Establishing a Mathematics Laboratory Program

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WHAT IS LABORATORY LEARNING IN MATHEMATICS?

The term *laboratory method* is commonly used today to refer to an approach to teaching and learning mathematics which provides opportunities for the students to abstract mathematical ideas through their own experiences. Laboratory programs can, in general, be characterized by the following features:

1. The student is actively engaged in the doing of mathematics; he is not a passive observer in the learning process.

2. Concrete materials, structured games, or environmental tasks are used to give meaning to mathematical concepts.

3. The students work much of the time individually or in small groups from written instructions.

WHY USE A LABORATORY APPROACH?

Some of the functions and aims of the laboratory method are:

1. to permit students to learn abstract concepts through concrete experiences and thus increase their understanding of these ideas;

2. to enable students to personally experience the joy of discovering principles and relationships;

3. to arouse interest and motivate learning;

4. to cultivate favorable attitudes toward mathematics;

5. to encourage and develop creative problem solving ability;

6. to allow for individual differences in the manner and speed at which students learn;

7. to enable students to see the origin of mathematical ideas and to participate in "mathematics in the making";

8. to enrich and vary instruction.

HOW CAN LABORATORY ACTIVITIES BE USED IN CONJUNCTION WITH A REGULAR MATHEMATICS PROGRAM?

Laboratory lessons can be inserted into a school's ongoing mathematics program in several ways. One period a [illegible] might be reserved for a laboratory program consisting of a series of speci[illegible] activities and games selected to supplement and to enrich the regular course. Such a program would function as as adjunct to the prescribed course of studies and would be aimed at fostering the development of independent and creative thinking and at improving attitudes, as well as providing a change from the textbook-chalkboard approach to learning mathematics.

Teachers can also use laboratory lessons to introduce new concepts and to review previously taught material. Students often require concrete experiences before a concept can be meaningfully developed in class at an abstract or symbolic level. Other ideas, first introduced in class, can be further explored and investigated in a laboratory setting. Topics such as probability, numeratior measurement, and geometry are particularly well suited for laboratory study.

HOW ARE LABORATORY LESSONS DEVELOPED?

Each of the activities later described in this paper is designed to lead to the development of a concept on the discovery of a relationship in mathematics.

CONCRETE MATERIALS: Each lesson is based on some type of physical material or manipulative device. Concrete materials serve several important functions:

1. They create interest and provide motivation.

2. They provide a real-world setting for the problem to be investigated or concept to be developed.

3. They provide a physical means by which the learner can begin to solve the problem or explore the concept.

4. They provide the learner with a way of verifying his hypotheses and checking his calculations independently of a teacher or textbook.

SEQUENCE OF LEARNING ACTIVITIES: In general, each lesson is structured to permit the students to arrive at the desired conclusion or generalization inductively. The students first familiarize themselves with the structure or operation of the apparatus or physical objects. They then use the material to gather information relating to the problem, recording this data in a table or on a graph, if possible. On the basis of these observations, hypotheses are formulated and tested. Finally, generalizations are stated. The newly discovered rule might then be used to answer additional questions or to do practice exercises.

WRITTEN INSTRUCTIONS: Although the lab lessons described later in this article can be presented to whole classes of students by their teachers (who would demonstrate with the concrete materials), the activities are most successful when performed by small groups of students working from written instructions. In preparing the "assignment cards", the teacher must consider the amount and kind of direction his students might require. It has been the experience of the writer that students new to a laboratory approach in mathematics usually want and need fairly specific instructions which indicate how they are to proceed and which provide feedback relating to their progress. However, after a short time, most students develop more confidence and can become independent to the point where they are willing to determine their own procedure for solving problems posed in rather broad terms. The lessons outlined in this paper are fairly detailed, but the instructions can be made more "open" or "closed" depending on the needs of the particular students who will be using them.

HOW IS A MATHEMATICS LABORATORY ORGANIZED?

GROUPING STUDENTS: Small group learning can offer several advantages over individual work or whole class instruction in mathematics. Most students (of both elementary and junior high school age) like school work which involves some physical activity and opportunity to talk to their classmates. In small groups, each pupil has an opportunity to work directly with the concrete materials and to take part in discussions relating to the activity and possible methods of solving the problem. Many laboratory activities must be performed by the students working as a team in which one person manipulates and another records. With activities of this type, if a group consists of more than two students, there is the danger that one of them will be "left out" of the activity and discussion.

Groups can be formed on the basis of friendship or ability. The writer suggests that to organize the pupils to perform the activities described in this paper, the teacher designate pairs of students who are compatible (but not "too" friendly) and who work at about the same speed and level of understanding. Placing a high ability student with one of low ability is not necessarily advantageous for either, as the two are likely to have different learning styles. The more able learner may wish to move from the concrete to the abstract more quickly than the student of lesser mathematical ability. Of course, the composition and size of the groups can be changed periodically.

ROOM: Almost any classroom can be readily transformed into a mathematics laboratory. "Stations" or "centers" for the activities are created by moving two or three desks (flattop are best) together in various locations around the room. Concrete materials and instructions for the activities can be kept on shelves or in cupboards where they will be easily accessible to the students on "lab days".

ROLE OF THE TEACHER: Although the teacher has a somewhat different role in a laboratory setting than in a more traditional classroom situation, he is still the key to a successful program. The teacher must first select or devise worthwhile activities which will be appropriate for his class. Students can often assist by bringing or making materials and by contributing task cards that they have developed.

During a lab period, the teacher acts as a guide or counselor, giving assistance when requested or needed, but encouraging the students to develop concepts through their own efforts. He must guard against giving students information that they are not ready to assimilate. On the other hand, a teacher would not allow students to become discouraged or waste time from lack of direction. Evaluation and recording of pupil progress is another important, although difficult, responsibility of the teacher.

HOW MIGHT STUDENTS BENEFIT FROM LABORATORY EXPERIENCES?

A recent study conducted by the writer to investigate the effects of implementing a mathematics laboratory at the Grade VII and VIII levels confirmed that students enjoy learning mathematics in this way, and indicated that such a program might be one way of achieving certain important objectives of teaching mathematics. Students in the laboratory group worked in pairs from written instructions, rotating through a set of 10 activity lessons based on concrete materials on a once-a-week basis.

Several measures were used to compare the lab students with students who had taken the experimental lessons in a teacher-directed class setting, and also with students who had not been exposed to the experimental materials but who had continued to study the regular program (*Seeing Through Mathematics*) full time. It was found that the use of 25 percent of class time in mathematics for informal exploration of new mathematical ideas did not adversely affect achievement in the regular program over a three-month period. In addition, tests of learning, retention, transfer, and divergent thinking indicated that students in both experimental groups had benefited mathematically from participation in the program. Although test scores were slightly higher for the students who had studied the lessons under a teacher in a class situation, the reaction of the lab group to their insturctional setting was more favorable. The lab students also rated higher than students in the other two groups (a) in feeling that learning mathematics is fun and enjoyable, and (b) in the view that mathematics is a subject which can be investigated and developed experimentally by using real objects rather than restricted to a textbook subject in which symbols are manipulated.

The most popular feature of the laboratory method as identified by the students was the opportunity which it provided for working independently of the teacher. The following are comments made by students in the laboratory group:

- I liked the privilege of working at your own speed and without a teacher always telling you what to do. It was fun and helped me a great deal. I think it is better than teaching from the book and is a lot more interesting.

- I liked where you could find out and prove things yourself so you would know for a fact that something is true.

SOME LABORATORY ACTIVITIES

The laboratory lessons now described have been used with students ranging from Grade IV to VIII and also by prospective elementary teachers. (It appeared that the latter category of students usually benefited from the concrete experiences leading to the mathematical ideas as much as the younger pupils.) It should be pointed out that these lessons deal mainly with abstract mathematical concepts. Other kinds of worthwhile laboratory tasks might be more real-life oriented or science-related.

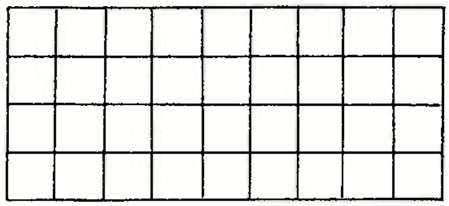
AREA AND PERIMETER

Purpose: To investigate the perimeters of the family of rectangles having a constant area.

Materials: 36 unit (one-inch) squares, graph paper.

Procedure:

1. Form a rectangle from the squares by arranging them in 4 rows and 9 columns as shown.



   Find the *area* - the number of squares needed to cover the rectangular region.
   Find the *perimeter* - the number of units around the outside edge of the figure.

2. Rearrange the squares to form another rectangle. (Make a different number of rows and columns.) Find its area and perimeter.

3. Form as many rectangles as you can using all 36 squares. Find the area and perimeter of each and record your answers in a table.

4. (a) Did you make a rectangle with 5 rows? Discuss.
   (b) Could you now find the area and perimeter of the rectangle with 3 rows and 12 columns without first making it and counting? How?
   (c) What did you notice about the area of each of the rectangles? Why is this so?
   (d) Which of the rectangles had the greatest perimeter?
   Which of the rectangles had the smallest perimeter?

5. Complete the table below. Use the information to make the graph showing the relationship of the perimeter to the number of rows.

No. of rows	No. of columns	Perimeter
1		
2		
3		
4		
6		
9		
12		
18		
36		

6. Use your graph to answer the following questions:

   (a) Give the dimensions of the rectangle having the smallest and greatest perimeters.
   (b) Find (approximately) the perimeter of the rectangle with a base of 15 units.
   (c) Find the dimensions of the rectangle having a perimeter of 50 units.

POLYHEDRA[1]

Purpose: To discover Euler's rule for simple polyhedra through examination of models of the regular polyhedra, prisms, and pyramids.

Materials: Models of various prisms and pyramids and the regular polyhedra (or materials and instructions for constructing them).

Procedure:

1. Find the model of the tetrahedron. It has 4 triangular *faces*. The line along which two faces meet is called an *edge*. The point or corner where three faces meet is called a *vertex*. How many edges and vertices (plural of vertex) does the tetrahedron have?

2. Count the number of faces, vertices, and edges of each of the solid shapes and record your findings in the table below.

Name of Shape	Number of Faces	Number of Vertices	Number of Edges
tetrahedron	4	4	6
cube			
square pyramid			
triangular prism			
octahedron			
pentagonal pyramid			
pentagonal prism			

3. Try to find a rule relating the number of faces, edges, and vertices, which holds for each of the above solid shapes. (Hint: Add the number of faces and vertices.)

4. See if your rule holds for the other solid shapes.

5. Examine the pyramids and prisms again. What is the difference between a (pentagonal) prism and a (pentagonal) pyramid?

6. In what way are the tetrahedron, cube, octahedron, dodecahedron and icosahedron special? Why are they called the *regular* polyhedra?

[1]Suggested by F. Bassetti *et al.*, *Solid Shapes Lab.* (New York: Science Materials Center, 1961).

7. Examine all the shapes again and fill in the table below on the faces of the solid shapes.

NAME	NUMBER OF FACES			
	TRIANGLES	SQUARES OR RECTANGLES	PENTAGONS	HEXAGONS
PYRAMIDS tetrahedron				
square pyramid				
pentagonal pyramid				
hexagonal pyramid				
PRISMS triangular prism				
cube				
pentagonal prism				
hexagonal prism				
REGULAR POLYHEDRA tetrahedron				
cube				
octahedron				
dodecahedron				
icosahedron				

INTERSECTING SETS[2]

Purpose: To find the number of members in the union of two intersecting sets. $[N(A \cup B) = N(A) + N(B) - N(A \cap B)]$

Materials: Set of 24 attribute blocks (3 colors, 4 shapes, 2 sizes); 2 hula-hoops.

Procedure:

1. Arrange the blocks in piles according to (a) shape, (b) color, (c) size.
   How many shapes (colors, sizes) are there?
   How many blocks are there of each shape (color, size)?

2. (a) Place all the red blocks inside one of the hoops. Count them.
   (b) Place all the square blocks in the other hoop. Count them.
   (c) Now arrange the two hoops so that all the red blocks are in one hoop and all the square blocks are in the other hoop (at the same time).
   How many red square blocks are there?
   How many blocks are there altogether in the two hoops (Number which are either red or square or both red and square)?

3. Answer the following questions. Try to find a way of answering the questions without placing the blocks in the hoops and counting.

   Find the number of blocks which are:

   (a) circular
       large
       both circular and large
       either circular or large or both

   (b) small
       blue
       both small and blue
       either small or blue or both

   (This activity can be extended to consider 3 intersecting sets by using the complete set of 48 logic blocks, with the additional attribute of thickness.)

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[2]Suggested by Z. P. Dienes and E. W. Golding, *Learning Logic, Logical Games.* (New York: Herder and Herder, 1966).

MATHEMATICAL BALANCE[3]

Purpose: To investigate certain properties of the whole numbers and to solve simple linear equations using a balance beam.

Materials: Balance beam.

Procedure:

A.

1. On the right hand side of the balance beam, place one ring on hook 10. In how many ways can you place hooks on the left hand side to balance this? Write a mathematical sentence for several ways.

2. Can you balance 10 using only a single hook other than hook 1 or hook 10? In how many ways can you do this?

3. Find which weights (from 3 to 25) can be balanced by placing rings on a single hook other than hook 1 or the hook corresponding to the weight. Record your findings in a table similar to the one below.

Weight	Yes or No	How
3	No	
4		
⋮		
10	Yes	2 rings on hook 5, 5 rings on hook 2
⋮		
25		

B.

1. 3 rings on hook 5 balance 5 rings on hook __________.
   Do some other examples like this.

2. 3 rings on hook 4 and 3 rings on hook 6 balance 3 rings on hook __________.
   Do some other examples like this.

C. Use the balance beam to find the solution set of each of the following:

1. $3 + n = 8$
2. $n + 2 < 6$
3. $3n = 6$
4. $2n > 8$
5. $2n + 7 = 15$
6. $4n + 6 = 3n + 8$

[3]Suggested by Z. P. Dienes, *Task and manual for use with the algebraic experience materials*. (Harlow: The Educational Supply Association Limited, Schools Material Division).

HOW MANY SUBSETS?

Purpose: To determine the relationship between the number of members in a set and the total number of subsets of that set.

Materials: Five Cuisenaire rods (white, red, green, yellow and black rods) in a small box.

Procedure:

Consider the 5 rods in the box as a set {W, R, G, Y, B}. A *subset* of the set would be determined by drawing from the box and number of the rods.

For example, if you drew the red and yellow rods, you would have the subset {R, Y}. How many other subsets have two members? List them. How many subsets have 1 member?
One possibility is to draw *all* five rods. This is a subset of a set.
Another possibility is to draw *none* of the rods. This is also a subset, called the empty set - ϕ.

So subsets may have 0, 1, 2, 3, 4 or 5 members. How many subsets are there althogether? List all the subsets you can and then guess how many you think there are altogether.

To help answer this rather difficult question, first consider a series of related, but easier questions.
Find all the subsets of the sets {W}, {W, R}, {W, R, G} and {W, R, G, Y}.

Record your findings in the table below:

Set	No. of Members	Subsets: 0 Members	1 Member	2 Members	3 Members	4 Members	Total No. of Subsets
{W}	1	ϕ	{W}	-	-	-	
{W, R}					-	-	
{W, R, G}						-	
{W, R, G, Y}							

Can you see a pattern? How many subsets will {W, R, G, Y, B} have?

Can you determine how many of these subsets have 0, 1, 2, 3, 4 and 5 members?

How many subsets would a set with 6 members have? With 10 members?

THE CIRCUMFERENCE OF A CIRCLE[4]

Purpose: To discover the relationship between the diameter and the circumference of a circle.

Materials: Wooden discs, tin cans, and so on, of various diameters; string; ruler.

Procedure:

1. Take a disc from the box. Using only the string and ruler, what distances can you measure on the disc?

2. Measure the *diameter* (distance across the disc) and the *circumference* (distance around the outside of the disc) and record your findings in the table below. Do this for each circular object.

Diameter	Circumference

3. Examine the table. Can you write a rule which approximately relates the diameter and the circumference of a circular object?

4. Use your rule to predict the circumference of a circular object of diameter (a) 1 inch (b) 10 inches (c) 15 inches.

5. Predict the diameter of a circular object which has a circumference of (a) 19 inches (b) 44 inches (c) 1 inch.

6. The radius of a circular object is the distance from the center to the outside edge. How is the radius related to (a) the diameter (b) the circumference.

7. If a circle has a radius of 5 inches, find (a) its diameter, (b) its circumference.

8. Find the radius of a circle having a circumference of (a) 25 inches (b) 63 inches.

[4]Adapted from The Madison Project's Independent Exploration Material, *"Discs,"* distributed by Math Media Division, H & M Associates, Danbury, Connecticut.

PROBABILITY[5]

Purpose: To introduce basic concepts in probability through experiments with coins, dice and sampling urns.

Experiment 1: Flipping a coin.

Materials: A penny.

Procedure:

Flip a coin ten times and record the number of times it lands heads and tails. Repeat this procedure five times and record your findings in the table below.

Trial	1	2	3	4	5	Total
No. of heads						
No. of tails						

What would be your best guess as to the number of times a coin would land heads out of 10 flips? 50 flips? Why?
How close were your results to what you expected?

Experiment 2: Rolling a die.

Materials: A die, paper cup.

Procedure:

Roll the die 30 times and record your findings in the table below (under "First Trial").

Outcome	First Trial	Second Trial	Total
one			
two			
three			
four			
five			
six			

[5]Adapted from E. C. Berkeley, *Probability and Statistics - An Introduction Through Experiments.* (New York: Science Materials Center, 1961).

How many one's, two's, etc. would you expect to obtain in 30 rolls of the die?
Did you get exactly 5 one's, two's, etc.? Discuss.

Repeat the experiment and record your findings under "Second-Trial". Total the results for the two trials. Compare your results with what you would have expected to obtain in 60 rolls of the die.

Experiment 3: Guessing the Urn

Materials: 4 sampling urns labelled A, B, C, and D, each with 20 beads as follows:

1. 18 black, 2 white
2. 14 black, 6 white
3. 10 black, 10 white
4. 4 black, 16 white

(Boxes or cups containing marbles, which are replaced after each draw, may also be used.)

Procedure:

Find Urn A. Can you determine which of the urns (1, 2, 3 or 4) this is? Perform the following experiment to help you make your decision. Shake a bead in the bubble ten times and count the number of times a black bead appears and the number of times a white bead appears. Now which of the four urns do you think is Urn A?

Identify B, C, and D in the same way.

Urn	Black	White	Guess
A			
B			
C			
D			

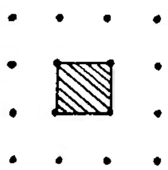
To see how well you guessed, remove the lids from the urns.

THE AREA OF A TRIANGLE[6]

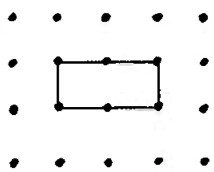
Purpose: To determine methods of finding the area of triangular regions which can be formed on a geoboard.

Materials: Geoboard and rubber bands.

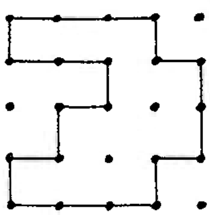
Procedure:



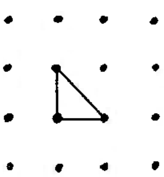
Let the shaded area represent 1 square unit on the geoboard.



How many square units are within this figure? ________

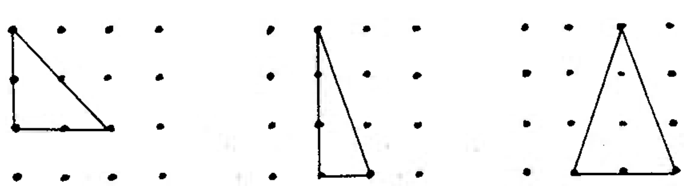


What is the area of this shape? ________ square units



How many square units are within this triangle? ________

What is the area of the following triangles?



________ ________ ________

Did you find an easy way to figure out the areas of triangles?

Explain how you did this.

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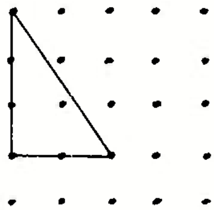
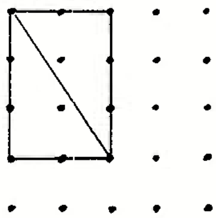
[6]Adapted from Donald Cohen, *Inquiry in Mathematics Via the Geo-Board.* Teacher's Guide. (New York: Walker, 1967).



Make this triangle on the geoboard.

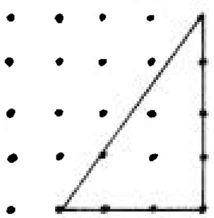


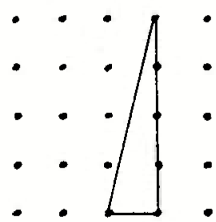
With another elastic band make a rectangle.

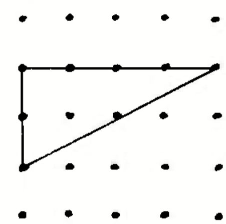
The area of the rectangle is _________ square units.

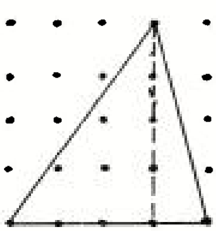
The area of the triangle is _________ square units.

Use this method to find the areas of the triangles below:

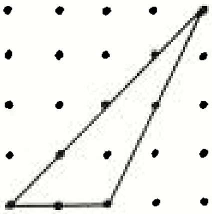




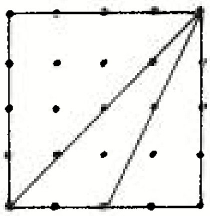




The last triangle can be thought of
as being made up of two triangles.



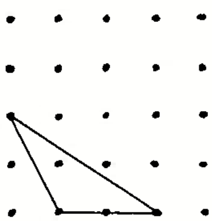
How would you find the area of this triangle?

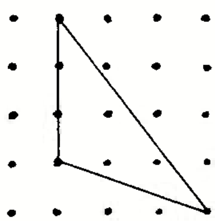


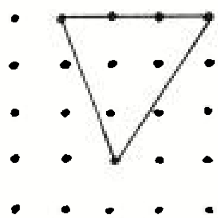
One way would be to complete the rectangles as shown and subtract two triangular areas.

Area = 16 sq. units - 8 sq. units - 4 sq. units = 4 sq. units.

Find the area of these triangles:



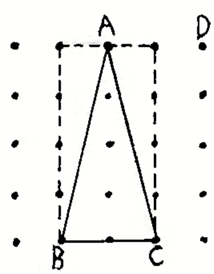




Given a triangle, choose a side you can determine the length of and call this side the *base* of the triangle. The *height* of the triangle is the length of the other side of the rectangle enclosing the triangle.

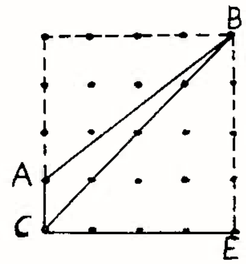
Example 1

In triangle ABC, if BC is the base, CD is the height = 4 units.

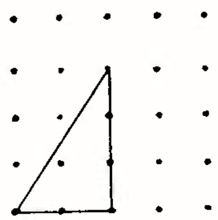


Example 2

In triangle ABC, if AC is the base, CE is the height = 4 units.



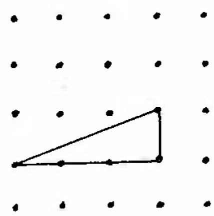
For each of the triangles below, find the lengths of the base and the heigh
Also find the area.



base ________

height ________

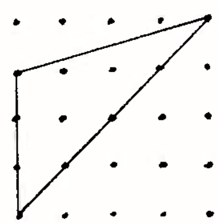
area ________



base ________

height ________

area ________



base ________

height ________

area ________

If you know the length of the base and the height of a triangle, how can you find its area?