Active Learning in Mathematics

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Everyone in Education knows that there are ebbs and flows, waves, kicks, the latest method, gimmicks, projects and innovations. There are discovery methods, rote drill, team teaching, open areas, traditional ways, my approach and yours, and one great difficulty in penetrating this maze to select what each of us does, is that the words are discrete and life is not. I mean, the mere saying or writing of the phrase "active learning in mathematics" is easily done and when it is said, it is finished. The phrase has four words, each is easily understood, it heads a piece of writing and because of such simplicities we can easily jump to the conclusion that similar clarity will automatically follow. More, it can easily be concluded that the simple ideas simply described can be as easily made active within classroom situations. Oh, if it were so easy!

Then, too, the words 'active learning' suggest that there exists something which is 'non-active learning'. I, for one, don't know what that could be. *All* learning is surely an activity, or it involves activity. It may be of the mind or it may be otherwise, but it must necessarily include a minimum of physical activity even if it be eye movements or the finger and hand movements needed to hold a pencil. However, although the word 'active' is strictly redundant, the phrase is one which is frequently used these days so there is value in pursuing some meanings for which the words might stand.

I am trying to suggest that what is important to discuss is not 'this' as opposed to 'that' in simple terms, but rather to ask *what* activities are relevant to the learning of mathematics, what *quality* of activities do we want, activities with *what?* What activities of the traditional kind of mathematics lessons are going to be continued regardless of our opinions, our preferences and current fads or styles? What traditional activities have we the power to retain or discard, yet wish to keep? What activities, under-emphasized in previous years, should we increase in density in the future? And, lastly, are there any activities never used before which need to be introduced? I suggest that careful blendings of many possible activities will lead to the best learning and to the fastest progress.

What components exist from which the blends can come? What will give us guidance in the choice of activities? Clearly it must be from the consideration of, firstly, the nature of the human beings involved in the learning situations; secondly, the physical situation of schools, classrooms and so on; thirdly, the nature of what it is we intend to communicate - that's the mathematics; and fourthly, some of the application requirements of the students' future.

Here we choose to deal only with the first and the third, for it is those that are frequently neglected in other than obvious ways. With regard to the other two, it can be said that the physical situations are usually adequate for most activities once the determined teacher is convinced of their value, while applications tend to look after themselves when deep understanding is present.

What follows, then, is an introduction to some of the important aspects of active learning in mathematics. They apply to all students at any grade, whether they be deemed by some to be slow or fast, dull or bright. The examples quoted can be multiplied almost without limit and they have to be for adequate classroom implementation. Here, however, hints only can be given.

THE DISCOVERY ASPECT

In a sense, every student has to discover everything. The mere projection of words, explanation, facial looks or activity by a teacher, does not necessarily imply or get a reaction by a student, let alone a hoped-for reception. The text may be handled and so may physical aids, but there have to be continual acts of volition by every student to trigger some kind of reception in his eyes or mind.

There are, however, varying levels of reception needed for understanding. That a fraction is conventionally written in the form $\frac{a}{b}$ needs little discovery. It is seen and almost immediately accepted by the youngest of students in school. That it represents not only a particular element as well as the whole of a particular class of ordered pairs of whole numbers, this, on the other hand, needs more and deeper discovery with a different quality and quantity of what has to go on within the person to make it as secure as the symbol seen with the eyes.

In mathematics teaching, we have traditionally acted as though simple surface-level discovery is all that is needed. We explain, we do examples, show how and get the children to practice on paper. We try to define subtleties and when we see the pupils failing we tend to the conclusion that it is because some are not capable. We call then "inattentive", "lazy", "slow" and by using more simple words, seem to suggest that it is they who are totally at fault. Seldom do we think that maybe it is we teachers who have not yet attained the highest forms of communication and activities to induce what is needed to have all students learn the mathematics.

If, therefore, discovery is a fact in learning processes, not just the name of a method, we surely have to enquire more into the ways and means by which students are compelled to take responsibility for their own learning, for no one ever did anyone else's. Teachers must minimize correction from an apparent authoritative standpoint and only insist on being an authority where the student cannot be. That can be condensed to just two functions: orchestrating the activities of the group and telling the children what are the agreed conventions of symbols and form. In all other matters - in the widening consciousness of the fundamental facts - the pupils must be helped to discover at first hand, and although little attention has been paid to this under the weight of verbal and written tables and formulae to be learned, every man and woman knows the effect of a general understanding into which many details fit, compared with the knowledge of many details with no connecting substratum of understanding.

Teachers need to be learning facilitators, creators of learning environments, flexible ones, appropriate to the tasks, to the actual students involved and capable of minute to minute changes. As a result of his wisdom and knowledge, using the art and craft of his trade, the teacher of mathematics in particular initiates activities which are such that anyone engaged in them can hardly help but be affected by the concentration of the latent mathematical ideas and concepts built in. It is arranged deliberately that the chances are very high that no student can avoid the inundation and, what's more, enjoy it!

The environment will include the use of physical aids, the writing on paper, the usual symbol work and texts, but it will also take into consideration the inner consciousness of every person in the room. No system can be ignored. Inner systems of humans must be recognized as essential concomitants of the learning environment - thoughts, wishes, day dreams, mind wanderings, feelings. It is these that constitute the stuff of existence as much, or more than, the manipulative aids, the books, the diagrams and the mathematics which we aim for.

> **It is the teacher's responsibility to watch the learning process and to introduce creative conflict at appropriate moments.**

THE INDIVIDUAL ASPECT

Every student has to do his own learning, with his insights, his past, and his present as he sees it. These are the filters through which pass all that occurs in our math lessons. We had better be aware that this is so.

Each student is unique and may differ much in his reception of something public from all the other receptions of the individuals in the group. It is impossible that two people receive the same thoughts, the same understanding, no matter how the teacher tries. Whereas one child happily accepts the following sequence:

$$\begin{aligned} \log 20 &= 1.3010 \\ \log 2 &= 0.3010 \\ \log .2 &= \bar{1} + .3010 \\ \log .02 &= \bar{2} + .3010 \end{aligned}$$

another's difficulty may lie in his worry that in the third and fourth lines there are + signs but in the other lines there are not. His difficulty evaporates as soon as he sees that he *could* write the equivalents:

$$\begin{aligned} \log 20 &= 1 + .3010 \\ \log 2 &= 0 + .3010 \\ \log .2 &= \bar{1} + .3010 \\ \log .02 &= \bar{2} + .3010 \end{aligned}$$

Now there is a balance of the forms which satisfy him. If the teacher does not understand a possible obstacle here and cannot help, the student may well feel he is just incapable of understanding where apparently the others do.

Another example, from elementary school:

Student A.	B.	C.	D.
2.48	2.48	2.48	2.48
-1.69	-1.69	-1.69	-1.69
1.21	.81	$1.\bar{2}\bar{1}$	.79

Because child D is "right" it does not follow that we know what he did, how he thinks. Correct answers are frequently written for the "wrong" reasons. What is needed is a suspended judgment, not a stamp of approval or disapproval. If we can find out what each child really did (and a good way of doing this is to ask them and *listen* to replies!) this is what might emerge:

A says "9 from 8, can't, so 8 from 9 is 1 ("always do what you can")
6 from 4, can't, 4 from 6 is 2 . . . etc."
B says "9 from 8, can't, so 8 from 9 is 1.
16 from 24 is 8."
C says "9 from 8 is $\bar{1}$, so 6 from 4 is $\bar{2}$ . . . etc."
D says "9 from 48 is 39; put down 9 and carry 3.
3 from 20 is 17, but 1 from 17 is 7.
Put down 7."

That's the tip of the iceberg of what goes on *within* individuals. It occurs all the time, in all lessons, with all students. As the understanding that this is so grows (together with an increase in knowledge of more and more possibilities within this vast, commonly unexplored field), every teacher can open for himself new vistas in his approach to teaching. The consequences are greater self-respect and successful learning for all students.

So then individualism also is a fact, not a method, not an opinion. We cannot choose to treat humans non-individually and succeed. If an "individual approach" appears to be only a fad in which it is assumed that it is organization which is at the bottom of individualism and if this passes next year to some new urge which has all students reacting the same way, people will still go on acting individually. Even if they are overwhelmingly herded and made to jump through the same hoops of authority, they still won't oblige except for a short while. Man's eternal quest for freedom will be maintained despite temporary setbacks.

THE GAMES ASPECT

Children and adults like playing games. Games are concentrated, challenging, enjoyable activities which can be given up at will, whereas work is more unwilling and there is more accepted pressure before it can be abandoned.

Children often like to play games with physical objects and they have show this in their preschool experience. However, games are not restricted to the use of physical objects nor competition against others. Word games, written games, games with oneself and with textbooks can all be rewarding.

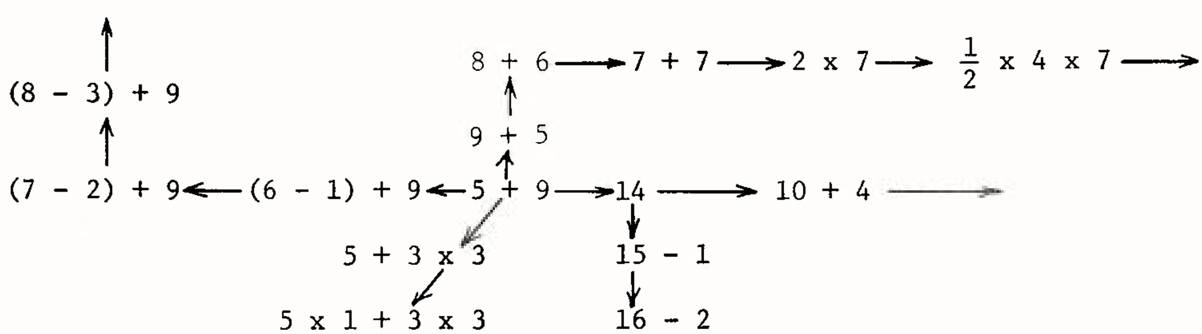
Some of the physical games are quite new in mathematics lessons. There are teachers who for years have had children apply mathematical and arithmetical principles to measuring, shopping and surveying, but today there are also many physical materials for use whose importance is not in application but in the lead they provide to the awareness of mathematical principles by the very act of using them and becoming more conscious of what one is doing. The colored rods, attribute blocks, multibase blocks, geometry models, geoboards, pebbles and counters, mirrors, cut-out figures, films, one's own body and body movements - these are some of the newer opportunities to be taken by good teachers.

As an example, consider the use of the so-called attribute blocks which are of different shapes and different colors. A red triangular block and a red square are alike because they are both red but they are different in shape. When we use blocks of other shapes, other colors, and use some with different thicknesses and even different materials, then we have pieces for many games which demand, in the very act of playing, understandings of differences, equivalences, intersections and union, elements and subsets. And although the use of such is by no means the only way, it can hardly be denied any longer that such notions are essential to a proper understanding of numbers, addition and multiplication, and more of the familiar list of what we want children to know.

Moreover, by the use of games, improved mastery of the traditional mathematics and arithmetic can be attained. Instead of laborious, non-related facts which somehow have to be memorized, the games approach involves thoroughness and intensity and subtly demands every student's on-going dedication.

Games with physical objects are only one type of game. There are many others so here we illustrate one which seems quite traditional:

CONTINUE THIS "WEB": From any name draw an arrow and write another name for the same number which comes to *your* mind. Have a reason why you went from one name to the next.



Do not ally the word "games" with something frivolous, a pastime, something we do "after work". One look around at man playing games shows clearly that the most played games are very demanding in energy, devotion, thought, and time and that they increase learning and powers. To try to meet mathematics as a series of games is not to drag it down to the wasteful, trivial, filling-time attitude. Rather, it is to elevate certain mental activities to the level of some of man's finest activities - the game of space exploration, of serving others or of life itself.

THE MATHEMATICAL ASPECT

Mathematics is concerned with the dynamic use, in the mind, of relations and relations of relations, with some applications to social and economic life.

Arithmetic is one of its branches having a main emphasis on the renaming of numbers by means of algorithms, tables, figuring, calculations and other devices. To learn that 2 + 3 = 5 *may* only be a matter of remembering that "5" is a word which is an acceptable substitute for the phrase "3 + 2", and "6" is not the accepted word. To recognize, however, that 3 + 2 = 2 + 3 implies much more. Now we get a hint of a basic principle (commutativity) and this may lead to other interests. Alternatively, it gives us a pattern to get new names for numbers we have not met before. (4576 + 687 surely must equal 687 + 4576 even if this - 4576 + 687 - has not been confronted previously).

To write or say 2 + 3 = 5 is a convention, a *convenience*. The symbols as such give no clue to what they stand for. It is just as easy to accept the writing 2 + 3 =7, but historically that is not what was developed, so we think that 2 + 3 = 5 is correct and find 2 + 3 = 7 uncomfortable. The meanings behind the symbols, however, are *not* convention. They are based on fact over which man has no control. He can only be ignorant of the fact. The fact is embedded in life and everyone of us is capable of firsthand experience and enlightenment. We therefore encourage mathematics partly to understand and use the ideas, the concepts, the relations, the facts and partly to help us maneuver the old arithmetic which is still vital and valuable.

On the whole, we do not question the kinds of activity suggested by the se of mathematics text books so long as the teacher appreciates that the one rinted form, the book order and the style of presentation is not necessarily ight for any student whatsoever! Those mass produced aids show the sophisticated utcome of years of thought and centuries of argument and symbol evolution. They ust be seen as such, used for the embodiment of desirable aims maybe, but approached along different and unsophisticated paths. They rarely exhibit the form f progress precisely needed by any child to reach many of the same conclusions.

EXAMPLE: Usually in text books the addition of fractions is dealt with before the operation of multiplication.

One student pointed out that addition is so:

$$\frac{2}{3} + \frac{5}{7} = \frac{(2 \times 7) + (3 \times 5)}{(3 \times 7)} = \frac{29}{21}$$

and multiplication is thus:

$$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

Addition, it seems, implies three multiplications and one addition. Multiplication needs only two multiplications. On what grounds, he asked, is *addition considered easier than* multiplication and taught first? Had he a point?

THE INTEGRATIVE ASPECT

If math is indeed concerned with relations and activities with relations, then it is essential that every student constantly has experience which relates things for *him*. It is *his* meaning which is vitally important, his continuous progress will be unique for *him*, and the relations which he explores will not by any means be confined to that portion of his school day called the "math lesson". Relations are abstracted by all humans in all kinds of situations. The math lesson will emphasize them, show what can be done with them, and greater power in their use will be gained. But the existence of relations must be from all subjects studied, from all life. Appreciation of integration across the curriculum is therefore important.

Integration is also needed within the math study itself so that growing experience is had in relational activities which will become second-nature and pervade all the educational scene.

Within the math study, such integrating, creative aspects can be dealt with in many ways. Here are two, briefly:

1. Reverse the usual task of getting an answer to a problem. Instead, get a problem to fit the answer: $x = 7$. Write some other equations of which $x = 7$ is the solution.

   The sum of four numbers is 2489. What are they?

2. Rather than set word problems, give an equation and from it invent an appropriate social situation.

   $3y + 4 = 19$. What situation in a grocery store might fit this?

Across the curriculum, apart from some obvious use of number work and equations, there are the rich, powerful mathematical ideas of relations, differences, attributes, sets and subsets, transformations, symbolisms, reversals and repetitions to explore and use. All of these occur and have their importance in social studies, art, science and music.

GRADE I

If a child can read and write in some form the words "step up pat" he can also read and write all six permutations of these same words.

He can be asked to do so therefore with teacher *expecting* him to do it. He can also be expected to read every one of the six sentences and know whether to him they make sense or not.

GRADE VI

In discussing "Man", students can be helped to list equivalences and differences of men and see intersections of attributes. This leads to an improved awareness of racial similarities and differences, all within the same human species.

THE HUMAN ASPECT

No mathematics lesson ever attended took place without human beings being present! Even if it is true that mathematics is devoid of feelings, opinions and other vagaries of everyday life, it is not true of mathematics learning situations, for the humans present have such attributes all the time. In the learning of the subject, therefore, there are facts of this kind to be taken into full consideration by the teacher: the need for understanding, respect for others, tolerance, frustration, tiredness, hope, lack of communication, annoyance, satisfaction. Unless teachers are aware of these as being present in every lesson, in every human, then a large part of what is really going on in the lesson will be completely missed and the teachers will be forced to resort to the traditional rationalization of "the students won't attend", or "they are slow and incapable".

It is not that we necessarily wish to change such feelings, but these are very real parts of everyone's life and we must certainly recognize their existence and the part they may be playing. Because of such awareness and allowance, we shall help the students commune with each other and with ourselves, since we shall no longer be dependent on outward signs alone which give little clues to the thinking and the perceptions going on within.

Perhaps the future progress of all learning depends solely upon the harmonizing of what has to be passed onto the new generation within the context of the "species specific mode" of every learner human in this case. It is fruitless to try to teach a dog to do what a dog is incapable of doing. The same holds for humans. We assume that telling is sufficient, whereas if we pause and think for a moment, each of us has plenty of evidence that telling is seldom sufficient for communication for anything other than trivialities. *It is not a human mode that learners automatically learn by being told.* Humans do not pay much attention to what others are saying. The mode surely suggests that we have to encourage activities in which it does not matter if the learners attend or not - except to themselves, which they cannot escape.

If a 5th grader wants to add fractions thus:

$$\frac{3}{7} + \frac{4}{9} = \frac{7}{16}$$

let him and encourage him to do more like it.

$$\frac{3}{4} + \frac{4}{12} = \frac{7}{16} \qquad \frac{2}{5} + \frac{5}{13} = \frac{7}{16}$$

$$\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{4}{13} = \frac{7}{16} \qquad \frac{7}{16} = \frac{3}{1} + \frac{2}{1} + \frac{1}{1} + \frac{1}{13} \quad (!)$$

When he has developed other names for $\frac{7}{16}$ according to the same rules, he most probably will begin to sense a contradiction with other thoughts he has about fractions. It is that contradiction within himself that will provide motivation and at least in this case the realization that that is not the way fractions can be combined.

SUMMARY

What we suggest, therefore, is a "systems approach" - one which takes into consideration *all* the systems which are active within the learning situation, outward and inward. Much cannot be avoided, some of it can be and some can be initiated by the teacher. Every learning situation is a very complex set of circumstances, whereas the history of education has tended to act more and more as though the learning is simple even if the organization is complex. A simple solution or a straight-forward outward appearance can invariably be assumed to be misleading and wrong. There has to be discovery and non-discovery; rote learning will still play a role; there will be partial understandings and superficial insights along a spectrum. There will not be artificial discussion about the group as opposed to the individual, for every group is composed of individuals, and every individual is a member of many groups. The variation of possible individual responses to every tiny situation, the variation of movements, thoughts, things to do and the way for people to do them - all contribute to a fantastic number of alternatives that in practice, whether we like it or not, make every lesson different, no matter how hard we try to repeat or homogenize.

Most of what goes on, most of the active learning, will be unseen by the members of the group. Each individual will have a concentrated view only of his own incoming sensations and inner thoughts, and on top of all this there will be the mathematics, the activity with relations, numbers, space, sets, operations, functions and the arithmetic, as a by-product of some of this. The complexity is inevitably so great that it makes one wonder how we ever got the belief that we can determine in any but rough form how some piece of learning *should* take place. Why do teachers plan lessons rather than prepare themselves to meet their students as they *will* be by the time of the next lesson?

A balance is needed, not one which attempts to remain still, but one which is nevertheless stable. Sometimes the kinds of activity will seem to be of one kind but because of the teacher's innate stability and his understanding, purposes and leadership, there will be some vaguely perceived fulcrum about which the kinds of activity will oscillate. The mathematics classroom will sometimes appear to be a hive of physical activity. Children will be measuring, weighing, or using blocks and ropes to discuss sets. Others may be constructing tetrahedra and investigating their rotations. Computers may be in use or some pupils may be outside the school with angle meters doing elementary surveying. Sets of cubes, colored or plain, may be seen, although the students using them may really be discovering the relations inherent in successive cube numbers. All the fun of the fair may seem the order of that day!

On other days, or with some children perhaps during the same time period, the activity will be with paper and pencil and the appearance will be traditional.

The facts concerning the consciousness of the different systems extant must lead to variety for each learner, for the whole class, and for the teacher, too. Underlying all, however, will be the commonality of an increasing mathematical consciousness for each child within a human community. This will result from the effects of an environment purposely engineered by the knowledgeable teacher in which the learning takes place intensely, individually, integratively and joyfully. That is the "active learning in mathematics" to aim for!