History of Numeration Systems

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The basic activities of this paper are adapted from Osborne, Roger et al, *Extending Mathematics Understanding* (Columbus, Ohio: Charles E. Merrill Publishing Company), 1962, and the paper formed the basis for a talk entitled "A Guided Discovery Approach to the History of Numeration" presented by Dr. Neufeld at the Winnipeg Meeting of the NCTM in October, 1970.

The topic of numeration is receiving increased emphasis in contemporary curricula for elementary school mathematics. Teachers are generally encouraged to use concrete materials in helping children to understand the decimal system of numeration. Some curricula suggest the inclusion of certain non-decimal numeration concepts. The curriculum advocated by Dienes states that children should be introduced to numeration by manipulating concrete materials involving base groupings from two to ten. The study of certain historical systems of numeration, such as the Roman system, has long been included in mathematics curricula. Some contemporary programs extend this coverage to brief treatments of Egyptian and Babylonian systems.

The purpose of this paper is to present a sequence of pencil and paper activities which will integrate four historical systems of numeration. Included in the four systems are examples of both decimal and non-decimal numeration. The activities are written to challenge teachers. This does not mean that upper elementary or junior high school students could not work at the same material. In fact, it is often found that students find activities such as these easier to understand than do teachers. This is probably due to the "unlearning" which adults must do in order to accommodate new ideas. The activities in their present format are designed for teachers who will hopefully revise them to suit the needs of their own classes.

The activities are presented at several cognitive levels. Interpreting the symbols of a particular system is somewhat comparable to Bloom's comprehension level. Constructing new systems parallel to those systems previously presented would correspond to the synthesis level. The last section of the activity requires the participant to work at the evaluation level.

ACTIVITIES

Introduction (Earliest times - conjecture only)

1. symbol (numeral) 1 from
   (a) 1 finger
   (b) 1 stick } (Likely horizontal)

2. 2 was $=$ or $//$; 3 $\equiv$ or $///$

3. regarding 4, 5 and so on - WHO KNOWS?

4. development of four systems
   (a) Additive
   (b) Multiplicative
   (c) Ciphered
   (d) Positional (place-value - "ours")

Additive Systems

1. Tally idea ~~||||~~ ~~||||~~ || for 12.

2. Egyptian example
   (a) Basic digits

1	𓏺	staff
10	𓎆	heel bone
100	𓍢 OR 𓍢	scroll or coiled rope
1000	𓆼	lotus flower

   (b) 2347 𓆼𓆼 𓍢𓍢𓍢 𓎆𓎆𓎆𓎆 𓏺𓏺𓏺𓏺𓏺𓏺𓏺

3. Roman example (additive with subtractive element)

   I II III IV *for* IIII *etc.*

Multiplicative Systems

1. One set of symbols for the basic digits

2. Another set for the powers of the base

3. Synthetic example

KEY

(a) Basic Digits

(1) /

(2) //

(3) ✳

(4) ▭

Powers Digits

(5^1) F

(5^2) T

(5^3) H

(b) 97 is (3 × 25+(4 × 5)+ 2

symbolized ✳T ▭ F //

(c) This is a base five example.

(d) Translate ▭ H/F is 505

▭ T is 100.

4. Synthesis - Construct a base three system:

(a) Basic Digits

/

//

Powers Digits

(3^1) T

(3^2) N

(3^3) S

etc.

an example of a possible system.

(b) Prepare examples below as in 3 (b) (d) above:

36 is (1 × 27) + (1 × 9)

Symbolized 1S/N

Translate - //N/T/ is 22

//S/N/T is 66

5. Chinese Multiplicative System

Given the synthetic example (3a), the following "addition" facts ("addition" symbol is /), and the clues, DISCOVER:

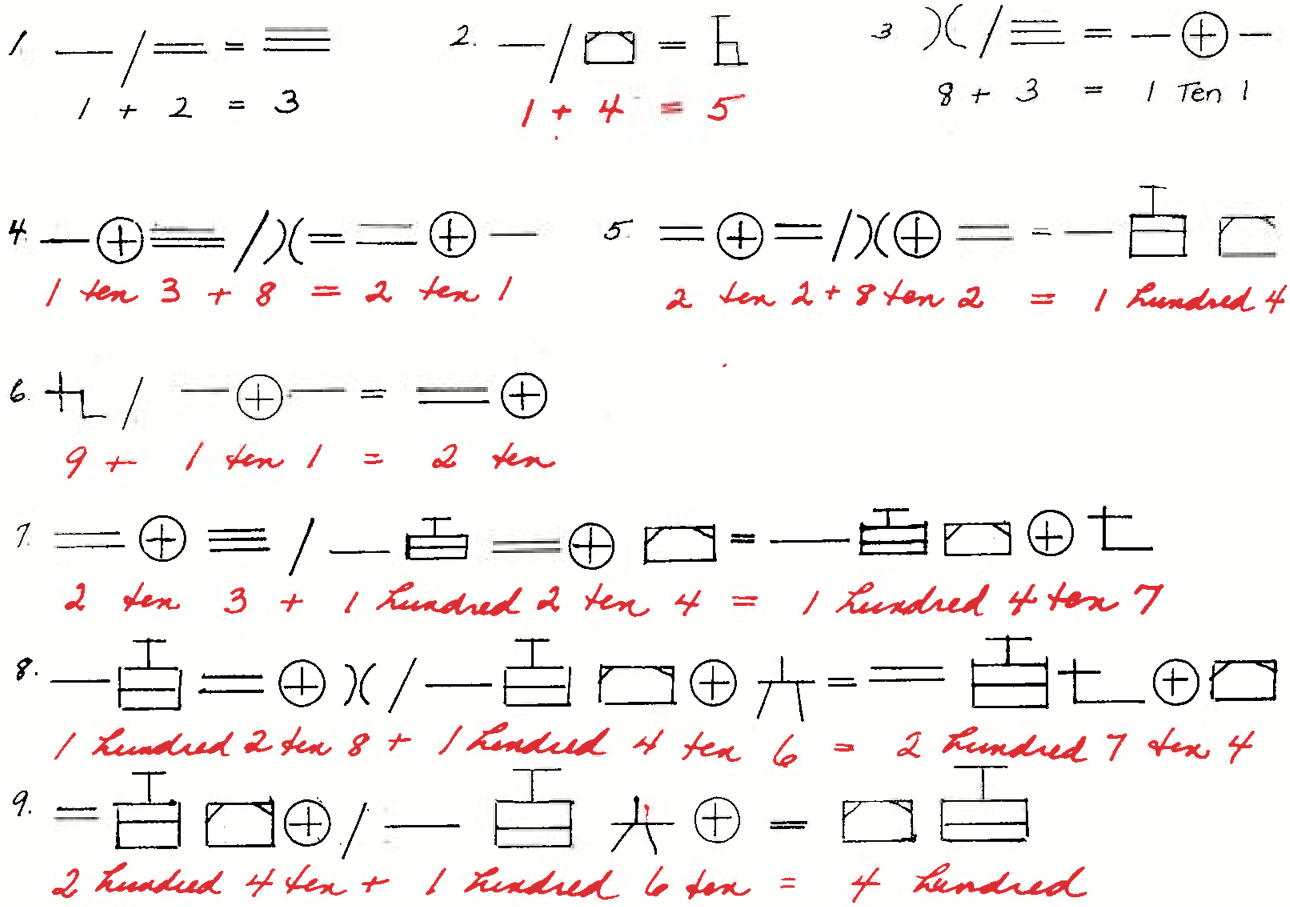
(a) the translation of problems 2 to 9.

(number 1 and 3 given)

(b) the notation system (what symbols are used comparable to our 0, 1, 2 . . .)

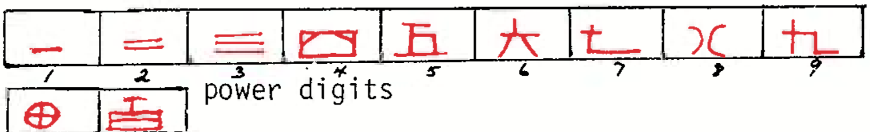
(c) the base

"Addition facts"



Answer space

(b) notation system



basic digits

power digits

(c) base ten

Ciphered Systems

1. A new symbol exists for each *multiple* of each *power* of the *base*.

2. Synthetic example (Ionic Greek model)

(a) symbols

1 - a	10 - j	100 - s
2 - b	20 - k	200 - t
3 - c	30 - l	300 - u
4 - d	40 - m	400 - v
5 - e	50 - n	500 - w
6 - f	60 - o	600 - x
7 - g	70 - p	700 - y
8 - h	80 - q	800 - z
9 - i	90 - r	900 - *

(b) This is a base ten or j example.

(c) The first power of the base is 10^1 or j ;

multiples are 1×10^1 j) -------- 9×10^1 r
1×10^2 s) -------- 9×10^2 *
etc.

(d) Translate 15 is je
347 is umg
650 is xn
999 is *ri
784 is yqd

Positional Systems

1. Place value comes into existence.

2. Zero is needed.

3. Babylonian example (partly positional) (Base 60)

   (a) Simple grouping to 60 (eg. 32 is <<<//),
   positional beyond 60.

(b) 18,812 is $\underbrace{/////}_{5(60)^2} + \underbrace{< ///}_{13(60)^1} + \underbrace{<<< //}_{32(60)^0}$

(c) Translate

58 is <<<<< ////////

27,362 is 7 /////// $-\times(60)^2$ + 36 <<< ////// $-\times(60)$ + 2 // $-\times(60)^0$

27,480 is /////// <<< ////// //

458 is /////// <<< ////// //

NOTE: Except for spacing, one numeral for three different numbers. The spacing of the answers is purposely identical to point up a basic source of confusion in this particular system.

4. Mayan example (Base 20)(actually 20 x 18)

(a) 360 (20 x 18) corresponds to the number of days in their year.

(b) Vertical positioning (in contrast to our horizontal positioning).

(c) 26,656 was

•••	$3(18 \times 20^2)$	=	21,600
•••• (two bars)	$14(18 \times 20^1)$	=	5,040
(shell)	0(20)	=	0
• (three bars)	16	=	16
			26,656

5. Mayan *Positional* System

Given Mayan example (4c), the following "addition" facts ("Addition" symbol is /), and the clues, DISCOVER:

(a) the translation of problems 2 to 9 (number 1 and 4 given)

(b) the notation system (What symbols are used comparable to our 0, 1, 2 . . .)

(c) the base

"Addition" facts

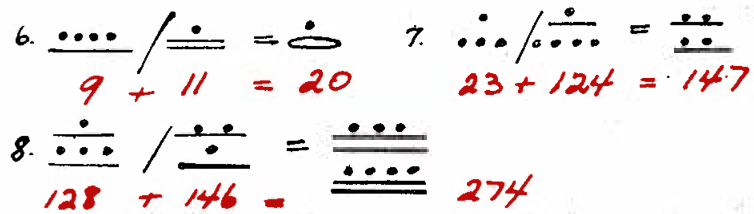
1. • / •• = •••
$1+2 = 3$

2. • / •••• = ____
$1 + 4 = 5$

3. ••• (bar) / ••• = • (two bars)
$8 + 3 = 11$

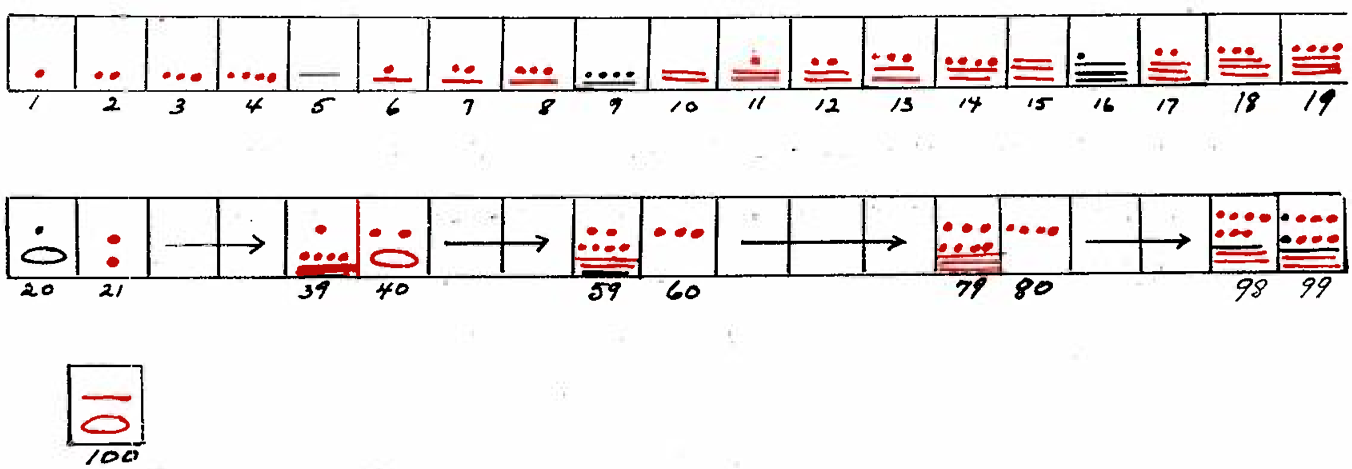
4. ••• (two bars) / ••• (bar) = :
$13 + 9 = 21$

5. :• / •••• •• = (bar) ••••
$28 + 92 = 104$



Answer space

(b) notation system (5, 9, 16, 20 are given as clues)



(c) base twenty

6. Our *Positional* System

Symbols of other systems seem strange but our symbols:

(a) are strange in that many have little or no connection to the cardinal value they represent (eg. 5 for ∴̇)

(b) are changing, for example, the development of "2" and "5".

(c) are unusual in that they aren't completely standard over the English-speaking world:

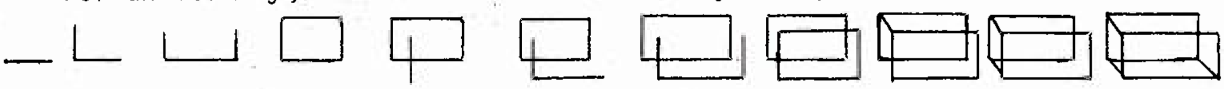
eg. in North America billion means 1000 million while in England billion means one million million.

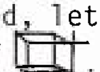
omparison of Systems

1. Numeration systems are man-made. Our system has a long history of development. IS OURS THE BEST? To answer this we must compare. So that our comparison isn't biased, we must use NEW SYMBOLS,

- not 1, 2, 3, 4, . . . (Arabic Positional)
- not I, II, III, IV, . . . (Roman Additive)
- not 一, 二, 三, 四, 五, ... (Chinese Multiplicative)
- not ciphered Greek symbols.

For uniformity, let us use a new set of symbols,



and a new base. Thus far it looks as if the base is twelve. If powers of the base are needed, let us use the convention of superscripts (Base)[1] =  , (Base) [2] =  , etc. If multiples of the base are needed (Greek ciphers) use subscripts 2 x (Base)[3] = , and so on.

2. Questions

(a) Why have any base at all? The number of basic symbols is reduced.

(b) If we must choose a base, what qualities should it have?

1. easily remembered
2. average number factors — nine (three); ten (five, two); twelve (six, three, two)
3. reasonable number of basic symbols.

(c) Which of the four basic systems utilized a ZERO?

1. Additive? No
2. Multiplicative? No
3. Ciphered? No
4. Positional? Yes

(d) All symbols in any positional system are placeholders. What is the basic function of any symbol (3, for example, in the numeral 4136)?

3 represents the number of tens

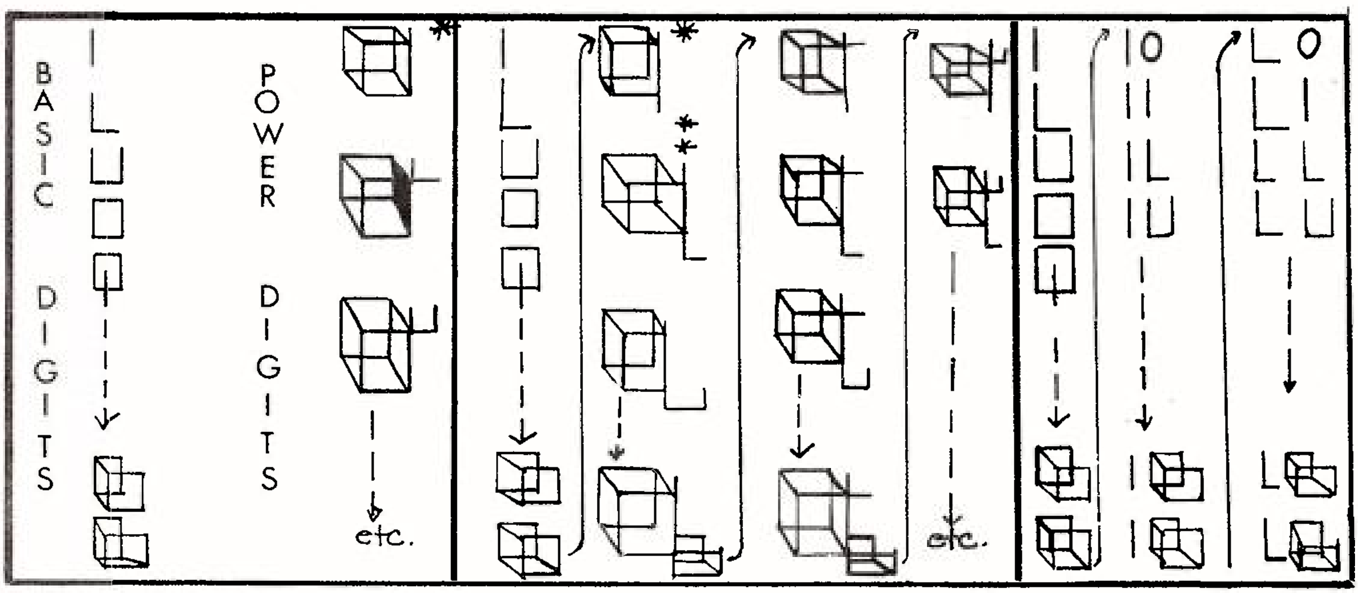
(e) Does the symbol for ZERO have a similar function in the numeral 608?

Yes, the number of tens is zero.

3. Comparison. Now for comparison purposes, let us use our *new symbols* and *new base* with all *four systems*.

(a) Numeration systems.

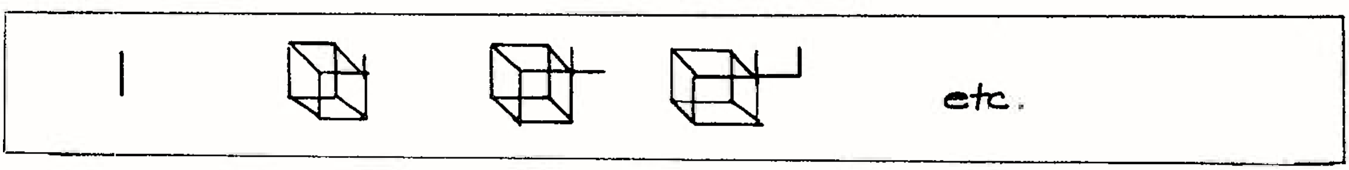
MULTIPLICATIVE CIPHERED POSITIONAL



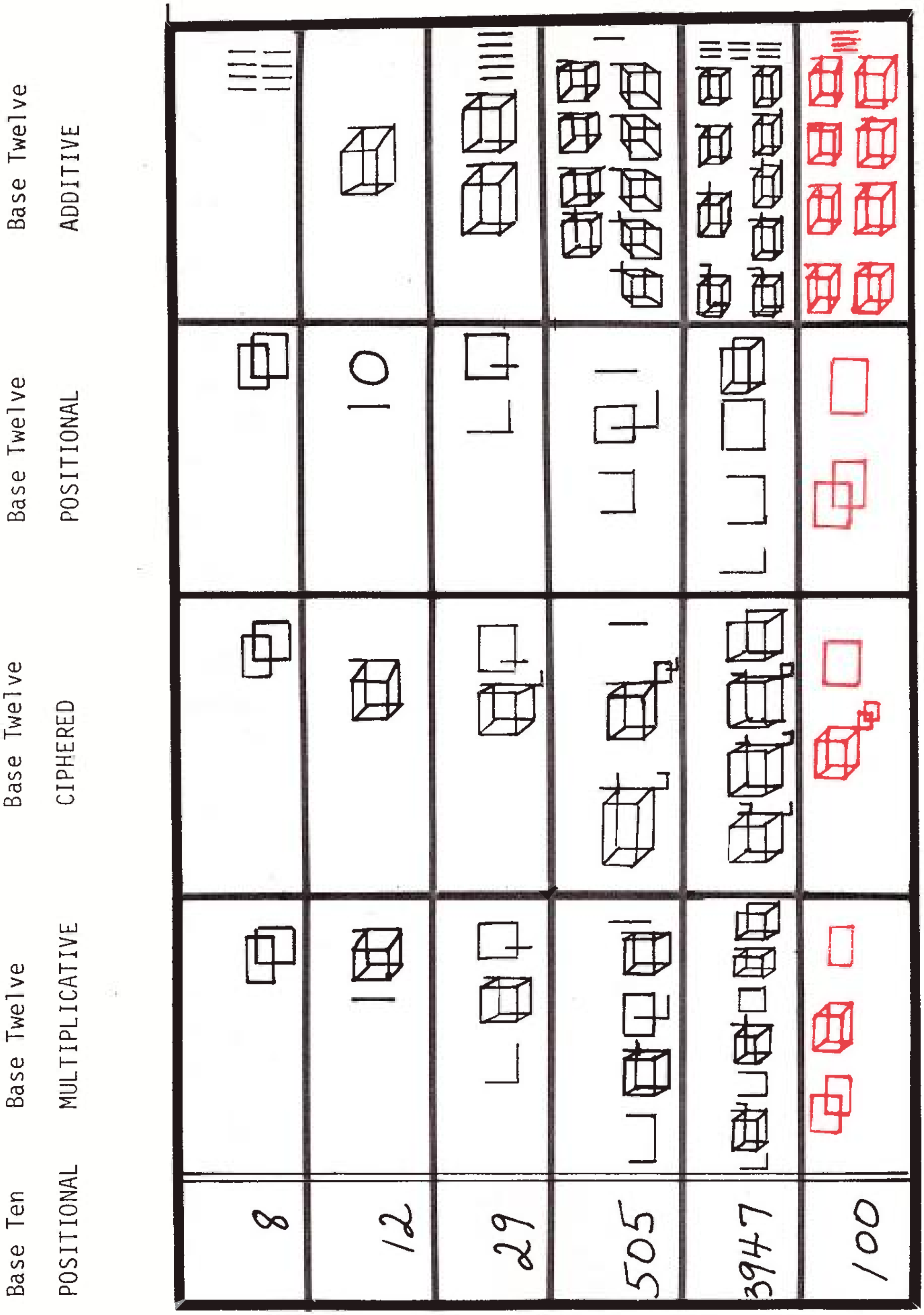
ADDITIVE

* Power of base TWELVE

** Multiple of the base TWELVE



3. (b) Translations - each of six numbers is symbolized as a numeral in five systems of numeration.



3. (c) Comparison of numeration systems.

Suppose you were able to choose one of the four numeration systems, each utilizing a base of twelve and the "block digit" symbols, to replace the system with which we are most familiar:

1. Indicate your preference by ranking the systems 1, 2, 3, or 4.

   Multiplicative -

   Ciphered -

   Positional -

   Additive -

2. What are some advantages of your highest-ranked system?

   Mult. - Place value name is symbolized
   Ciph. - much information in a single symbol
   Pos. - fewest symbols used
   add. - easiest to figure out - just add.

3. What are the disadvantages of the other systems of numeration?

   Mult. - not too concise (almost as bulky as Additive)
   Ciph. - Complicated Individual symbol.
   Pos. - place value not symbolized.
   add. - very "bulky"

4. What is the value in studying other systems of numeration?

   1. Put ourselves in the "shoes of children." We realize the problems children have in learning the base ten positional system.
   2. Understand how other systems have contributed to the development of our system.
   3. Understand the basic ideas of base, place value and symbols by seeing these ideas applied to a variety of parallel systems.
   4. Realize the "man-madeness" of systems in that numerals and arrangement can be arbitrarily chosen.
   5. Examples in everyday life - clock, eggs, computers, etc.