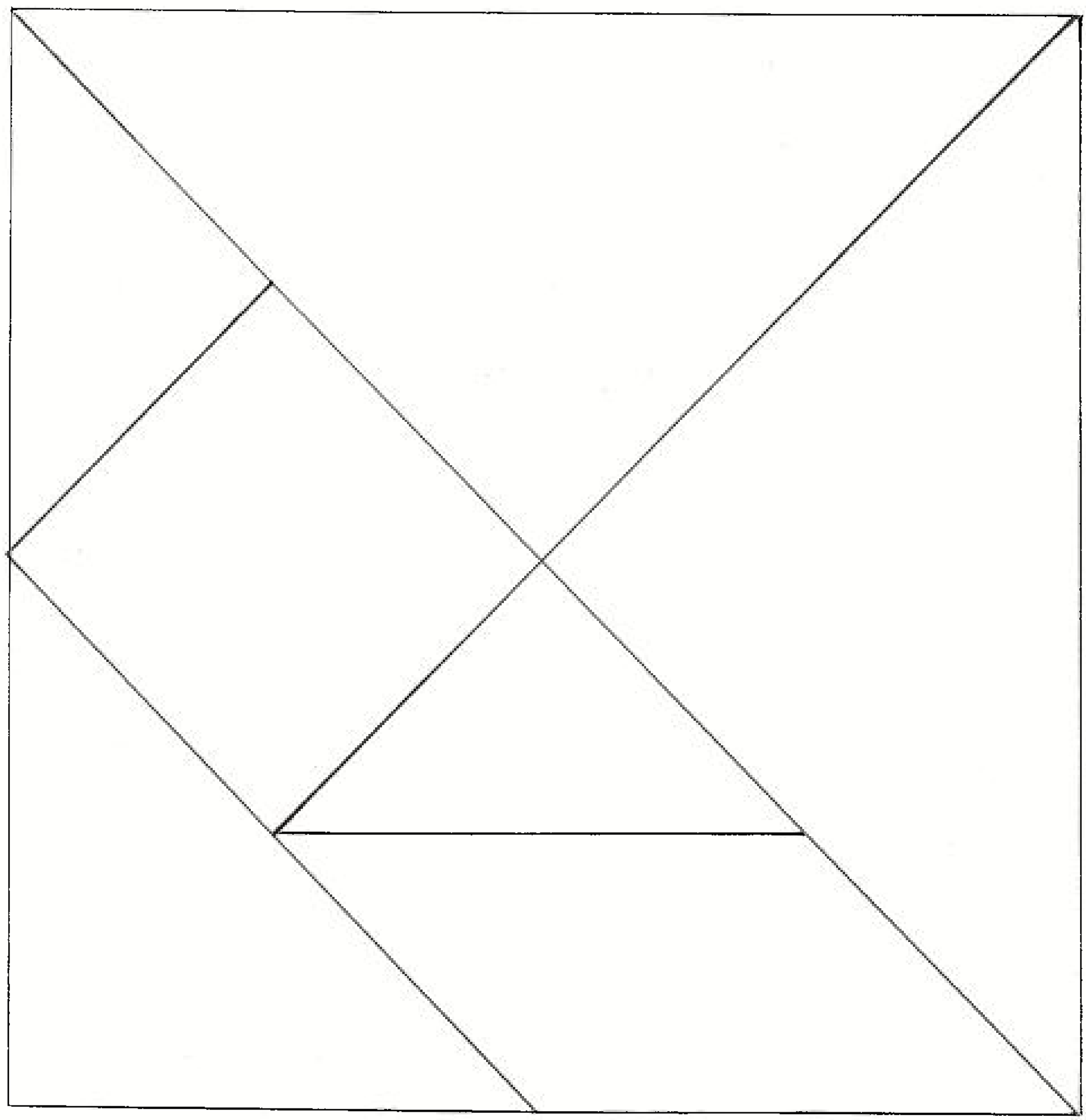
   Compare your shapes with those of your friend. There are over 175 different shapes which can be made from the seven pieces.

3. Try to put the pieces together so that they form the original square that you started with.

4. How many four-sided figures can you make using two pieces? Keep a record of your shapes by tracing them onto a sheet of paper.

5. Do the same for three pieces, four pieces, five pieces, six pieces, seven pieces.

6. Choose a partner and play the following game. Each player chooses a number of pieces from his Tangram puzzle and makes a geometric shape. Each player traces his shape onto another piece of paper and then the players exchange papers. The first person to make the other player's design from the Tangram pieces is the winner.



Chinese Tangram

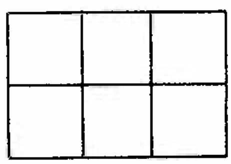
Number Theory

The last sample assignment card involves the concept of prime and composite numbers. It could very easily be expanded to include the concepts of square numbers, perfect numbers and factors of a number. The only material needed is a set of 24 squares. Squares 1" x 1" are probably convenient.

Prime and Composite Numbers

There are 24 cardboard squares in the pile in front of you. You will be asked to take some of these squares and construct rectangles or squares with them.

1. Take six squares. Make all the rectangles or squares you can. (Your figure should not have any holes in it.) For each rectangle write down the length and width as shown below.



2 x 3 (The 2 means 2 rows and the 3 means 3 columns)

Need we make a 3 x 2 rectangle? Why not?

What other rectangles did you make with 6 squares?

2. Use the squares to help you complete the following table. For eacn set of squares make and then record the size of all the rectangles or squares you can make. Two and six have been done for you. Remember 2 x 3 is the same as 3 x 2.

Number of Squares Used	Size of Rectangle or Square	Number of Different Rectangles or Squares
2	1 x 2	1
3		
4		
5		
6	1 x 6; 2 x 3	2
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		

3. For which numbers could you make only one rectangle?

   These numbers are called *prime* numbers. A prime number can be divided only by one and itself without a remainder.

4. *Composite* numbers are numbers which can be represented by more than one rectangle. They have more than 2 divisors. List all the composite numbers from 2 to 24.

5. Is 27 a prime or composite number? 29? Why?

CONCLUSION

Assignment cards such as those illustrated above can be developed for any part of the curriculum. The prescribed curriculum need not be ignored in a laboratory approach. The textbook and curriculum guide are valuable sources of ideas for activities.

It doesn't take long for children to become enthused about mathematics when they can learn by actively participating in a laboratory program. Such enthusiasm may or may not result in more significant learning but it will certainly improve children's attitude toward mathematics. Enthusiasm and active learning will result in a higher noise level. However the noise results from meaningful activity and therefore is not opposed to learning.

The laboratory approach to teaching mathematics has much to offer. Why not give it a try?

Arithmetic Accuracy Activities

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Edmonton Public School Board

Skill in handling numbers is an important part of the arithmetic program in the elementary school, and it seems that drill is still necessary, but drill need not be boring and can be incorporated into a laboratory or activity program. This point is made by Edith Biggs,[1] who has been very influential in the development of the activity approach to the teaching of mathematics.

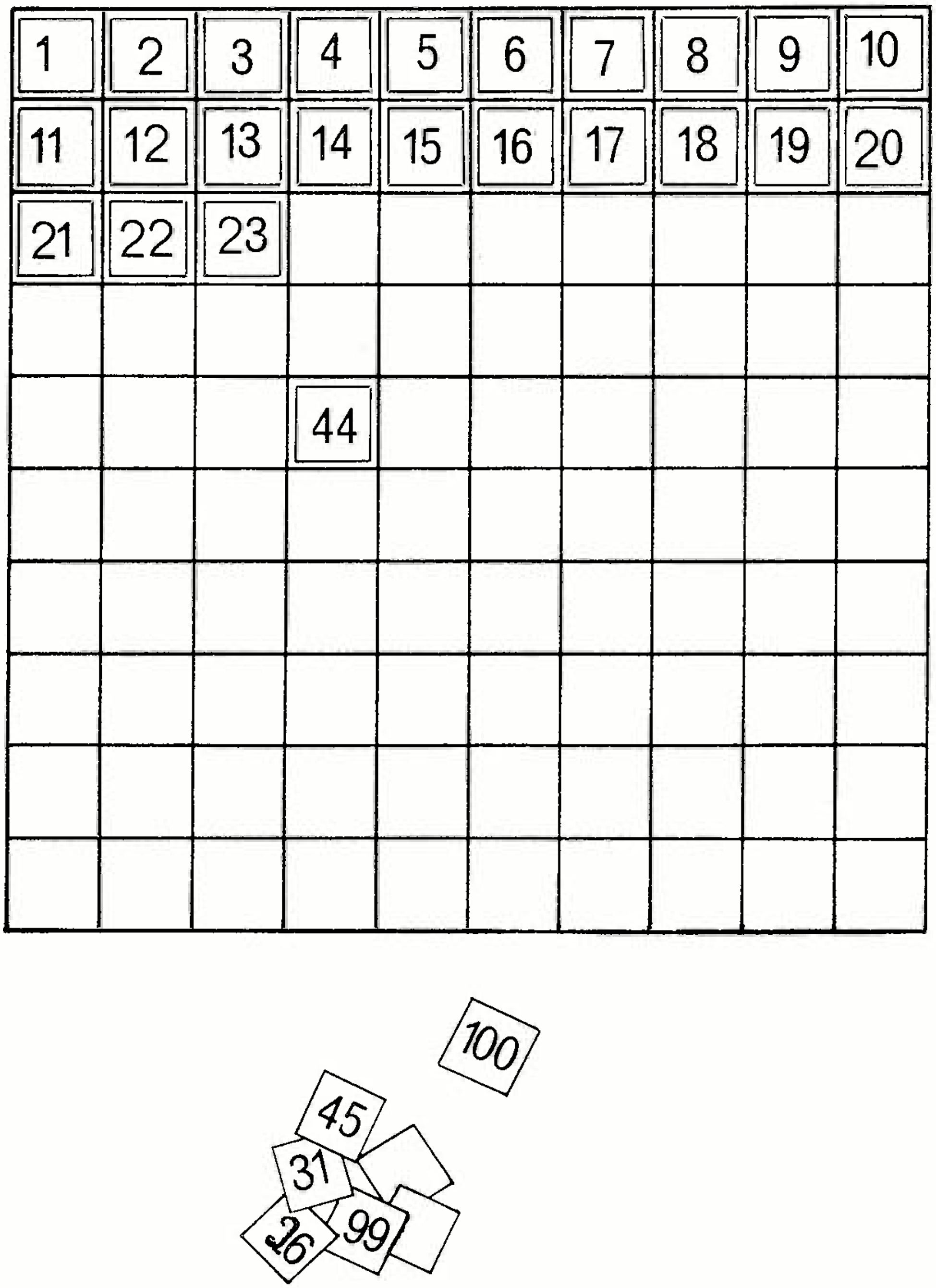
This article contains a few suggestions for drill activities which have been tried in the classroom and were enjoyed by the pupils. Included also are some suggestions for handy games materials and some recommended commercial materials relating to number facts. Grade III and IV pupils have been kept in mind in developing the following activities although the application may be broader.

SKILL-DRILL GAMES

High Times (2 - 4 players)

Materials - deck of playing cards (face cards removed), or make a set of 40 cards using the numerals from "1" through "10"
- score paper and pencil for each player
- 100 board with removable numerals, or make a 10 x 10 board (grid) and the numerals "1" through "100" on smaller cards to fit within the squares.

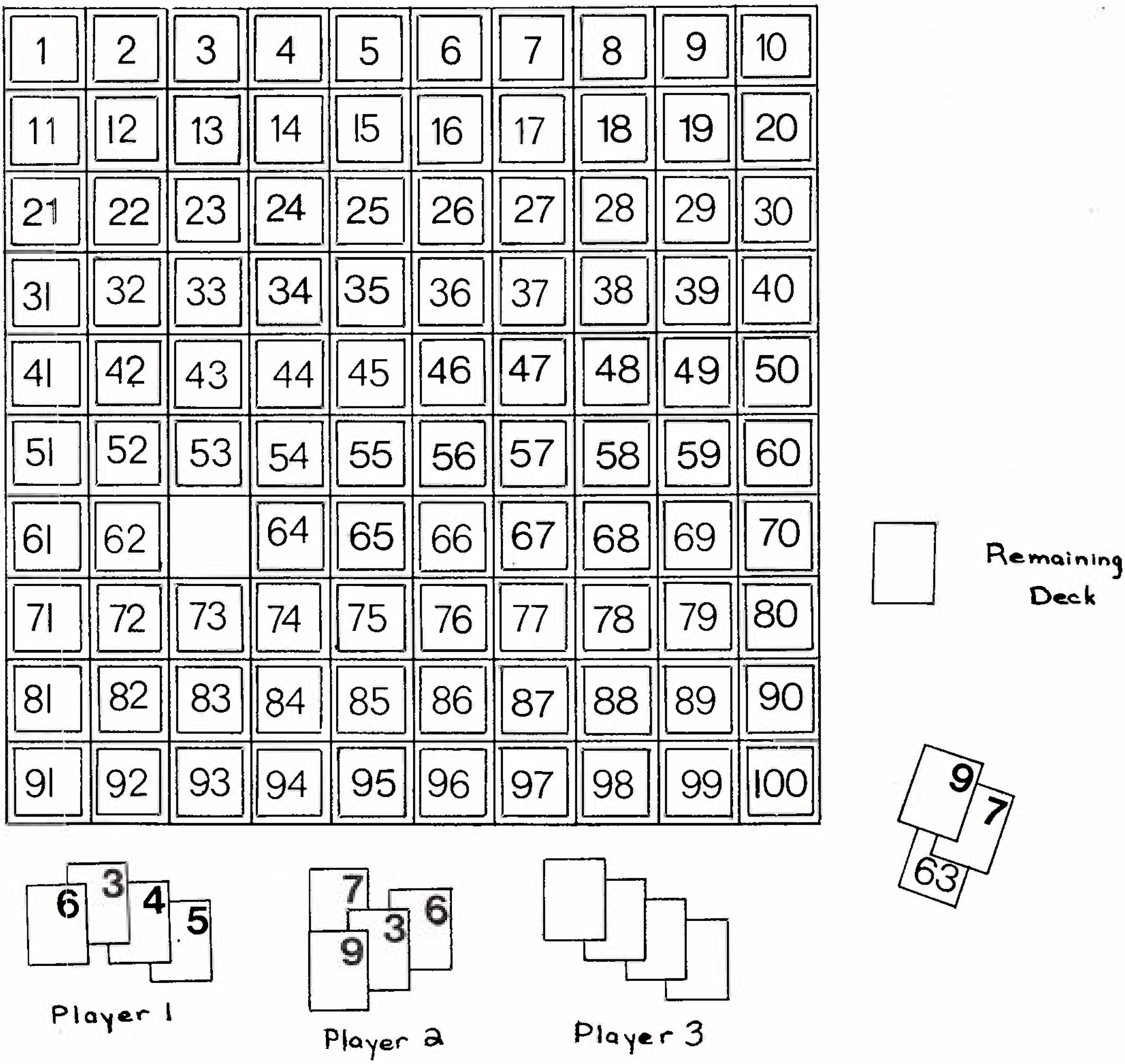
[1]Edith E. Biggs. "The Teaching of Mathematics at Primary Level". (Lead paper presented to the Commonwealth Conference on Mathematics in Schools, Trinidad, September, 1968).



Procedure: Position the numeral cards from "1" through "100" as begun in the above illustration. (Vertical arrangement is equally acceptable.)

Shuffle the playing cards and deal four cards to each player. Place the remaining cards face down in a central location.

The first player selects two cards from his hand and multiplies the numbers shown to find the correct product. He then removes from the hundred board the numeral card which shows his product, and places the three cards face up in front of him. He then draws two more cards from the top of the remaining deck.



Player One records 63 on his score sheet for his first turn.

It is now Player Two's turn. He cannot play his 9 and 7 as the product card 63 is not on the hundred board. He should play his 9 and 6 to get 54 which is the highest score possible with his hand. He would then record his score, select the next two cards and play continues.

After each player has had three turns, this round is complete and the scores are totaled to see which player has won the round. (Note: At this point you may wish to check the accuracy of your pupils by glancing at their three sets of cards and checking their addition computation.) The winner of each round scores one point (or all players can score the number of points corresponding to the number of players whose score was lower than their own). The produc cards are then replaced on the 100 board, the deck shuffled and another round started.

A game is composed of five rounds, or a time limit may be imposed.

This activity encourages pupils to learn the higher number facts. If desired more of the higher number cards could be made and·the lower number cards omitted.

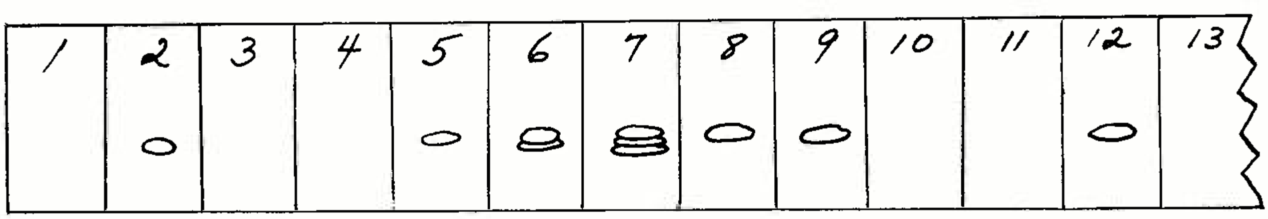
Dice Facts

Materials: two dice
number line
discs or some type of stacking markers (100).

Procedure: Two pupils can work together at one number line (or they can work alone) and compare their results after using 100 discs with the results of another group on another number line. If only one number line is being used, pupils can record their results, repeat, and compare.

If pupils work in pairs, one can handle the dice, the other the discs, and both can check for the answer.

Now let's begin. Shake the dice. Add the numbers which are face up. Place a disc on the number line in the space of the resulting sum. Repeat 100 times.



When comparing and analyzing the results, Grade III and IV pupils should be able to draw some conclusions and give some reasons for the frequency with which certain sums appear.

Variations:

(a) Multiply and mark the resulting product. What is the largest number covered? The smallest? How many numbers are covered? Not covered? Which numbers are covered? Not covered? Why? Which numbers are covered most often? Why?

(b) What happens when you use dice whose faces are numbered with numbers other than 1 through 6?

(c) Try using three dice and adding.

Number Boards (2 players per board)

Materials: number boards made on manila tag.
number squares in two colors for covering (from construction paper).

Procedure: Each player takes all the number squares of one color. The squares are placed, number down, in a pile in front of the player. Players take turns picking up one number square, looking at the number and finding a position on the number board to correctly place the number. If no place can be found, that number square is placed face up in front of the player to count against him and the other player takes his turn.

72	6x7	7x4	5x9	3x9	49	4x9	72
7x5	8x3	7x9	6x4	3x7	8x6	4x5	7x6
9x5	8x7	36	8x8	4x4	3x8	9x6	35
4x8	6x9	5x5	4x7	5x4	6x5	7x3	4x6
6x6	9x3	6x8	81	8x4	7x8	9x7	8x5

unplayable square

unplayable squares

patterned number squares

bordered number squares

PLAYER 1

PLAYER 2

Play continues until all the number squares have been turned up and positioned on the board or in front of the players. The player with the fewest unplayable cards wins.

To be sure that all the squares have been correctly placed on the board, an answer card to match the board can be made, or the teacher or a knowledgeable child can check the board. Any incorrectly played squares count two points against the player.

You may design boards with fewer cells or with the same facts repeated to provide practice with the facts you are presently introducing.

An egg timer could be used to insure that the game moves along quickly.

Here are three examples of boards you could design. Also illustrated are answer cards for each board.

8+5	7+7	6+9	5+6	8+6	9+6	8+7	9+4
7+4	5+9	6+5	7+5	4+9	4+7	6+6	7+6
7+9	6+8	9+8	8+4	8+9	3+8	9+5	8+8
9+9	9+3	5+8	7+8	9+7	4+8	6+7	8+3

8x9	6x7	7x4	5x9	3x9	7x7	4x9	9x8
7x5	8x3	7x9	6x4	3x7	8x6	4x5	7x6
9x5	8x7	9x4	8x8	4x4	3x8	9x6	5x7
4x8	6x9	5x5	4x7	5x4	6x5	7x3	4x6
6x6	9x3	6x8	9x9	8x4	7x8	9x7	8x5

40÷8	45÷9	56÷7	64÷8	36÷6	24÷8	72÷9	35÷7
42÷7	48÷6	30÷5	36÷4	36÷9	56÷8	28÷4	63÷7
40÷5	32÷4	81÷9	27÷3	20÷4	72÷8	21÷7	25÷5
24÷6	35÷5	24÷3	45÷5	24÷4	21÷3	54÷6	42÷6
54÷9	48÷8	28÷7	32÷8	63÷9	49÷7	30÷6	27÷9

Answer Cards

Since a piece may fit in more than one cell, the board will not likely come out with this pattern arrangement, but should have these answers. The numbers in the "extra" row will be those left over.

13	14	15	11	14	15	15	13	
11	14	11	12	13	11	12	13	
16	14	17	12	17	11	14	16	
18	12	13	15	16	12	13	11	
14	18	15	17	16	17	13	18	EXTRA

72	42	28	45	27	49	36	72	
35	24	63	24	21	48	20	42	
45	56	36	64	16	24	54	35	
32	54	25	28	20	30	21	24	
36	27	48	81	32	56	63	40	
49	35	24	72	81	56	64	36	EXTRA

5	5	8	8	6	3	8	5	
6	8	6	9	4	7	7	9	
8	8	9	9	5	9	3	5	
4	7	8	9	6	7	9	7	
6	6	4	4	7	7	5	3	
7	9	9	3	7	8	7	6	EXTRA

Snakes and Ladders - Division (2 - 4 players)

Materials: set of division (or subtraction) flash cards.
colored markers (one for each player).
Snakes and Ladders board.

Note: Have your pupils design their own board. The measuring, calculating, and logical thinking that goes into making a board is a mathematics project in itself.

Procedure: A pupil draws a flash card from a facedown pile, reads the question and supplies the answer. If he is correct he moves the number of spaces equal to his answer. If his answer is incorrect or if he is not able to answer he is told the correct answer but he does not move. The next player then takes a turn.

You will likely want a judge to verify answers and operate a timer.

HANDY GAMES MATERIALS

Here is a list of some materials which should be found in every arithmetic classroom and can be used in a variety of ways.

Playing Cards

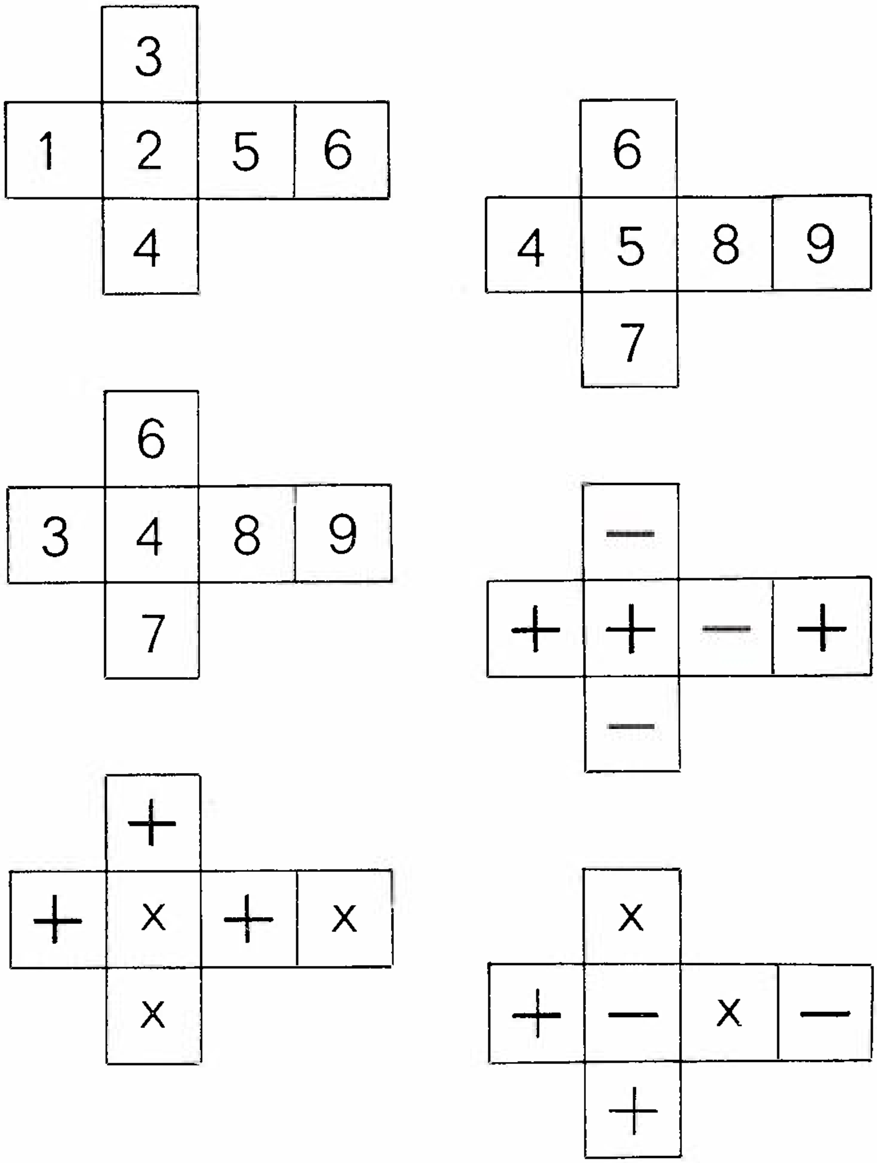
A deck of playing cards with the face cards removed can be used for many mathematical games such as Krypto, Rule Rummy, High Times, etc.

Arithmetic Flash Cards

These serve as a source of questions, especially for games involving division and subtraction where dice do not provide the desired facts.

Dice

You will want several types of dice in addition to the usual one to six dot type. These can be made of one-inch or 3/4 inch wooden cubes, marked with permanent ink marking pen. Sugar cubes will serve as a temporary dice. Indicated on the flats below are some of the markings you may find useful.



Squared Paper and Markers

One-inch and 1/2 inch squared paper is very useful for pupils to use in designing their own drill-type games. Pupils will also want a supply of colored discs (bingo markers) or some type of markers, as well as dice.

Challenge your pupils to design mathematical games. They may surprise you.

RECOMMENDED COMMERCIAL GAMES

Winning Touch by Ideal

This game provides practice with multiplication facts.

Orbiting the Earth - Multiplication by Scott Foresman (Gage).

Orbiting the Earth - Division by Scott Foresman (Gage).

Quinto by 3M Company.

A very enjoyable family game which works on multiples of five (or six, seven, eight or nine if desired) and requires addition and subtraction skill. It is similar to the familiar word game, Scrabble.

Tuf by Encyclopaedia Britannica Publications.

This game involves work with the four fundamental operations and the building of mathematical sentences.

Many other games and activities are suggested in *MATHEX*, available from Encyclopaedia Britannica Publications Limited, Montreal.

Guessing and Testing - A Sequence of Fallibilistic Lessons

A.J. Dawson
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This paper grew out of a talk entitled "Guessing and Testing" presented by Dr. Dawson at the Winnipeg Meeting of the NCTM in October, 1970.

It seems to be a characteristic of the current activity in mathematics education to focus on the active learning approach to the teaching of mathematics. Several subgroupings of methods relevant to an active learning orientation can be identified. For example, the *games approach* is designed to foster enjoyable concentrated activities by children in the classroom frequently motivated by the use of physical materials. The *discovery approach* is based on the concept that students must accept new experiences rather profoundly in order to be able to build on these discoveries in a meaningful fashion. Moreover, one must not forget the individual in the many activities being designed for the mathematics classroom. Consequently, the *individual approach* recognizes the fundamental hypothesis that every student has to do his own learning in his own unique way. As a result, it must also be recognized that individualization is a reality and not a method open to choice. Our only choice is as to how we identify this "species specific mode".

There is a danger in each of these approaches that the mathematics being taught may be lost in the methodology. As a consequence, we can see the development of the *mathematics approach*, an approach which sees mathematics as the dynamic use in the mind of relationships, and of relationships of relationships. In this view, mathematics is not an end-product to be presented to students as a finished science in a colorfully bound and illustrated book, but rather mathematics is a process, a process in which students can be involved and in which they can produce products. The *integrative approach*, on the other hand, focuses on the need for relevancy of the subject matter to every child, student and teacher involved in the study of mathematics. This approach involves an integration of mathematics with other school subjects, integration of operations, transformations, attitudes of discovery, self-respect and responsibility. One may also identify a rising concern with children as people in the classroom.

*Dedicated to E. T. Nepstad whose compassion for, interest in, and involvement with students has been a constant source of inspiration for many years.

This may be termed the *human approach*, for according to this view, no matter what goes on in classrooms, it goes on with human beings. This implies that tolerance, compassion, frustration, tiredness, hope, lack of communication and a lack of attention are attributes of every lesson - mathematics or otherwise - for these are attributes of human beings. Therefore, teachers and students need to develop attitudes of understanding, acceptance, and respect for others; they need to allow each other to utilize their own freedom; and they need to develop skills of learning how to learn. Finally, the *systems approach* realizes that human beings, mathematical concepts, educational institutions and classrooms are all very complex systems. Each student and teacher has to work within each of these systems. An alteration in any one of these systems produces changes or responses in each of the other systems. Therefore, all systems must be recognized and appreciated so that educators may advance in the direction of maximizing the efficiency of these systems in producing environments where children may learn and grow in an atmosphere which is fun, exciting, challenging and rewarding.[1]

The thrust of the above description of approaches to the teaching of mathematics is that there is no *one* best way to foster the learning of mathematics - there is not *the* approach. None of these approaches is a panacea. Consequently, arguments as to which approach is best are sterile. Instead, it is the writer's contention that the focus should be on selecting the methods and materials and on devising the classroom climate which seems most appropriate or potentially most fruitful for aiding students to learn mathematics. It should be noted at this point that the approaches described above are by no means mutually exclusive nor exhaustive.

Within any one or all of these methods, it is possible to develop various teaching strategies, strategies which are not necessarily approach oriented. The writer has recently developed strategies of teaching which seem appropriate to an active learning approach. However, these are not appropriate to just the games approach or the discovery approach or the mathematics approach. Indeed, it can be argued that the strategies to be discussed here are appropriate to many approaches.

The strategies of teaching developed by the writer spring from a philosophical foundation. A basic philosophical position was chosen and the logical implications this position has for classroom activities in mathematics were explored. The philosophical position adopted is called CRITICAL FALLIBILISM. It was developed by Karl R. Popper during the last four decades.[2] Moreover, the application of Critical Fallibilism as a foundation for describing the nature of mathematical inquiry has been explored by Lakatos.[3] The writer has extended

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[1]The reader is directed to John Trivett's article in this publication for a further expansion and description of these approaches or aspects of an active learning approach.

[2]The development of Critical Fallibilism is documented and presented in two books by Karl R. Popper, namely, *Conjectures and Refutations*, Basic Books Inc., 1962, and *The Logic of Scientific Discovery*, Basic Books Inc., 1958.

[3]Imre Lakatos, "Proofs and Refutations", *The British Journal for the Philosophy of Science*, Vol. 14, 1963. Pp. 1 - 25, 120 - 139, 221 - 245, and 296 - 342.

Lakatos' work (and to some degree Polya's also) to the point that strategies of teaching mathematics have been developed.[4]

The purposes of the paper are to briefly explain the underlying assumptions of Critical Fallibilism, to then describe the model of instruction derived from this position, and finally to provide illustrations of some of the teaching strategies inherent to this model of instruction.

The Philosophical Position

It is the contention of those who espouse Critical Fallibilism that knowledge grows by means of conjectures and refutations. In endeavoring to solve some mathematical problem, the mathematician would conjecture a possible solution to the problem. However, simply conjecturing possible solutions is not good enough, the mathematician would then attempt to either prove or refute his conjecture.

If he is successful in refuting his conjecture, then he must seek a new conjecture guided by his new knowledge of a conjecture which was not satisfactory. On the other hand, if he proves his conjecture, it still behooves him to analyze his proof in order to identify hidden lemmas and weaknesses in his proof. In applying this philosophical position to teaching strategies, it would seem that children should be allowed the freedom to conjecture or guess hypotheses as possible solutions for their mathematical problems, to severely test these conjectures, and to prove these conjectures.

From a Fallibilistic viewpoint, then, one is able to identify three phases of mathematical inquiry: origination, testing and proving. A consideration of the possible permutations of these three phases leads to the development of a model of instruction.

A Fallibilistic Model for Instruction

If the origination phase or conjecturing phase is denoted by "O", the testing phase by "T", and the proving phase by "P", then there are six possible permutations of these phases. They are the following:

1. O - T - P
2. O - P - T
3. P - O - T
4. P - T - O
5. T - O - P
6. T - P - O

If one studies these six cycles, it becomes evident that the last three cycles are not individually possible, although they may occur as parts of longer patterms of instructional sequences. For example, if cycle one above was followed by cycle two so that the flow of cycles was the following:

(O - T - P) - (O - P - T),

[4]A. J. Sandy Dawson, "The Implications of the Work of Popper, Polya, and Lakatos for a Model of Mathematics Instruction", Unpublished Doctoral Dissertation, The University of Alberta, Edmonton, Fall 1969.

then a rearrangement of the parentheses would yield cycle six as shown below:

$$O - (T - P - O) - P - T.$$

However, even in this case, the initial testing phase is preceded by an origination phase. The point is that cycles four to six could not be the initial cycle in a longer sequence of cycles because it is necessary to have originated a conjecture before one can test it or prove it. Consequently, this leaves one with the first three cycles as being the only viable cycles. Even then, the question may be asked as to why cycle three is a viable strategy. This question will be answered below.

Cycles one and two represent two of the three strategies of teaching derivable from a Fallibilistic orientation. For ease of reference, cycle one is denoted by TP and called the testing-proving strategy. Cycle two is denoted by PT and called the proving-testing strategy. Taken together, these two cycles are called the *naive* instructional pattern because the first phase of both strategies is that of origination by a conjecturing process. Cycle three is called the *deductive* instructional pattern and is denoted by DED.

The distinguishing characteristic of this latter pattern is that origination is a deductive process or a proving process in the following sense: a conjecture could be originated in a deductive or proving fashion if one begins with a set of axioms, definitions and undefined terms and proceeds to derive consequences from these axioms. These consequences could be thought of as conjectures which were being proved, in the sense of logically derived, as they were being originated. Therefore, cycle three is a viable strategy of teaching.

From a Fallibilistic standpoint, then, origination is of two types, naive and deductive. Naive origination is characterized by a guessing procedure, a procedure which seeks to propose plausible solutions to problems. Deductive origination proves a conjecture as it is being originated.

The distinctions among these three strategies of teaching stems from the order in which particular phases are utilized. The distinctions do not arise from basic differences among the phases themselves. One exception does exist however, for as noted previously there is a difference between naive and deductive origination. Figure 1 depicts a Fallibilistic model for instruction in skeleton form, identifying the two patterns of instruction and the two strategies within the native instructional pattern.

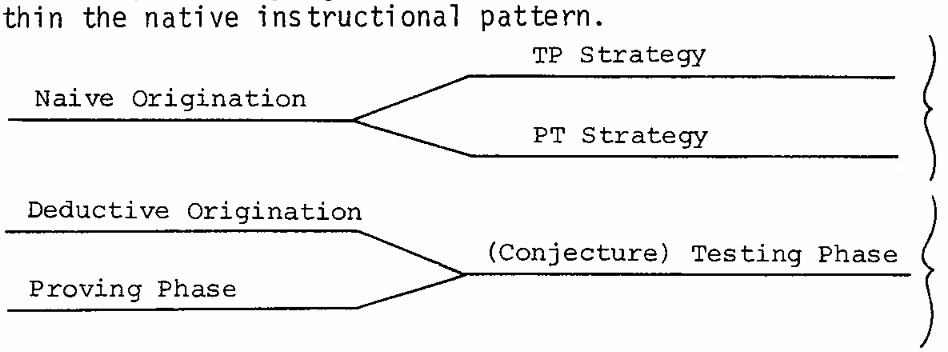


Figure 1
A Fallibilistic Model for Instruction

The next section of the paper is a record of a sequence of lessons in which the writer had students focus on the naive instructional pattern.[5]

A Sequence of Fallibilistic Lessons

A number of matters should be noted at the outset of this section of the paper. First, the word *lesson* is not meant to mean a specified block of time. Indeed, with some groups of students the sequence of lessons to be described below actually took only 45 minutes. With other groups of children, the sequence was carried out over a period of a week or more. The amount of time taken depended entirely on the group of students with whom the writer was working Generally speaking, however, the older the group of students who were involved, the shorter the period of time which was required. Nevertheless, it must be recognized that all of the mathematical topics considered in this sequence of lessons were new to the students and hence they were treated only at an introductory level.

Second, every attempt was made in all lessons to remove the teacher from the center of attention in the classroom in order that the students could interact with their peer group and the mathematics under discussion. As a result, the teacher's role becomes that of the creator of the learning environment rather than that of an authority figure in the classroom. Finally, the names of the students in the following record are not the names of students who have actually partaken of this sequence of lessons.

Lesson One: The main objective of this activity was to acquaint students with a guessing and testing strategy as a means of acquiring knowledge. As a corollary of this objective, it was desired that students would come to treat *wrong* guesses on a cognitive level rather than an affective level, the point being that even so-called wrong guesses contribute to one's knowledge. Since this lesson serves as an introduction to the guessing and testing strategy, it was desirable that the lesson be fun, exciting, challenging and rewarding. Let us join the lesson as the teacher sets the problem:

TEACHER: I would like you to try to determine the rule I am using to generate the following sequence of numbers: 4, 16, 37 . . . . Can anyone guess what the next number in the sequence would be and hence how the sequence is obtained?

DOUG: I think the next number is 1369.

SCOTT: That can't work, because 16 squared isn't 37. (Scott realizes that Doug looked at the first two numbers in the sequence, guessed the squaring hypothesis, and then squared 37 to obtain 1369.) Is the next number 49? (Scott seems to have focused on the difference between 16 and 4, namely 12, and the difference between 37 and 16, namely 21. He is guessing that perhaps these differences alternate. Incidentally, as each number is suggested, the teacher records them on the board so that there is a growing list of numbers which do not work - refuted conjectures.)

[5] For illustrations and explanations of the deductive instructional pattern, the reader is directed to the dissertation noted in the previous footnote, especially chapter five.

TEACHER: No, the next number is not 49. I'll tell you the next number in the sequence. It is 58. (The sequence is now 4, 16, 37, 58 . .)

SUSIE: Well, the differences now are 12, 21, and 21. Is the next number either - let's see - 70 or 79. (Susie is guessing naively that there may be a pattern of differences which is either 12, 21, 21, 12 or 21, 21, 21.)

TEACHER: No, neither of those is the next number. The next number in the sequence is 89. (General puzzlement usually follows. The sequence is now 4, 16, 37, 58, and 89. The teacher by giving additional numbers is attempting to provide a wider basis on which the students can test their guesses. As a result of the addition of this last number, Susie's conjecture seems to be refuted. In the actual classroom setting, the teacher would usually seek many more guesses before revealing new numbers in the sequence. The process has been shortened here for the purposes of writing the sequence up for this paper.)

JEFF: Does it (the pattern) have anything to do with the squaring of the numbers? (Jeff is looking for patterns not just numbers. The actual numbers serve only to *test* the pattern and the pattern is the real conjecture.)

TEACHER: Perhaps. (He's not *too* helpful.)

SUSIE: Is the next number 120 or 102? (Susie has many conjectures, but she remains focused on "differences" between numbers in the sequence. In the first case, she guesses the sequence of differences to be 12, 21, 21, 31, 31, and in the second case that it might be 12, 21, 21, 31, 13.)

TEACHER: No, neither of those is the next number.[6]

Several things should be noted in this illustration. First, the conjectures put forth by the students are naive; that is, they are guesses which are not deductively obtained, but rather which seem plausible on the basis of the sequence of numbers. Second, the students are able to test their conjectures by using their proposed "rule" to determine the next number in the sequence. They then ask the teacher for refutation or corroboration. This refutation is devoid of personal criticism of the merit of the student's guess.

The role of the teacher is to set the problem and then to inform the students if their guess as to the next number in the sequence is satisfactory or not. The teacher also acts as the recorder of these guesses so that the students may know what numbers have already been suggested. The former role of the teacher is performed without penalty or praise in order that the students may interact with the mathematical problem without attempting to conform to some preconceived behavior patterns established by the teacher.

[6] Those readers who have not already *guessed* the sequence for themselves are encouraged to send their guesses to Sandy Dawson who will refute or corroborate the reader's conjecture. His address is Professional Development Centre, Simon Fraser University, Burnaby 2, British Columbia.

Some readers may be saying something like: "Yes, that is interesting, but the mathematical problem being considered is not really of any significance." This is not the point, however. The goal of the lesson was to have the students develop attitudes which made guessing and testing an accepted classroom activity as well as to get the teacher out of the center of attention in the classroom. Consequently, the particular mathematical topic chosen is not of particular importance; the attitudes and strategies to be developed are important. This is an introductory lesson and as such it should be fun and exciting and at the same time it should present a problem which is challenging to the students.

Lesson Two: Once the teacher feels his students are comfortable with a guessing and testing orientation, it is possible to move to the next series of activities. Lesson two is the first of two lessons dealing with mathematical topics commonly found in a secondary school mathematics program. The relationship between lesson two and lesson three is considered in lesson four. The ideas for lessons two and three were obtained from the Madison Project materials, which are reported on in two films produced by the Project, "Guessing Functions" and "A Lesson with Second Graders".[7]

The situation the students are presented with in lesson two is as follows: three students (the *team* of experts) are asked to make up an open sentence of the form $mx + b = y$, that is, a linear equation. The remainder of the class is charged with the task of "guessing" what equation these students are using in creating a table of values for the equation. Members of the class suggest values for "x" and the team of experts using their equation determine the corresponding value of "y". In the Madison Project form of equation writing, the equation under discussion might be as follows: $(\square \times 9) - 6 = \triangle$.

A table of values is created from the suggested numbers and the team's response to these suggestions. It is usually the case that the first such rule or equation would probably be set by the teacher. However, once the students understand the situation, the teacher steps out of the situation and simply acts as the recorder of values for the table. Let us join such a class which is just beginning to discuss the open sentence given above.

TEACHER: Does the team have a rule in mind?

TEAM: Yes we do. Would someone please suggest a number?

ANITA: Try 6 please.

TEAM: 6 would give 48.

$\square$	$\triangle$
6	48

BETH: Would you apply your rule to 1 please?

TEAM: 1 yields 3 when we apply our rule.

$\square$	$\triangle$
6	48
1	3

[7]These two films are available on a rental basis from the Madison Project, 918 Irving Avenue, Syracuse, New York. The reader is also directed to two books written by Robert B. Davis, Director of the Madison Project. The two books are *Discovery in Mathematics*, Addison-Wesley Company, Don Mills, Ontario, 1964, and *Explorations in Mathematics*, Addison-Wesley Company, Don Mills, Ontario, 1966.

DAPHNE: Try 0 please.

TEAM: (After some discussion over this request, the team responds.) Yes, 0 gives us -6.

$\square$	$\triangle$
6	48
1	3
0	-6

JOHN: I think I know the equation. Is the equation you are using this one: $(\square \times 10) - 12 = \triangle$?

TEACHER: Can you test your guess, John? (The teacher has re-entered the situation now, not by saying "Yes, John is right", or "No, John is wrong", but by encouraging John to test his guess. This requires John to first determine how he might test his guess and then actually test it.)

JOHN: Well, if I put 6 in the box, I get 48. Now if I put 1 in the box I get - let's see - -12. Oh! I guess that doesn't work.

NEIL: Would you please try 2?

TEAM: 2 gives 12.

$\square$	$\triangle$
6	48
1	3
0	-6
2	12

(The rule is then applied to the numbers 3, 4 and 5 obtaining respectively the responses from the team of 21, 30, and 39 so that the table of values now appears as follows:

$\square$	$\triangle$
6	48
1	3
0	-6
2	12
3	21
4	30
5	39

Another student guesses the rule to be $(\square \times 7) + 2 = \triangle$. This is quickly refuted. Finally, we witness the following dialog.)

JOHN: Look here now. The triangle numbers are increasing by 9 each time the box numbers increase by 1. When the box number is 0, the triangle number is -6. Let's see then, is the equation $(\square \times 9) - 6 = \triangle$? (This is quickly checked for each of the values already in the table and found to be satisfactory.)

JOHN: Alright then. If this is the rule then if I tell the team 8, they should tell me 66. Is that right team?

TEAM: Yes, we would tell you 66 and you do have the rule we were using.

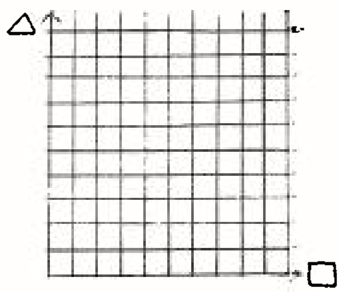
Several ingredients of this lesson must be noted. First, the teacher in responding to John's first suggestion as to what the equation was, asked John to test his guess. Obviously, the guess was made on the basis of looking at only the first pair of values in the table. What is of importance is that the teacher does not discourage John from guessing. Moreover, the guessing is on a naive level in that a solution is proposed and then tested to determine its validity. The student makes a guess, tests it, and revises it if it is refuted. In such a classroom climate, the situation is one which encourages guesses, one in which no stigma is attached to guesses which do not happen to work. There is nothing wrong with being wrong. Indeed, as Davis points out, everything one knows is to some extent wrong.[8]

Second, the means of finally arriving at the conjecture which proves to be the equation the team of experts is using begins to take on the characteristics of deductive origination in that John identifies certain patterns and bases his guess on these patterns. One can see here the beginnings of a deductive strategy of conjecture origination.

Third, we see again that the role of the teacher in this lesson is minimal once the lesson is underway. He acts as a recorder for the table of values, as one who sometimes makes a suggestion or a comment, but who does not interfere in a major way with the interaction among the students and between the students and the mathematics.

Lesson Three: A different guessing and testing situation is created in this lesson. On this occasion the mathematical topic is that of naming points in the usual manner in the first quadrant of a Cartesian coordinate system. The goal is to have the students identify for themselves the method or pattern which is utilized to plot points given a pair of numbers. The teacher begins the lesson by setting some limitations to the problem.

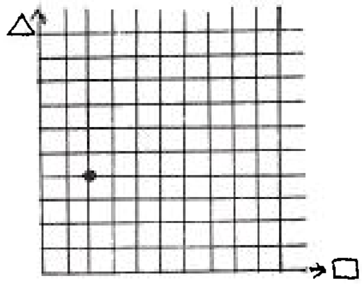
TEACHER: I have drawn a grid or set of crossed lines on the board. I would like you to try to determine how I am placing points on this grid. To do that, I would like you to tell me two positive whole numbers The first number you tell me I shall call the box number. The second number you tell me I shall call the triangle number. Would you please use numbers less than 10 for the time being. Alright, who will tell me two numbers? (The teacher would start with a grid much like the one below.)



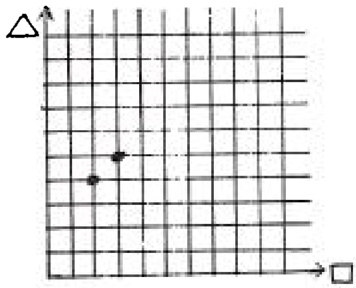
[8]Robert B. Davis, "The Madison Project's Approach to a Theory of Instruction," *Journal of Research in Science Teaching*, Vol. 2, p. 155.

COLLEEN: 2 and 4.

TEACHER: The box number is 2 and the triangle number is 4. (The teacher marks the appropriate point on the grid.)

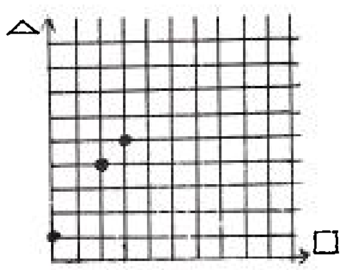


DONNA: 3 and 5. (The teacher plots the point 3, 5)

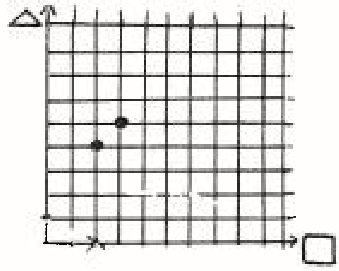


ALICE: 0 and 1.

TEACHER: That's a bit tricky, but let's see where that one goes. (The teacher plots this point after some consideration.)



TEACHER: Who thinks they know how I am placing the points on the grid? (Several students raise their hands.) Fine, will those students please come up to the board and be the team of experts. (They do so.) The team is now going to place points on the grid. The rest of you have to now tell them what two numbers would name that point. (The team place a point on the grid.)



HEATHER: That's 2 and 0.

TEAM: Yes, we agree.

TEACHER: Heather can join the team.

And so it goes. Without the teacher ever saying a word about the x-axis and the y-axis, the origin, or where to begin counting, the class is able to determine by guessing and testing how the various points are plotted and named. As a follow-up activity for this lesson, the class could be divided into two groups so that a game of tic-tac-toe (x's and o's) could be played but using a five-by-five board or a six-by-six board or whatever size board the teacher thinks appropriate. This is actually a drill activity, but it is a drill activity which in the writer's experience the students find to be fun, exciting, challenging and rewarding.

It is evident that in lesson three the teacher again removes himself from the center of the classroom situation. The students are left to guess and test relative to the mathematics under consideration. They are not guessing and testing as to what answer the teacher is wanting or seeking, a game which is too common in classrooms today.

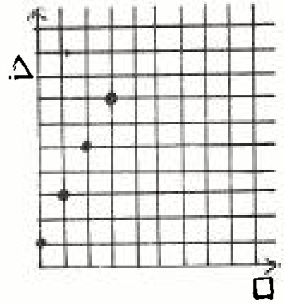
Lesson Four: This lesson is designed to bring together the mathematical topics the students have been investigating in the previous two lessons. We join the lesson at the point where the students have established the following table of values from an equation which a team of experts has created.

□	△
1	3
3	7
0	1
2	5

Several possible equations have been suggested and quickly refuted. The teacher makes the following suggestion:

TEACHER: Could you use the grid system we talked about recently to assist us in finding the equation the team is using?

BETH: Well, we could mark the points on the grid for each pair of numbers in the table. (This is done. It yields the following grid.)



DAPHNE: I can tell by looking at the grid what the value will be if I tell the team 4. It will be 9. Is that right?

TEAM: Yes, that is right.

JOHN: Yes, look every time we go over one on the graph, we go up two. If you look at the table of values, you see the triangle numbers increase by two every time the box number increases by one. I think I know the equation. Is it ($\square$ x 2) + 1 = $\triangle$?

TEAM: Yes, that is the rule or equation we were using.

ANITA: Why did you guess that you add one to the ($\square$ x 2) John?

JOHN: Because I noticed last time that every equation we worked with gave the number you add or subtract when the box number is zero. (John certainly was looking for patterns and relations, and relation of relations, in order to make such an observation.)

In this fourth lesson, one can see how two mathematics topics which may seem unrelated when presented in isolation can be brought together in order to give the students greater power and range or scope for their mathematical guessing and testing techniques. By applying the graphing techniques explored in lesson three to the linear equations of lesson two, students were able to see the patterns created in the table of values in a graphical dimension.

These four lessons are but a very brief description of how Fallibilistic strategies of naive origination and testing could be introduced to students in your classroom. Some of the more fundamental issues involved in utilizing such an approach are discussed next.

Summary and Conclusions

At the outset of the paper, a brief description of a variety of current approaches to the teaching of mathematics was provided. The purpose of this discussion was two-fold. First, it was contended that none of the approaches mentioned (and even more could be added) was a panacea for our problems in teaching mathematics. The usefulness, fruitfulness and validity of any approach depends on the goals of a particular lesson, the students involved in the lesson, the teacher guiding the lesson, and the physical environment in which the lesson occurs. Second, it was argued that no one approach is inherently any better than any other approach. However, this does not mean the teacher should only utilize one method. Indeed, a teacher must use a variety of approaches in order to create an environment in the classroom which is conducive to students' learning of mathematics.

The sequence of Fallibilistic lessons and strategies of teaching must be viewed in this more global context. From a Fallibilistic viewpoint, mathematical knowledge was seen to grow as a function of conjectures and refutations. Within this overall orientation, three strategies of teaching were identified, based on the three phases of mathematical inquiry. The three phases of inquiry were those of origination, testing and proving. The naive instructional pattern was composed of two teaching strategies: the TP strategy in which naive origination was followed by a testing phase and then a proving phase; and the PT strategy where the phases of proving and subsequently testing followed the origination phase. The deductive instructional pattern (DED) was distinctive in that the origination and proving phases proceeded simultaneously giving rise to a deductively generated conjecture, a conjecture which was proven while it was being originated.

The Fallibilistic model of instruction and the strategies of teaching described were offered as being applicable to many of the approaches discussed earlier. The illustrations provided in the form of a sequence of lessons were examples of only the TP strategy of teaching. Furthermore, there were a number of fundamental characteristics of this strategy focused on in the illustrations. Among these was the desire to remove the teacher from the center of attention in the classroom in order to allow the students to interact with their peer group (the human approach) as well as to interact directly with the mathematics being studied (the mathematics approach). The teacher's role thus became one of being the creator of the learning environment. As a result, it is the teacher's job to manipulate the physical setting of the classroom, to select the mathematical topics for consideration, and with the aid of the students to develop ways of testing student conjectures independently of the teacher. It should be recognized that the decisions the teacher might make regarding one of these components would have effects on all the other components (the systems approach). The very act of removing the teacher from the focus of attention in the classroom will produce varying responses from students and will alter the learning environment in the classroom.

One of these responses, it was contended, was that students' guesses could be treated on a cognitive level rather than on an affective level. The students would cease playing the game of trying to out-guess the teacher. Instead, they could direct their energy and attention to the mathematics and to their colleagues' responses to the mathematics. For those readers who are concerned about the students being allowed to guess at answers, it must be remembered that a guess is usually called wild only if it fails; if it succeeds the guess is usually called a daring one. Furthermore, the testing situation which is created does act as a force which fosters responsible guessing rather than irresponsible guessing in the classroom.

It was suggested throughout the sequence of lessons that the tenor of the lesson should be one of fun and excitement, one which is challenging yet rewarding to the students. The creation of knowledge is a tremendously exhilarating experience whether the creator is a child or an adult. One only has to witness the joy expressed by children or the sense of accomplishment and pride exhibited by adults when they create or discover something to become convinced that learning is fun, exciting, rewarding and challenging. Why, then, cannot this also be the feeling demonstrated by students in mathematics classrooms? If the many wrong paths and false steps all of us take in learning are treated cognitively rather than affectively, if we are not branded as failures for wrong guesses, if the teacher becomes a learning facilitator rather than a knowledge transmitter, then perhaps the joy and anticipation of learning which children have when they enter school will remain with them throughout their lives.

Finally, it probably goes without saying that a teacher would not utilize a guessing and testing strategy at all times in the classroom. It is but one strategy that seeks to accomplish specific goals, but these goals are by no means all encompassing. Consequently, teachers are encouraged to have a Fallibilistic approach become part of their repertoire of teaching strategies. However, as with all such strategies, the Fallibilistic approach is not a panacea and should not be viewed as such.

History of Numeration Systems

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The basic activities of this paper are adapted from Osborne, Roger et al, *Extending Mathematics Understanding* (Columbus, Ohio: Charles E. Merrill Publishing Company), 1962, and the paper formed the basis for a talk entitled "A Guided Discovery Approach to the History of Numeration" presented by Dr. Neufeld at the Winnipeg Meeting of the NCTM in October, 1970.

The topic of numeration is receiving increased emphasis in contemporary curricula for elementary school mathematics. Teachers are generally encouraged to use concrete materials in helping children to understand the decimal system of numeration. Some curricula suggest the inclusion of certain non-decimal numeration concepts. The curriculum advocated by Dienes states that children should be introduced to numeration by manipulating concrete materials involving base groupings from two to ten. The study of certain historical systems of numeration, such as the Roman system, has long been included in mathematics curricula. Some contemporary programs extend this coverage to brief treatments of Egyptian and Babylonian systems.

The purpose of this paper is to present a sequence of pencil and paper activities which will integrate four historical systems of numeration. Included in the four systems are examples of both decimal and non-decimal numeration. The activities are written to challenge teachers. This does not mean that upper elementary or junior high school students could not work at the same material. In fact, it is often found that students find activities such as these easier to understand than do teachers. This is probably due to the "unlearning" which adults must do in order to accommodate new ideas. The activities in their present format are designed for teachers who will hopefully revise them to suit the needs of their own classes.

The activities are presented at several cognitive levels. Interpreting the symbols of a particular system is somewhat comparable to Bloom's comprehension level. Constructing new systems parallel to those systems previously presented would correspond to the synthesis level. The last section of the activity requires the participant to work at the evaluation level.

ACTIVITIES

Introduction (Earliest times - conjecture only)

1. symbol (numeral) 1 from
   (a) 1 finger }
   (b) 1 stick } (Likely horizontal)

2. 2 was = or // ; 3 ≡ or ///

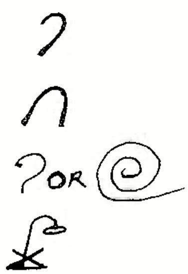
3. regarding 4, 5 and so on - WHO KNOWS?

4. development of four systems
   (a) Additive
   (b) Multiplicative
   (c) Ciphered
   (d) Positional (place-value - "ours")

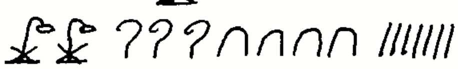
Additive Systems

1. Tally idea 𝍸 𝍸 || for 12.

2. Egyptian example
   (a) Basic digits



1		staff
10		heel bone
100	OR	scroll or coiled rope
1000		lotus flower

   (b) 2347 

3. Roman example (additive with subtractive element)

   I II III IV *for* IIII *etc.*

Multiplicative Systems

1. One set of symbols for the basic digits

2. Another set for the powers of the base

3. Synthetic example

KEY

(a) Basic Digits

(1) /

(2) //

(3) ✳

(4) ▭

Powers Digits

(5^1) F

(5^2) T

(5^3) H

(b) 97 is (3 × 25+(4 × 5)+ 2

symbolized ✳T ▭ F //

(c) This is a base five example.

(d) Translate ▭ H/F is 505

▭ T is 100.

4. Synthesis - Construct a base three system:

(a) Basic Digits

/

//

Powers Digits

(3^1) T

(3^2) N

(3^3) S

etc.

(b) Prepare examples below as in 3 (b) (d) above:

36 is (1 × 27) + (1 × 9)

Symbolized 1 S/N

Translate - //N/T/ is 22

//S/N/T/ is 66

an example of a possible system.

5. Chinese Multiplicative System

Given the synthetic example (3a), the following "addition" facts ("addition" symbol is /), and the clues, DISCOVER:

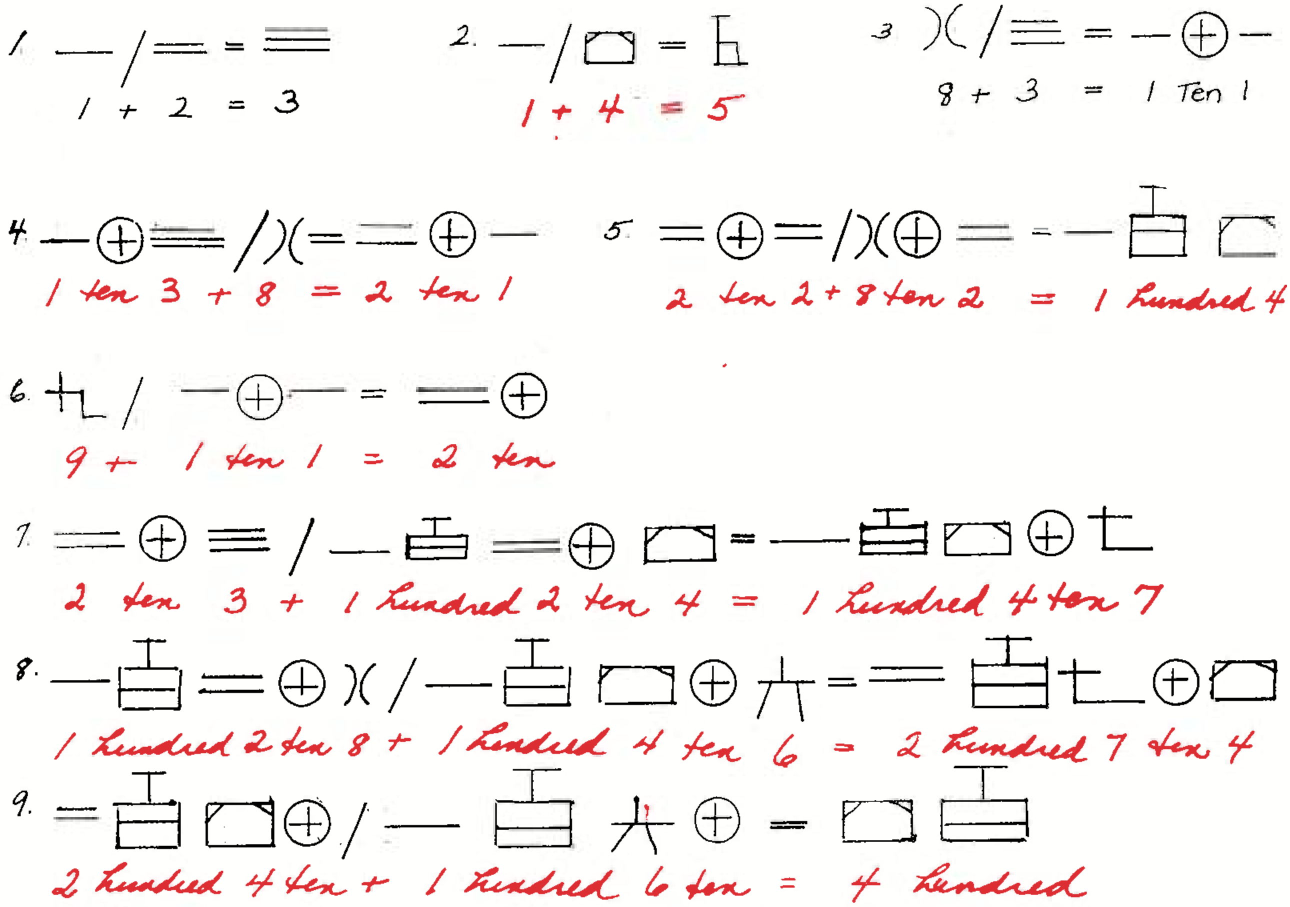
(a) the translation of problems 2 to 9.

(number 1 and 3 given)

(b) the notation system (what symbols are used comparable to our 0, 1, 2 . . .)

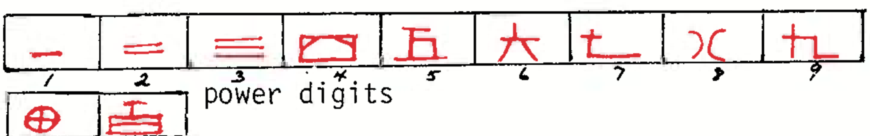
(c) the base

"Addition facts"



Answer space

(b) notation system



basic digits

power digits

(c) base ten

Ciphered Systems

1. A new symbol exists for each *multiple* of each *power* of the *base*.

2. Synthetic example (Ionic Greek model)

(a) symbols

1 - a	10 - j	100 - s
2 - b	20 - k	200 - t
3 - c	30 - l	300 - u
4 - d	40 - m	400 - v
5 - e	50 - n	500 - w
6 - f	60 - o	600 - x
7 - g	70 - p	700 - y
8 - h	80 - q	800 - z
9 - i	90 - r	900 - *

(b) This is a base ten or j example.

(c) The first power of the base is 10' or j ;

multiples are 1(10' j) ------- 9(10' r)
1(10² s) ------- 9(10² *)
etc.

(d) Translate 15 is je
347 is umg
650 is xn
999 is *ri
784 is yqd

Positional Systems

1. Place value comes into existence.

2. Zero is needed.

3. Babylonian example (partly positional) (Base 60)

(a) Simple grouping to 60 (eg. 32 is <<< //), positional beyond 60.

(b) 18,812 is ///// < /// <<< //

$5(60)^2 + 13(60)^1 + 32(60)^0$

(c) Translate

58 is <<<<< ////////

27,362 is 7 /////// $-\times(60)^2$ + 36 <<< ////// $-\times(60)$ + 2 // $-\times(60)^0$

27,480 is /////// <<< ///// //

458 is /////// <<< ////// //

NOTE: Except for spacing, one numeral for three different numbers. The spacing of the answers is purposely identical to point up a basic source of confusion in this particular system.

4. Mayan example (Base 20)(actually 20 x 18)

(a) 360 (20 x 18) corresponds to the number of days in their year.

(b) Vertical positioning (in contrast to our horizontal positioning).

(c) 26,656 was

•••	$3(18 \times 20^2)$	=	21,600
•••• (2 bars)	$14(18 \times 20^1)$	=	5,040
⬭	0(20)	=	0
• (3 bars)	16	=	16
			26,656

5. Mayan *Positional* System

Given Mayan example (4c), the following "addition" facts ("Addition" symbol is /), and the clues, DISCOVER:

(a) the translation of problems 2 to 9 (number 1 and 4 given)

(b) the notation system (What symbols are used comparable to our 0, 1, 2 . . .)

(c) the base

"Addition" facts

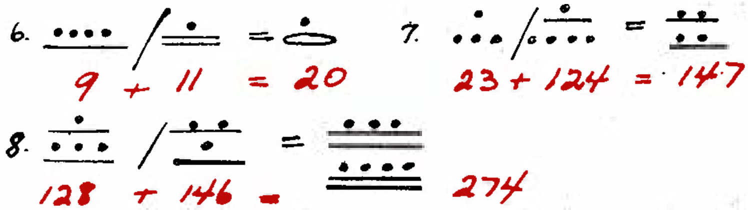
1. • / •• = •••
   1+2 = 3

2. • / •••• = ____
   1 + 4 = 5

3. ••• (bar) / ••• = • (bars)
   9 + 3 = 11

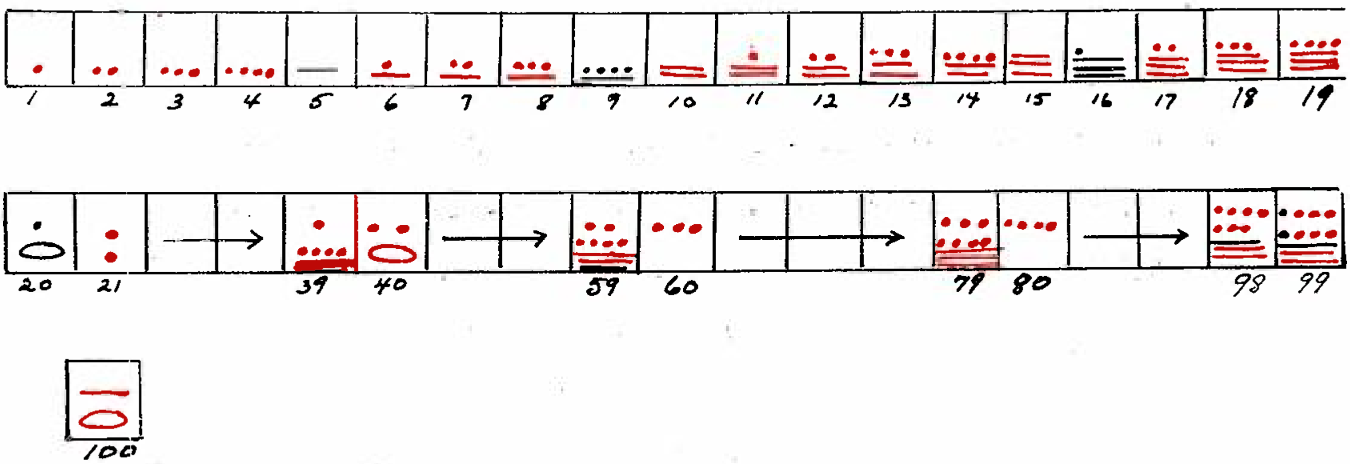
4. ••• (bar) / •••• (bar) = :
   13 + 9 = 21

5. •:• / ••••• •• = — ••••
   28 + 92 = 104



Answer space

(b) notation system (5, 9, 16, 20 are given as clues)



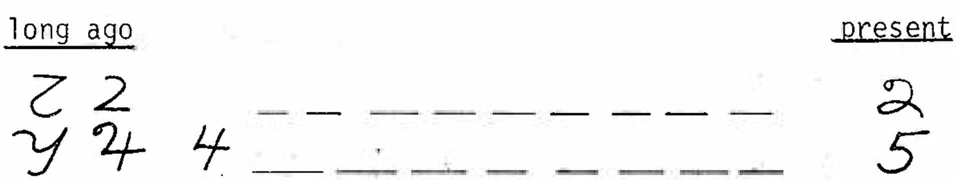
(c) base twenty

6. Our *Positional* System

Symbols of other systems seem strange but our symbols:

(a) are strange in that many have little or no connection to the cardinal value they represent (eg. 5 for ⁙)

(b) are changing, for example, the development of "2" and "5".



(c) are unusual in that they aren't completely standard over the English-speaking world:

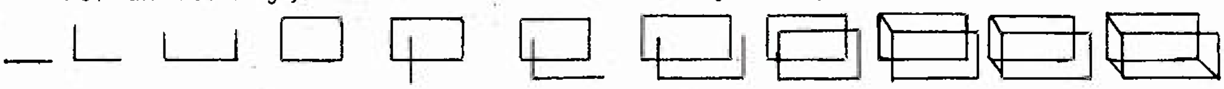
eg. in North America billion means 1000 million while in England billion means one million million.

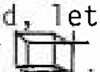
omparison of Systems

1. Numeration systems are man-made. Our system has a long history of development. IS OURS THE BEST? To answer this we must compare. So that our comparison isn't biased, we must use NEW SYMBOLS,

- not 1, 2, 3, 4, . . . (Arabic Positional)
- not I, II, III, IV, . . . (Roman Additive)
- not 一, 二, 三, 四, 五, ... (Chinese Multiplicative)
- not ciphered Greek symbols.

For uniformity, let us use a new set of symbols,



and a new base. Thus far it looks as if the base is twelve. If powers of the base are needed, let us use the convention of superscripts $(\text{Base})^1$ =  , $(\text{Base})^2$ =  , etc. If multiples of the base are needed (Greek ciphers) use subscripts $2 \times (\text{Base})^3$ = , and so on.

2. Questions

(a) Why have any base at all? *the number of basic symbols is reduced.*

(b) If we must choose a base, what qualities should it have?

1. easily remembered

2. average number factors — nine (three); ten (five, two); twelve (six, three, two)

3. reasonable number of basic symbols.

(c) Which of the four basic systems utilized a ZERO?

1. Additive? *No*
2. Multiplicative? *No*
3. Ciphered? *No*
4. Positional? *yes*

(d) All symbols in any positional system are placeholders. What is the basic function of any symbol (3, for example, in the numeral 4136)?

3 represents the number of tens.

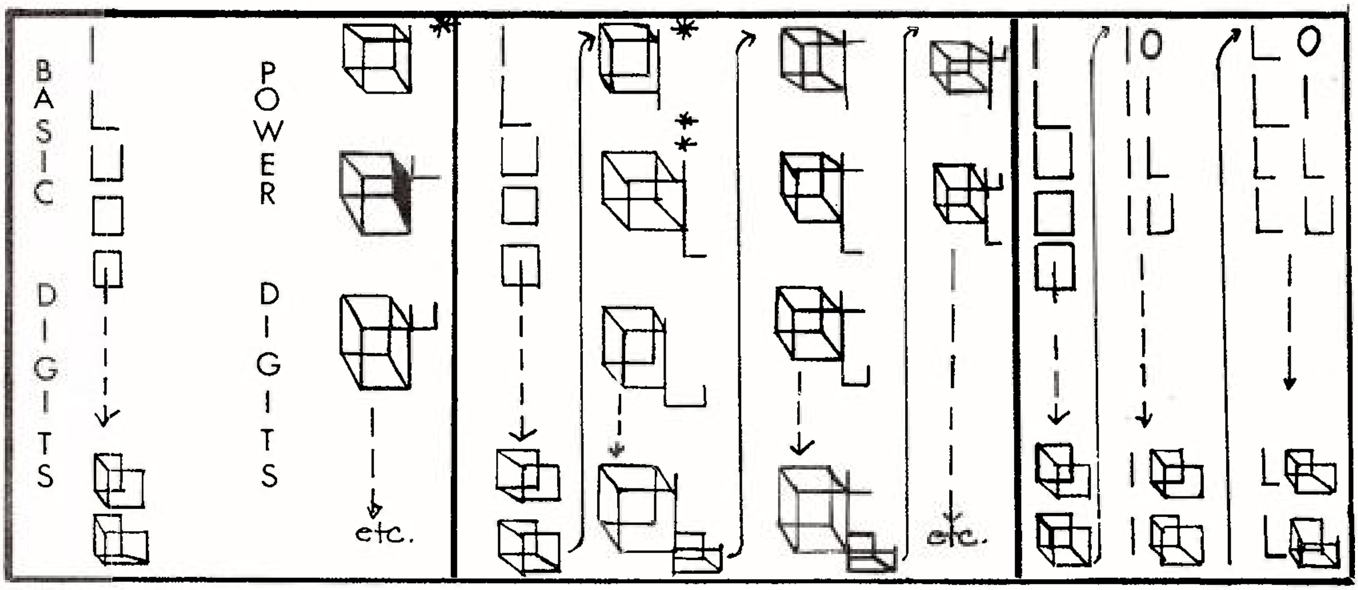
(e) Does the symbol for ZERO have a similar function in the numeral 608?

Yes, the number of tens is zero.

3. Comparison. Now for comparison purposes, let us use our *new symbols* and *new base* with all *four systems*.

(a) Numeration systems.

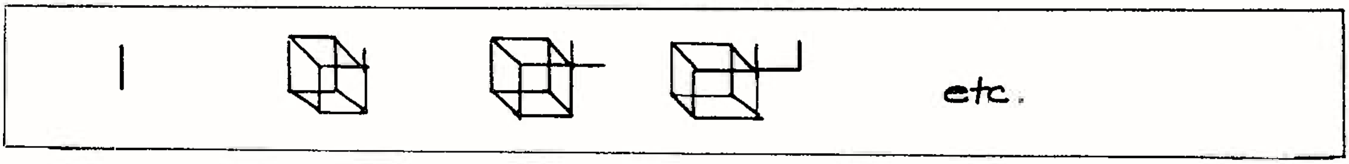
MULTIPLICATIVE CIPHERED POSITIONAL



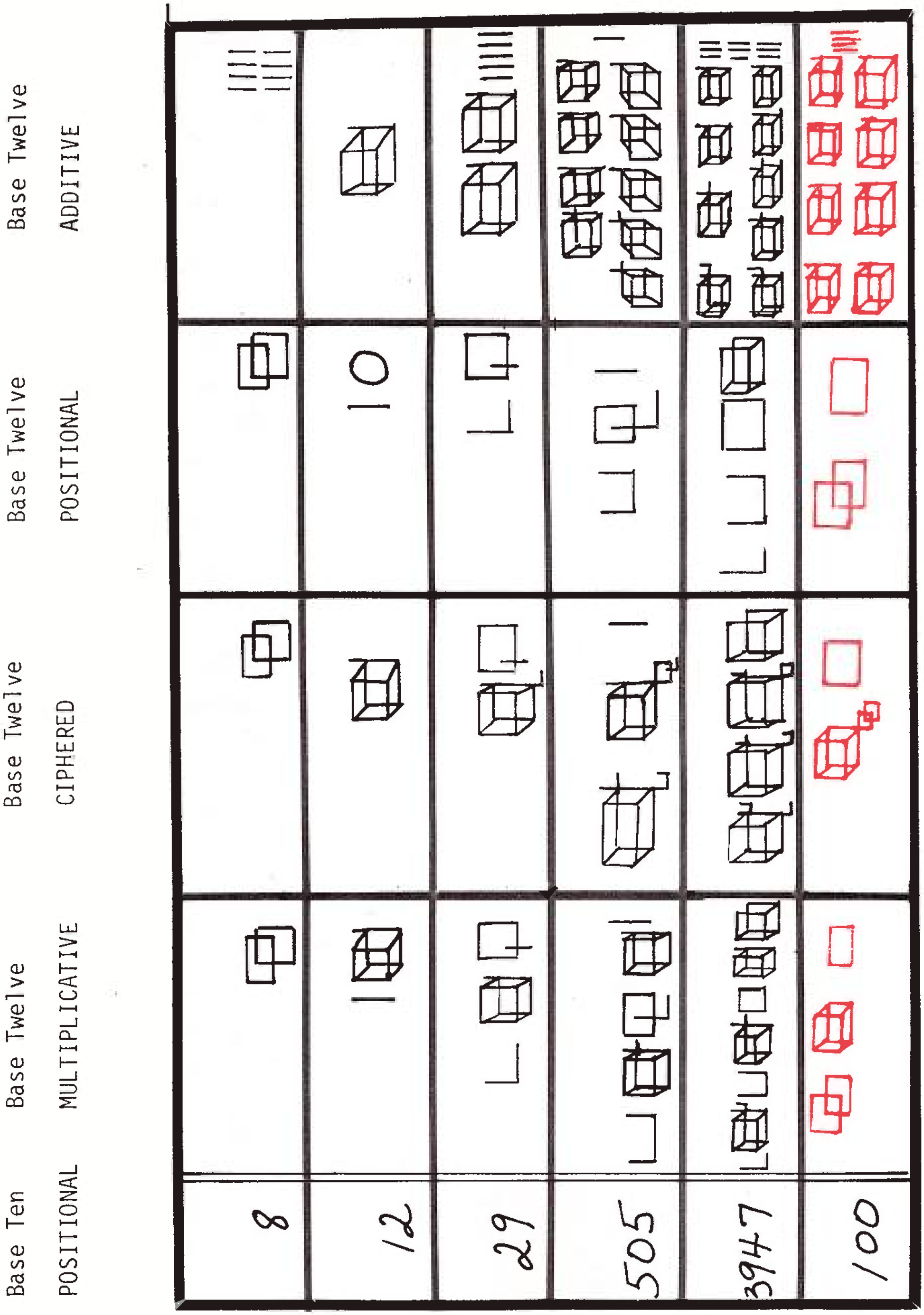
ADDITIVE

* Power of base TWELVE

** Multiple of the base TWELVE



3. (b) Translations - each of six numbers is symbolized as a numeral in five systems of numeration.



3. (c) Comparison of numeration systems.

Suppose you were able to choose one of the four numeration systems, each utilizing a base of twelve and the "block digit" symbols, to replace the system with which we are most familiar:

1. Indicate your preference by ranking the systems 1, 2, 3, or 4.

   Multiplicative -

   Ciphered -

   Positional -

   Additive -

2. What are some advantages of your highest-ranked system?

   Mult. - Place value name is symbolized
   Ciph. - much information in a single symbol
   Pos. - fewest symbols used
   add. - easiest to figure out - just add.

3. What are the disadvantages of the other systems of numeration?

   Mult. - not too concise (almost as bulky as Additive)
   Ciph. - Complicated individual symbol.
   Pos. - place value not symbolized.
   add. - very "bulky"

4. What is the value in studying other systems of numeration?

   1. Put ourselves in the "shoes of children." We realize the problems children have in learning the base ten positional system.
   2. Understand how other systems have contributed to the development of our system.
   3. Understand the basic ideas of base, place value and symbols by seeing these ideas applied to a variety of parallel systems.
   4. Realize the "man-madeness" of systems in that numerals and arrangement can be arbitrarily chosen.
   5. Examples in everyday life - clock, eggs, computers, etc.

An Inventing Unit on Area for Grade VII

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In another article in this issue, the author talks about a "discovery unit" dealing with senior high school mathematics content. I chose to use "Inventing Unit" in the title of this article because, since writing the previous article, I have come to the conclusion that mathematics, in the true sense, is not "discovered", rather it is "invented". Scientific laws are discovered because these laws are inherent in the nature of things. However, the mathematician invents mathematical structures to help him solve problems. The distinction is not of critical importance, but I will use the word "invent" throughout the article in the hope that it will give the reader a slightly different slant to the instructional method proposed herein.

The theme of this article is to suggest a method of teaching a unit of two weeks duration on finding areas of simple geometric figures to either Grade VII or VIII students. I am sure the reader is familiar with discovery teaching and I suggest he can't go too far wrong if he thinks of this in the discovery sense as opposed to the newly coined "inventing" sense. Before a teacher sets out to use this method, he must first convince himself that there is something more to teaching than simply giving students the knowledge of how to find areas of various geometric figures. While eventually he wants his students to be able to manipulate formulae, he also wants them to know what area is, to be able to solve some unusual problems and, in general, to have a well-grounded intuitive knowledge of area. If you think that some of these latter notions are important in learning about area, then continue to read. I cannot emphasize too strongly that the proposed method of teaching area (and indeed, it can be applied to any content), is predicated upon the idea that the traditional objectives for teaching mathematics are not broad enough. So let us admit that we are prepared to spend two weeks having fun learning mathematics and also that the criterion for success which we will use will not be the kind of question normally found on a final examination in mathematics.

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*Dr. Sigurdson has a video tape of many of the student activities generated by the lessons he refers to in the article, and he invites any interested teachers to enquire about viewing the tape at the University Education Building in Edmonton. This paper grew out of a talk entitled "Teaching a Unit Through Discovery", presented by Dr. Sigurdson at the Winnipeg Meeting of the NCTM in

Many advocates of this method will insist that I am being too hard on "discovery teaching." They insist that one can attain all the traditional achievement objectives and get many additional benefits. I partly agree with this but I think when one is beginning to use the method he should not get "hung up" on achievement as the criterion for success. After a teacher becomes particularly accomplished in using the method, there are no limits to the benefits which may accrue from it. Also, we must not forget that students are not accustomed to the method. They are more used to teachers who are directive and willing to tell them answers. So we should not naively expect that the method will achieve overnight success. I do believe that you will have fun teaching this way and that the students will enjoy "learning" this way.

STRUCTURING THE UNIT

The first and most important general rule to follow in setting up a unit along discovery lines is to present the broadest possible description of the task. Tell the students that for the next two weeks they will be trying to develop ideas for coping with a general type of problem. Once you allow the students to begin working at the problem they will specify certain aspects on which they want to concentrate first. Each of these individual aspects can result in a specific activity. The previously mentioned quadratic unit resulted in 10 of these activities. At first, the teacher will have to help students isolate the problem areas. But the ultimate goal in such an "inventing" unit is that the students will be able to invent their own activities. They should be able to specify the areas they wish to work on and develop ways of attacking these areas.

INSTRUCTIONAL PROCEDURES

In addition to the teacher's setting up a general type of problem for the students to work at, he must learn certain instructional procedures which will promote discovery or "inventing". I will now suggest some guidelines for the teacher to follow in an "inventing" situation. First, do not be concerned with using precise terminology, but rather give the students an intuitive feeling for the problem. The idea is to begin by giving the class a very poorly defined statement of the problem. This instructional pattern is extremely important to follow because it will force the students to determine for themselves exactly what the problem is. And it is well understood in mathematics that half of the work in solving a problem is over once the problem has been defined.

Phase One

The initial exploratory period of working on the problem should be done by the students working in pairs or small groups. The reason for this is that an individual student who doesn't know clearly what the problem is might easily focus on an inappropriate aspect of the problem or else he may simply run out of ideas. The chance of either of these things happening when a small group is working on the problem is much less. The noise level of 30 students working in small groups is rather high, especially if they are more or less noisy to begin with. Every teacher must find his own means of coping with this problem. During the exploratory period the teacher can supply the students with ideas to help them get started. However, the teacher must be careful not to assist directly in solving the problem; he should, rather, suggest apparently productive lines of thought which the students can investigate. Also during this instructional phase, the teacher should be accepting and encouraging toward any ideas whether they are "correct" or not. I placed the word "correct" in quotation marks because anything is "correct" if it gets a student closer to the solution. Even a completely wrong approach can help in arriving at a solution if the student finds out where the approach is wrong and can modify it accordingly. In the traditional classroom a student feels badly about making a wrong answer. The teacher in the discovery class must help the student overcome this feeling, especially during the exploratory phase.

The problem of students running out of ideas quickly can also be alleviated by making the initial problem a very "primitive" one. That is to say, the problem should be approachable by the slowest pupils in the class and yet still be full of potential for the best students. Of course, if the range of ability in your classroom is extreme, it will be difficult to find such problems. The ordinary problem at the end of a chapter in a mathematics textbook is usually not "primitive", that is, either you know how to do it or you don't. The problem which structures the unit on area described in the following pages is a good example of a "primitive" problem.

Phase Two

After a certain period of exploratory work, which may be as long as a whole 45-minute mathematics period, the teacher will lead the whole class in a group discussion. The idea is to have the students hypothesize solutions. Some of the statements will take the form of identifying the importance of working on a certain aspect of the problem. Here again, all hypotheses should be accepted with the idea that the teacher will not evaluate. This will probably confuse the students, especially if they have had a teacher who normally only writes *correct* statements down on the blackboard. The teacher must remind the class that he will not give answers and that they, themselves, are completely responsible for agreeing as to what is acceptable or unacceptable (I hesitate to put "right or wrong"). It is extremely important for the teacher to help the class keep track of these hypotheses. A good technique for keeping track of them is to give the hypothesis the name of the student proposing it, for example, "Dwight's Hypothesis". At some stage the teacher must say, "Okay, I think we have enough hypotheses. Let's start to evaluate them". At this stage it might be appropriate to let the students work in pairs or small groups or again it may be fruitful to keep the whole-class discussion going.

During the second part of phase two, in which the evaluation of the hypotheses takes place, a teacher might suggest alternative proofs. One form of alternate proof uses a counter-example to show that the hypothesis in question does not cover a particular situation. Another idea to use at this time is that two hypotheses cannot both be correct. Such a conflict situation can be extremely helpful for motivational purposes. During this phase of the instructional procedure, the teacher must become slightly more directive. The amount of direction the teacher gives is determined by many factors in the classroom situation. Knowledge of both the subject matter and the student helps the teacher in making this decision. In actual practice, one of the obvious indices to look for in determining the amount of guidance is the frustration level of the class. They want to determine if something is correct or not and they don't know how to do it. They feel the importance of a situation that they can't cope with. This can be a very valuable learning situation, but beyond a certain point, it is simply maddening.

Phase Three

The last phase of the instructional process is that of consolidating the ideas that have been evaluated as being useful. It is not important that the solutions to problems be stated in the normally acceptable form. The criterion should be that the solutions are understood by the students in their own way. In certain instances the teacher may want formulae stated in the usual way simply for the sake of convenience. This final stage, which may be called a stage of closure, is a difficult one to handle because the teacher must try to preserve the students' feeling that they have come up with these answers. Here it becomes obvious to the students that the teacher does know the answers. But if one can give the students the feeling that the answer is not the important thing, rather that the process of arriving at the answer is the important thing, then this last stage of closure is simply the "icing on the cake" - the "cake" being what went on in the first two stages.

The three phases mentioned here will undoubtedly repeat themselves many times during the unit; that is, the sequence of exploratory work, hypothesizing and evaluation, and closure will repeat themselves as often as the teacher and the class think it necessary. It cannot be emphasized too strongly that during the exploratory and the hypothesizing phases the teacher must strive to create an accepting atmosphere in the class. Students should be encouraged to think out loud and share their ideas with everyone. When 30 minds share ideas on the same problem, the results can be not only very interesting but also very productive. The three critical factors in determining the effectiveness of the method are the ability of the teacher, the quality of the general problem, and the personal characteristics of the students. By personal characteristics of the students I mean not only their intelligence but their willingness to share ideas, and their general manners in the classroom.

In the paragraph above I referred to the "effectiveness" of the method. One might ask "What is the criterion in determining effectiveness?" "What do we mean by effectiveness?" I would first of all suggest it is not the "speed" with which the students arrive at a solution. It is perhaps the "quality" of the ideas that come up during the course of the unit. Or perhaps it is the degree of interest shown by the students in mathematics.

I suggest that we now take a "discovery" approach to learning how to teach an "inventing unit". Let's do it and then discuss what we have done. The following is what happened when I tried the unit with a Grade VII class.

CLASSROOM PROCEDURES

First Class Period

I began the class by giving out the sheet of paper identical to that shown on the following page. We spent 10 minutes talking about the kinds of figures and their names. Each of the figures could be referred to by a letter so the name of the figure wasn't really essential. The class was told that the numbers along the sides of these figures were the measures in centimeters. The first question I asked the class was "Which of these figures is the biggest?" Of course, a number of different answers were forthcoming. In fact, the class was using three different ideas of "bigness": perimeter, area, and dimensions. So we narrowed down the loosely-defined problem which I had asked and I said that what I had really meant was "Which is the biggest in area?"

The next few minutes were spent discussing area. The students were able to give me many intuitive notions of area: "amount of space within a closed region", "the surface of something", "contents inside something". I accepted these comments and said, "Now that you have an idea of what area is, the task you have is to place these figures in order from the one with the biggest area to that with the smallest area." At this point I handed out centimeter rulers and some one-centimeter cubes which I said they might find useful.

The students were asked to work in groups of three and to convince each other that the order they had was correct. They then began to work at their task. I stressed that they could cut up the sheets, that they should draw lines on the pages, or use them in any manner they wished. After about 10 minutes of the exploratory period, one student raised his hand and said: "I know what area is but how do you find it?" After another minute he said, "I know how to find perimeter. Is it the same as the area?" I responded, "That's an interesting hypothesis and, indeed, the question is, "How do you find the area?" It was clear that the class had indeed identified what the problem was.

After another 20 minutes in which the students worked on their own, I asked the whole class to pay attention because I wanted to discuss some of their ideas. The purpose of the discussion was to spread their ideas around and certainly not to evaluate their ideas. During the following 20 minutes, which brought us to the end of our first class period, a number of ideas came up. As it turned out, the students did not want to talk about the order, they were merely interested in focusing on figures that were particularly interesting (to them). I labeled these ideas by using the names of the students.

Figure C: Dwight's hypothesis - You measure the perimeter. Then you change the perimeter into a standard shape like a square or a rectangle and you find the area of this shape. (He apparently thought he knew how to find the area of a square or a rectangle.)

Figure H: Steven's hypothesis (for a trapezoid) - Cut it in half and flip it onto the other end; then you have a rectangle of which you can easily find the area.

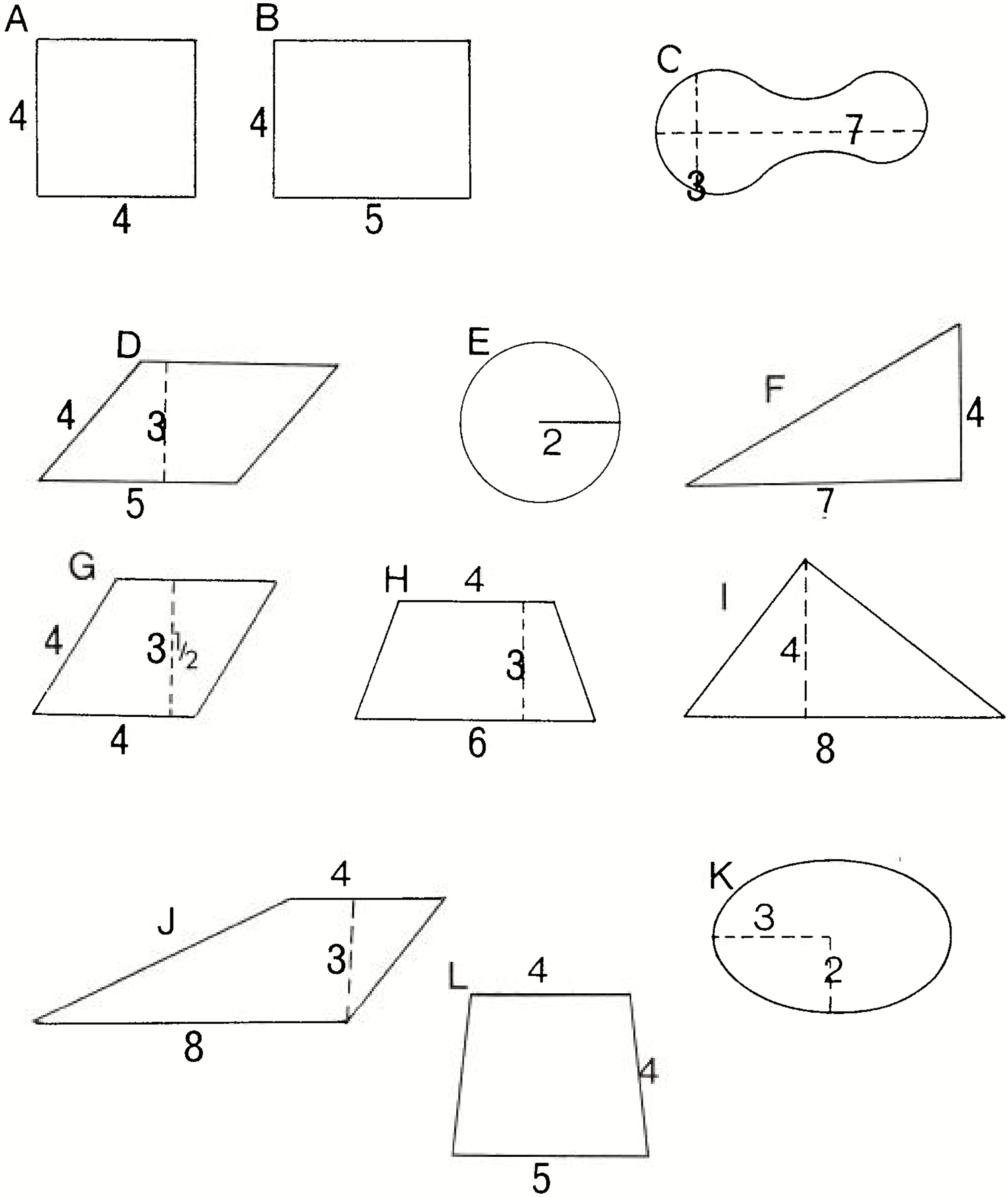

A
4
4
B
4
5
C
7
3
D
4
3
5
E
2
F
4
7
G
4
3½
4
H
4
3
6
I
4
8
J
4
3
8
K
3
2
L
4
4
5

Figure I: Ronnie's method of squaring - Draw lines on top of the triangle, both vertical and horizontal, marking out the area of the triangle in squares; then you count the number of squares. (He argued that if you do it this way it usually comes out even.)

Figure J: John's method of doubling - You add equal triangular halves to both sides of the triangle. You then have a rectangle of which you find the area by using Ronnie's method of squaring.

There were other suggestions that came up in this period but the ones listed seemed to be the most clearly stated and they seemed to be the most productive of further ideas. Dwight's hypothesis drew very little criticism. In fact most people agreed with it. I did not discourage them from using it. Steven's hypothesis was criticized because the two sides of the figure were not on the same slant. It is perhaps unfortunate that Steven did not state his hypothesis as a solution to the parallelogram because there it works beautifully. Ronnie's method was accepted as being interesting but rather inaccurate, while John's method was literally greeted with cheers and statements like: "Hey, ya, it works." "Man, that's neat!" In spite of these reactions I felt it was still too soon to ask them all to copy John's method down in their notebooks as a correct solution for finding the area of triangles. However, after this session I felt that the students had a number of productive ideas which they could take back with them to the problem of ordering the areas.

I would like to re-emphasize the reaction to John's method. The reaction illustrates clearly that many other students in the class were completely ready for John's invention (not to say discovery). Critics have said that the only person benefiting from a discovery is the discoverer. But I suspect many students in the class were saying to themselves: "Man, that is so easy; why didn't I think of that?" So in a sense the discovery was theirs. Another point of John's method deserves mention: the class (except for three or four students) was incidentally agreeing that you could find the area of a rectangle by squaring it off in one-by-one centimeter squares. We had not yet talked about the areas of squares or rectangles.

Second and Third Class Period

The next period the class was allowed to work on their own. I gave some suggestions to help them along. The 45-minute period passed quickly. The The third period began with

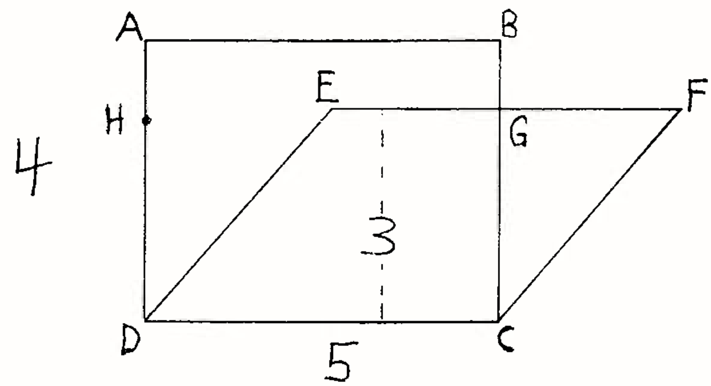
> Gord's Hypothesis - The perimeter can be big or small. It does not depend on the area.

This of course was a contradiction of Dwight's hypothesis. Gord's hypothesis was simply noted; no one seemed anxious to talk about it. The area of concern once again became Figure I. Using the perimeter method as proposed by Dwight the answer turned out to be 25, but using John's method the answer was 16. At least two of the students refused to go on to new work until we had settled the problem once and for all. Then came the embarassing question: "Well, do *you* know the answer?" I tried to avoid an answer but finally said I probably could figure it out but that I wanted them to convince each other as to the correctness of any answer.

We finally got around to discussing the parallelogram, Figure D. Dwight took the initiative and restated his hypothesis:

Dwight's second hypothesis - When a rectangle is pushed over to make a parallelogram, the area of the parallelogram has the same area as the original rectangle.

This hypothesis brought about much disagreement. Dwight's point was that the area wouldn't just evaporate or go away; therefore, it must still be within the parallelogram. Many arguments were presented. Two were especially effective. First, someone cited extreme case of pushing the parallelogram so that it would be "just about" flat and the area would be very small. The other argument is illustrated by the figure below. (For the sake of clarity I will put the argument in my own words.) Rectangle ABCD has an area of 20. You can find the area of the parallelogram CDEF by taking triangle CGF and moving it into position of triangle DHE. And "it just fits exactly". So we reach the conclusion that the area of the parallelogram is smaller than the area of the rectangle. A further argument was proposed by calculating the area of the parallelogram to be 15, "So the amount of area that spills over is 5 centimeters".



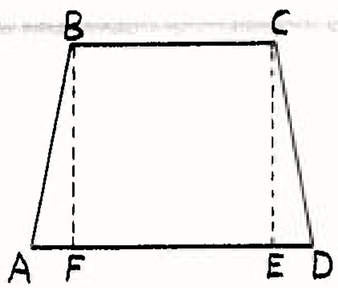
As the reader will readily notice the arguments are completely intuitive. Dwight and his supporters still would not agree. I realized that they had a certain amount of personal commitment to their ideas and, besides, it is difficult to change an idea that you really believe in. So I tried to leave it by saying: "Okay, just think about it for a while." But again two or three students would not go on unless "Dwight agrees." And again they asked me if I knew the answer. Some teachers would consider Dwight's refusal to change his mind in view of the facts, a failure of the method, but it is very pleasing for a teacher to see a student committed to a significant idea - it was all his! Within a few days Dwight was completely convinced his original hypothesis was incorrect.

The reader will also notice that units of square measure were not used at this stage. Area was simply a number. By the end of the unit we had agreed to use square centimeters as the units of area measure.

Sixth and Seventh Class Periods

After about five 45-minute classes we had agreed on almost all the figures and the order in which they came, except for C, E, H, J, and K. I led a fairly directed discussion on solving H and J. Directing the class at this point seemed appropriate because they had tried many approaches to finding these areas. And, if nothing else, they were convinced that they couldn't do the problem. In addition to this I told them the formulae for figuring out areas for the circle and the ellipse. One student's response: "Oh, ya, does it always work?" convinced me that they were appreciative of the answer. We also discussed the problem of finding the area for C and agreed that the best we could do was to get an approximation, by using some squaring method.

I am sure a teacher reading the last paragraph is saying: "Why didn't you set up some discovery activities for the students to work on the trapezoid and maybe even the circle?" In answer I would have to say that this would have been completely possible, if not desirable, but we had spent considerable time on apparently important side hypotheses and I did not want to spend more than two weeks on this unit. Another class using this unit solved the trapezoid problem by finding the area of the rectangle BCEF (see the following figure), erasing the rectangle, pushing the two triangles ABF and ECD together and after finding the dimensions of the new triangle, calculating its area. The area of the trapezoid was then easily found.



Ninth Period

The last period was spent in consolidating the solutions and hypotheses that we had worked on. Each of the students wrote down the method used for finding the areas of the different figures. I personally find this consolidation stage of the method very important. A student will not remember a solution to a problem just because he discovered or invented it. He must use it and be reinforced through its use and by the teacher. When John invented his method of finding the area of the triangle, I did not say: "Yes, John, you are an excellent student and the solution you proposed will always work. Let us try it out in this other case." If I had said this, I think John would have remembered it for a long time. However what I did say was: "That's very interesting John. Valerie, what do you think of it?" This was not very reinforcing for John, but I am more interested in the students appreciating the process of evaluating any hypothesis and, in fact, receiving their reinforcement from the mathematics.

This, then, in general, was the kind of activity that went on in a Grade VII classroom for a period of two weeks. During this time, the students worked on their own or in groups for approximately 60 percent of the time, with the rest of the time being spent in class discussion. The teacher played the role of "chairman of the meeting" in which, so to speak, he did not have a vote. I have not included everything that went on during these periods, although I do have a complete record of it. I hope this has given you some desire to try out this unit. I can almost guarantee that, as long as you are able to create a responsive atmosphere, your class will come up with as many good ideas as are mentioned here.

OTHER SIMILAR ACTIVITIES

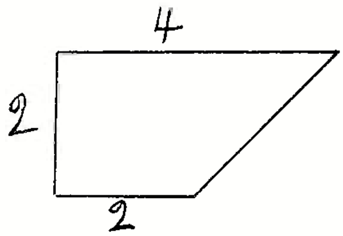
Two other closely related classroom activities come to mind. First, it would be very easy to reproduce an identical sheet with letters beside the line segments instead of numbers. This could lead to the students discovering (inventing) formulae. It would be especially interesting to see how quickly they would identify the generalization that the same formula is always used for triangles and so on. This would be especially interesting if you made one of the triangles obtuse.

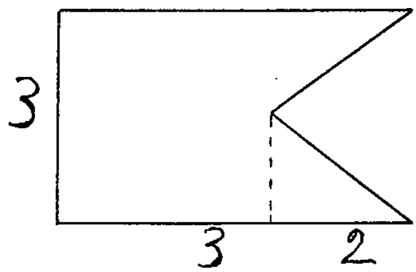
The second activity is really another unit. The task could be to order the figures according to perimeter. The problem of perimeter is simple for Grade VII students, but Figure C would prove interesting, as would the circle and any figure that requires measurement. You might pose the problem of which perimeters can be found without measuring.

The last sentence brings to mind a thought that needs to be reiterated. The problem it identifies is interesting in-and-of-itself, but the problem becomes even more relevant in that it contributes to a much larger objective, that of finding perimeters. The whole idea behind structuring a unit is to allow the student to make discoveries or inventions that relate significantly to a larger framework.

EVALUATION

In teaching any unit, we have to ask, "Did we achieve that which we set out to achieve?" In answer to this, I asked the students three types of questions on a one-period test. The first type was to find areas of figures similar to those that made up the unit. The second type of question dealt with finding the areas of complex figures. Since many of their approaches to the problems in the unit consisted of cutting figures apart it seemed that the students would do especially well at finding the areas of the following figures:





The third type of question concerned the hypothesis that had been made:

1. Do you think Ronnie's method of squaring is a good way of finding the area of a triangle? Write a brief comment on your answer.

2. Gord hypothesized that the perimeter of a figure might be very large while the area is small. Do you agree or disagree with this hypothesis? Why?

I felt questions of this type were important in order to determine how many students had actually been paying attention and relating to the classroom work.

Actually, the real test of the unit would be to see if the students became better hypothesizers and better evaluators of mathematical statements. Such tests are not easily constructed, and to detect student growth in these areas after a two-week treatment would really be a marvel. The best I could do was to see if they were relating to the treatment as such. The conclusion that I arrived at was that most students were relating at a significant level.

CLOSING REMARKS

I said earlier in the paper that I was going to take a discovery approach to this paper, namely, I would do the thing first and then discuss it afterward. We are all interested in, "What did the students learn by doing this?" My evaluation showed that they learned the material covered in the unit. By learning the material of the unit, here are some of the things, I think, they learned:

1. They learned what it is to make a mathematical hypothesis, a guess, and what to do with it when it is not completely correct.

2. They learned that something is correct when you can convince others by logical argument that it is correct.

3. They learned that you can do mathematics without using symbols but that symbols which everyone understands aid mathematical communication.

4. They learned that the way you evaluate a mathematical idea is to collect the necessary data and check to see that the idea gives useful answers.

5. They learned that mathematics as a basic human activity is fun in and of itself.

6. They learned that there are a variety of methods of solving mathematical problems and that some methods are better for different purposes.

7. They learned that when you have ideas that are your own, no one else's, you become committed to them, and that this commitment to an idea is a lot different from just knowing a formula for solving a problem.

I have not listed these outcomes in any order of priority. Which do *you* feel is the most important? I probably feel that number five comes first. And number seven reminds me of Mark Twain's comment about his wife's swearing: "She knows all the words but she ain't got the tune." Is there a difference between knowing something and having a "feeling" for something?

An outcome that I did not list because it seemed not to fit into the scheme is that this method of teaching by "inventing" or "discovery", whichever term you like, gives the student an opportunity to think out loud. Thinking is just talking to yourself. You can become a much better thinker if you begin by talking to others, then, eventually, you don't need others. But the only way you can talk to others productively is if you trust them completely.

This last statement has to do with the teacher creating a responsive atmosphere in the classroom, an atmosphere where the student will dare to make any mathematical statement whatsoever.

I hope I have given you enough material in this article to prompt you to try teaching this "inventing" unit. Make 100 copies of the page of figures which I gave my students, and hand it to your class. I would be very interested to hear of any reactions you have, especially positive reactions. If your reaction is negative I will try to visit your school and show you "how it is done!" That is what is commonly called a *challenge!*

Designing Action Mathematics for Low Ability Students

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INTRODUCTION

All of us probably have felt the frustrations of working with low ability groups of mathematics students. They seem frustrated, almost ready to fail, and hence often become resentful of the authority a teacher standing in front of them represents. Yet if we consider ourselves teachers, we have to believe that such students want to learn and in particular want to learn mathematics if given a real opportunity.

What is the nature of an instructional environment that provides this "real opportunity"? It will be the purpose of this paper to describe the basis for one such environment - a mathematics laboratory - to give extensive samples of materials actually used with 14- to 15-year-old students and to comment on the effects of the use of such materials.

The mathematics laboratory seems to be in vogue today. Actually, the laboratory idea can be traced back to the turn of the century and is frequently related to the methodology of progressive educationalists. Whereas the laboratory previously was used to show the social utility of already learned mathematical notions, today's mathematics laboratory is thought of as a vehicle for the actual learning of mathematical ideas.

There is considerable theoretical justification for using activity approaches as instructional vehicles in mathematics. Piaget (1967) suggests that there are two modes by which young children develop new ideas: imitation and play. Piaget (1967) and other researchers (Sutton-Smith 1968, Vance 1969) find that play also has the effect of allowing for new uses of previously learned ideas and that play seems to sponsor acts of idea generation in children. It would seem that as children enter school however, the instructional activities in mathematics seem to emphasize imitation and, in fact, deny this tremendous learning resource of purposeful play. We feel this denial is particularly acute for the low achiever in mathematics. It would seem that the bright successful student in mathematics, particularly in the secondary school, can mentally "play" with the ideas of mathematics. However, the low achiever, particularly those of lower ability, appears to have few symbolic mental images available and seems in need of some structured concrete experiences.

The word "structure" in the last sentence brings to mind ideas of Bruner (1966). Bruner suggests that students need to be faced with problems as a learning tool. Yet these problems cannot be so vaguely stated as to allow rejection by students nor so difficult as to create frustration for students. For the low ability student these two considerations are very important. The materials and instructional procedures must be designed to provide adequate structure and feedback to the student and yet allow for the "playing" with mathematical ideas, and provide motivation for student construction of mathematical ideas.

One such source of motivation arises from the work of Z. P. Dienes. Dienes (1967) suggests that the proper learning of mathematical ideas requires experience with examples of the idea which allow for both perceptual and mathematical variations of the idea. It would seem for the student of low ability that a variety of teaching techniques and settings would also prove valuable.

The above discussion provides the following principles for designing instructional settings for low ability mathematics students:

1. The situation must allow for concrete "play" with mathematical ideas.

2. The situation must pose interesting questions but these must be developed in steps which these students can handle.

3. The material used must be readable.

4. An idea must be developed in *several* concrete settings.

5. The materials and setting must provide adequate feedback to the student.

6. To provide for the above, and to give pedagogical variety, several teaching settings should be used.

The sample materials on the area of a circle which follow are part of a complete package on measurement which the authors developed on the basis of the above principles. They were used with classes of low ability students in the following manner.

A. Parts 1, 2 and 3 were used in mathematics laboratory sessions. In these sessions small groups of students (usually 2 or 3) worked with concrete materials following the direction on the activity sheets provided. These students can be assigned to groups on several bases. Although the students in our experiment did not experience reading difficulties, it is advisable to have at least one reader in a group. Usually the group functioned in such a fashion that one student performed the activity while another recorded results. It should be noticed that while questions are posed, they are in small steps. The idea, in this case the formula for area of a circle, is currently defined in each setting.

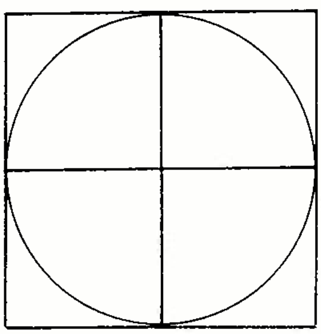
B. Prior to and during the use of part 4 of the material, class sessions were held to review the accomplishments of the labs and to obtain closure on the central ideas. Part 4 of the materials allowed for supervised practice items.

C. Part 5 (a) of the materials again was designed for use by individuals and small groups. In the case of every topic, interesting practical applications were provided. These problems appeared highly motivating and also provided a further application-oriented variation of the major mathematical topics involved.

D. Part 5 (b) is an active extension of part 5 (a). Since many low ability students seem to like problems presented in a large and/or realistic setting, part 5 (b) allowed for some simple but meaningful applied projects.

AREA OF A CIRCLE, PART 1

1. In the diagram below, a circle has been drawn inside a square that just fits around it.



2. Notice the four smaller squares that are formed by the outside square and the four radii drawn in.

(a) Is the side of each small square the same length as the radius of the circle? ______________________

(b) What is the area of each small square if the radius of the circle is 5 units? ________________ r units? ________________

(c) Is the area of the circle less than the area of the four small squares? ________________

(d) About how many times as large as the area of one small square is the area of the circle? ________________

AREA OF A CIRCLE, PART 2

1. Take a sheet of graph paper and the four wooden discs.

(a) Place one of the discs on the paper and draw a circle by tracing around the disc with a pencil.

(b) Draw four radii in the circle and a square that just fits around the circle, as shown in Question 1, page 1. This will give four smaller squares, as before.

(c) How many units long is the radius of the circle? ________________ (Count the spaces; each space on the graph paper is 1 unit.) Round this number to the nearest whole unit and record it in the second column of the table below.

(d) Is the side of each of the four squares the same length as the radius of the circle? ________ Therefore, the area of each square is ______ x ______ or ________ square units. Record this number in the third column of the table.

(e) Find the approximate area of the circle by counting the number of whole squares within the circle and adding to this number an estimate for the partial squares within the circle. Round this result to the nearest whole number and record it in the fourth column of the table.

(f) Divide the number in the fourth column (A) by the number in the third column (r^2). Round this result to the nearest whole number and record it in the last column of the table.

2. Repeat the above work with the three remaining discs.

	Radius (r)	Area of Square (r^2)	Area of Circle (A)	$A \div r^2$
Disc 1				
Disc 2				
Disc 3				
Disc 4				

3. Look at the numbers in the last column of the table. Are they all the same?____________ Is the number 3 in every case? ____________________

4. Do these results suggest that the area of a circle is about 3 times the square of its radius? ________________________________

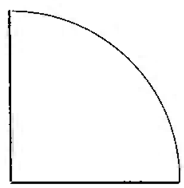
5. In actual fact, the area of a circle is π times the square of its radius. (You will recall π from the formula $C = 2\pi r$ for finding the circumference of a circle.) We can therefore find the area of a circle by multiplying the square of its radius by π. This suggests the formula $A = \pi r^2$ where A is the number of square units in the area of the circle and r is the number of units in the radius. As before, the value of π is taken as 3.14 or 22/7.

AREA OF A CIRCLE, PART 3

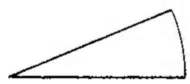
1. Take the plastic disc and a piece of paper.

(a) Using the disc, trace out a circle on the paper and cut along this circle to make a paper disc.

(b) Fold the paper disc along a diameter and then fold it again so that it looks like this:



(c) Fold it a third time and then a fourth, so that after the fourth folding it looks like this:



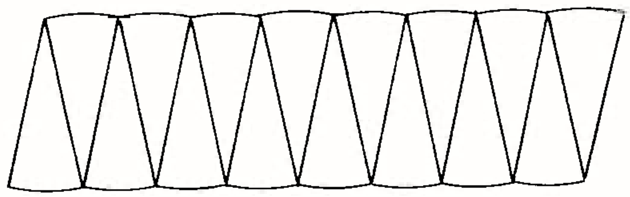
2. Unfold the paper, Sketch a picture below to show what the disc looks like with the fold lines.

3. Cut the disc along the fold lines.

(a) How many separate pieces do you have? ______________________

(b) What shape does each piece resemble? ______________________

4. Arrange 16 pieces as shown below.

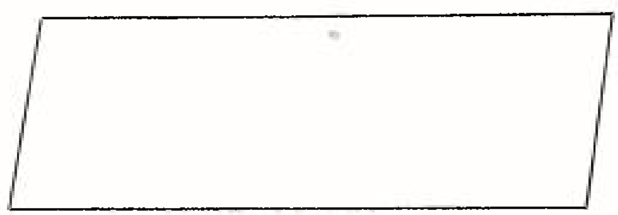


5. Does this arrangement look like a figure you know?_____________Does it look like a parallelogram?_______________________

6. (a) Which dimension of a circle is the height of this "parallelogram"?______ Is it the radius, r?_____________

(b) Which dimension is the length? _________________
Is it one-half the circumference? ___________________
Since the circumference of a circle is given by $C = 2\pi r$, then one-half of the circumference is $\frac{1}{2} \times 2\pi r$ or πr. Therefore, the length of this "parallelogram" is πr units.

(c) Show these dimensions in the diagram below.



(d) Can you find the area of this "parallelogram"?__________Is it $\pi r \times r$ or πr^2 square units?__________________

(e) Is this also the area of the circle?_______ Therefore, the area of the circle is___________square units.

7. Can we therefore find the area of a circle by using the following formula?

$A = \pi r^2$

AREA OF A CIRCLE, PART 4

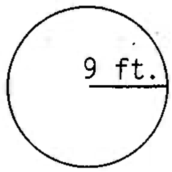
1. Use the formula $A = \pi r^2$ to find the area of each circle shown below. (Use 3.14 or 22/7 for π.) If the diameter is given, first find the radius by dividing the diameter by 2.

   Example: If a circle has a diameter of 8 in., then $r = 8 \div 2 = 4$. Therefore

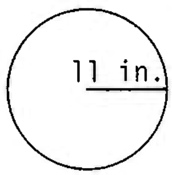
   $$A = 3.14 \times 4 \times 4$$
   $$= 50.24$$

   The area of the circle is 50.24 sq. in.

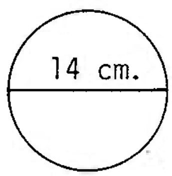
   (a)

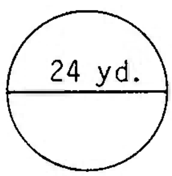
   (b)

   (c)

   (d)

2. Find the area of a circle having

   (a) radius 4.5 cm.

   (b) radius 7.7 ft.

(c) diameter 26 in.

(d) diameter 13 yd.

3. Is the area of a circle with diameter 20 cm. the same as the area of a circle with radius 10 cm.? ______________

AREA OF A CIRCLE, PART 5(a)

1. Find the area of a circle having a radius of

   (a) 13 in.

   (b) 2.5 cm.

2. Find the area of a circle having a diameter of

   (a) 7 ft.

   (b) 22 yd.

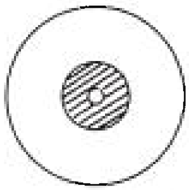
3. The free throw circle on the basketball court has a radius of 4 ft. What is the area of the circle?

4. What is the area of a circular patio, 10 ft. in diameter?

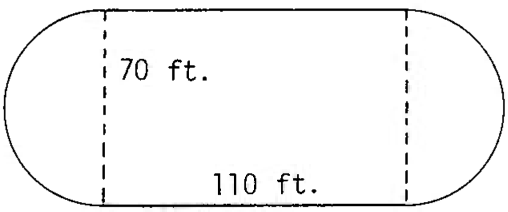
5. A rotating lawn sprinkler can water a lawn for a distance of 35 ft. in every direction. How large an area can the sprinkler cover?

6. How many square feet of cloth are needed for a circular table cover if the table is 3 ft. in diameter and the cloth hangs 6 in. all around?

7. The picture below shows a 12 inch phonograph record. The label in the center is 4 in. in diameter. The rest of the record is playing surface. What is the area of the playing surface?



8. What is the area of the running track pictured below?



9. A tinsmith cut a circular piece of metal 6 in. in diameter from a square piece 6 in. on each side. How much metal was wasted?

10. The girls in the cooking class each rolled out some dough into a rectangle that was 14 in. by 10 in. Their cookie cutter was 2 in. in diameter. (a) How many cookies was each girl able to cut out? (b) What area of dough was left for rolling out again?

11. The boys in the gardening class are planting flowers in a circular flower bed, 18 ft. in diameter. (a) How many flowers should they order if each flower requires 2 sq. ft. of garden space? (b) How much will the flowers cost at 60¢ each?

12. A circular rug, 8 ft. in diameter, is placed on a rectangular floor, 10 ft. by 12 ft. The part of the floor not covered by the rug is to be tiled. (a) How many square feet of tile will be needed? (b) About how many tiles will be needed if each tile measures 9 in. by 9 in.?

13. The pressure on the piston of a sports car engine is 70 lb. per sq. in. What is the total pressure on the piston if it has a diameter of 3.4 in.?

14. A circular wading pool, 28 ft. in diameter, is surrounded by a concrete walk 7 ft. wide. Which do you think is larger in area, the pool or the walk? ________________ Prove your answer.

15. At the center of one side of a house 30 ft. on a side, a dog is tied by a leash 40 ft. long. What is the total area over which the dog can play?

AREA ACTION, PART 5(b)

1. Find the area of the surface of the circular flower bed in the school courtyard.

2. Find the area of the base of the storage tanks at the Esso terminal.

3. Find the area of the "bullseye" and of the successive rings of a dart board.

4. Find the area of the bottom and the top of an angel-food cake pan.

SUMMARY

In studying the materials and the discussion of their use the design principles should be apparent. The materials provide students the opportunity to "play" with mathematical ideas in a non-complex feedback-rich setting. The instructional settings are varied; there are concrete materials labs, class discussions, problem lab sessions and project labs each with a particular purpose. There are two important by-products of the use of such materials and settings. The low ability student gets a rare opportunity to learn on his own. Further, the teacher assumes non-authoritarian roles as guide, instructional manager and small group tutor.

The authors were responsible for a large scale controlled experiment, testing the materials and design principles discussed herein. The results were very encouraging. As well as reacting very favorably to the pedagogical variety, students in the lab groups were significantly higher than a non-lab group in achievement, in attitude towards mathematics and in feeling that they had learned to work independently.

It is hoped that the reader will try to apply the design principles outlined above in developing instructional settings for your low ability students. Perhaps you will find your teaching frustrations turned to teaching fun.

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General Guidelines for Creating Mathematical Experiences

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This paper formed the core of a talk entitled "Do-It-Yourself Mathematics" presented by Professor Bale at the Winnipeg Meeting of the NCTM in October, 1970.

INTRODUCTION

The original title of this paper, "Do-It-Yourself Mathematics," has a double meaning. In the first place, it refers to mathematics the student can discover and create for himself. In the second place, it is concerned with the mathematical experiences that the teacher can create for this purpose. There are many such published experiences, but most of these involve two problems. The first is that the search for and implementation of the published experiences may sometimes be as difficult as creating one's own experiences. The second problem is that of finding suitable experiences. In particular, most published experiences are concerned with skills in the basic operations, or measurement and geometry concepts appropriate for the elementary school, and few cover the other kinds of topics of the regular junior high school mathematics programs.

The intent of this paper is to indicate a few general and widely applicable guidelines for creating mathematical experiences on any desired topic. These guidelines are at present in an embryonic stage, as they have been gleaned from recent analysis of personally created mathematical experiences for junior high school students. In other words, there are no claims of finality. The guidelines are open to considerable modification. They are merely a beginning.

How does the teacher devise these experiences? With great difficulty. There is more truth than humor in this remark. In this kind of activity an element of art is invariably involved. This art will often manifest itself as the experience of the teacher grows. Hopefully, the guidelines will help the teacher get off the ground.

GENERAL PROCEDURES

Assuming the teacher has decided that he would like to create a mathematics laboratory experience for a given topic, he should first list the concepts and skills involved. These should be closely studied in order to determine behavioral objectives and to generate necessary subordinate concepts and skills. The next step is to list general and specific applications. This is the point at which the first real difficulty arises. For many topics, appropriate specific applications are not easy to find. Although it is desirable to have these, the teacher should not waste too much time in their search because there is a feasible alternative, the contrived experience. The main discussion in this paper will be concerned with the contrived experience. Let us assume that the teacher has chosen this kind of experience. The next section will deal with general guidelines for creating contrived experiences. A useful summary of the general procedures appears in the form of a check list as below:

1. Choose topic.

2. List concepts and skills in hierarchical form.

3. List subordinate concepts and skills in hierarchical form.

4. Determine general and specific applications.

5. Decide between application or contrived experience.

GUIDELINES FOR CONTRIVED GAMES

At the outset, a decision between investigative or game experiences must be made. Although either or both types can be used, it is generally more appropriate to use investigative activities for applications and game activities for contrived experiences. Examples of both these types will be given in later sections.

The first guideline is to *ensure that constructive experiences precede analytic ones*. Constructive experiences are synthetic as opposed to analytic. The student constructs or builds the concepts for himself in an informal way. Keep in mind that this is only a general guideline. If the experience is designed for post-conceptual practice in skills, this point will not apply, provided the teacher is sure that the concepts involved are intuitively understood. Furthermore, if the teacher is certain that the maturity or richness of the experiential background of the student is adequate, he may disregard this point. But if in doubt, apply it.

Aim at simplicity. It is all too easy to involve difficult concepts other than those for which the experience was devised. This is particularly true for game activities which easily can involve difficult procedures or rules. A danger is that the teacher may abandon devising laboratory experiences merely because he has made them more difficult than other methods he might utilize. One way to further the goal of simplicity is to *aim at only one game* or investigation for each unit may be considered a coherent idea. Later, in the light of experience, the teacher may opt to combine several of these.

Consider the possible experiences on an abstract-concrete dimension. Some students may be ready for the abstract at the outset, whereas others may need to work with manipulative experiences at first and graduate to abstract experiences later.

Divulge only the procedures for playing the game, not the concepts to be learned. For each move in a game there should be several *alternatives*. Application of the concepts to be learned should ensure making the best move. *The instructions should be minimal* and allow for student choice and modification.

A checklist of general guidelines for contrived games appears below. These guidelines also apply to application games, and by substitution of the word "investigation" for "game", one can equally well apply them to application or contrived investigations. In addition, for investigations one should attempt to make the outcome important for the students.

1. Constructive precedes analytic.
2. Aim at simplicity.
3. Aim at only one game for each idea.
4. Devise several concrete and abstract games.
5. Divulge only procedures, not concepts to be learned.
6. Provide moves or strategies which constitute alternatives for which application of the concepts to be learned ensure the best strategy.
7. Provide minimal instructions and allow for choice and modification.

SAMPLE CONTRIVED GAME EXPERIENCE

TOPIC: Factorization (*Seeing Through Mathematics, Book One,* Van Engen, et al., ch. 70)

CONCEPTS: Factor, prime, prime factor, unique factorization, g.c.f.

SKILLS: Recognition and production of above.

SUBORDINATE CONCEPTS: Product, quotient, divisibility, powers, factor, property of the number "1".

APPLICATIONS: General - dividing and subdividing situations.
Specific - many for factors, but none feasible for other concepts.

DECISIONS: 1. Contrived experiences.

2. Game activities

3. Assume knowledge of subordinate concepts sound.

Game 1 - Factors and Primes

Procedure

One player takes a pile of chips (or counters). Each player takes these in turn and makes his move. This procedure continues until no further moves are possible. The last player to make a move picks a new quantity. The game ends after a pre-arranged quantity of different piles has been used.

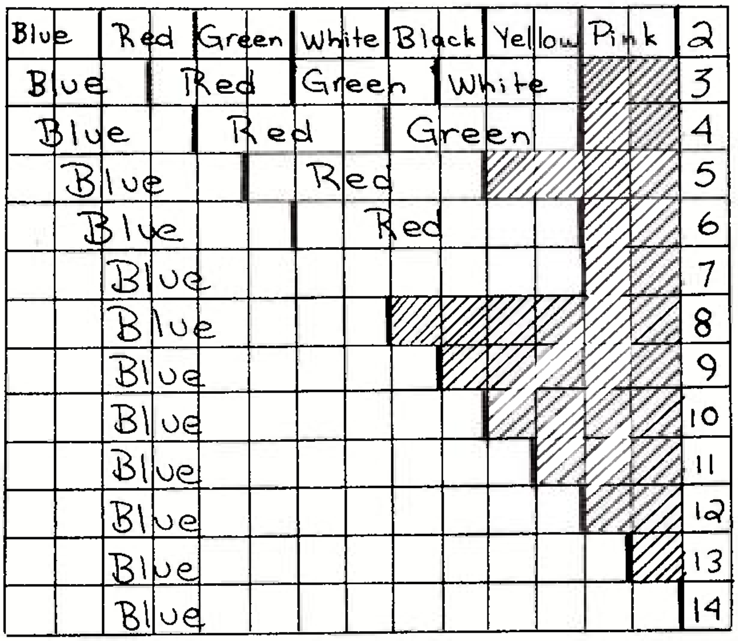
Moves

Place the chips in one row on squares of one color so that there are at least two on each square and none left over. Each succeeding move is up or down the board, but never to a row already used.

Score

One point for each successful move. Player with most points at the end of the game wins.

Game Board



Game 2 - Prime Factors and Primes

Same as game 1 except scoring and move changes as follows:

1. If a player can make a move (as previously described) of the chips on one square to a lower numbered row, he takes another move and the previous player loses one point.

2. If a player can make a move (as previously described), of all the chips in a row to a lower numbered row not already used for that pile, he takes another move and scores two points.

3. These new moves should be only demonstrated, not made.

Game 3 - Prime Factorization

The player may use as many squares in a row as he desires, but he must place the same number of chips (at least two) in each square, and this number must be equal to the row number. The other rules are the same as for game 2 except that each player can take one chip from each square in a row for a type 1 demonstration move.

Game 4 - Common Factors

Use two piles of different quantities, and two boards. Same rules as game 1 except that each player must make the same moves on both boards to score points.

Game 5 - Greatest Common Factor

Use the two board procedure of Game 4. Only moves of all chips from a row to a higher numbered row are allowed. A player scores one point for each of these moves unless it is the last move for a particular pile, in which case the score is two.

SAMPLE APPLICATION INVESTIGATION

TOPIC: Compound Conditions for More Complex Problems
(*Seeing Through Mathematics*, ch. 39)

CONCEPT: To solve conditions like $b + g = 11 \wedge b - 3 = g$

Application Investigation - To Solve the Condition Mentioned

The students are asked to undertake the following activity:

From their class of 15 boys and 9 girls (the numbers in the condition can be slightly varied to fit the situation of a particular class) the students are asked to compose two teams for a proposed ball game (if possible the game should actually be played at a later date). There must be two umpires, the same number of girls on each team, and three more boys than girls on each team.

In accord with the abstract-concrete guideline, the students could be allowed to set up the condition and solve it, or write the names on paper, or each name of class members on separate sheets of paper, or, for an even more concrete experience, physically attempt to select boys and girls for the teams.

The investigation can be converted into a game by using the teams proposed by the first student who arrives at a correct solution, and allowing choice of position on team in order of completion of correct solutions.

THE TEACHER'S ROLE

Finally, a few words concerning the teacher's role. The amount of intervention by the teacher will vary. It will likely be considerable, but it should take place in a special way. The teacher has already intervened by devising the experience. Further intervention should take only the following forms:

1. Clarification of procedures, but not concepts to be learned.

2. Evaluation and diagnosis of student progress.

3. Modification of the experience, direction to other experiences, or encouragement of the student to modify the experience himself.

Arbitration disputes arising from games should be handled tactfully, most likely in a mode in which the teacher has achieved previous success.

Finally, the ultimate aim of the teacher should be that the student react with nothing else but the mathematics of the situation.

An Active Learning Unit on Real Numbers

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A complete 160-page copy of Mr. Fisher's *Active Learning Unit on Real Numbers*, containing detailed descriptions of, and teachers' guides for, the 46 student activities developed is available to members on request from Barnett House, 11010 - 142 Street, Edmonton 50. Non-members may obtain a copy at a cost of $1 from Barnett House. Mr. Fisher is presently teaching in Nigeria on a Canadian external aid program.

INTRODUCTION

The active learning unit on real numbers described in summary form on the following pages was developed to cover all of the mathematical concepts listed in the unit entitled "Introduction to Real Numbers" in the *Program of Studies for Junior High Schools of Alberta* (Alberta Department of Education, 1969). Activities that would foster an understanding of the concepts identified above and that would promote active learning on the part of students were collected from a wide variety of sources and modified to suit the needs of the unit or, where necessary, the author created original activities to fill a particular requirement. Many more activities were collected (46 in all) than would be needed by any particular student. The idea was to provide the student and teacher with a choice of activities based on individual interests and needs.

While the author attempted to provide students with many types of materials to provide a variety of experiences, the mathematical goals of the unit were kept in mind. It was easy to find new materials that were interesting, stimulating, and easily learned, but unless they served purposes other than just motivation and pleasure, they were not included.

Every effort was made to make the activities "ready-to-use" for both student and teacher, subject to the teacher's judgment as to whether, or in what way, any particular activity would be used.

The activity unit was tried out with eight classes of Grade VIII students from three Calgary Junior High Schools during April, May, and June, 1970. Evaluation of the feasibility of the unit was based on interviews with each of the four participating teachers as well from a checklist evaluation of specific activities and from student comment sheets. Using criteria that included differences in content, instructional demands, instructional effectiveness, and student attitudes, the real numbers active learning unit was judged by the teachers as preferable to the conventional textbook oriented approach. The four teachers indicated that they would use a majority of the activities in the real numbers unit in following years. Data collected on student achievement and attitudes indicated achievement not significantly different from that of control classes (following the conventional approach) and significantly more positive attitudes towards mathematics in the active learning classes as compared with the control classes.

What follows is a summary of some of the activities included in the active learning unit on real numbers.

SQUARES AND SQUARE ROOTS

The reason for including the activities on squares and square roots was to help introduce the primary new concept in promoting an understanding of the real numbers, namely the concept of an irrational number. The specific objectives of the activities were to have the students know how to square a rational number and how to find the square root of a rational number.

The student activities on squares included discovering number patterns based on square numbers or perfect squares, matching a rational number with its square, and graphing the relationship between a number and its square for the integers from 1 to 10. By the drawing of a continuous line plotted from the latter activity, the student was able to use the graph to find the approximate square of rational numbers such as $4\frac{1}{4}$ and $7\frac{1}{2}$. This same graph was used in the following section to find square roots. In total, there were eight activities on squares. This did not mean that all students were expected to complete all of these activities but that the student was able to exercise some freedom of choice based on his interest and abilities.

Finding the square root of a rational number presented more of a challenge to the student. The initial activity introduced the concept that finding the square root of a number was the opposite to finding the square of a number. This was accomplished through the use of a number of cutouts of perfect squares with only the area labeled on the paper square. The student's task was to select a paper square and then deduce the square root of the number (area) by guessing the length of the side of the square. By checking his answer, he quickly confirmed the correctness of his guess. No rulers or measuring instruments were permitted.

Following this, the symbol for square root ($\sqrt{\ }$) was introduced along with a simple exercise on finding the square root of perfect squares. How to use a table of squares and square roots was included, and the student used the one found in the back of the text. (Please note that all references to the student text refer specifically to *STM2*, Van Engen *et al.*, 1964). To this point, the student had not been taught how to calculate square root and the challenge to discover a method on his own was initiated by asking him to find the exact square root of 2. While this is impossible, the intention of the activity was not to frustrate the student but to acquaint him with the problem of calculating the square root, in addition to giving him the opportunity to devise his own method. Three different methods for calculating square root were presented to student and teacher: the guess method, an adaption of Newton's method, and the algorithmic method. Based on his previous experience of trying to calculate $\sqrt{2}$, the student was free to choose the method he preferred. The final activity in this section offered the student the opportunity to calculate $\sqrt{2}$, again using one of the previously mentioned methods of calculation as opposed to the trial and error method previously used. Emphasis in these latter activities was on understanding the concept of square root and not expertise in calculating square root.

IRRATIONAL NUMBERS

The activities on the square root of a number were introduced primarily to indicate the need to develop a number system that is an extension of the rational number system. In the previous unit, the students studied the rational number system and discovered that all rational numbers could be expressed as repeating, infinite decimals. However, the activities on finding the square root of rational numbers introduced numbers that could not be expressed as repeating, infinite decimals (or rational numbers). Thus the students were introduced to irrational numbers.

The specific objectives of this part of the unit were to establish student understanding of the difference between a rational and an irrational number and that the set of real numbers is the union of the rational and irrational numbers. The initial activity dealt with the first objective by presenting the students with a mixture of infinite decimals and asking them to sort them into two groups. Hopefully the student would discover that some of the infinite decimals repeated and some did not, leading him to sort the two groups on this basis. The existence of irrational numbers was further illustrated with a copy of a computer calculation of π to two thousand decimal places. Students were challenged to find a repeating pattern among the decimals. Further evidence of the existence of irrational numbers was supplied by means of a fairly elementary proof that numbers like $\sqrt{2}$ are not rational. The proof was available to the interested student and was not used as a classroom exercise. A story about the discovery of irrational numbers by the Pythagoreans and their attempt to keep this discovery a secret was also included for historical background and interest. Opportunity for students to create their own irrational numbers was provided by means of two activities, one of which involved students rolling numbered cubes and recording the numbers on the top face after the placing of a decimal point. By continuing this procedure, the student generated his own non-repeating decimal.

The extension of the set of rational numbers to include the set of irrational numbers produced what was defined as the set of real numbers. This concept was reinforced with a game that could be played with a small group of students. Cubes from the game of *TUF* (Avalon Hill Company, 1969) were used to play a game whose purpose was to review previously studied number systems, in addition to learning more about irrational numbers and the real number system. Following rules based on *The Real Numbers Game* (Allen, 1966), the students took turns shaking the five cubes of dice. After rolling the cubes, each student playing tried to detect and write down as many numbers of a particular kind as he could from the symbols that appeared on the top faces of the cubes. The winner was the player who identified the most correct numbers of the kind specified. For example, players might be asked to identify integers, positive rational numbers, negative irrational numbers, or real numbers, to name a few. Since the main purpose of the game was to learn about real numbers, one of the rules stated that at least one of the rolls of the cubes must specify irrational numbers. As a special challenge, students were invited to use the materials supplied to make up new rules or a different game of their own.

For example, imagine that the following symbols appeared on the top faces of the cubes rolled on the playing surface:

$$4 \qquad 7 \qquad \frac{1}{2} \qquad - \qquad \sqrt{}$$

If the set of irrational numbers was called, each of the following numbers could have been listed:

$4 - \sqrt{\frac{1}{2}}$,	$\sqrt{\frac{1}{2}} - 4$,	$\sqrt{7} - \frac{1}{2}$,	$\frac{1}{2} - \sqrt{7}$,	$\sqrt{7\frac{1}{2}}$,	$\sqrt{4\frac{1}{2}}$,
$\sqrt{47\frac{1}{2}}$,	$\sqrt{74\frac{1}{2}}$,	$-\sqrt{7}$,	$-\sqrt{\frac{1}{2}}$,	$\sqrt{7 - 4}$,	$-\sqrt{47}$,
$\sqrt{7}$,	$\sqrt{\frac{1}{2}}$,	$-\sqrt{7 - 4}$,	$\sqrt{47}$,	$\sqrt{74}$,	$\sqrt{47 - \frac{1}{2}}$,
$\sqrt{7 - \frac{1}{2}}$,	$\sqrt{4 - \frac{1}{2}}$,	$\sqrt{74 - \frac{1}{2}}$,	$7 - \sqrt{\frac{1}{2}}$,	$-\sqrt{74}$,	$-\sqrt{7\frac{1}{2}}$,
$-\sqrt{47\frac{1}{2}}$,	$-\sqrt{74\frac{1}{2}}$,	$4 - \sqrt{7}$,	$-\sqrt{4\frac{1}{2}}$,	$\sqrt{\frac{1}{2} - 7}$,	$47 - \sqrt{\frac{1}{2}}$.

We have listed only 30 of the possible irrational numbers. The reader might want to determine the composition of the entire set. In the class, the winner would be the student with the highest point total based on a scoring system of "right subtract wrong".

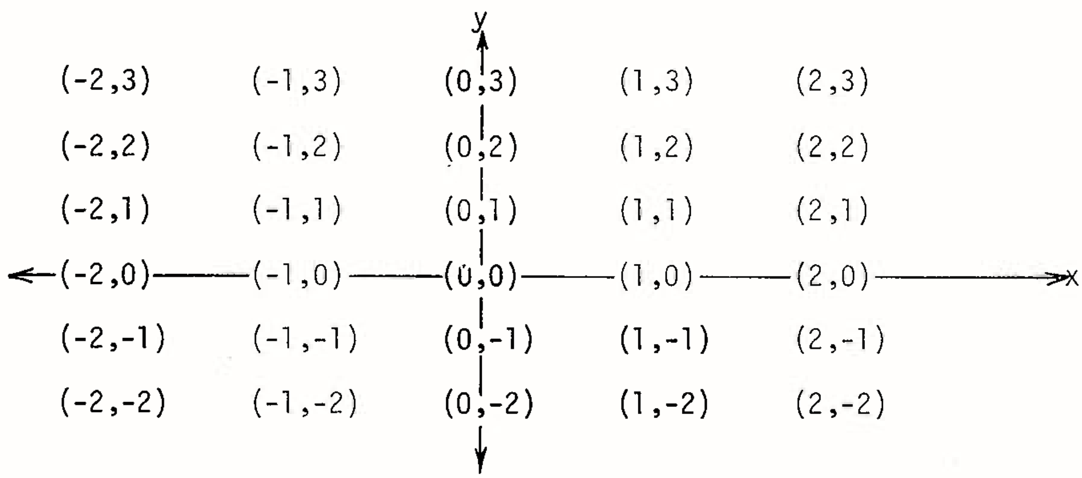
Activities based on the real number line completed the main study of this section. The problem of fitting an irrational number onto the rational number line was dealt with. One method used was based on a geometric exercise involving a triangle with a hypotenuse of measure $\sqrt{2}$. By measuring the hypotenuse with a compass, the student could establish the approximate position of $\sqrt{2}$ in relation to the origin on a number line by describing an arc. To challenge the better students, activities on the Pythagorean Theorem and the construction of measures of irrational numbers were included. One of these involved the construction of a very artistic geometric figure called Archimedes Spiral.

GRAPHING IN THE REAL NUMBER PLANE

The specific objective of this section was to extend the student's understanding of the rectangular coordinate system by introducing the real number plane which included all four quadrants and the real number lines separating these quadrants. The student's previous background had largely been limited to plotting ordered pairs of natural numbers in the first quadrant. Behavioral objectives were specified in terms of associating points on the real plane with ordered pairs of real numbers.

This section of study provided teachers with some excellent opportunities to encourage active learning among their students. Avoiding a traditional, expository approach to an extension of the rectangular coordinate system, one teacher played the game of Tic-Tac-Toe with the students as they had previously done, and then extended the coordinate system to encourage the students to discover how to name points in the other three quadrants. Once the students had discovered the system, their enthusiasm was promoted further with games like "Battleship" and "Go". Both of these games are excellent for practicing the use of the rectangular coordinate system and can be played with two teams of one or more players a side. "Battleship" is a very good activity for students of all abilities, and any player quickly catches on to the naming of points in the coordinate system if he wants to sink the enemy navy and win the battle (Biggs and Maclean, 1969: 51). "Go" is more correctly called "The Point-Set Game" (Davis, 1964b: 55-66) and is a more sophisticated game perhaps more suitable to the keener mathematics student, but still a good class activity.

Another introductory activity to the rectangular coordinate system was called "Living Coordinates" (School Mathematics Study Group (SMSG), 1965b: 109-112). An excellent activity involving every student in the class, each student was identified by an ordered pair of numbers according to his position in the classroom seating plan. For example, with a class of 30 students arranged with 6 students in each row, the following coordinate system might be used:



Each student identified himself by one of the above ordered pairs. The activity consisted simply of students standing in response to specific questions, such as:

1. Will all of the students in the third quadrant stand?

   e.g. (-2,-1), (-1,-1), (-2,-2), (-1,-2) only.

2. Will the Y axis stand?

3. Will those students whose x coordinate is negative stand?

4. Will all those students whose x and y coordinates have opposite signs stand?

After students have been introduced to linear equations and inequalities, the living solutions of these can also be graphed.

Plotting a series of ordered pairs and joining them with line segments to produce caricatures of Snoopy or Mickey Mouse was also a motivating activity for most students. The students were encouraged to try to make up their own plotting activity based on some figure of interest to them, perhaps Yogi Bear or Robin Hood. A coloring book is helpful in making up these ordered pair sketches. As further enrichment for this type of activity, students were shown the technique for "transforming" or changing Snoopy by performing different operations on the ordered pairs. For example, Snoopy's face can be distorted sideways by doubling all the x values of each ordered pair and leaving the y values the same.

The major activity introduced in this section was a biological experiment based on the growth of a culture of mold (SMSG, 1965a: 137-148; SMSG, 1965b: 112-118). The purpose of this experiment was to show the student how certain mathematical procedures could be used to study basic biological processes. In this particular experiment, the growth pattern of mold was the biological process studied and all data necessary were gathered and analyzed by the students. The experience and knowledge gained from the previous plotting activities provided sufficient mathematical background for this activity.

Materials needed for the experiment included aluminum pie tins, graph paper, gelatin, a bouillon cube, "Saran Wrap", scissors and rulers.

Working in groups of two to four, students proceeded according to a set of instructions which included the preparation of the pie tin with graph paper on the bottom, and the mixing and pouring of the gelatin mix onto the graph paper in the pie tin. The gelatin mix was left to set while exposed to the air where it became contaminated with mold spores. After approximately five to ten minutes, the tin was covered and sealed with Saran Wrap and stored in a warm, dark cupboard.

As soon as the mold became visible (perhaps two to three days later) the student's task was to identify the position of each dot of the mold on the coordinate grid at the bottom of the tin and transfer its position to a separate piece of graph paper. This was to be done every day, and the accumulated data were tabled and analyzed according to a detailed set of instructions. A graph of the growth of the mold based on the student data was plotted, and students were encouraged to describe and predict the growth of the mold on the basis of their results.

To this point, the student had completed the major portion of the mathematics involved. The further study of the biological process illustrated by the growth of mold was encouraged because of the many pertinent and interesting topics that were related to this experiment. These topics included the study of the growth curves of other living organisms such as gourd fruit, bacteria, chickens, corn plants, the liver and brain of human boys and girls, as well as the population growth curves of the United States and Canada. In addition, the study and analysis of these growth curves contributed not only to an understanding of the biological phenomena of growth, but also served as an opportunity to discuss and study the problems of population explosion, birth control and pollution.

THE REAL NUMBER SYSTEM

The solution of problems is often a major source of frustration for both teacher and student. It is also one of the most important parts of any secondary mathematics course. This section of study attempted to alleviate student difficulty in problem-solving with many different methods and activities.

To begin with, the students were introduced to an optional review of the field properties, but with non-numerical examples instead of the exclusive use of abstract numbers. The non-numerical examples were used to make the properties more meaningful to the student through seeing their use in a different context. The commutative property, for example, might be illustrated by putting on the left shoe and then the right shoe, or vice versa, with the same result each time. A similar example would be putting on your coat and hat. The two operations of going through a doorway and opening the door provide a non-commutative example. In addition, students were encouraged to exercise their own imagination by making up some examples of their own to illustrate the properties studied.

The real number field is a mathematical number system and the concept of a number system is relatively abstract to the majority of students. For this reason, a number of experiments dealing with finite mathematical systems were suggested to the teachers. These non-numerical mathematical systems served to illustrate some of the real number properties through student discovery activities with concrete materials. As Mansfield (1966: 5) has pointed out ". . . the best way to grasp an idea is to examine many different manifestations of it." Considering the emphasis on number properties in recent mathematics curricula, any attempt to make the properties more meaningful to the student seemed in order. In addition, the activities or experiments chosen served to promote an active learning approach.

For example, a version of command arithmetic (Cleveland, 1969) used four commands: right face, left face, about face, and front face. By using a student "soldier", the command right face followed by about face was found to be equivalent to the command left face. Through completing an operation table, the students worked out all possible combinations of the above four commands. The activity then became more of a challenge, with the student facing the task of determining which of the properties under consideration were true for this system. Students tested the closure, commutative, associative, identity and inverse properties. Similarly, they tested the same properties in different systems based on activities that included rotating hexagons, flipping pennies, manipulating colored rods and flipping rectangular cards.

Solving algebraic conditions is of primary importance in the Grade VIII course, and the learning of algebraic properties should help most students with this problem. To motivate students to learn the many properties used (about 18), a game called the "Property Bee" was created. This was a class competition between two teams with the winner being determined by the side with the most points. Points were awarded for each correct property identified in solving a particular condition and a point was subtracted for each wrong answer. Most teachers played the game using an overhead projector and transparencies of chains of conditions. For example, the following solution might have been displayed:

A.	$X - 3.7 = {}^{-}17.6$	Given
B.	$X + {}^{-}3.7 = {}^{-}17.6$	________
C.	$(X + {}^{-}3.7) + 3.7 = {}^{-}17.6 + 3.7$	________
D.	$X + ({}^{-}3.7 + 3.7) = {}^{-}17.6 + 3.7$	________
E.	$X + 0 = {}^{-}17.6 + 3.7$	________
F.	$X = {}^{-}17.6 + 3.7$	________
G.	$X = {}^{-}13.9$	________

To derive condition (B.) from (A.), the Difference Property was used. If the student correctly identified this property, one point would be awarded his team. If his answer was incorrect, his team would lose a point and a student from the opposing team would try to provide the correct answer. This procedure would continue until the property was identified correctly, after which someone would attempt to identify the next equivalent condition in the chain. Obviously, a student needed to know the algebraic properties if he hoped to make points for his team and this fact, plus the enjoyment of competition, served to motivate students to learn their properties.

The property bee served as an excellent activity for introducing three new algebraic properties of inequalities. Students had not yet learned to solve inequalities like the following by using the less than or greater than properties. For example:

A. $\frac{3}{5}W > {}^{-}7$

B. $\frac{5}{3}(\frac{3}{5}W) > \frac{5}{3} \cdot {}^{-}7$

C. $(\frac{5}{3} \cdot \frac{3}{5}) W > \frac{5}{3} \cdot {}^{-}7$

D. $1W > \frac{5}{3} \cdot {}^{-}7$

E. $W > \frac{5}{3} \cdot {}^{-}7$

F. $W > {}^{-}11\frac{2}{3}$

In the above example, condition (B) was derived from condition (A) using the "positive multiplier property of less than". To this point in their course, students had not learned this property or the other two properties, "the sum property of less than" and "the negative multiplier property of less than". By learning these properties, students were introduced to a more sophisticated approach to solving inequalities, which would be invaluable in solving inequalities of greater difficulty.

The method suggested for learning the new properties was to introduce an inequality like the above in the middle of competition in the property bee. Unless there was a well-read scholar in the class, the game would become stalled. At this point, the teacher could suggest that the class find out about this unknown property that had spoiled their game. This was an ideal opportunity to present the class with some simple conditions of inequality whose solution depended on the unknown property. Rather than show the students how to solve these conditions and tell them the name of the property, a discovery approach was suggested and students were encouraged to come up with their own method of solving conditions involving inequalities. If, for example, a student called Gary worked out a method for solving an inequality, the method might be called "Gary's Rule". If Gary's Rule was equivalent to the unknown property, then the problem would be solved and, once the students agreed to and understood Gary's Rule, the property bee could resume. Similarly, the remaining two properties of inequality could be taught. By actively involving the students in the discovery of these new properties, the students were likely to enjoy and understand their use and meaning.

Prior to engaging in the solution of a given condition on their own, students were presented with a variety of puzzles based on chains of equivalent conditions. The puzzle would be to sort out a mixture of equivalent conditions and put them in the correct order. For example, the student might pull out the following assortment of conditions from a particular envelope labeled $4w = 3(w + 8)$:

$$
\begin{aligned}
(^{-}3 + 4)W &= 24 \\
4W &= 3W + 24 \\
{}^{-}3W + 4W &= 0 + 24 \\
1W &= 24 \\
{}^{-}3W + 4W &= {}^{-}3W + (3W + 24) \\
4W &= 3(W + 8) \\
{}^{-}3W + 4W &= 24 \\
W &= 24 \\
{}^{-}3W + 4W &= ({}^{-}3W + 3W) + 24
\end{aligned}
$$

The student task was to put these equivalent conditions in the correct order, starting with the given condition $4W = 3(W + 8)$. The student needed a knowledge of the properties used to derive the equivalent conditions in addition to a familiarity with the technique of solving conditions. An answer key was provided with each envelope to assist the student in checking his work. The teacher's task was simply to assist any students having difficulty.

Another use for these puzzles was to challenge the student to solve the given condition without opening the envelope. If a conscientious effort by the student was unsuccessful, the student could seek teacher assistance or simply open the envelope and check the answer key. Thus the students were free to work and learn on their own, and the teacher was involved only with students who needed him.

A different approach, and perhaps more helpful to some students having difficulty with solving conditions, was an inquiry method based on the teacher encouraging the students to seek their own methods for solving conditions. The use of a number line to graph their solutions was also suggested. In all previously mentioned activities involving properties and conditions, the ultimate objective was for the student to be able to solve conditions on his own. While the chains of equivalent conditions used in the property bee and the chain puzzles might have served as an example of a method of solving these conditions, there was no intention of restricting students to a particular method. Students were encouraged to discover their own ways and look for short-cuts in the case of a lengthy solution in the process of learning to solve algebraic conditions.

The translation of an English sentence to a mathematical sentence is a major student difficulty in solving word problems. Opportunity to develop this skill was provided in one of the activities which also included suggestions for students to reverse the translation process by making up a problem for a condition like $2x + 5 = 10$. Another activity of a similar nature, but of far more interest and challenge to the students, was "magic algebra" (Glenn and Johnson, 1961). In addition to preparing the student for word problems, magic algebra provided him with valuable practice in solving algebraic phrases and creating his own algebraic tricks.

Consider the following example: The student is given these instructions.

Step 1 - Pick a number

Step 2 - Add 5

Step 3 - Multiply by 2

Step 4 - Subtract 8

Step 5 - Divide by 2

Step 6 - Subtract the number you started with.

The "magician" in charge announces that his "mystical powers" have resulted in everyone finishing with the number 1 (unless the student has made a mistake).

The teacher's task is to stimulate his students to solve his magic trick. One suggestion, if needed, is to use a variable, letter, or place-holder to represent the original number chosen. The following solution should be typical of the end result.

Student Instructions	Algebraic Solutions	
Step 1 - Pick a number	N	N
Step 2 - Add 5	$N + 5$	$N + 5$
Step 3 - Multiply by 2	$2N + 10$	$2(N + 5)$
Step 4 - Subtract 8	$2N + 2$	$2(N + 5) - 8$
Step 5 - Divide by 2	$N + 1$	$\frac{2(N + 5) - 8}{2}$
Step 6 - Subtract the number you started with.	1	$\frac{2(N + 5) - 8}{2} - N$

Once the student understands the "magic" involved, he can be encouraged to make up an algebraic trick of his own, perhaps ending with the number 25 or 13, for example. The secret is to design a set of operations that will eliminate the original number chosen and replace it with a number that you have chosen.

Many other magical tricks can be introduced with the "magician" amazing everyone by telling a student his age, the amount of change in his pocket, the number in his family, the house number, his birthdate, and many others, all based on similar algebraic manipulations. Again, the creative task for the teacher is to encourage his students to not only solve these magic tricks but to try and make up ones of their own.

Perhaps the best example of active learning is the activity card approach to problem-solving. Pethen (1968: 11) has created an excellent card program for the elementary grades with the purpose of "providing practical mathematical experience for 12 or 15 or 20 pairs of children, with each pair able to proceed at its own best speed and at its own most suitable level ...". With the same intent, more than 50 problems were provided for each classroom. The problems varied in degree of difficulty, as well as the type of activity or experience intended, from simple pencil and paper exercises to the gathering of scientific data based on classroom or environmental experiments. The students could work on the problem card of their choice individually or in small groups. In most cases, the pupils were able to read the card, gather any information or material necessary, and solve the problem with little or no help from the teacher. However, it is essential that the teacher discuss the results of many of these problems with the group to ensure a real understanding.

More specifically, the purpose of the activity card approach was to allow children to:

1. Discover for themselves the essential methods of problem solving.

2. Work at their own ability level and rate of learning.

3. Work on problems of particular interest to them.

4. Learn from each other and work cooperatively with their peers.

A good introduction was necessary to initiate a high standard of thinking, working, and discussion in addition to arousing interest. Pethen (1968: 12-13) suggests the following guidelines for students in order that they know what is expected of them:

1. The card is the student's problem, the student's challenge.

2. The card should be read carefully and discussed by both partners.

3. If the students are clear about what is to be done, they should proceed on their own (including any collection of materials and information).

4. The student should ask the teacher for assistance only as a last resort.

5. He should show and record his work on paper.

6. He should make sure the card is completely answered.

7. He should check with the teacher when finished. If the teacher is busy, then he should proceed with another card and check his previous problem later.

The choice of problems for these cards was not intended to limit their use in any way. Teachers were free to adapt, add to, or delete from the set of cards as they saw fit, according to the needs and interests of their students and available materials. In fact, teachers were encouraged to do this to prevent the activity cards from losing their interest and effectiveness.

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A Discovery Unit on Quadratics

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Although it has been common practice to develop a theory of learning or concept development and then to suggest implications from the theory regarding teaching practices, it is possible to proceed in the reverse direction, namely to identify good teaching practices and, on the basis of these, to generate theory - not only a theory of instruction but also of the psychology of learning and motivation. It is the present authors' contention that this is the more fruitful approach to developing new ideas about instruction. Perhaps the best example of such an approach has been the work of Robert B. Davis in the Madison Project.

Davis has, over the past 10 years, developed mathematics teaching materials (1) (2) for use in classrooms with children from Grades III to IX. From the interesting classroom situations that arise from his teaching approach - a "discovery" approach - Davis has more recently developed a portion of a "theory of instruction" (3). The present paper will follow a similar pattern in relegating theory to a secondary position and practice to the forefront. In fact, why do we need theory at all if we can develop the practice without it? This is a highly relevant question but perhaps even more crucial is the question: "What kind of theory do we need?" The answers to these two questions could fill volumes but perhaps it will suffice to say that general principles (theory) will aid in developing new practical classroom experiences, and, in our opinion, the kind of theory we need is not psychological theory but what might be called instructional theory which would consider groups of students with one or more teachers in a classroom engaged in learning complex concepts.

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Because of this orientation toward instruction and theory, we have chosen to present an actual classroom unit of discovery teaching and then try to make theoretical statements about why and in what areas the unit was successful. The basic notion behind the development of this unit is to fill a void in current teaching practices. The extreme emphasis in our mathematics curriculum at all levels is on content; we do not spend enough time encouraging the student to think creatively about mathematics. A comment Piaget made at a conference in the United States is certainly relevant in this regard.

> The goal in education is not to increase the amount of knowledge but to create the possibilities for a child to invent and discover himself.[1]

In recent years, attempts to overcome this deficiency have been made by an incorporation of discovery methods into our instruction. Although thousands of pages have been written on discovery teaching, few attempts at a systematic development of a unit have yet appeared. This is especially true at the senior high school level. The present paper intends to begin to fill this void by presenting a discovery-oriented unit at the high school level. The activities on the quadratic presented here have been tried in classrooms and another report expanding on some aspects of these ideas can be found in a master thesis written at the University of Alberta (4).

Most often today we think of discovery lessons in connection with the elementary school or at most the junior high school. Perhaps it is because we believe that the discovery method of teaching is at best inefficient and we don't think it appropriate to waste the time of students in the senior high schools. Or perhaps we feel that the senior high student isn't prone to the same type of playful, non-purposive activity as is the elementary school student, the kind of activity that is essential for any discovery lesson. Or perhaps we feel that the type of mathematical concepts learned in senior high school do not lend themselves to discovery learning. The authors are prepared to argue against any of these suggestions, but theoretical arguments are not what we need, but rather practical examples. The intent of the next few pages of this article is to show the reader how it is possible to organize a unit of about 10 activities on the quadratic function in a manner that will allow students to discover relationships pertaining to the quadratic equation and the quadratic formula.

A VIEW OF DISCOVERY

But first, what is discovery? In a mathematics learning context, it involves putting students in a situation to which they can react so that, hopefully, they will get feedback from mathematics. That is the essential element. It is most important that the teacher keep out of the learning situation. The student should arrive at a conclusion because the mathematics has suggested it to him, not because the teacher has suggested it. So we ask ourselves the question: "Is it possible to engage the student in mathematical activity that will provide him with feedback about the quadratic equation?" However, rather than look for one activity, we will look for a series of activities or a series of problems to present to the class.

[1]Eleanor Duckworth (citing a verbal comment made by Piaget), "Piaget Rediscovered", *The Arithmetic Teacher,* 11: 498, November, 1964.

The rationale for breaking the unit down into a number of activities rather than one is threefold. First of all the objective of a discovery lesson is to structure the students' activities so that a discovery is within their reach; that is, the objective is to structure the environment so that relationships are seen more readily. In order to do this for the quadratic function and the amount of learning expected, we have to think of more than one activity. The second reason for introducing a number of activities is a practical one of keeping the class of 30 students together. Some undoubtedly would question the validity of keeping 30 students together, thinking students should be allowed to work at their own rates. All we say to this is that our school system is not set up for this type of instruction; that is, we operate with 30 students in one instructional unit. A third reason for structuring the lessons in a series of activities is to allow the students to suggest some of the next activities. Clearly, as the unit proceeds, it will become more obvious to the students where the conceptual developments are leading and consequently more obvious what should follow. This last reason relates to the ultimate goal of discovery as viewed by the authors - if, in fact, it is possible to give the students practice in thinking up activities so that they become skilled in doing this, then the objectives of mathematics teaching have been reached. If students can think up their own discovery (learning) activities, there are no bounds to the amount of mathematics they can learn. Here may well be the crux of discovery. Can students be taught to recognize and invent activities that will result in new information about mathematics? This indeed is creating mathematics and will surely be viewed by everyone as, even if idealistic, the desired end-product of mathematics teaching.

In structuring the activities for the discovery classroom sessions from which students could get feedback about mathematics, it was necessary to pose the initial problems of the activities in broad terms. In fact it was deemed desirable to encompass the whole unit in a broad frame of reference. The graph, that is, the picture of the quadratic function, seemed ideally suited for this purpose because not only is it a very primitive, and all encompassing notion, but also the students were already familiar with the Cartesian coordinate system and with plotting points and linear functions. The graph also has the advantage of being easily displayed by means of an overhead projector and as such is easy to talk about.

Brief mention should be made of the instructional processes[2] involved in the lessons. Each activity described below began with a broad question which the students were asked to answer. The usual procedure in dealing with the activity was for the students to explore on their own for a while, then hypothesize solutions, evaluate the hypotheses and finally engage in a summing up and practice session to formulate precisely any rules and to develop skill in any techniques that appeared to be useful. Perhaps the most important guideline used by the teacher was to let the mathematics speak for itself and allow students to evaluate each other's hypotheses and solutions to problems.

[2]An extended treatment of the instructional processes is available in Johnston's master's thesis. (4)

THE QUADRATIC

Activity One

The graph of the parabola, $y = x^2$, was displayed on the overhead projector. The parabola was placed on the points (0,0), (1,1), (-1,1), (2,4), (-2,4), (3,9), (-3,9), and the students were asked to come up with a rule relating the coordinates of the points.

The students were allowed sufficient time to formulate their ideas and the following suggestions were among those offered:

1. The space between any two points parallel to the X-axis is the same distance from the Y-axis, and they have the same height.

2. $x^2 - y = 0$

3. $y = x^2$

4. Other points can be predicted by the rule: "up by two's, that is, 1, 3, 5, 7,... along the Y-axis, and over one each time along the X-axis".

5. The y value for any point on the graph is never smaller than 0.

A discussion of each hypothesis followed, and the class agreed upon the rule:

$$\{(x,y) \mid y = x^2, x \in R, y \geq 0\}$$

Activity Two

The second activity was to find the effect on the rule of moving the original graph (same shape as $y = x^2$) to various points in the coordinate plane. The students were asked to try to generalize the formation of new rules when the graph was moved in a certain fashion consistently.[3] They moved the graph around and made observations as to new points and formulated new rules.

Some of the hypotheses resulting from this exploratory period were as follows:

1. From the origin, if you move the graph up on the Y-axis, the equation changes from $y = x^2$ to $y = x^2 + 1$, $y \times x^2 + 2$, and so on.

2. If you move the graph down the Y-axis, the equation changes from $y = x^2$ to $y = x^2 - 1$, $y = x^2 - 2$, . . .

[3]The instructor gave each student a piece of onionskin paper and a piece of graph paper. A graph of the parabola $y = x^2$ was drawn on the onionskin as related to the graph paper underneath. The students were told that they could move the vertex of the graph to any now point provided that they kept the graph in the vertical or horizontal position. They were able to see through the onionskin and observe points on the graph in the new position.

3. From origin, if you move the graph up or down (change in y), you add that change to the x^2.

4. From the origin, if you move it to right or left (change in x), you add or subtract that change to y.

5. If you rotate 90°, you set $y^2 = x$, (and the *tip* of the graph doesn't leave the origin).

6. If you rotate 180°, you set $y = -x^2$.

7. If you rotate 270°, you set $-y^2 = x$.

8. If you move it over one space on the X-axis, you get $x^2 - 2x = y - 1$.

9. If you move n spaces to the right on the X-axis, you get $y = (x - n)^2$.

10. If you move n spaces to the left on the X-axis, you get $y = (x + n)^2$.

A discussion revealed that hypotheses 8, 9, 10 were really the same. Hypothesis 4 was exemplified by $y + 1 = x^2$ to represent the graph with vertex at (1,0) which was shown to be wrong. Several good questions arose from the discussion.

1. How do you make a graph wider or narrower?

2. How do you move a graph up or down and sideways at the same time?

3. Do these rules work when you invert the graph, and move it around?

Activity Three

For the third activity, the instructor chose to investigate the student-initiated problem, "How do you make a graph wider or narrower?" Two graphs (not the rules) were presented to the pupils on the overhead projector.

1. $y = \frac{1}{2} x^2$

2. $y = 2x^2$

The first graph was clearly on the points (0, 0), (1, ½), (2, 2), (3, 4½), (-1, ½), (-2, 2), (-3, 4½). The second graph was clearly on the points (0, 0), (1, 2), (2, 8), (-1, 2), (-2, 8).

The students were asked to find a rule for each of these graphs, and after some time spent in personal inquiry, they offered the following suggestions:

For graph 1, $2y = x^2$ or $y = \frac{1}{2}x^2$

For graph 2, $\frac{1}{2}y = x^2$ or $y = 2x^2$

Further practice at testing hypotheses was hindered because no wrong conjectures were offered by the students. This is a typical example of a situation in which the immediate response of only correct hypotheses actually impedes the development of the lesson.

The graphs $y = 2x^2$, $y = x^2$, $y = \frac{1}{2}x^2$ were then superimposed one upon the other and projected on the screen by means of the overhead. An observation of these graphs confirmed the pupils' intuitive feeling about the effect of the coefficient of x^2 upon the shape of the graph - as the coefficient of x^2 becomes larger, the graph becomes narrower.

As a sub-activity, the students were asked to move these graphs, $y = \frac{1}{2}x^2$ and $y = 2x^2$, around, and see what relationships held. The following conclusions were among many suggestions offered:

1. The general method of moving the graph $y = x^2$ about, has the effect of changing the rule $y = x^2$ to $y = (x - \Delta x)^2 + \Delta y$.

2. This same general rule applies to moving the graphs $y = \frac{1}{2}x^2$ and $y = 2x^2$.

   $y = \frac{1}{2}x^2$ becomes $y = \frac{1}{2}(x - \Delta x)^2 + \Delta y$

   $y = 2x^2$ becomes $y = 2(x - \Delta x)^2 + \Delta y$

It was clear from the discussion following that the students had a fair understanding of the quadratic function, and the role of the coefficients in determining the shape of the graph. They also had a good working knowledge of the "vertex form" of a quadratic function. Significantly, to this point no reference had been made to the general form of the quadratic function.

Activity Four

The fourth activity was designed to foster an understanding of the standard form of the quadratic function, and once again the graph technique was used. At this point the instructor chose to structure the activity considerably by presenting the students with the following series of functions to be graphed:

1. $y = x^2 + 2x + 1$
2. $y = x^2 + 2x + 2$
3. $y = x^2 + 4x + 4$
4. $y = x^2 + 4x - 5$
5. $y = x^2 - 6x - 3$
6. $y = x^2 - 7x + 2$
7. $y = 2x^2 + 4x + 6$
8. $y = 2x^2 - 6x -9$

During the exploratory period that followed, there was considerable class interaction: One student wanted to just plot a few points and fill them in to complete the graph. Another disagreed because "sometimes you can't find the vertex exactly," and besides, "there must be an easier way to do it." Still another became quite excited because she knew how to predict other points if she only knew where the "tip" was. "Tell me how to find the tip?" A student who had obviously been examining the textbook suggested using the formula for finding the vertex, $-x = \frac{b}{2a}$

During the graphing of these functions, several students, in addition to solving the original problem, developed on their own the technique of "completing the square." They also became increasingly aware that the "vertex form" of the quadratic function was decidedly more useful than the "standard form." The class then held a discussion on how to find the "completed square" form from the "standard" form. The class arrived at some rules. Basically, the students just guessed, and then tried to work it out. Here it was necessary to give the students practice in applying the technique of completing the square. It is important to emphasize that the students need to recognize the various methods used and to become skillful in using various techniques.

Activity Five

The fifth activity was aimed at arriving at a method for finding the vertex, hopefully a formula, and also at the significance of the negative coefficient of x^2. The activity is really a rephrasing of the previous activity, but it did, however, present a little variety in dealing with the vertex of the quadratic. The problem posed was that of finding the highest (or lowest) point on the graphs of the following quadratic functions.

1. $y = (x + 2)^2 - 7$
2. $y = -x^2 + 3$
3. $y = =2x^2 + 9$
4. $y = (x - 2)^2 - 7$
5. $y = x^2 - 8x + 27$
6. $y = -x^2 - 12x - 3$
7. $y = 2x^2 - 8x + 112$

The students were given time to work on the above exercises. They had no problem applying the results of the first activity in knowing that graphs of 1, 4, 5, 7 opened upwards and y had a smallest value, whereas the graphs of 2, 3, 6 opened downwards and y had a highest value. From the discussion which followed, these observations resulted:

1. The students determined that $(\ )^2$ was always positive.
2. The highest or lowest value of y would be when the $(\ )^2$ term was zero.
3. The value of y at this point was just the constant term.

Some students wanted to solve this problem of finding the highest or lowest point on the graph by finding the vertex using the old format

$y = (x - \Delta x)^2 + \Delta y$. This of course is really all that was being done. Once this observation had been made the activity clearly became one of getting practice in applying the old knowledge and skill to solve a new problem.

Activity Six

The sixth activity constituted a review on finding the vertex and plotting the quadratic function. However, in addition, the students were to find:

1. axis of symmetry
2. range
3. x and y intercepts

The definitions of these terms given in class were not by means of formulae, but rather in terms of what these concepts mean in relation to the graph. The students had little previous knowledge of these concepts, and as they were working they were asked to look for possible formulae for finding them. The meaning of formula was identified as a short-cut to getting answers.

Problems were assigned from the text with the directions to plot and find 1, 2, 3 above.

1. $y = x^2 - 6x + 8$
2. $y = x^2 - 4x + 3$
3. $y = 4x^2 - 4x - 15$
4. $y = x^2 - 10x + 9$
5. $y = -x^2 + 2\sqrt{2}\,x - 3$
6. $y = -3x^2 - 2x$
7. $x^2 + 4x + 4$
8. $y = x^2 + 4x + 6$

Several students found the x intercepts by getting the quadratic function in the form:

$y = (x - k)^2 + m$, setting $y = 0$,

that is, $(x - k)^2 + m = 0$

$(x - k)^2 = -m$ and so on.

This is, of course, the handiest way of solving the corresponding quadratic equation. Other students were simply graphing the quadratic function and observing the points of intersection with the X-axis.

After allowing sufficient time for the students to examine these problems on their own, the following generalizations were made by the students:

1. The y-intercept is the value of the absolute term in the expression.
2. The range is associated with finding the vertex and especially the y coordinate of the vertex.

3. The axis of symmetry is $x = -b/2a$, and the vertex was formed by associating this value of x with the maximum or minimum value of y.

4. Various methods can be employed in finding the x-intercepts:

   i. plotting points, drawing the graph, and then estimating the x-intercepts.

   ii. finding the vertex, using the formula, then using the format 1 over, 1 up, etc., for plotting the graph, and then estimating the x-intercepts.

   iii. setting $y = 0$ in the "standard" form, then factoring to find the x-intercepts.

   iv. putting the quadratic in the form, $y = (x + k)^2 + m$, setting $y = 0$ and then solving for x.

Finally the formulae for range, intercepts and axis of symmetry were established.

An important end was served by this exercise in that it allowed the students to appreciate that the "thing" exists by itself and that a formula is merely a means of finding it quickly. This set of problems, besides serving the function of summation activity led directly to the problem of finding the roots of quadratic equations and the character of these roots.

Activity Seven

The seventh activity consisted of finding the roots of the equations:

1. $x^2 + 4x + 2 = 0$
2. $(x + 2)^2 - 3 = 0$
3. $-4x^2 + 7x - 2 = 0$
4. $4x^2 + 7x + 6 = 0$
5. $4\ (x + 8)^2 + 6 = 0$
6. $x^2 + 2\sqrt{6}\ x + 6 = 0$

They were asked to also make a graph of the equations that had no roots. Problems arose when the graph did not cross the x-axis or when surd values occurred, or when the graph was tangent to the x-axis, as is the case in "6". In some cases they couldn't find roots; they were sure there weren't any; yet by plotting the graph they could see that there really were. Here is an example of a real problem arising. They knew roots existed, and they had necessary skills to find the roots but in many cases they could not find them. Mathematics was now providing the problem and the feedback for the students.

Some of the questions generated by this activity were:

1. How do you tell when an equation will have only one root?

2. When is the equation a perfect square?

3. Can you tell when the graph is tangent to the x-axis?

4. Can you tell by looking at the equation what kind of roots it will have?

5. Why is it that when you multiply $x^2 + 3x + 2 = y$ by 2 to get $2x^2 + 6x + 4 = y$ you get a different function but when you multiply $x^2 + 3x + 2 = 0$ by 2 to get $2x^2 + 6x + 4 = 0$ you get the same equation?

Most of these problems were resolved by class discussion in terms of the coefficients of the quadratic equation in the standard form. However this is an illustration of students generating not solutions but actual problems. The problems were especially good because they had the knowledge to solve them at their finger tips.

Activities Eight, Nine and Ten

The three activities to complete the unit were presented in a similar manner. They centered on the problem of

1. Finding the quadratic formula for the roots of the quadratic equation.

2. Finding a relation between the roots, sum and products of roots, and the quadratic equation.

3. Application of maximum and minimum problems, the solution of which was attempted, again, by reference to the graph.

The reader of these pages may find it an interesting exercise to set up activities which will assist students in arriving at solutions to these problems. By this time in the unit the students should be almost ready to suggest their own activities to handle such problems. Perhaps one difficulty is worthy of mention. It is the question of in what terms the quadratic formula is desired. For example, if the equation is in the form $(x + p)^2 - q = 0$ the formula is simply $x = -p \pm \sqrt{q}$. In fact this suggests an approach to the activity. But the difficulty also points out that asking for the quadratic formula in terms of the coefficients of the standard form is a rather arbitrary request. The rationale for wanting the formula in this form can provide very interesting class discussions.

Activity Eleven

The final activity of the unit was a summing-up activity and as such consolidated all the discoveries of the previous 10 activities. The importance of this activity cannot be over-emphasized. It should provide a resume of all generalizations and formulae and pick up pieces of information that are important, but, because the method is not highly structured, do not receive enough emphasis. Although the ideal would be for every activity to tie in with other activities, this does not occur generally and so the summary acitvity may be remedial in this sense.

INSTRUCTIONAL GUIDELINES

Content Organization

From the point of view of the content, the idea of function or graph of the function was used as a thread throughout the unit. It served as the unifying element and the pieces of information picked up during the activities merely had to be incorporated into the quadratic function-graph schema. Even the roots of the equation could be interpreted as being related to the graph, while the nature of the roots take on a "physical" meaning in this context. The approach can clearly be labeled as the "function approach to the quadratic".

Teacher and Student Behavior Patterns

Equally as important as the content are the teacher's and students' behavior during these activities. Such statements as are made below could be taken as the beginning of a theory of instruction. All statements are made about observed teacher and student behaviors which resulted as they endeavored to carry out the discovery activities.

1. The teacher should state broad, poorly defined problems in order that the students can not only solve the problem but also before solving it discover its boundaries and character.

2. The teacher must accept, hopefully without evaluation, all related problems and solutions for consideration by the class.

3. The teacher must not insist on precise terminology or on one approved method of solution at the beginning of a unit or activity. It is quite natural for the vertex of the graph of a quadratic function to be called the "tip".

4. The teacher must attempt to keep the class together by keeping everyone informed of the discussion and by setting up appropriate summing up activities.

5. The teacher must not be too concerned with achieving closure on each lesson. During some lessons, little progress, in the sense of arriving at a definite conclusion, may have been made and it therefore becomes necessary to leave the activity for next period. This is often difficult to do with a highly motivated class which is used to receiving answers upon demand.

6. The students must be given the mind set that they are to continually search for short cuts, patterns, formulae and generalizations as solutions to problems.

7. The students must understand that unless they contribute to the class discussions they are not likely to learn much. A teacher using this method can really notice a Monday or Tuesday morning when the class is tired and not particularly motivated to discuss. In many traditional classrooms this lack of desire to discuss is welcomed.

8. The students must learn to evaluate every statement that is made by another student or the teacher. The problem here is that a student must listen carefully to his classmates knowing full well that the comment being made is incorrect to some degree.

Some Special Obstacles

Every teacher of mathematics knows the importance of formulae for solving problems and for generating new ideas in mathematics. Usually the students in mathematics want to work only at the formula level. But one of the problems that arose during this unit was that once the students learned how to solve problems by some basic means, they objected to learning a formula. For example, why learn the quadratic formula to find roots when roots can be found by simply getting the equation in vertex form! Well, why? This is not an easy question to answer but it is an extremely important one.

There is also the problem of the advanced pupil. He is liable to give away the solution before half the class understands the problem. This is a serious problem which must be dealt with in different ways, depending on the classroom. One solution to the problem is to send the student to the library with his mathematics text for individual study. The solution may seem rather drastic but it does work.

A related problem is the one of the student who looks in the textbook. For the most part, it should be possible to discourage the students from consulting the book unless they are absolutely "hung up" on some problem. In this case, the textbook can be a valuable source of information. The textbook is also a good source of exercises which may be used for practice or for exploratory questions to begin an activity. The textbook, if used appropriately, can be very helpful in this method.

The basic idea behind the use of discovery teaching as proposed in this paper is that students, in learning a "complex coding system" as exists in mathematics learn as much from their mistakes as they do from their correct responses. If a student makes an hypothesis and discovers it is incorrect, he should have a greater understanding of the problem and its dimensions. Here, in fact, we seem to have generated a theoretical principle that contradicts some learning theories which advocate the reinforcement of correct responses. However, learning will result from an incorrect hypothesis if the student appreciates where the hypothesis is in error and how it might be improved upon, if not corrected. But in order to achieve this desired goal of learning through mistakes, it is important to make accurate note of all hypotheses and their evaluation. This keeping track of all solutions to a problem can become a messy business for the teacher but some system must be worked out for this all-important function. One such system would be to have the students keep an account of all major hypotheses, right and wrong, in notebooks. This will provide a means for them to keep track of ideas and concepts as developments evolve through the unit.

Goals of Teaching by Discovery

The first goal is that students should learn mathematics. Many argue that the discovery approach is not an efficient method of learning mathematics. If learning mathematics means learning solutions to problems by means of formulae, then it is possible that this approach is not very efficient. But if one requires learning at a level where a student can attack unusual problems, that often do not have formula solutions, if one requires learning where the student is familiar enough with the content to hold intelligent discussions, if one requires learning where the student becomes excited about all the possibilities in a given problem, then the discovery method must certainly rank high in efficiency.

A second goal is that the student should learn how to create mathematics. If mathematics is created by hypothesizing, evaluating, and rejecting or accepting, and many mathematician would agree that it is, then practice in these activities can only improve the students' ability to handle mathematics creatively. Along these lines, many educators insist that a discovery style of teaching is most valuable for the above average student capable of making discoveries. However, if the goal is to help students become more creative in their use of mathematics, then surely the below average students stand to gain more than the already creative students. The argument continues that creative students are already getting the benefit of creative experiences in their everyday life; indeed, they can get creative experiences from expository teaching. However, for the less creative students it becomes necessary for the teacher to structure their environment so that their creative abilities can find expression. So, from the point of view of the second objective, discovery teaching is more valuable for the below average student than for the above average. Again the complication arises in that the traditional teacher is so concerned about getting the below average students through a maze of formulae that he has no time for other considerations.

CONCLUSION

Associations between the preceding practical description and cognitive learning theory are many. But basically the whole approach - an underlying theme for the unit - can be conceived of as establishing an accommodating structure or schema which, once understood, can be used in assimilating the various aspects of the unit. As each student verifies or rejects an hypothesis, he modifies his cognitive structure, which each time comes closer to the total schema of the quadratic function and equation in its entirety. If such a process is to be approximated in the classroom, it is of vital importance that students display their cognitive structure in the form of hypotheses to be tested. The more conscious they are of their cognitive structure and the more they subject it to testing, the more likely it is to be shaped efficiently and correctly. In making these statements, we are relying on the premise that the cognitive structure of the learner is best varied not by the authoritarian teacher but through encounters with the structures it is trying to approximate, which in this case is mathematics.

It would be possible to establish other points of agreement between modern cognitive learning theory and the discovery method, but, as we stated at the beginning of the paper, it is not our aim to seek guidance from psychological theory. Our concern is to devise procedures that "work" in the classroom of 30 more-or-less intelligent, motivated, concerned students. We are not too hopeful that any theory derived from such endeavors will form a nicely integrated framework. The most we can hope for is to make some general statements about poorly defined situations that can hopefully be interpreted by teachers who will be operating in at least as poorly defined situations. Some may think this is being overly pessimistic about the state of affairs but the alternative is to try to find nice theoretical positions on a matter which is complicated beyond comprehension. Teaching a pigeon in a cage to turn a somersault is one thing, but teaching 30 pupils in a social environment not far removed from the discothèque to comprehend and create a "complex coding system" like mathematics is quite another.

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Annotated Bibliography of Resource Materials for Promoting Active Learning in Mathematics

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University of Calgary

Association of Teachers of Mathematics, *Notes on Mathematics in Primary Schools.* London: Cambridge University Press, 1967.

An excellent source of practical suggestions for mathematical activities for elementary school children and spanning a wide range of basic concepts.

Bates, John, and Irwin, Donald, *Teachers' Guide for Developmental Math Cards* (Kits A to L). Don Mills, Ontario: Addison-Wesley (Canada), Limited, 1970.

A series of sets of colorfully illustrated activity cards and accompanying teacher's guides spanning Grades I to VI. The activity cards are organized around number, measurement, geometry, games, and notation concept strands. The teacher's guides give detailed suggestions as to how to use the cards and how to initiate an active learning program in elementary school mathematics. The cards are cross-referenced and immediate activities and enrichment activities are suggested for each card.

Biggs, Edith E., and MacLean, James R., *Freedom to Learn. An Active Learning Approach to Mathematics.* Don Mills, Ontario: Addison-Wesley (Canada) Limited, 1969.

A very practical source of materials for anyone planning to initiate an active learning approach to mathematics. Contains detailed descriptions of mathematics workshops for teachers, suggestions for countless student activities for learning a broad range of mathematical concepts and skills, and comments on administrative, teacher, and student roles as well as evaluation procedures. A distillation of workable procedures from years of classroom experience based on Piagetian ideas about how children learn.

Central Iowa Low Achiever Mathematics Project (CILAMP) Activity Booklets:

Gimmicks
Roman Numerals
Introduction to Flow Charting
Lost in Space
How it Might Have Been
Measurement
Area Measurement
The Protractor
Tangrams
LAMP
Enrichment Student Projects (ESP)
Graphing Pictures
The First Probability Programs
Math in Sports
Road Map Math

A well worked out series of activity booklets aimed at low achieving junior high school mathematics students but suitable for the lower grade levels as well. The complete set of mimeographed booklets can be obtained for $15.00 from: Jack R. Williams, Director, Central Iowa Low Achiever Mathematics Project, 1350 E. Washington, Des Moines, Iowa, 50316.

Copeland, Richard W., *How Children Learn Mathematics, Teaching Implications of Piaget's Research.* New York: The Macmillan Company, 1970.

Covers the usual topics in a course on methods of teaching elementary school mathematics, but places emphasis on how children learn mathematics rather than "techniques of teaching". The book is interwoven throughout with numerous practical illustrations of how to use laboratory and manipulative materials to help children learn mathematical concepts at the concrete operational level.

Davis, Robert B., *Discovery in Mathematics, A Text for Teachers.* Reading, Massachusetts: Addison-Wesley Publishing Company, 1964.

________________, *Explorations in Mathematics, A Text for Teachers.* Palo Alto: Addison-Wesley Publishing Company, 1967.

________________, *Matrices, Functions, and Other Topics, Teachers' Commentary,* The Madison Project, 1963. (918 Irving Avenue, Syracuse, N.Y., 13210).

These Madison Project publications provide detailed question sequences and descriptions of activities designed to promote "active learning" on the part of students. They contain many excellent examples of "advanced" topics that can successfully be handled by young children.

Dienes, Z. P., and Golding, E. W., *Exploration of Space and Practical Measurement.* New York: Herder and Herder, 1966.
Describes numerous games for children leading to an understanding of geometry, measurement, time, capacity, weight and area.

___________, *Learning Logic, Logical Games.* New York: Herder and Herder, 1966.

A teachers' handbook describing the kinds of experiences in logical thinking which children could be exposed to in the first two grades.

___________, *Modern Mathematics for Young Children.* New York: Herder and Herder, 1966.

Contains many suggestions based on Dienes' theory for presenting such topics as operations on sets, logical operations, number, numeration, place value, and number operations to young children.

___________, *Sets, Numbers, and Powers.* New York: Herder and Herder, 1966.

Contains lessons and games for elementary and junior high school students leading to an understanding of sets and numbers in terms of transformation "machines".

Elliott, H. A., MacLean, James R., and Jorden, J.M., *Geometry in the Classroom.* New Concepts and Methods. Holt, Rinehart Winston, 1968.

An integrated, thoroughly modern activity-oriented approach to geometry from beginning elementary to late secondary school levels - suggestive of many geometric concepts that can and should be introduced much earlier than has normally been the case.

Fisher, Dale, *A Feasibility Study on Active Learning with Real Numbers.* Unpublished Master's Thesis, The University of Calgary, Calgary, 1970.

The appendix of this thesis contains a detailed description of the teacher's guide for 46 student activities designed to provide a complete introduction to the system of real numbers, covering topics found in the real numbers unit in the Alberta Grade VIII mathematics curriculum but developing these topics entirely on the basis of student activities. Members are entitled to one copy free of charge, upon request. Non-members may obtain the appendix, entitled "An Active Learning Unit on Real Numbers", at a cost of $1 from Barnett House, 11010 - 142 Street, Edmonton 50.

Johnson *et al., Activities in Mathematics, First Course - "Patterns".* Glenview, Illinois: Scott, Foresman and Company, 1971.

This is the first in a set of four consumable workbooks (the others deal with "Numbers", "Measurement", and "Probability") that make up the AIM First Course. The series is full of interesting and challenging activities, complete with punch-out materials, aimed at building positive attitudes with slow learners in mathematics. The first course was written for the Grade VII level but can be used at a variety of levels. The topics covered by the second course are: "Graphs", "Statistics", "Properties", and "Geometry". Teacher's editions and spirit duplicating masters and overhead visuals books are available for both sets.

Jones, H. R., Hancox, W. J., and Shephard, J. G., *Mathset*. W. J. Gage, 1967.

A series of student assignment cards on measurement and graphing.

Kieren, T. E., and Vance, J. H., "The Theory of Active Learning: Its Application in a Mathematics Workshop", in *The Manitoba Journal of Education*, Vol. IV, No. 1, pp. 33 - 40, November, 1968.

An article describing some features of the theories of learning developed by Bruner and Dienes and the creation of and student reactions to a mathematics laboratory based on these ideas.

MATHEX, *Basics of Mathematics Experiences*. Toronto: Encyclopedia Britannica of Canada, 1966.

A description of techniques and ideas from classroom practice for providing children with stimulating experiences in mathematics.

Nuffield Mathematics Teaching Project - Teachers' Guides. John Wiley and Sons, New York (In Canada: Longmans Canada Limited, Don Mills, Ontario).
I Do and I Understand, 1967.
Mathematics Begins, 1967.
Beginnings, 1967.
Pictorial Representation, 1967.
Shape and Size (2), 1967.
Shape and Size (3), 1968.
Computation and Structure (2), 1967.
Computation and Structure (3), 1968.
The Story So Far, 1969.
Environmental Geometry, 1969.
Problems, Green Set, 1969.
Desk Calculators, 1967.
Probability and Statistics, 1969.
Graphs Leading to Algebra, 1969.

Forthcoming Nuffield Titles (In Press):

Mathematics in Practice
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Guide to the Guides
Into Secondary School
Red Problems
An excellent series of teacher's guides detailing activities suitable for elementary school children to learn actively a wide range of basic mathematical ideas (strongly influenced by Piagetian notions about the nature of children's learning processes).

Pethen, R. A. J., *The Workshop Approach to Mathematics.* Macmillan, Toronto, 1968.

_______________, *Mathematics Workshop - Complete Set: Cards 1 - 224.* The Macmillan Company of Canada Limited, 70 Bond Street, Toronto 2, Ontario.

A thoroughly well-organized elementary school mathematics workshop program consisting of student "open-ended" activity cards and a teacher's guide.

Sawyer, W. W., and Nelson, L. D., (Editors). *MATHEX:* Mathematics Experience Materials for Canadian Schools. Levels 1 to 6 Teacher and Pupil Bulletins. Toronto: Encyclopedia Britannica of Canada, 1966.

A series of pupil activity sheets and teacher commentaries written by Canadian mathematics teachers to enable children to discover mathematical concepts for themselves. Mathematical games, puzzles, problems and experiments for kindergarten through junior high school.

Seymour, D. G., and Gidley, R., *Eureka* (Palo Alto, California: Creative Publications, 1968).

A unique publication filled with offbeat, interesting mathematical problems, puzzles, cartoons, jokes and similar material.

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