An Inventing Unit on Area for Grade VII

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In another article in this issue, the author talks about a "discovery unit" dealing with senior high school mathematics content. I chose to use "Inventing Unit" in the title of this article because, since writing the previous article, I have come to the conclusion that mathematics, in the true sense, is not "discovered", rather it is "invented". Scientific laws are discovered because these laws are inherent in the nature of things. However, the mathematician invents mathematical structures to help him solve problems. The distinction is not of critical importance, but I will use the word "invent" throughout the article in the hope that it will give the reader a slightly different slant to the instructional method proposed herein.

The theme of this article is to suggest a method of teaching a unit of two weeks duration on finding areas of simple geometric figures to either Grade VII or VIII students. I am sure the reader is familiar with discovery teaching and I suggest he can't go too far wrong if he thinks of this in the discovery sense as opposed to the newly coined "inventing" sense. Before a teacher sets out to use this method, he must first convince himself that there is something more to teaching than simply giving students the knowledge of how to find areas of various geometric figures. While eventually he wants his students to be able to manipulate formulae, he also wants them to know what area is, to be able to solve some unusual problems and, in general, to have a well-grounded intuitive knowledge of area. If you think that some of these latter notions are important in learning about area, then continue to read. I cannot emphasize too strongly that the proposed method of teaching area (and indeed, it can be applied to any content), is predicated upon the idea that the traditional objectives for teaching mathematics are not broad enough. So let us admit that we are prepared to spend two weeks having fun learning mathematics and also that the criterion for success which we will use will not be the kind of question normally found on a final examination in mathematics.

*Dr. Sigurdson has a video tape of many of the student activities generated by the lessons he refers to in the article, and he invites any interested teachers to enquire about viewing the tape at the University Education Building in Edmonton. This paper grew out of a talk entitled "Teaching a Unit Through Discovery", presented by Dr. Sigurdson at the Winnipeg Meeting of the NCTM in

Many advocates of this method will insist that I am being too hard on "discovery teaching." They insist that one can attain all the traditional achievement objectives and get many additional benefits. I partly agree with this but I think when one is beginning to use the method he should not get "hung up" on achievement as the criterion for success. After a teacher becomes particularly accomplished in using the method, there are no limits to the benefits which may accrue from it. Also, we must not forget that students are not accustomed to the method. They are more used to teachers who are directive and willing to tell them answers. So we should not naively expect that the method will achieve overnight success. I do believe that you will have fun teaching this way and that the students will enjoy "learning" this way.

STRUCTURING THE UNIT

The first and most important general rule to follow in setting up a unit along discovery lines is to present the broadest possible description of the task. Tell the students that for the next two weeks they will be trying to develop ideas for coping with a general type of problem. Once you allow the students to begin working at the problem they will specify certain aspects on which they want to concentrate first. Each of these individual aspects can result in a specific activity. The previously mentioned quadratic unit resulted in 10 of these activities. At first, the teacher will have to help students isolate the problem areas. But the ultimate goal in such an "inventing" unit is that the students will be able to invent their own activities. They should be able to specify the areas they wish to work on and develop ways of attacking these areas.

INSTRUCTIONAL PROCEDURES

In addition to the teacher's setting up a general type of problem for the students to work at, he must learn certain instructional procedures which will promote discovery or "inventing". I will now suggest some guidelines for the teacher to follow in an "inventing" situation. First, do not be concerned with using precise terminology, but rather give the students an intuitive feeling for the problem. The idea is to begin by giving the class a very poorly defined statement of the problem. This instructional pattern is extremely important to follow because it will force the students to determine for themselves exactly what the problem is. And it is well understood in mathematics that half of the work in solving a problem is over once the problem has been defined.

Phase One

The initial exploratory period of working on the problem should be done by the students working in pairs or small groups. The reason for this is that an individual student who doesn't know clearly what the problem is might easily focus on an inappropriate aspect of the problem or else he may simply run out of ideas. The chance of either of these things happening when a small group is working on the problem is much less. The noise level of 30 students working in small groups is rather high, especially if they are more or less noisy to begin with. Every teacher must find his own means of coping with this problem. During the exploratory period the teacher can supply the students with ideas to help them get started. However, the teacher must be careful not to assist directly in solving the problem; he should, rather, suggest apparently productive lines of thought which the students can investigate. Also during this instructional phase, the teacher should be accepting and encouraging toward any ideas whether they are "correct" or not. I placed the word "correct" in quotation marks because anything is "correct" if it gets a student closer to the solution. Even a completely wrong approach can help in arriving at a solution if the student finds out where the approach is wrong and can modify it accordingly. In the traditional classroom a student feels badly about making a wrong answer. The teacher in the discovery class must help the student overcome this feeling, especially during the exploratory phase.

The problem of students running out of ideas quickly can also be alleviated by making the initial problem a very "primitive" one. That is to say, the problem should be approachable by the slowest pupils in the class and yet still be full of potential for the best students. Of course, if the range of ability in your classroom is extreme, it will be difficult to find such problems. The ordinary problem at the end of a chapter in a mathematics textbook is usually not "primitive", that is, either you know how to do it or you don't. The problem which structures the unit on area described in the following pages is a good example of a "primitive" problem.

Phase Two

After a certain period of exploratory work, which may be as long as a whole 45-minute mathematics period, the teacher will lead the whole class in a group discussion. The idea is to have the students hypothesize solutions. Some of the statements will take the form of identifying the importance of working on a certain aspect of the problem. Here again, all hypotheses should be accepted with the idea that the teacher will not evaluate. This will probably confuse the students, especially if they have had a teacher who normally only writes *correct* statements down on the blackboard. The teacher must remind the class that he will not give answers and that they, themselves, are completely responsible for agreeing as to what is acceptable or unacceptable (I hesitate to put "right or wrong"). It is extremely important for the teacher to help the class keep track of these hypotheses. A good technique for keeping track of them is to give the hypothesis the name of the student proposing it, for example, "Dwight's Hypothesis". At some stage the teacher must say, "Okay, I think we have enough hypotheses. Let's start to evaluate them". At this stage it might be appropriate to let the students work in pairs or small groups or again it may be fruitful to keep the whole-class discussion going.

During the second part of phase two, in which the evaluation of the hypotheses takes place, a teacher might suggest alternative proofs. One form of alternate proof uses a counter-example to show that the hypothesis in question does not cover a particular situation. Another idea to use at this time is that two hypotheses cannot both be correct. Such a conflict situation can be extremely helpful for motivational purposes. During this phase of the instructional procedure, the teacher must become slightly more directive. The amount of direction the teacher gives is determined by many factors in the classroom situation. Knowledge of both the subject matter and the student helps the teacher in making this decision. In actual practice, one of the obvious indices to look for in determining the amount of guidance is the frustration level of the class. They want to determine if something is correct or not and they don't know how to do it. They feel the importance of a situation that they can't cope with. This can be a very valuable learning situation, but beyond a certain point, it is simply maddening.

Phase Three

The last phase of the instructional process is that of consolidating the ideas that have been evaluated as being useful. It is not important that the solutions to problems be stated in the normally acceptable form. The criterion should be that the solutions are understood by the students in their own way. In certain instances the teacher may want formulae stated in the usual way simply for the sake of convenience. This final stage, which may be called a stage of closure, is a difficult one to handle because the teacher must try to preserve the students' feeling that they have come up with these answers. Here it becomes obvious to the students that the teacher does know the answers. But if one can give the students the feeling that the answer is not the important thing, rather that the process of arriving at the answer is the important thing, then this last stage of closure is simply the "icing on the cake" - the "cake" being what went on in the first two stages.

The three phases mentioned here will undoubtedly repeat themselves many times during the unit; that is, the sequence of exploratory work, hypothesizing and evaluation, and closure will repeat themselves as often as the teacher and the class think it necessary. It cannot be emphasized too strongly that during the exploratory and the hypothesizing phases the teacher must strive to create an accepting atmosphere in the class. Students should be encouraged to think out loud and share their ideas with everyone. When 30 minds share ideas on the same problem, the results can be not only very interesting but also very productive. The three critical factors in determining the effectiveness of the method are the ability of the teacher, the quality of the general problem, and the personal characteristics of the students. By personal characteristics of the students I mean not only their intelligence but their willingness to share ideas, and their general manners in the classroom.

In the paragraph above I referred to the "effectiveness" of the method. One might ask "What is the criterion in determining effectiveness?" "What do we mean by effectiveness?" I would first of all suggest it is not the "speed" with which the students arrive at a solution. It is perhaps the "quality" of the ideas that come up during the course of the unit. Or perhaps it is the degree of interest shown by the students in mathematics.

I suggest that we now take a "discovery" approach to learning how to teach an "inventing unit". Let's do it and then discuss what we have done. The following is what happened when I tried the unit with a Grade VII class.

CLASSROOM PROCEDURES

First Class Period

I began the class by giving out the sheet of paper identical to that shown on the following page. We spent 10 minutes talking about the kinds of figures and their names. Each of the figures could be referred to by a letter so the name of the figure wasn't really essential. The class was told that the numbers along the sides of these figures were the measures in centimeters. The first question I asked the class was "Which of these figures is the biggest?" Of course, a number of different answers were forthcoming. In fact, the class was using three different ideas of "bigness": perimeter, area, and dimensions. So we narrowed down the loosely-defined problem which I had asked and I said that what I had really meant was "Which is the biggest in area?"

The next few minutes were spent discussing area. The students were able to give me many intuitive notions of area: "amount of space within a closed region", "the surface of something", "contents inside something". I accepted these comments and said, "Now that you have an idea of what area is, the task you have is to place these figures in order from the one with the biggest area to that with the smallest area." At this point I handed out centimeter rulers and some one-centimeter cubes which I said they might find useful.

The students were asked to work in groups of three and to convince each other that the order they had was correct. They then began to work at their task. I stressed that they could cut up the sheets, that they should draw lines on the pages, or use them in any manner they wished. After about 10 minutes of the exploratory period, one student raised his hand and said: "I know what area is but how do you find it?" After another minute he said, "I know how to find perimeter. Is it the same as the area?" I responded, "That's an interesting hypothesis and, indeed, the question is, "How do you find the area?" It was clear that the class had indeed identified what the problem was.

After another 20 minutes in which the students worked on their own, I asked the whole class to pay attention because I wanted to discuss some of their ideas. The purpose of the discussion was to spread their ideas around and certainly not to evaluate their ideas. During the following 20 minutes, which brought us to the end of our first class period, a number of ideas came up. As it turned out, the students did not want to talk about the order, they were merely interested in focusing on figures that were particularly interesting (to them). I labeled these ideas by using the names of the students.

Figure C: Dwight's hypothesis - You measure the perimeter. Then you change the perimeter into a standard shape like a square or a rectangle and you find the area of this shape. (He apparently thought he knew how to find the area of a square or a rectangle.)

Figure H: Steven's hypothesis (for a trapezoid) - Cut it in half and flip it onto the other end; then you have a rectangle of which you can easily find the area.

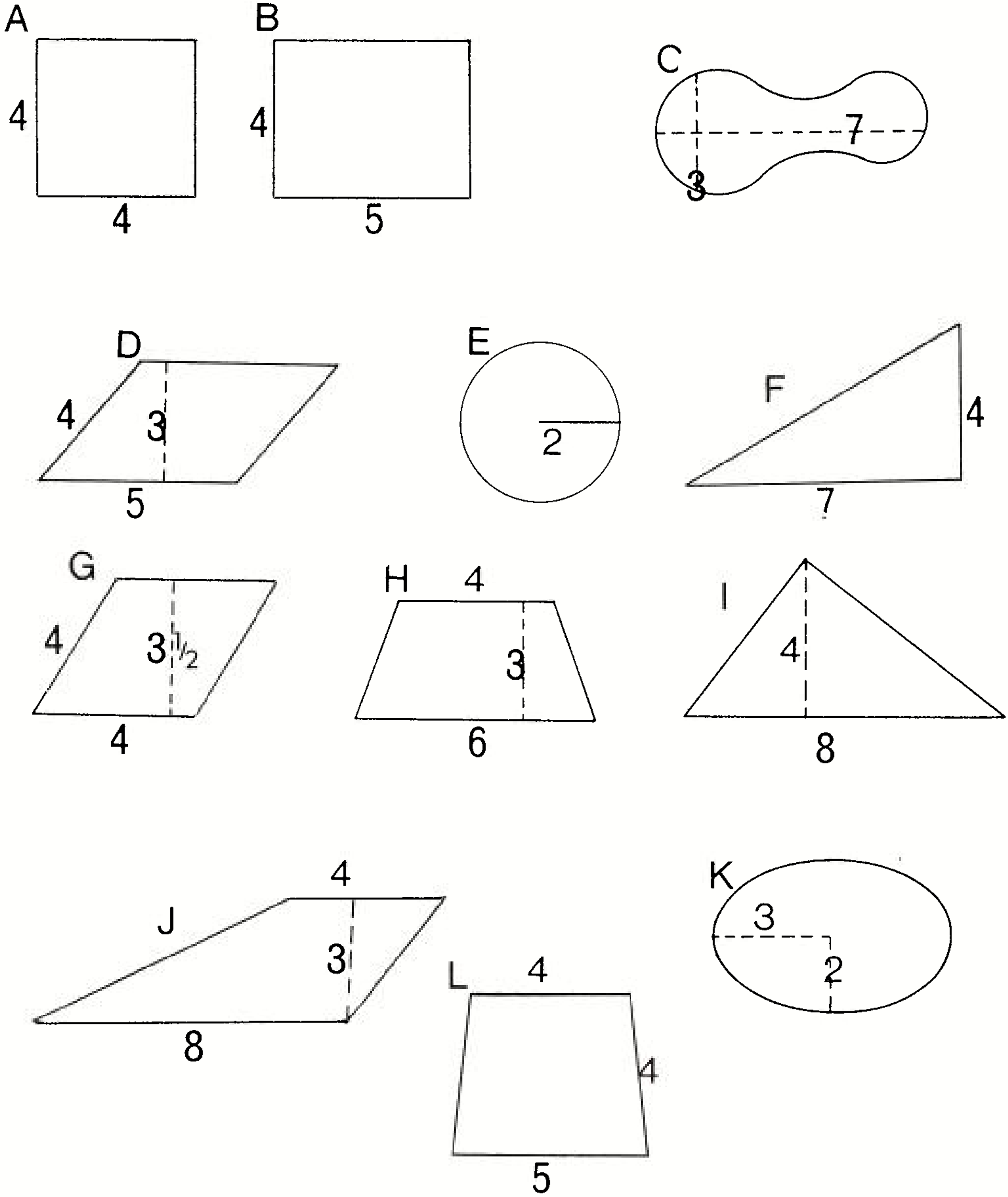
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Figure I: Ronnie's method of squaring - Draw lines on top of the triangle, both vertical and horizontal, marking out the area of the triangle in squares; then you count the number of squares. (He argued that if you do it this way it usually comes out even.)

Figure J: John's method of doubling - You add equal triangular halves to both sides of the triangle. You then have a rectangle of which you find the area by using Ronnie's method of squaring.

There were other suggestions that came up in this period but the ones listed seemed to be the most clearly stated and they seemed to be the most productive of further ideas. Dwight's hypothesis drew very little criticism. In fact most people agreed with it. I did not discourage them from using it. Steven's hypothesis was criticized because the two sides of the figure were not on the same slant. It is perhaps unfortunate that Steven did not state his hypothesis as a solution to the parallelogram because there it works beautifully. Ronnie's method was accepted as being interesting but rather inaccurate, while John's method was literally greeted with cheers and statements like: "Hey, ya, it works." "Man, that's neat!" In spite of these reactions I felt it was still too soon to ask them all to copy John's method down in their notebooks as a correct solution for finding the area of triangles. However, after this session I felt that the students had a number of productive ideas which they could take back with them to the problem of ordering the areas.

I would like to re-emphasize the reaction to John's method. The reaction illustrates clearly that many other students in the class were completely ready for John's invention (not to say discovery). Critics have said that the only person benefiting from a discovery is the discoverer. But I suspect many students in the class were saying to themselves: "Man, that is so easy; why didn't I think of that?" So in a sense the discovery was theirs. Another point of John's method deserves mention: the class (except for three or four students) was incidentally agreeing that you could find the area of a rectangle by squaring it off in one-by-one centimeter squares. We had not yet talked about the areas of squares or rectangles.

Second and Third Class Period

The next period the class was allowed to work on their own. I gave some suggestions to help them along. The 45-minute period passed quickly. The The third period began with

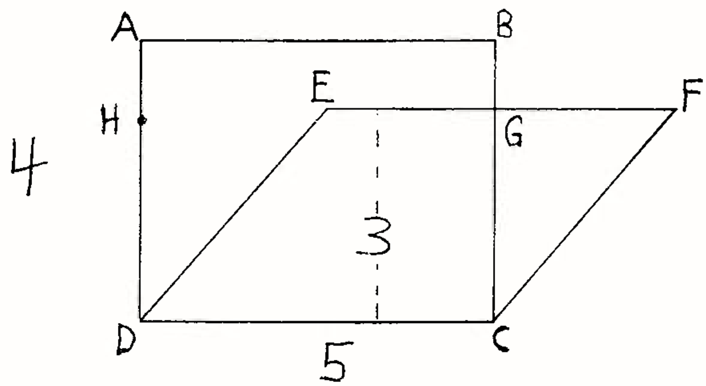
Gord's Hypothesis - The perimeter can be big or small. It does not depend on the area.

This of course was a contradiction of Dwight's hypothesis. Gord's hypothesis was simply noted; no one seemed anxious to talk about it. The area of concern once again became Figure I. Using the perimeter method as proposed by Dwight the answer turned out to be 25, but using John's method the answer was 16. At least two of the students refused to go on to new work until we had settled the problem once and for all. Then came the embarassing question: "Well, do *you* know the answer?" I tried to avoid an answer but finally said I probably could figure it out but that I wanted them to convince each other as to the correctness of any answer.

We finally got around to discussing the parallelogram, Figure D. Dwight took the initiative and restated his hypothesis:

Dwight's second hypothesis - When a rectangle is pushed over to make a parallelogram, the area of the parallelogram has the same area as the original rectangle.

This hypothesis brought about much disagreement. Dwight's point was that the area wouldn't just evaporate or go away; therefore, it must still be within the parallelogram. Many arguments were presented. Two were especially effective. First, someone cited extreme case of pushing the parallelogram so that it would be "just about" flat and the area would be very small. The other argument is illustrated by the figure below. (For the sake of clarity I will put the argument in my own words.) Rectangle ABCD has an area of 20. You can find the area of the parallelogram CDEF by taking triangle CGF and moving it into position of triangle DHE. And "it just fits exactly". So we reach the conclusion that the area of the parallelogram is smaller than the area of the rectangle. A further argument was proposed by calculating the area of the parallelogram to be 15, "So the amount of area that spills over is 5 centimeters".



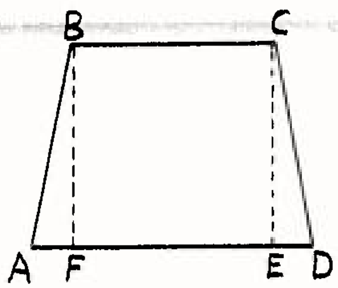
As the reader will readily notice the arguments are completely intuitive. Dwight and his supporters still would not agree. I realized that they had a certain amount of personal commitment to their ideas and, besides, it is difficult to change an idea that you really believe in. So I tried to leave it by saying: "Okay, just think about it for a while." But again two or three students would not go on unless "Dwight agrees." And again they asked me if I knew the answer. Some teachers would consider Dwight's refusal to change his mind in view of the facts, a failure of the method, but it is very pleasing for a teacher to see a student committed to a significant idea - it was all his! Within a few days Dwight was completely convinced his original hypothesis was incorrect.

The reader will also notice that units of square measure were not used at this stage. Area was simply a number. By the end of the unit we had agreed to use square centimeters as the units of area measure.

Sixth and Seventh Class Periods

After about five 45-minute classes we had agreed on almost all the figures and the order in which they came, except for C, E, H, J, and K. I led a fairly directed discussion on solving H and J. Directing the class at this point seemed appropriate because they had tried many approaches to finding these areas. And, if nothing else, they were convinced that they couldn't do the problem. In addition to this I told them the formulae for figuring out areas for the circle and the ellipse. One student's response: "Oh, ya, does it always work?" convinced me that they were appreciative of the answer. We also discussed the problem of finding the area for C and agreed that the best we could do was to get an approximation, by using some squaring method.

I am sure a teacher reading the last paragraph is saying: "Why didn't you set up some discovery activities for the students to work on the trapezoid and maybe even the circle?" In answer I would have to say that this would have been completely possible, if not desirable, but we had spent considerable time on apparently important side hypotheses and I did not want to spend more than two weeks on this unit. Another class using this unit solved the trapezoid problem by finding the area of the rectangle BCEF (see the following figure), erasing the rectangle, pushing the two triangles ABF and ECD together and after finding the dimensions of the new triangle, calculating its area. The area of the trapezoid was then easily found.



Ninth Period

The last period was spent in consolidating the solutions and hypotheses that we had worked on. Each of the students wrote down the method used for finding the areas of the different figures. I personally find this consolidation stage of the method very important. A student will not remember a solution to a problem just because he discovered or invented it. He must use it and be reinforced through its use and by the teacher. When John invented his method of finding the area of the triangle, I did not say: "Yes, John, you are an excellent student and the solution you proposed will always work. Let us try it out in this other case." If I had said this, I think John would have remembered it for a long time. However what I did say was: "That's very interesting John. Valerie, what do you think of it?" This was not very reinforcing for John, but I am more interested in the students appreciating the process of evaluating any hypothesis and, in fact, receiving their reinforcement from the mathematics.

This, then, in general, was the kind of activity that went on in a Grade VII classroom for a period of two weeks. During this time, the students worked on their own or in groups for approximately 60 percent of the time, with the rest of the time being spent in class discussion. The teacher played the role of "chairman of the meeting" in which, so to speak, he did not have a vote. I have not included everything that went on during these periods, although I do have a complete record of it. I hope this has given you some desire to try out this unit. I can almost guarantee that, as long as you are able to create a responsive atmosphere, your class will come up with as many good ideas as are mentioned here.

OTHER SIMILAR ACTIVITIES

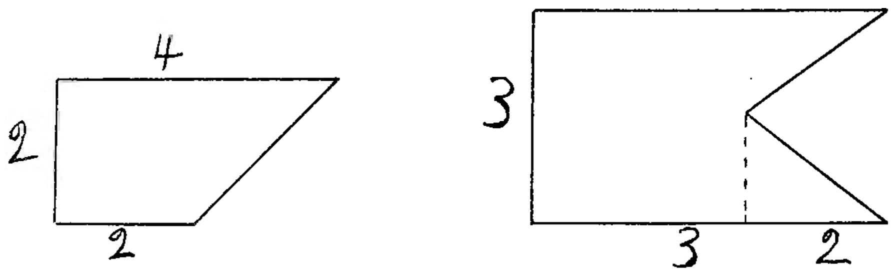
Two other closely related classroom activities come to mind. First, it would be very easy to reproduce an identical sheet with letters beside the line segments instead of numbers. This could lead to the students discovering (inventing) formulae. It would be especially interesting to see how quickly they would identify the generalization that the same formula is always used for triangles and so on. This would be especially interesting if you made one of the triangles obtuse.

The second activity is really another unit. The task could be to order the figures according to perimeter. The problem of perimeter is simple for Grade VII students, but Figure C would prove interesting, as would the circle and any figure that requires measurement. You might pose the problem of which perimeters can be found without measuring.

The last sentence brings to mind a thought that needs to be reiterated. The problem it identifies is interesting in-and-of-itself, but the problem becomes even more relevant in that it contributes to a much larger objective, that of finding perimeters. The whole idea behind structuring a unit is to allow the student to make discoveries or inventions that relate significantly to a larger framework.

EVALUATION

In teaching any unit, we have to ask, "Did we achieve that which we set out to achieve?" In answer to this, I asked the students three types of questions on a one-period test. The first type was to find areas of figures similar to those that made up the unit. The second type of question dealt with finding the areas of complex figures. Since many of their approaches to the problems in the unit consisted of cutting figures apart it seemed that the students would do especially well at finding the areas of the following figures:



The third type of question concerned the hypothesis that had been made:

1. Do you think Ronnie's method of squaring is a good way of finding the area of a triangle? Write a brief comment on your answer.

2. Gord hypothesized that the perimeter of a figure might be very large while the area is small. Do you agree or disagree with this hypothesis? Why?

I felt questions of this type were important in order to determine how many students had actually been paying attention and relating to the classroom work.

Actually, the real test of the unit would be to see if the students became better hypothesizers and better evaluators of mathematical statements. Such tests are not easily constructed, and to detect student growth in these areas after a two-week treatment would really be a marvel. The best I could do was to see if they were relating to the treatment as such. The conclusion that I arrived at was that most students were relating at a significant level.

CLOSING REMARKS

I said earlier in the paper that I was going to take a discovery approach to this paper, namely, I would do the thing first and then discuss it afterward. We are all interested in, "What did the students learn by doing this?" My evaluation showed that they learned the material covered in the unit. By learning the material of the unit, here are some of the things, I think, they learned:

1. They learned what it is to make a mathematical hypothesis, a guess, and what to do with it when it is not completely correct.

2. They learned that something is correct when you can convince others by logical argument that it is correct.

3. They learned that you can do mathematics without using symbols but that symbols which everyone understands aid mathematical communication.

4. They learned that the way you evaluate a mathematical idea is to collect the necessary data and check to see that the idea gives useful answers.

5. They learned that mathematics as a basic human activity is fun in and of itself.

6. They learned that there are a variety of methods of solving mathematical problems and that some methods are better for different purposes.

7. They learned that when you have ideas that are your own, no one else's, you become committed to them, and that this commitment to an idea is a lot different from just knowing a formula for solving a problem.

I have not listed these outcomes in any order of priority. Which do *you* feel is the most important? I probably feel that number five comes first. And number seven reminds me of Mark Twain's comment about his wife's swearing: "She knows all the words but she ain't got the tune." Is there a difference between knowing something and having a "feeling" for something?

An outcome that I did not list because it seemed not to fit into the scheme is that this method of teaching by "inventing" or "discovery", whichever term you like, gives the student an opportunity to think out loud. Thinking is just talking to yourself. You can become a much better thinker if you begin by talking to others, then, eventually, you don't need others. But the only way you can talk to others productively is if you trust them completely.

This last statement has to do with the teacher creating a responsive atmosphere in the classroom, an atmosphere where the student will dare to make any mathematical statement whatsoever.

I hope I have given you enough material in this article to prompt you to try teaching this "inventing" unit. Make 100 copies of the page of figures which I gave my students, and hand it to your class. I would be very interested to hear of any reactions you have, especially positive reactions. If your reaction is negative I will try to visit your school and show you "how it is done!" That is what is commonly called a *challenge!*