1966

ANNUAL

NCTM Calgary Summer Meeting

MCATA Publication of the Mathematics Council of The Alberta Teachers' Association

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PREFACE

The purpose of this *Annual* is to bring into focus the highlight of the year in mathematics education in Western Canada - the regional conference of the National Council of Teachers of Mathematics held at Calgary in August, 1966. Everyone interested in mathematics education in Western Canada is indebted to the Mathematics Council of The Alberta Teachers' Association for organizing the conference.

The editors of the MCATA *Annual* saw fit to gather papers presented at the conference as they were available and appropriate for publication. The following pages are, then, merely a sample of the many excellent presentations given in Calgary.

The papers represent a range of opinion and content. Adrien Hess has mixed feelings about the "Revolution in Mathematics", while William Steeves shows concern over the recent developments including "discovery teaching". Agnes Rickey outlines a program developed in Florida for slow learners in Grade VII. Irvin Brune takes a hard look at geometry as it should be taught in the elementary school - from "op art" to proof. Murray McPherson indicates how one Canadian province met the challenge of teacher education for the new mathematics programs. Tom Atkinson has some mathematical ideas for the teaching of problem-solving.

Algorithms and computers can be taught as a course in high school, and David Alexander has done just this in Toronto. Arnold Harris suggests transformation geometry for secondary school as he has taught it in Ontario and observed it in Great Britain and Denmark. Allan Gibb presents possible uses of TV in teacher education, while Douglas Crawford, another university professor, discusses teaching and learning mathematics from a theoretical psychological point of view. An actual discovery lesson is described by Solberg Sigurdson; and Father Egsgard concludes the publication with important reminders that we are, in spite of the new mathematics, still teaching individual human beings.

A spectrum of important topics is presented by people who spend a great deal of time thinking about teaching and learning mathematics. If their thoughts make your classroom activities more meaningful, then the *Annual* has served its purpose.

- The Editors

WHAT - ANOTHER REVOLUTION?

Adrien L. Hess
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In this article, Dr. Hess is referring to the American (United States) high schools, where, in many localities, no mathematics is required for high school graduation. We have not yet reached this point anywhere in Canada.

At the time when I was trying to choose a title for this paper, I had just read an article in the *Mathematics Teacher* on "The Second Revolution in Mathematics". The most appropriate question then seemed to be "What - Another Revolution?" I was rather sure that I knew how to answer this. As time passed and I did more reading and studying, I became more uncertain of my answer. Many articles have appeared in newspapers and magazines which give some light on different aspects of the question.

As a starting point, I turned to the dictionary for help. I looked up the meaning of the word "revolution". As usual, Webster came through in good style. Here is what I found:

(a) Revolution is a progressive motion of a body around a center or axis such that any line of the body remains throughout parallel to its initial position to which it returns on completing the circuit. Revolution is often combined with rotation.

(b) Revolution is a total or radical change - as a revolution in thought. In political science it is a fundamental change in political organization or in a government or constitution; the overthrow or renunciation of one government and the substitution of another by the governed.

Thus, my title allowed me to take a text and depart from it. As the occasion demands, I will use either of the definitions. In the first part of my paper I will employ the idea of a revolution as progressive motion of a body round a center and the return of that body to its initial position.

In a chapter entitled "What Shall We Teach in Geometry", which appeared in the *Fifth Yearbook of N.C.T.M.*, the lead paragraph's heading was "Revolution in Mathematics". The writer discussed the pressure from many sources to cut down on the time and material devoted to arithmetic. It was felt that much of the work with common fractions could be omitted since one used common fractions very little in everyday life. The same kind of attack was directed at secondary mathematics, particularly algebra and solid geometry.

The attacks continued. Certain concepts or topics were found to be too difficult at a certain grade level. Therefore, they were delayed to the next grade. In fact, it was thought by some that one could delay much of arithmetic to the seventh and eighth grades. In algebra, topics such as l.c.m., h.c.f., and certain kinds of factoring were dropped and inequalities treated lightly.

Beginning in the 1950's with the advent of "new" or "modern" mathematics, it has been found that youngsters can learn much more mathematics than was ever believed possible. Concepts and topics have been moved down from grade to grade. It has been found that youngsters in the primary grades can do ruler and compass constructions, that youngsters in the intermediate grades can do algebra. Inequalities have been given added emphasis. The l.c.m. and h.c.f. is again in good standing. At the high school level, statistics and calculus are being taught. Do you see that we now have a complete revolution?

Algebra has long been one of the stumbling blocks for freshmen in high school. General mathematics has been used as a way to alleviate this trouble. A next step was to relax the requirements of mathematics for graduation from high school. This received great impetus from the "Eight Year Study" which was launched in 1932. The study was designed to see whether the nation's colleges could relax their admission requirements and accept students from high schools who did not have the traditional pre-college course of study. This meant that mathematics, as well as many other subjects, was optional. The results of the study showed that no single set of academic studies insured success in college. This led many secondary schools to either abandon requirements of so many units of mathematics for graduation from high school or to materially lessen the number of units required. In June, 1952, Benjamin Fine reported in the *New York Times* that the 30 schools who had participated in the study were essentially right back where they stood in 1932. In 1956, the State Board of Education in Montana once again made two units of mathematics a requirement for graduation from high school. Nationwide, schools - elementary and secondary - are requiring mathematics. Yes, another revolution.

In the *Fifteenth Yearbook of N.C.T.M.*, published in 1940, Appendix III is entitled "Terms, Symbols and Abbreviations in Elementary Mathematics." In this, certain terms are listed. Among these terms are three: equivalent, ray, half-line. The Commission recommended that the word "equivalent" be abandoned because "this is an unnecessary substitute for the more precise expressions 'equal in area' and 'equal in volume', or (where no confusion is likely to arise) for the single word 'equal'." Concerning the words "ray" and "half-line", the Commission wished to be non-committal. Today we talk about equivalent sets which are not necessarily equal. The terms "ray" and "half-line" are used extensively. This should be a constant reminder to each of us that we do not know what youngsters will need 25 years from now. When some expert decides that a certain concept or topic is of no use and should be eliminated, we should be reminded of the amount of time that transpired between George Boole's "Laws of Thought", written about the middle of last century, and the application of Boolean Algebra to electrical circuits by Shannon during the second quarter of the present century.

Another place where the wheel seems to have failed to turn is what people often call field work in mathematics or practical applications. Some writers include this under mathematics laboratories. In this paper I wish to extend the usual notion to include mathematics library. Modern works in psychology by Piaget and others point out the value of physical apparatus in the mathematics classroom. Writers in England have recommended physical experiences for over a quarter of a century, yet the movement does not seem widespread even in that country. In the United States, textbook writers and contributors to current periodicals have devoted much effort to this phase of mathematics. With the present emphasis in the United States on education of culturally deprived children, the need for physical objects for manipulation is being stressed: various teaching aids, models and devices are needed. Classrooms are still being designed without thought for small group arrangements. Unfortunately, many mathematics teachers must share their classroom with other teachers. This hardly lends itself to the development of a mathematics classroom. There is no revolution here by either definition of the word revolution.

During the 1930's and early 40's one finds several forces at work shaping the place of mathematics in secondary schools. Since there was little else to do, more and more children attended school. This heterogeneous group presented many problems. There was much emphasis on "child-centered" schools and the core curriculum. The feeling was that a class of children should assist the teacher in selecting what went into a given course. Concepts and activities included in a course were scrutinized for their social applications. Some writers based the topics included in their books on the criterion of social application. Mathematics was considered as a tool subject. Structure seemed to be an unheard-of word.

During World War II the schools were informed that inductees knew no mathematics. Any discerning teacher of mathematics could have given his information at a much earlier time. Schools were exhorted to put in crash programs. Suddenly mathematics became important.

Following World War II and the attendant explosion of mathematics, mathematicians, businessmen, cartoonists and others turned attention to the question of mathematics. Writers found that youngsters loved mathematics if it were the right kind and if it were carefully and precisely written. Structure became the key word - understanding was most important. In some experimental programs, social applications were given slight consideration or were entirely omitted. In a NCTM publication, *An Analysis of New Mathematics Programs*, one of the things considered was "How much emphasis should be placed on social applications of mathematics?" The committee found that some programs used social applications to develop a clear concept of mathematics, while other programs omitted them because they tended to cloud the clarity of a subject's structure. Apparently more attention is presently being given to social applications. We are still on the merry-go-round.

It seems generally agreed that man's need for mathematics led him to create mathematics. When a need arises today, mathematics is created to meet the need. Earlier mathematical systems were models of physical phenomena. There was a time when leading mathematicians like Klein, Hilbert, Poincare, Weyl, and Von Newman made notable contributions to the field of physics as well as to mathematics. Some of the experimental school programs have given little consideration to such applications. In fact, some new programs in mathematics tend to separate mathematics and science to an even greater degree. There are those writers who feel that the application of mathematics may, in the near future, lead the field in restoring the contact between theory and practice. Even much more than ever before applications depend on the support of even the most abstract theories. One can find little evidence of any revolution at this point.

For some time, drill has been a word with a shady reputation. Earlier in this century a certain psychology of learning made it seem desirable to accentuate drill. It was felt that enough drill would fix the path from the stimulus to the response so that a youngster would never forget a given concept or fact. Whether a youngster understood the whys and wherefores of a concept or whether the concept or fact was of any use seemed to be unimportant.

Suddenly in the 1950's it became evident that if a pupil understood what was being done, particularly if he discovered it himself, old-fashioned drill was not needed. In fact, according to an early bit of reporter fantasy, drill was no longer needed. There were some who did not want to break entirely with tradition; so drill was included, not in the main stream, but before a child could have his candy he had to take his sulphur and molasses. Others maintained that if the right kind of problems were used, enough drill would result. All in all, drill became a word of questionable character. Within the past year we find that there are those who say that we should not throw the baby out with the bath. The fact that some things are old is no reason to discard them. There exists a sinking feeling among some that we may have a generation of youngsters who can associate and commute but who find it difficult to actually solve a problem because of weakness in fundamentals. The wheel may have revolved too far. However, Heaven forbid that we ever return to the starting point on the drill question.

In Montana there has been a great emphasis on reading for many years. You will find a section on reading at every teacher's convention as well as summer conferences at the various colleges. Administrators have been very much aware of the reading problem in the schools. More power to them! For it was - and still is - a serious problem. If progress is made for reading in general, it cannot but help reading as it pertains to mathematics. It is my personal opinion that the emphasis on speed reading has been very detrimental to reading in mathematics as well as science. Reading 1,600 words a minute may be a worthy ideal in some fields, but it certainly is not for mathematics.

The Arithmetic Teacher, Vol. 59-157, February, 1966, contains an article by Call and Wiggin on "Reading and Mathematics". I will quote some very pertinent remarks from this article:

1. There is some merit in teaching special reading skills for the solution of mathematical problems.

2. Even very good readers, as measured by the Cooperative Reading Test, have difficulty in the interpretation of the kind of reading found in word problems.

3. Part of the difficulty which teachers encounter in the teaching of mathematics is that they are not equipped to teach reading.

4. Part of the difficulty which teachers encounter in the teaching of mathematics comes from a special kind of reading disability which does not appear on standard testing instruments.

5. If, by teaching reading instead of mathematics, we can get better

results, it seems reasonable to infer that the competent mathematics teacher might get considerably better results if he were trained to teach reading of the kind encountered in mathematics problems.

Some of the modern programs place considerable emphasis on reading. It appears to me that pupils who have trouble reading old mathematics will have as much trouble reading modern mathematics. As far as I can ascertain, this is one portion of the mathematical world that has been untouched by revolution in any sense of the word. It is what one might call an unexplored wilderness.

Once a boy who was herding sheep became lonesome. One day it occurred to him to call "wolf, wolf". Upon hearing this call, his neighbors rushed to his aid only to find that he had perpetrated a fraud. Later, a wolf did actually appear, but when he called "wolf, wolf" no one came to his rescue. It seems that our use of the word revolution leaves us in about the same position. As I commented earlier, the heading of the first paragraph in Chapter II of the Fifth Yearbook (1930) of NCTM is "Revolution in Mathematics". There is little in the paragraph which leads one to believe that the upheaval was drastic. The title of a 1961 NCTM publication is *The Revolution in School Mathematics*. In a recent article in the *American Mathematical Monthly*, there is a statement "on the wake of the revolution represented by the Bourbaki attempt". When I was preparing this paper I noticed a book with a 1963 copyright in the library entitled *Stillborn Revolution - The Communists Bid for Power in Germany 1921-3*. More recently there has been talk of "A Second Revolution" in mathematics. One writer talks of three revolutions in American education. It is my contention that overuse of the word has made it too commonplace to be really effective.

One of the often mentioned characteristics of modern mathematics is the unambiguity and preciseness of the language. None of us would care to quarrel with such worthy aims. On occasion it seems that the aim may be overworked. For example, let us consider the following: "Given the equation $F(x,y) = 4x + 3y = 12$, determine the set $A = \{(x,y) \quad x \in N,\ y \in N,\ F(x,y) = 12$, where N is the set of all integers$\}$." I believe that we will all agree that this is logically clear. Now let us compare this with: "Determine the integral roots of the equation $F(x,y) = 4x + 3y = 12$." While it is desirable and necessary to introduce and use symbols in some mathematical situations, it is neither desirable nor necessary to introduce and use symbols where there is a simpler, more straight forward way of writing the same idea. Modern mathematics has introduced an open season for inventing and using new symbols. Each time the urge arises to invent a new symbol, the question should be asked "Is this symbol necessary?" Sometimes, in an effort to make a definition of a term precise and unambigious, it becomes cumbersome and wordy. One reads the definition and then wonders how to use it. There are times when it is better if the wheel does not turn.

I am sure that all of us eagerly peruse any report of a study of comparison of achievement in modern mathematics versus traditional mathematics. I must confess that one of the earlier reports contains a result which is disconcerting. It was found that students taught by teachers who had no previous experience in that particular experimental program did better than students taught by teachers with such experience. This is indeed a revolution in every sense of the word! I will let you persue this particular point a little further in your own thinking. Remember, you experienced teachers are now expendable!

There is another point in these reports that has always intrigued me. In these reports the results are usually varied. It may be that achievement in the experimental program is not statistically different from achievement in the control program. One program may show gains higher than the other. In any case, one is likely to find the statement "they are getting something in the experimental program that we cannot measure". In all fairness, it is true that the tests used, particularly in the early studies, were devised for traditional mathematics. Might one not apply the above statement equally well to traditional mathematics?

Now let me turn my attention to the second meaning of the word revolution. This meaning is a total or radical change, or, as in political science, a fundamental change in government and substitution of another by the governed. In the usual political revolution today, this means that some people are hurt.

Most of us have read *Goals For School Mathematics.*[1] Some consider this as the opening salvo in the second revolution in mathematics. A recent article stated as a heading that modern mathematics fears it is getting old. There is evidence that the end is not in sight.

It is probably safe to say that the first revolution in mathematics was concerned about college-capable students. It is true that many experimental programs want to be included in the programs concerned with students who are not college-bound. Some persons have asked just how widespread this first revolution is. The question has also been raised as to how well acquainted writers of experimental programs are with the results of research in the psychology of learning. Materials are written to be logically sound, but are they psychologically sound? A return to an old question! Again, what is to be done with the lower one third of the school population as far as mathematics is concerned? With these unanswered questions, I will direct my attention to some other aspects of the revolution.

I mentioned earlier that in most political revolutions people get hurt. In the present mathematical revolution people were hurt. One group of people who have been hurt are the mathematics teachers. In the days of drill - ask no questions, do the problems as I tell you - the teacher soon achieved a mastery of the situation. With the textbook in one hand and the answer book in an open drawer, the teacher soon achieved confidence in his or her ability. Certainly, anyone could teach arithmetic. If one had had a course in algebra, this qualified him to teach the subject.

With the revolution, teachers were called on to teach new material in a way new to them. When one is unacquainted with something, there is always fear, suspicion and lack of confidence. To add to this, one must lead youngsters to discover concepts and facts. Incidentally, the so-called discovery method is far from new. Socrates, by questioning, tried to lead people to truth.

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[1]Cambridge Conference on School Mathematics, *Goals for School Mathematics* (Boston: Houghton-Mifflin Company, 1963).

I trust that all of you know what happened to Socrates. Back to the mainstream. Above all things, encourage youngsters to ask questions! "Now see if you can do the problem another way!" "Make generalizations; what guesses can you make?" "But my answer book shows only one way - and it does not provide answers to all questions that youngsters can ask." No wonder teachers were upset and driven almost to nervous prostration. Institutes, inservice courses, and reading have become the order of the day. A perusal of the Cambridge Report should leave everyone with the feeling that there is no end to this revolution as far as teachers are concerned. The textbook and the answer book is not enough. Here is a score for the scoreboard of the revolution!

The question of homework has been troublesome for many years. There was a time when homework in prodigious quantities was assigned students. The students reacted to this sort of thing in various ways. Some students blithely ignored the assigned work. Other students did the work faithfully. The next day they achieved a degree of popularity with a certain group of students who did not do the homework but wished to make the teacher believe that they had. Still another group called on Dad or Mother for assistance. It was a matter of paternal pride that Dad knew how to do the problems. However, some parents resented the fact that they had to do work which, in their opinion, teachers should have done. Studies were made on achievement in mathematics with and without homework. Since there seemed to be some question about its good effect, this helped usher in an era of no homework in mathematics in many schools.

With the advent of the new mathematics and urgency of more youngsters to study mathematics, the problem again came to the front but in a different guise. Again, homework became an acceptable state of affairs in mathematics. But something had happened. Dad could not do Johnnie's homework. Yes, he might get the answer - "but that is not the way we do it." Now it was the parents' turn to be hurt. They were frustrated because they saw no reason for the change. For their children to be able to do mathematics that they did not understand was adding insult to injury. Some school systems, in order to forestall some of this trouble as well as criticism of the new mathematics, did not let the youngsters take papers home. After all, if one adds 2 and 2 it is 4 and not 10. Why this nonsense? What use is this in business? One sure way to whet curiosity is to keep things a secret. In recent years numerous books and booklets have been written for parents explaining the new mathematics. Many parents have attended evening classes.

All of this should make it quite clear that parents are interested in schools and in what their youngsters are being taught. Informed parents are the best supporters of education that one can get. Another score for the mathematics revolution!

One innovation that has been the result of the mathematics revolution is the way in which textbooks, teaching aids, and materials are prepared. Before M.R., some person or two persons, occasionally three persons wrote a book. Each person concerned was usually considered an authority in mathematics or the teaching of mathematics. Due attention to the various committee recommendations was included - at least in the preface.

Various approaches have been used in the preparation of the new programs.

In some programs the writing was done by a team consisting of lay persons, teachers at level the material was being prepared, university mathematics professors, educators, psychologists, consultants, etc. In other programs a group of teachers were given training in the subject area. After this, they prepared materials to be taught. In other cases it was done by a team at the university level. Occasionally, the program was the result of the leadership of a single person. Coming out of such varied background, one would expect the programs to differ in many points.

The approaches described above indicate that a great number of persons are acquainted with a given program. Any such program is not the work of one person but the pooled efforts of the team. As a by-product of this cooperation, persons at the university level are welcomed in the elementary and secondary schools. Persons from the elementary and secondary schools are often called on to teach a course or so, particularly in the summer sessions. Speaking as one who has participated in a writing team, I believe that this is a third score for the mathematics revolution. This is a change from one form of government to another - with no one being hurt. Furthermore, this method stresses the need for the training of teachers in the new materials and methods.

A survey reported in *Fortune* Magazine a number of years ago showed mathematics as one of the most disliked subjects in school. My experience at the university level with persons going into elementary education - and who will be teaching arithmetic - bears out this finding. Somewhere back in the past many of these persons had acquired a dislike and fear of mathematics. The unfortunate fact is that these persons are inclined to continue to dislike and fear mathematics and even carry over this attitude to anything or anyone connected with mathematics. My experience over the past few years has led me to believe that the modern mathematics can be so taught as to dissipate the dislike and fear already present. This experience has convinced me that we can repair some of the damage done at an earlier time and dissipate the fear of many persons. Do not assume that we win them all - neither has any of us been nailed to the cross. Score four for modern mathematics - many persons who formerly feared and disliked mathematics gain an understanding with new mathematics.

One problem of every teacher of old mathematics as well as new mathematics was and is the slow learner. It is quite evident to anyone who has observed a group of persons that some learn faster than others. In school with the "pages to be covered" per day this means that the slower learner gets farther and farther behind. Eventually, this person turns to something else which is less frustrating. At an earlier point in this paper I suggested that the experimental programs in modern mathematics were generally aimed at the college-bound student - we hope this group includes the faster learners. Nevertheless, the question kept arising as to how slow learners would fare with modern mathematics. Some reports exuded a mild amount of optimism, but generally the results were summed up somewhat as follows: "The slow learners do no worse with modern mathematics than they do with traditional mathematics." Of course, if one is at the bottom of a given heap, then there is actually only one way to go.

I would like to dwell on the topic of the slow learner just a little longer. There are writers who contend that when properly presented, the contents of modern mathematics can be understood and enjoyed by many of the slow learners. Allowances need to be made in terms of the individual student's rate of progress. I am sure that reading is a decided factor in success. In my own mind, I am convinced that there are some students with a blind spot for certain colors. Some recent work with slow learners indicates that the number of persons who cannot learn mathematics if properly taught is much smaller than is generally assumed. It takes time for ideas to grow. It is true that NCTM and other groups have been giving some attention to the development of materials for the slow learner.

Let us remember that a slow learner will eventually be a voting citizen. If we continue to neglect this group of students, we can hardly expect royal treatment from them when they sit on school boards and in other seats of authority. The least we should try to accomplish is a friendly attitude on their part toward mathematics. Mark up a goose egg on the scoreboard for modern mathematics and its help to the slow learner.

MODERN MATHEMATICS IN ELEMENTARY SCHOOL - A CRITICAL VIEW

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If I were to choose a text for my remarks, it would be an observation by Harold Wilson, Prime Minister of Great Britain: "We have had far too many words chasing too few ideas." I shall now proceed to live up to that statement, "... too many words chasing too few ideas."

Since 1959, or a little earlier, there has been a startling revolution in mathematical content of textbooks, in teaching methods, and in teacher attitude. All the mathematical thinking of many interested educators seems to have burst forth all over this continent and indeed the whole world. Public demands for better mathematical and scientific education gave an impetus to this and literally everybody has felt the impact. Publishers, authors, many teachers at the elementary, secondary, and university levels of education seized on the movement to attempt to revitalize the mathematics programs. Psychologists have restated the laws of learning and propounded new ones based on recent studies and research. The schools have been swamped with all sorts of new texts, materials, and devices, most of which have said or implied that here is something new which every school must have or it will be hopelessly behind. Who wants to be behind the times?

It should be remembered, of course, that changes now affecting mathematics are part of a wider movement enveloping the whole pattern of education.

I shall not attempt to give what I think should be a new program for the elementary school, Kindergarten to Grade VI. There are many new programs which are done much better than I could do them. Let me just make two statements.

Firstly, I do not subscribe, emphatically do not subscribe, to developing a program or flow chart of elementary school mathematics based on any one set of materials, be that set of materials blocks, rods, apples, oranges, or rabbits. Perhaps I lean a little toward rabbits because they can multiply quickly. I do believe that teachers should have available whatever materials there are and should use those which they can use best. Two years ago when I visited several Canadian cities, I observed teachers in Vancouver using rods, in Richmond - blocks, in Burnaby - various materials - charts, blocks, rods, and unifix, in Edmonton - circles on the chalkboard, and in Toronto - workbook material.

Secondly, I subscribe substantially to the program suggested by Dr. H.F. Fehr in his article "Sense and Nonsense in a Modern Mathematics Program", to be found in *The Arithmetic Teacher*, February, 1966.

Dr. Fehr says that, "If at the end of sixth grade almost all the children know the decimal system of notation; can read and write the numerals for whole and fractional numbers; can at an adult level of performance do all four computational operations on whole numbers and fractions, both in common and decimal notation; have an intuitive understanding of the rationale and structure underlying these computations; can apply this knowledge meaningfully to the solution of problems involving measure and per cents; and know and recognize common geometrical figures and relations among them; we shall have achieved an outstanding and notable advance in elementary school mathematical education. This is something for us to aim at."

Now as to method - today the so-called "discovery method" is in vogue. If you are not using the discovery method of teaching elementary school mathematics, you are behind the times. We tend to go in circles; progress is slow. I am reminded of the story of the snake. The head probes ahead of the tail. It keeps probing this way and that, and long after the head has passed a certain point, the tail arrives. Naturally, the tail is a little out of sorts about this until it discovers that the head has come around to the same spot. I would remind you that in this action the tail, in trying to keep up with the head, has made some progress and that the whole snake has moved ahead a little. So it is with education; we never go back quite to where we were; we simply change the names and use different language.

Discovery is not new. The vast scientific and mathematical knowledge of today came to us largely through discovery. Men discovered, tested, and fought to have their discoveries accepted. The Church condemned Galileo's discoveries, but you recall the Pope wouldn't sign the condemnation document because he was infallible and was smart enough to realize that if at some future time Galileo was proved right, it wouldn't look well for the infallible Pope to be proved wrong.

Today the discovery method means letting the child discover things for himself. This is good, but the teacher must provide the environment and be the guide and leader. The guiding philosophy today is that the child should think and figure things out for himself. As Sophocles said long ago, "One must learn by doing things, for though you think you know, you have no certainty until you try."

Let us see how it works: A group of children are seated around a pile of blocks. Each child selects a certain number. The teacher asks, "How many blocks have you?" One child replies, "two". "Would you like some more blocks?" The child takes three more blocks. "How many blocks did you take this time?" The child counts and says, "three". "Good, how many blocks do you have altogether?" The child counts again and says, "five blocks". And so he discovered that a group of two blocks and a group of three blocks gave him a group of five blocks. "Can you arrange the five blocks in different groups?" He discovers that he can have a group of four blocks and one block by itself, a group with only one block, a set of one element.

Another child selects three blocks and three more blocks. He finds that he has six blocks and discovers that two groups of three blocks made six blocks; that he can rearrange them in three groups of two blocks. And so he learns the beginning of addition and multiplication and, if you like, subtraction and division. He writes what he has learned on the chalkboard.

All through his school life the child should have opportunities of learning by finding out. What is needed if democracy is to survive is a type of mind which demands to hear both sides of every story and weighs one against the other objectively. This is the essence of democracy.

When children explore for themselves, they make discoveries which they want to communicate to their teachers and to other children, and this results in discussion. It is this changed classroom relationship which is the most important development of all.

The teacher sets the stage and provides the opportunities, the motivation, and the necessary stimulation. I recall being in a Grade III class recently. The children were doing subtraction with borrowing with three place numbers. After one child had explained and demonstrated one way on the chalkboard, four children each worked on the question in a different way and then the class discussed which way seemed best. These were children being taught to figure things out for themselves and to evaluate their work.

Textbooks today are full of examples which give the pupils an opportunity to think for themselves if the teacher will let them. One trouble is that teachers insist on reading the examples to the children and in so doing interpret the problem. Usually they are interested only in the answer and not in how the child thought in order to get the answer.

In problem solving, the child must learn to restate the problem in simple mathematical form; he must also learn that the unknown may be represented in this form by a symbol. In interpreting problems, the child must learn to state mathematically the information given in the problem. In other words, he must learn to think logically, and although we do not mention the word logic to him, we must endeavor to be logical even if the elementary school child himself is not logical.

While it is our hope that the child will learn to think and discover, it is the teacher's responsibility to see that he gains a wide variety of experiences with concrete material, semi-concrete material, and abstractions. New textbooks are full of many ideas for varying the pupil's work and giving him the opportunity and guidance he needs to help him develop his ability to think independently. Every effort has been made to help the teacher to give the child the variety he needs in his work. Experimental approaches to multiplication and division, finding area by counting squares, different methods of dividing a fraction by a fraction, "what's my rule" questions, cross number puzzles, experiments in design with geometrical figures, are but a few. Have you ever asked a Grade VI class to see in how many ways they can divide a two-inch square into eight equal parts?

Teaching the new elementary school mathematics program requires that the teacher be continuously conscious of the child's question, "Why?" This question must be answered for the child, either by the child's experimenting or by the teacher. Therefore, every effort must be made in our teaching to help the child to know why our numbers are the way they are, why we carry, why we divide whole numbers and fractions the way we do, why we multiply the number of square inches in a row by the number of rows to find area, and why we multiply the diameter of a circle by pi in order to find the circumference. These are a few of the child's "whys".

I mentioned earlier that the discovery method is very popular in educational thinking today and that its use will prove most helpful in teaching new programs in elementary mathematics. However, be assured, it is no panacea; the millenium has not arrived. It is still necessary that children learn and practise so that their responses to many facts will be automatic. They must, over the years, gain a storehouse full of information which will be ready for use instantly. While all our new methods are aimed at motivating the child, holding his interest, and giving him the satisfaction of achievement, our efforts will be largely fruitless if the child does not gain precise knowledge and the ability to use it. To accomplish this, we must give the child an opportunity to check his work or the teacher must check it so that the pupil knows that he has achieved and that he has accomplished something now. It is not good enough for the child to wait for a day, or even much longer, before he knows what he has accomplished. We can learn from industry. Some of the in-service training programs of the Northern Electric Company, for example, are so planned that the trainee knows not only whether his responses are correct or not but he knows what effect his mistakes will have on production, and he knows it immediately.

This brings us right up to the teacher. It makes little difference how much money we spend, how good our buildings are, how good our equipment is, how many TV's and radios we have; the education of our children will be as good as are our teachers. The fact of the matter is that the only really important and indispensable factor in helping children to learn is a good teacher. No devices, whatever they may be, will replace him. It is still true that a good teacher on one end of a log and a student on the other is the best learning situation.

Fehr says that in order to implement a sensible program we need two essentials: suitable textbooks and good teachers. He goes on to say, "But a good mathematics program depends not so much on a textbook or a syllabus as on the teacher and the methods of teaching. Let us make no mistake! Any program, no matter how sensible, modern, and balanced it may be, can degenerate into mere dogma in the hands of a dognatic teacher." Have you ever heard a teacher putting a class through "rod drill"? White is 1, red is 2, light-green is 3, and so on. I have!

There are certain basic qualifications which every teacher should have:

1. the basic teacher training and education offered by various teachers' colleges and universities (I have said 'training and education' because I believe that each teacher requires training in the routine of class management in its broadest sense);
2. fundamental education in the subjects taught in school and as broad a knowledge as possible in the special subjects to be taught (for the elementary teacher it may be enough to have studied only high school mathematics successfully, but I doubt it; the teacher needs a knowledge of the number systems, the theory of sets, mathematical structure and relationships and much more if he is to teach the new programs in mathematics well);
3. the teacher also needs a good knowledge of the elementary school course of study (it is not enough for him to know the work of just the grade

which he is teaching - a Grade III teacher needs to know the work of Grades I and II and that of Grades IV to VI, yes, and some high school work also).

These three elements - teacher training and education, a liberal education with a depth of knowledge in the special subject field, and knowledge of the school curriculum - are basic. In addition, each teacher must be a student. The teacher must not be content with preparing his lessons from the material in the textbook or manual; he must constantly be searching for new ideas, for variations in method, and for enrichment materials.

The teacher must teach in the classroom. It is quite impossible now to have the pupils learn new programs in mathematics by assigning work and forgetting about the children for the next 20 or 30 minutes. Teachers just must not do this any more, not even in the high school grades. Why?

Firstly, the pupils in any class, be it large or small, vary in ability and must be taught in groups or given individual attention.

Secondly, an assignment that can be done without the teacher's guidance is too easy. It is necessary for the teacher to go around to see whether or not the children are having difficulties and, if they are, to give them the necessary assistance. Recently I visited a class in which the pupils had been given a sheet of questions including about 10 subtraction questions with borrowing. The teacher allowed some children, and one in particular, to work all these 10 questions incorrectly and thus waste time when a little help at the beginning would have put the child on the right track.

Thirdly, the new programs are activistic in nature. They require discussion, probing, eliciting answers from the pupils, evaluation of answers, checking of written work, preparation and use of all sorts of illustrative materials. The teacher just cannot do this from behind the teacher's desk, that stronghold of ignorance.

Some of you may be wondering just what this has to do with the new programs in elementary school mathematics; after all, this talk was supposed to be a critical view of the new programs. I want to tell you that it has everything to do with the new programs. The new programs are almost useless without the teacher. We have in our educational setup today eminently qualified educators who can devise wonderful procedures set forth in manuals and we have these programs available for teachers, but the biggest problem of all is to improve what goes on in the classroom. Unless this is done, the new programs will be largely ineffective. The new textbooks and manuals are very complete. They set forth suggested methods, activism, or discovery, but the teacher must make himself thoroughly familiar with them so that he can guide the children in their learning.

If the teacher is to help the children to learn or think more clearly, he must be concerned with what they are thinking and how they think. Writing tests and checking answers are not enough. The teacher must talk to individual pupils and find out how they are thinking. Ask them to explain what they are doing. Now you may say, "How can a teacher with 25 or 30 wiggling boys and girls find time to interview each one separately?" Well, fortunately, this is not necessary with all pupils but only those who are having trouble. For example, take the Grade III child mentioned earlier who experienced difficulty in subtraction with borrowing. I observed that the child was following the correct procedures except that she forgot she had borrowed from the tens place. I sent her to the chalkboard and asked her to explain each step of her work. This she did quite well and loud enough so that any pupil who wished to listen could hear. It was only necessary to ask her how many tens she had left after she borrowed to draw her attention to her error. She corrected the question, worked a couple more at the board and was on the right track. Other members of the class having the same or a similar difficulty learned from the discussion, and the child at the board showed that her thinking was, on the whole, satisfactory.

One of the most difficult problems we face in the improvement of elementary mathematics have to do with the indifferent or lukewarm teacher. The inservice education of our teachers is a vast undertaking. Fortunately, most teachers are very anxious to do a better job and they, in turn, will (so we hope) infect those whose work is less effective with a desire to improve. During the past five or six years we have in fact witnessed the most unbelievable change in the attitude of teachers of mathematics and their desire to teach more effectively. They are vitally interested, full of enthusiasm, keen about learning the new programs, and becoming increasingly effective teachers. There is also evidence that the children, too, have caught the spirit.

Research is another great need. To my knowledge, there has been no research which has established whether or not the new programs are better than the traditional programs, but the research undertaken during the past 20 years into the psychology of learning provides unchallengeable evidence that sound and lasting learning can be achieved only through active and continued participation.

There has been and is much criticism of educational research. S.L. King reported that at the 50th Anniversary Meeting of the American Educational Research Association there was now "the conscious emphasis on processes of teaching and learning rather than on their end products". Much of the research reminds me of how the hippopotamus got its name. It seems that when God was naming the animals his assistant asked him why he called this one the hippopotamus. God replied that he thought it looked as much like a hippopotamus as any animal he had ever seen.

The individual interview technique, although slow and fraught with many variables, has promise, but it must be very carefully planned. Also, we know so little about what or how children are thinking. Some of you may have read that charming collection of essays, *O Ye Jigs and Julips* by Virginia Hudson, written when she was 10 years old. In her essay *Etiquette in Church* she writes, "Etiquette is what you are doing and saying when people are looking and listening. What you are thinking, is your business. Thinking is not etiquette".

Brownell, (*The Arithmetic Teacher,* April, 1966), says that, "We tend to minimize the complexity of evaluative research." Glennon, (*The Arithmetic*

Teacher, May, 1966), quotes Cronback as suggesting that existing research on inductive teaching, the discovery method, "has not begun to give the answers needed for firm recommendation of the schools."

The researcher has great difficulty in trying to keep pace with the panacea-mongers. May I suggest to you, "Be hospitable to new ideas, but beware of panacea-mongers!"

There is great pressure on schools today to adopt all sorts of new devices. These must be evaluated, and decisions must be made as to whether or not they are suitable. Also, there is great pressure on schools to introduce new topics and to teach old topics earlier. Should we not always ask ourselves, "Why?"

It is my thesis that within the next few years you will see the present new programs in elementary school mathematics revised and revitalized. There will be many new procedures to be evaluated. There will be a greater emphasis on the learning of fundamental concepts and relationships relevant to the child's needs and environment as he grows and develops. There will be greater emphasis on how we analyze problems, the method of attack, and the selection of pertinent facts. Methods will improve and the activist approach will become the normal; future teachers, having been taught in school by this approach, will use it with greater ease and refinement. Notice that I said the 'activist approach', not method. There are many teachers, I fear, who think that the 'discovery method' means 'any' method, and this so often means 'no' method. While I believe that it is good for children to devise their own methods, I also believe that these methods should be evaluated and the best selected. Whatever method is used, it must follow sound mathematical principles and be well done.

To sum up, we have fine new programs of elementary school mathematics, but many of them emphasize pure mathematics to the exclusion of what is more practical for the average child. New programs must be examined carefully, and we must beware of new programs which claim to be the 'great answer' for which we are searching. Also, remember that no program replaces the competent and devoted teacher.

The so-called discovery method is very promising in the hands of the teacher with imagination and ability, but let us be honest with ourselves, too few teachers have the imagination and ability to use it really effectively. Also, let us remember that some things are not worth the time it takes to discover them.

There must be good inservice courses for teachers, but beyond these courses there must be small study groups in which the teachers have an opportunity to discuss their work and they themselves must study, experiment in their classrooms, and constantly and continually evaluate and reevaluate the effectiveness of their methods.

Teachers using the new programs must teach. They must plan their work very carefully and provide activities which are challenging and which take into account the individual differences of the children in their classroom. This means also very clearly that they must not be content with merely keeping children busy.

Above all, let us remember that we are teaching children, children who, we hope, will take places in a free society. However, freedom needs framework. For the individual that framework is character; for society it is the law. In education today, there is much bewilderment and some dismay. It is our duty to teach our children that they will respect the creative restrictions which make public order.

EXPERIENCING MATHEMATICS - A PROGRAM FOR SLOW-LEARNING SEVENTH GRADERS

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Many persons, more expert than I, have presented pros and cons on the subject of grouping. I shall not enter this debate. However, as an instructional supervisor who renders services and support to classroom teachers and assists them in maximizing their efficiency in meeting the individual needs of their students, I rendered my support in favor of grouping. The teachers of junior high school mathematics in the Dade County, Florida, public schools strongly voiced a need for a different curriculum organization, especially at the Grade VII level. The opposed organization included two groupings for instruction: a group of students recommended by feeder elementary schools for accelerated mathematics instruction, and a group not recommended for accelerated mathematics. Criteria that included standardized tests scores, arithmetic grades and recommendations by elementary school principals determined placement in classes for acceleration. For the most part, the group of "not recommended" students was randomly assigned in groups of 35 for instruction. Herein arose our problem. Many students and teachers faced failure. Too often the range in mathematical maturity frustrated and overwhelmend the most dedicated teacher. When newer texts began appearing on the market reflecting innovations in mathematics instruction at the pre-college levels, many of our teachers envisioned this answer: teach the mathematically talented the "new math", and teach traditional Grade VII mathematics to the others. This, of course, was doomed for failure. Finally, during the spring of 1965, it became administratively possible to provide four levels of instruction in the junior high school.

Mathematical maturity of students for these three levels could be spelled out. Teachers could also be provided with commercially produced instructional materials. But for our lowest level - Level 1 - appropriate instructional materials could not be located. In addition most of our junior high school teachers were not equipped to teach the arithmetic of the elementary grades. With the grouping of pupils who needed such basic instruction, the inadequacies of our teachers would become painfully obvious. Therefore, an NDEA grant was secured to

- develop instructional materials for Grade VII students who score at least below the tenth percentile on standardized tests;
- produce a television series for inservice training of teachers in teaching basic arithmetic.

This grant made possible the writing of the book *Experiencing Mathematics* during the summer of 1965. How, in reality, did this book evolve?

The team assigned to this task consisted of three secondary and three elementary teachers. The chairman was in contact with many of the team members during the spring of 1965. They had to establish some basic assumptions and arrive at a working philosophy. The work was dictated by these assumptions:

1. Mathematical retardation is not necessarily controlled by the economic conditions of the pupils' families.

2. Negative attitudes about mathematics and about probable success in this area will predominate.

3. Developmental activities (experiences for induction) must appeal to the social, emotional and chronological maturity of the pupils as well as to their mathematical maturity.

4. Pupil involvement and responsibility for self-enhancement can be maximized.

5. Short, purposeful and distributed drill activities are important fixatives.

6. Instructional time devoted to development of understandings and concepts should not outweigh instructional time devoted to drill.

Before a page was written, this team (a) identified pupils who would qualify for placement on Level 1; (b) administered diagnostic tests during the summer of 1965 to Level 1 students enrolled in summer school (care was taken to diagnose mathematical achievement of a sample that would realistically reflect the economic stratification of students who would be grouped together during the regular academic year); (c) analyzed the above data and other data supplied by the county testing program to ascertain areas of greatest deficiency; and (d) critiqued the sequence of arithmetic learnings to determine what reorganization, if any, was needed.

As a result of this background information, the team decided to prepare instructional materials that would provide opportunities for positive changes in attitudes, support teachers in their attempts to guide these students in "learning to learn", and reorganize the use of instructional time so that students could have a longer time exposure to rational members.

(Comment: From our study, we were not sure of the reasons that made many students with average and above intelligence score low on achievement tests. Was it because of the seasonal arrangement of arithmetic instruction, particularly in Grades IV to VI, namely - <u>fall</u> - add and subtract whole numbers; <u>winter</u> - multiply and maybe divide whole numbers; <u>spring</u> - make sure they pass all diagnostic tests on operations on whole numbers before introducing fractions; school is almost out so fractions should be introduced!?)

Now, what changes were incorporated in *Experiencing Mathematics?*

"Let's Talk" - teacher-directed activities that involve conversations with two characters - Alpha and Beta were created to guide students in talking about their learnings and in abstracting worthwhile generalizations. Many students in this population have language problems. They cannot look at an example or participate in an activity and see relationships. This skill, if we can call it that, must be developed. These students often fail to ask questions, simply because they perceived nothing about which to question. The dialogues between Alpha and Beta raise questions that the student might ask, and direct attention to specific relationships from which valid generalizations may be drawn. All "Let's Talk" sections are designed to reveal to the teacher the depth of pupil understanding. We conceive of this phase as analogous to the physician's case history work-up. The patient talks, he describes his complaints and relates the history of illnesses of his close relatives. From this, the physician gets cues on how he is to proceed. Similarly, the teacher should be cued into determining the nature and kind of additional experiences needed.

The organization of the text permitted a reversal in using instructional time. We begin with the set of non-negative rationals and attempt to create a sense of operational consistency between (a) operations on rational numbers in rational form, (b) operations on rational numbers in decimal notation, and (c) operations on rational numbers named as whole numbers.

The order of development is minutely sequenced with respect to skills and understandings these students bring to Grade VII.

Students are encouraged to determine for themselves from the algorithms presented the ones with which they feel most comfortable. This is by no means an innovation, since most recent texts include algorithms other than standard algorithms. However, we are deliberately encouraging teachers to accept the differences in children. This is extended to include acceptance of good non-standard algorithms. Simply because an algorithm can be concisely recorded does not mean that it is the easiest way for all children.

Teachers in 35 of Dade County's 42 junior high schools submitted these evaluations of *Experiencing Mathematics:*

1. Does this text appeal to the student for which it was intended?

Partially	5
Yes	27
No	3

2. What are the strengths of this text?

Charts, illustrations, visuals, and patterns.
Sequence of topics - development of concepts.
Working in the book.
Variety in practice problems.
Simplicity of material and presentation.
Readability.
Written on a very elementary level with appeal to these older children.
Multiple methods presented.
Dialogues between Alpha and Beta.

Experiencing Mathematics: Our ideas and philosophy have been tested. They appear to have value. We have attempted to build concepts from the ground up and win children to learning mathematics. Our level organization provides opportunity for greater individualization of instruction. Students are not only being taught on their achievement level, they are also finding their comfort level. With these divisions, we divided to conquer!

SOME K-6 GEOMETRY
MODERN MATHEMATICS

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What is modern mathematics?

We ask this question often to initiate a discussion wherever there is interest. Repeating it should not bore us, for, as yet, no one has found a concluding answer. And, as we get more insight, we tend to revise and improve our answers. To date, in the light of the work of the National Council of Teachers of Mathematics during the past 47 years, 'modern mathematics' means improved mathematics.

Continuing Concepts

In order to help the phrase 'modern mathematics' to signify improved mathematics, our comments will emphasize five geometrical concepts. These properly begin in early school days and continue through later school days. They include sets, forms, measures, illustrations, and proofs. They permeate geometry, and geometry permeates life.

Readiness

Indeed, the proper subject to study by all of us is geometry. It is everywhere. Forms, just as well as numbers, interest people of all ages. Even a pre-school child is shape-conscious. Interest in shapes, we believe, continues through the years.

Geometry not only appears in mathematical applications; it also underlies much of further work in mathematical theory. Yet, all too often in the past we have let children go on to secondary schools painfully innocent of geometry. This we now deplore. To defer geometry to the secondary school is sometimes to eliminate it entirely. For, being grossly ignorant of the subject, many pupils do not elect it. Instead, when they enter high school, our pupils should possess a rich knowledge of geometric facts, a thriving readiness for reasoning, and a strong desire to continue to learn geometry.

Sets

Mathematics treats sets - sets of numbers and sets of points. These, and the things that people can do with them, comprise the subject. Whereas the set of all numbers has no one-word name, the set of all points is called 'space'. Both the set of all numbers and the set of all points have infinitely many members, and both have numerous subsets. Through the years new numbers have produced an abundance of new algebras and new arithmetics. The idea of adding new points to the collection of all points may, at first thought, seem less likely than the idea of creating new numbers. But there is at least an abundance of new ways of looking at points, and the creation of new geometries has become commonplace.

So it is that studying sets of numbers and sets of points provides an excellent approach for pupils of all ages. Indeed, to summarize briefly what modern mathematics is all about, to impress your friends, to convince people that you are in the know, and to mystify those who query, you can simply say that mathematics today deals with "sets and stuff".

Subsets

Seldom do students of geometry investigate the set of all points. Rather, they look at interesting subsets. Hence the remainder of our discussion will focus on special subsets. These collections of points underlie the geometry of form, an especially exciting study for pupils in the grades. Note how simply the ideas evolve and how familiar the materials illustrating the ideas are!

We begin with a sheet of paper. What is its shape? What name does this shape have? If the sides all had the same measure, what would we call the shape? Is a square a rectangle? Sometimes this provokes a lively discussion. Squares and oblongs as special cases of rectangles and, possibly, rhombi and rhomboids as special cases of parallelograms might well ensue.

Definitions

Another outcome stresses the outline of the paper, be it an oblong (a rectangle with not all four sides having the same measure) or a square (a rectangle with all sides having the same measure). The set of points under consideration, not the interior of the figure, makes up the figure. This implies that, from kindergarten on, we try to use precise language, while the children are developing correct concepts. Of course, numerous activities to help children create clear conceptions should precede their memorizing of technical words. Once they know how squares differ from oblongs, and once they recognize how oblongs and squares are alike, the pupils then appreciate that the words 'square', 'rectangle', and 'oblong' may be useful additions to their vocabularies.

Definitions, just as well as technical terms, should grow from experiences rather than start the experiences. The idea of square corners, fundamental to rectangles, outweighs the word 'rectangle' in importance. Much of our discussion to follow in this article stems from the assumptions: (a) experience, concept, name; (b) activity, idea, definition.

Let us reassert, then, that geometry in 1967 emphasizes sets of points (outlines in such figures as squares, circles, triangles, et cetera) rather than portions of planes. Therefore, circles, ovals, triangles, rectangles, and other configurations can hardly be colored. To "color the triangles red" would mean literally to retrace the line segments that the pictures represent. To fill the interiors of triangles with a red crayon detracts from the concept we seek, namely, the border set of points.

More valuable than the tasks of "Color the triangles red", "Color the squares green", and "Put a cross in each circle" would be simply to "Change the paper with oblong shape to a paper with square shape", "How many parts with shapes like triangles can you make by folding a square paper once?" and "If you fold an oblong paper twice, how many oblong shapes can you get?" Similarly, the experience of "Fold a paper once in any way; then fold the paper again so that part of the folded edge fits along the rest of the folded edge. Then open the paper and tell about the folds" packs a greater potential than cutting and pasting disks and other parts of planes.

Creases

Let us return to our sheet of paper. It carries us on. Crease the sheet from one corner to the opposite corner. With squares this comes easily; with oblongs, not so easily. We urge the pupils to achieve a sharp crease. Indeed, we call for a perfectly sharp crease. Hopefully, some pupil(s) will demur at this. How can a crease be perfect?

Then we appeal to pupils' imagination. Suppose that the crease were perfect. How wide would the crease be? Further creases, such as the crease joining the other two corners, all illustrate the idea of length without width. While this is percolating for slow learners using a square paper, the sharpies can be working with an oblong paper. To name the idea which we have uncovered, we might suggest joining opposite corners of another paper with a mark made along a ruler. What can pupils call such a mark? A line. A ruler, incidentally, is a device for ruling (marking) lines. Since many rulers contain linear scales, we can also measure lengths with them. The key idea, however, is that of marking lines. A ruler primarily helps us to represent lines and segments of lines.

Lines

As pupils develop readiness for lines, we extend the idea that lines have length without limit. Our creases show parts (segments) of lines. Really, lines are neither creases, nor marks, nor anything physical. They are mental constructs - things with length, but no width. A bit of incidental learning involved, once pupils have the idea of opposite corners, arises in the word 'diagonal'. Possibly some have already learned "catty-cornered", "kitty-cornered", "cater-cornered", or some such expression, which possibly not only lacks elegance but also, etymologically, means four-cornered (French 'quatre'), rather than opposite.

Crosses

From sharp creases we focus attention on the place where two sharp creases cross. A pair of diagonals will do it, so will any pair of creases that cross. Then, - and please note the deduction and the imagination - if two creases, each perfectly sharp, cross, what has to be the size of the place where they cross? This leads to the idea of place without size. Pupils learn to call this idea 'point'. They learn further that a point, exacting as a line and as a number, stems strictly from the mind. Number, point, and line are ideas - not symbols, marks, or creases. With this in mind, if we move a paper containing a specific point, do we move the point? To answer this, pupils only need to consider the sole property that a point has position.

Points

Repeated foldings reveal that lines contain numerous points. There can be many crossings - in fact, an infinity. So a line is a set of points. Similarly, a paper represents another set of points, a plane. Furthermore, all points together make up space. So planes and lines are subsets of space. Occasionally, a mathematically mature youngster may think of lines as sets of points, planes as sets of lines, and solids as sets of planes. Rarely a pupil may inquire, "What does a set of solids make up?" This pushes the imaginations of most of us. We can use objects to represent point, line, plane, and solid. Theoretically, we can go on. However, we do not have models to represent four-space, five-space, et cetera.

Separators

This bears on various objects as separators also. A point separates a line into two half lines. A line separates a plane into two half planes. A plane separates space into two half spaces. What does a space separate into what?

Angles

The idea of sets of points goes farther. An angle is a set of points - two rays not on the same line, with a common endpoint. (A ray is a half line with a specific endpoint.) As you probably read in *The Mathematics Teacher* for January, 1967, not all mathematicians accept this definition for an angle. Such a definition does indeed raise questions about straight angles, angles of a triangle, and angles increasing or decreasing via rotation of one or both sides of the angles. As with the case of moving a specific point, can we rotate a set of points about a fixed endpoint?

The concept of an angle as a set of points, composed of two half lines and one common endpoint, probably suffices for K-6 geometry.

Measures

Except for the discussion about rectangle, all that we have thus far considered requires no measurement; much of geometry does not depend on size. Unfortunately, this very important idea in mathematics suffers from want of attention. Ironically, on the other hand, the idea of measurement also suffers from neglect. All of us know that quantity as well as number means much in life. All of us can recall in a moment numerous applications of mathematics that involve measurements. Yet the daily use of geometric forms and units of measure in traditional programs in elementary schools concentrates on computation, as if computation were the sole goal in mathematics for children!

Accordingly, we hope that all pupils have frequent opportunities to make rough, or unstandardized, measurements. How many steps from the throwing place should the target be? How many spans is your marble from Jack's marble? How many hands high is your horse? How long is your desk in pencil lengths? How many cupfuls does this jar contain? How many papers do we need to cover a desk? How many newspapers to cover the floor? How many wooden blocks to fill the box? Questions such as these can get at the much needed understandings.

Comparing an object with another object used repeatedly should happen early and often. It prepares the way for a sticky sort of question. How can a set of points, each having zero size, make up a line that has length other than zero length? The answer goes back to Archimedes, circa 250 B.C., who stated that a small segment laid down sufficiently often end to end can exceed a large segment. This assumption, which shifts from points to segments, is the key to measurement.

Number Lines

After pupils have had some experience with measuring lengths with understandardized units, they become ready to apply Archimedes' postulate in the making of number lines. (Strictly, they are number-line segments). On segments which pupils represent on papers or chalkboard they choose a starting point. To the right of this point they mark other points each at some chosen distance from the marked point nearest it. Then, using numerals, they assign numbers to each of the points marked. Helping children early to set up such a correspondence between points and numbers amounts to a tremendous accomplishment. This idea rates highly in importance geometrically. The same idea, by the way, promotes understanding in arithmetic. Order is basic. Here, early in the pupils' lives, arithmetic and geometry reinforce each other.

Paths

Unions of line segments become paths that interest most pupils of all ages. Making the transition from straight-line segments to curves leads the way to closed curves, which include polygons, such triangles, quadrilaterals, pentagons, hexagons, et cetera. The study of the properties of closed curves provides much interesting geometry. Even the girl who referred to "clothed curves" made a contribution to the day's work.

Points inside closed curves have significance, too. They constitute space of two dimensions, later to be measured and reported via a number, called 'area'.

Drawings

Along with our traditional neglect of geometry for the sake of geometry (even where some classes study geometry for the sake of drill in arithmetic via formulas not often comprehended) we tend also to neglect drawing-board geometry. Apparently we do not know for sure what the least age for ability for properly handling ruler and compasses may be. In some quarters people fear that pupils will use compasses as weapons of offense and defense. However, we doubt that we shall ever learn this minimum age for proper use by keeping compasses away from pupils until they reach high school. Numerous worthwhile projects on the drawing board conduce to enhanced geometrical insights. They contribute to the store of knowledge that pupils need for general education and for mathematical education. Besides numerous standard exercises such as bisections, parallels, and perpendiculars, teachers and pupils can improvise some.

Illusions

Drawing, folding, measuring, cutting, and observing provide numerous opportunities for pupils to learn fundamental lessons of life: Things are not always what they seem to be. No better way exists to get at this principle. When the teacher holds up a white cylindrical stick, asks the pupils what it is, and calmly unrolls the paper which they will take to be a piece of chalk, the lesson endures. Illustrations of numerous optical illusions should be in the teacher's supply of materials.

Op Art

Recently museums of art have become ever more fascinating. The collections of geometrics interest us especially. The popularity of this kind of art has widely spread to encompass illustrations in books, journals, and newspapers. Advertising matter of all kinds includes geometrics. Blurbs for books now contain geometric designs, whether the books contain mathematics or not. Illustrations of a special variety, Op Art, seem to predominate. They appear not only in galleries of art but also in shops and stores, especially in materials for women's clothing, wall hangings, upholstery, and floor coverings. The implications for interesting projects in art and mathematics have almost no limit. This aspect of modern mathematics helps us to emphasize the contemporary spirit, the feeling for discovery, the creative urge.

Proof

The gradual development of ideas of abstract geometry, such as point, line, plane, closed curve, interior point, exterior point, half line, half plane, angle, et cetera, encourage pupils to go beyond things visible to things invisible. We refer to mathematical constructs, ideas, and objects of thought which become what we make them. Physical objects help us to learn, but the elements of geometry are mental objects.

An aim of present-day mathematics emphasizes the importance of correct language. 'Number' differs from 'numeral'. 'Line' differs from 'line segment'. We represent length, area, and volume by using numerals, because we can count (directly or by rule of computation) the units. Yet the segments, surfaces, and solids that we draw, measure, and calculate contain points which have infinite numerosity.

We have seen that mathematics stems from well-taught imaginations. Points have position only; we do not actually use them; we use imagination to understand them; what we see only crudely represents them. Similarly, 'line', 'plane' and 'space' denote subsets of points that we can imagine perfectly but which physical objects represent imperfectly.

To communicate one's ideas (imaginations) to one's fellows, one needs to use language correctly. To convince one's fellows that one's conclusions merit acceptance, one needs to use proof correctly. As the child matures from pre-schooler to teenager, he adopts progressively more logical forms of proof. To expect him to produce full-blown proofs at some arbitrary age, say 15, is unrealistic. Rather, we might appeal first to authority; these two rectangles have the same size because Dad says so. Next, the pupil may depend on measuring; these two triangles measure the same along three pairs of sides. Next, he may depend on computing, which entails counting; these two oblongs contain equal numbers of square units employed as measuring sticks.

Reasoning appears in all such developments. Gradually, pure reasoning renders dependence on physical objects less necessary. The child reasons that, if sticks represented the idea of 40, then 40 means four tens of sticks. To answer the question "How many eights are in 40?" one needs only to imagine that two sticks removed from each 10 makes four twos, or one eight. This plus the four eights, made from the tens, produces five eights. One reason that there are five eights in 40.

Similarly, questions in geometry receive convincing answers via reasoning, with configurations needed only for more and more complicated questions. Indeed, sketches often suffice. One reasons correctly from incorrect figures.

Pure reason serves also to produce answers not entirely possible to achieve via physical objects. If one perfectly sharp crease crosses another perfectly sharp crease, the size of their intersection has to be zero. That is, a zero width (abstracted from a less-than-perfectly-sharp crease) crossing another zero width has to be a crossing of zero width.

Numerous reasonings such as these prepare pupils not only for more formal and more rigorous mathematical deductions but also for less formal and less rigorous everyday decisions. The upshot emerges as the transfer of some training for which teachers have quite specifically planned and taught. Proof of the sort appropriate to the maturity of the children belongs to every program in mathematics. To defer it to the period as late as high school is to rob pupils of their fair and proper intellectual birthright.

Summary

Proof lies at the heart of mathematics; if our endeavors do not lead to proofs, then they hardly merit the name 'mathematics'. Traditionally, in grade schools we have emphasized the computations of arithmetic. And we have shunned geometry. In so doing, we have erred - twice:

1. Geometry could not be separated from life; it is everywhere. To defer it was to delay an important part of general education. To postpone it was, in some cases, to eliminate it, for the students later elected not to take it.

2. Geometry could not be learned effectively in the traditional Grade X crash program. Pupils had to learn numerous facts and compose proofs at the same time. As a result, many resorted almost entirely to memorizing the facts and the proofs.

The approach in improved mathematical programs encourages pupils to learn geometric facts and geometric ways of thinking from kindergarten on. This does not mean moving the traditional, concentrated course to a lower grade. Rather, it signifies a gradual approach - spiral learning from simple essentials to more and more mature understandings. Intuition lies at the heart of the matter. Objects, pictures, and drawings illustrate geometric ideas. What could be simpler than sheets of paper to illustrate planes, or creases in paper to illustrate lines, or crossings of creases in paper to illustrate points?

Along the line teachers emphasize the importance of experimenting - drawing, observing, guessing, counting, measuring, and reasoning. Insight increases because the subject fascinates pupils, and interest underlies learning.

Outlook

The best time to study geometry reaches from the cradle to the grave. This makes sense to the teacher who, being a victim of a by-gone sink-or-swim course in high school, promptly sought other electives. This heartens the teacher who, despite the shock he had in Grade X, now confronts the suggestion (or, possibly the command) that he begin to teach geometry to pupils in the lower grades. This emboldens the teacher who, aware of the fact that his memory fails on some of the details, can recapture them in the bright line of good understanding. This light prevails as his pupils learn geometry in a gradual, intuitive program that emphasizes observations, inductions, and deductions.

Are you willing to embark or to enlarge your endeavors on such an important journey in the education of all children?

THE TRAINING OF TEACHERS IN NEW MATHEMATICS - TV AND REGIONAL SEMINARS

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Although the title of the paper does include the term 'New Mathematics', the first thing I wish to do is to assure you that I do not intend to contribute to the already over-abundant attempts to explain this term; and I do not propose to deal with a heavy mathematical problem. I simply want to make some observations on inservice training of teachers made necessary by the revolution known as 'New Mathematics'.

Only by way of introduction I should like to remind you that our society has recently made both increased and more diversified demands upon the mathematical skills of the graduates of our schools. So extensive and diversified have these demands become that it is necessary to re-examine the whole program of arithmetic and mathematics with a view to equipping students with those skills that they are apt to need in the immediate future.

This re-examination and revision of the school mathematics program have led to major problems in retraining of teachers. How are teachers to keep up to date? Who is responsible for retraining or inservice programs? These questions have led some teachers to wonder why they ever considered the profession.

Many suggestions have been put forward with respect to the background which a mathematics teacher should have - and frankly, in spite of some well developed programs such as the CUPM program (Committee on Undergraduate Program in Mathematics), I am deeply concerned about the lack of mathematical background of many of our beginning teachers. In my position in the Faculty of Education at the University of Manitoba I am involved in teacher training and am vitally concerned with the kind of preparation we are providing for our student teachers. From the standpoint of the topic of this paper I am doubly concerned, for if the preservice training is not adequate, the inservice must go on forever. However, whatever the duration of our efforts, my concern centres on the retraining of teachers already in the field.

During the past three years two committees (one for Grades I-VII, one for Grades VIII-XII) have been working on curriculum revision in mathematics under the Curriculum Director of the Manitoba Department of Education. These committees have become deeply involved with inservice training. It has been suggested that revision committees should not necessarily become involved in this type of "extra-curricular sport"; however, experience has pointed out that revision is not fruitful unless teachers are prepared and equipped to teach the new courses. It is my purpose to report on progress being made in so equipping the teachers of Manitoba.

Two approaches or methods have been tried: (1) regional seminars or workshop series consisting of 20 hours of instruction of discussion, and (2) a television series of 26 programs followed by a centrally administered examination. I will deal with these approaches in order, perhaps not in order of importance but certainly in the order in which they are used.

The Regional Seminars

At the outset, several problems became evident:

1. If the new courses which had been designed or selected from pilot studies were to be implemented successfully, a major training program was necessary to deal with both method and content.

2. There were insufficient competent instructors to meet with several thousand teachers.

3. The shotgun approach or effect of inservice had to be avoided (many inservice programs are ineffective because they do not reach or "hit" all teachers.)

4. There were general problems of communication with teachers in that (a) reasons for change had to be made clear, (b) the authority of what to change had to be established, and (c) teacher reaction and evaluation by teachers had to be gauged or appraised.

5. A general lack of concern was noticeable on the part of teachers at any one level for the problems of those at another level (for example, the university staff were not concerned about high school and high school not concerned about elementary).

Keeping these problems in mind, inservice seminars were set up in the following manner:

In the spring of 1965 the Department of Education made each school inspector with the local superintendent responsible for arranging a 20-hour seminar for the mathematics teachers of his division. This authorization gave the programs official sanction and thus encouraged all to attend. These seminars were to be arranged at suitable times but to be time-tabled outside school hours.

The instructors were selected from a corps of pilot course teachers and committee members. These teachers had completed one year of teaching new courses under the guidance of the mathematics department at the University of Manitoba. This group served as a sounding board for difficulties and also tended to make the university instructors and the school teachers work more closely together.

The instructors were paid by the provincial department of education, and in addition to conducting the seminars they were expected to act as consultants during the following year.

The programs were designed to deal only with the highlights of the new courses and to give experience in developing a "discovery" method.

The resulting seminars were considered quite successful in some areas, while in others the response was mediocre. In all, the majority of teachers involved in new programs did receive some introduction to modern mathematics.

Teacher Training by Television

In addition to regional seminars the Manitoba Department of Education agreed in 1965 to try television as a means of extending and reinforcing the new ideas in school mathematics. By television I am not referring to closed circuit instruction, rather to the use of commercial broadcasting facilities. TV was felt to be especially useful in Manitoba for two reasons: first, there were some early hours available (morning television time which would be relatively free from competition by other programs), and second, the programs could be made available to those who were not able to attend the seminars (the teachers of the far northern schools and those who, because of family commitments, were unable to be present). After some discussion with the Canadian Broadcasting Corporation, it was decided that 10:30 on Saturday morning would be the best time for an experimental program. A program guide was produced to indicate the topic to be discussed, the date, and to give the viewer some idea of the preparation necessary. These guides were distributed to all who applied for the course.

The title of the series "Conversations in Mathematics" was chosen by Dr. Bernard Noonan of the Mathematics Department, University of Manitoba. Dr. Noonan was the driving force behind the experiment, who performed a major part of the writing as well as the camera work. "Conversations in Mathematics" has been referred to by the Manitoba Department of Education as a successful experiment in inservice teacher training by means of telecasts. It was a series of 25 telecasts with the purpose of pointing out and explaining to teachers of mathematics the unusual and possibly unfamiliar topics as they appear in the mathematics texts adopted in Grades IV, V, VI, VIII and IX in the Province of Manitoba, September 1965. While the series had been designed for teachers, each telecast was self-explanatory and could be viewed with understanding by both the student and the layman.

One of the most immediate justifications for considering the experiment successful is the fact that when the series was originally planned, 500 registrants were hoped for, whereas their final count was 3,612. These registrants were divided as follows:

Portage la Prairie	54	(2%)
Brandon	182	(5%)
Winnipeg	1455	(40%)
Other	1921	(53%)

To obtain these and other data, a registration form was sent to each of the more than 9,000 teachers in Grades I to XII in Manitoba schools. Since the telecasts were meant primarily for teachers from Grades IV through IX, the greatest response was expected to come from teachers of these grades. This proved to be the case.

Numbers alone, of course, show only interest. To achieve success, interest must be justified and satisfied. Evaluation forms which were supplied to the viewers indicated a favorable response with 59 percent giving the programs a "very good" rating. Comments attached at the end of each evaluation sheet served to further convince us that the series had achieved it purposes.

In order to test further the effectiveness of the telecasts and also to provide the viewer with a record of achievement in the televised course, 55 examination centres were set up in the province in which teachers could write a three-hour examination based on the series limited to teachers. Of the 390 teachers who wrote,

29% achieved a mark of 80% or over
41% achieved a mark of 75% or over
60% achieved a mark of 65% or over
88% achieved a mark of 50% or over.

The distribution of teachers who wrote the examination was as follows:

Portage la Prairie	0.3%	65% outside urban area
Brandon	3.1%	indicated that it was
Winnipeg	31.1%	reaching those for
Other	65.5%	which it was designed.

Since "Conversations in Mathematics" has proved such an effective teaching instrument, a brief account of events leading to its production may serve as a guide to at least one way of achieving inservice training of teachers by means of television.

Early in the fall of 1963 a decision was made to test the acceptability of a carefully selected Grade VIII mathematics text by using it in several pilot classes. To familiarize the teachers with the nature and intent of the experimental text, a course of instruction based on the text was given at the University of Manitoba and Brandon College. From this first experience it became clear that an extensive inservice teacher training program would be an invaluable aid in establishing a contemporary curriculum in mathematics in Manitoba, not only in Grade VIII but also in Grades IV, V, VI and IX where contemporary tests were to be introduced and later, of course, in the remaining grades as the revision became complete.

The two committees of revision (one for Grades I to VII and the other for Grades VIII to XII) decided independently that, as texts which proved satisfactory in pilot classes were adapted for the entire province, television - among other means - should be used as a way of reaching the very large and far-flung audience. To this end, a joint committee was formed under the chairmanship of the Supervisor of School Broadcasts to determine the nature, extent and format of an inservice teacher training course by means of telecasts. The result was 28 topics which formed the basis for "Conversations in Mathematics". It should be noted at this point that behind the determining of the nature of the course were two years of experience of eminently qualified educators, experience with a great variety of contemporary texts and in preparing teachers to use them successfully. However, good material for a television program needs a meaningful and appealing presentation to make the program effective.

In seeking to achieve these characteristics, "Conversations in Mathematics" was fortunate in that (a) it made free use of the professional, technical and production facilities of the Canadian Broadcasting Corporation, thanks to the wholehearted cooperation of that body; (b) it drew on the wide experience in the fields of radio and television education of the Supervisor of School Broadcasts and her staff; (c) it was written, presented and reviewed by educators who had taken an active and responsible part in the selection of texts and in the inservice training courses given to teachers of pilot classes and who, at the same time, were mathematicians experienced in the use of the television medium; (d) a representative selection of its programs was previewed for comment and criticism by teachers, principals, inspectors, and representatives of CBC and various sections of the Department of Education.

In conclusion, it can be said that the contribution which "Conversations in Mathematics" has made to excellence in education in Manitoba is the consequence of a happy fusion of the experience and abilities of professional broadcasters with those of well-informed and eminently qualified educators.

From our experiment in television teaching I would make the following observations:

1. TV can be used as an effective means of carrying on inservice training, provided it is supplemented by some means of stimulating the teacher-student interaction of the classroom.

2. Careful planning of programs along with the development of a program guide makes for better viewing and learning.

3. The use of a limited number of instructors or assistants makes the series a more coherent package.

4. The topics and content must be planned and reviewed by a committee which includes teachers who are actively engaged in the classroom. I stress this point, as our experience would indicate that if the selection of material and the methods of presentation are left entirely with the professional mathematicians, the telecasts tend to become university lectures lacking in the techniques and teaching methods which are so important in the new mathematics courses. (Teaching by discovery will not be developed in a program given as a lecture.)

The classroom teacher is still the key to a good program in mathematics for our students. We must concern ourselves with the best methods of conveying the spirit of "New Mathematics" to these teachers who are very busy people. This is a big task - we cannot leave it to Dr. Donovan Johnson, to the NCTM, or to a good teachers' edition of a text; all who are involved in implementing new programs must be creative and diligent in the area of endeavor.

THE SITUATION PROCESS
IN PROBLEM-SOLVING:
HELP OR HINDRANCE?

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Among the objectives of teaching mathematics, we find two which are given considerable attention. With the risk of over-simplification, may I refer to them as

1. the content of mathematics,
2. its application to our environment.

I shall leave for you to decide what the first involves. I am concerned in this paper with the second; specifically, I propose to consider problem-solving, which is one phase of the teaching towards it.

The procedure which is referred to as the "situation process" technique of problem-solving requires basically four steps:

- Analyze the physical situation to determine the action that takes place.
- Write a mathematical sentence, using numerals for known numbers and placeholders for unknown numbers, insuring that the operations involving the numbers bear a direct relationship to the action identified in the first step.
- Do the computation necessary to determine the unknown number or numbers.
- State the answer to the question asked in the problem.

To illustrate the steps, consider a rather simple problem situation described as follows:

> Eight children were playing a game. Some more children arrived to play with them. Then there were 17 children playing together. How many more children had arrived?

1. The analysis of the situation reveals that two groups of children have been combined to produce one group. The addition of two numbers and their sum is suggested.

2. The mathematical sentence is $8 + n = 17$, because 8 is a numeral naming the number of the first group, n is a placeholder for a numeral since the number of the second group is not known and 17 is a numeral naming the number of the group resulting from the combining of the two groups.

3. To determine the number named by n, one subtracts 8 from 17. But why? The computation is justified by reasoning that when the first group (whose number is 8) is removed from the combined groups (whose number is 17), the group whose number is not known remains. Thus an auxiliary sentence $17 - 8 = n$ may be thought of, and the computation is evident.

4. The answer to the question is: "Nine more children arrived to play."

Discussion and argument among mathematically experienced individuals often develop because there are several sentences equivalent to $8 + n = 17$. (Sentences are equivalent if the replacement or replacements for n that make the sentence express a true statement are the same in both cases.) Here the sentences $17 - 8 = n$ and $n + 8 = 17$ are equivalent to $8 + n = 17$. Some teachers aver that the first of these may be used because it reflects the thinking which a child must use to justify the computation; others support the second in that the union of sets is commutative.

It is more difficult to adhere to just one point of view when one considers the comparison situation. Consider the example:

> Eight children were playing in one yard and 17 children were playing in another yard. How many more children were playing in the second yard than in the first?

The technique involved is the matching of the members of the set of 8 in one yard with a subset of 8 of the 17 in the other yard. When the subset of 8 is removed from the set of 17, the remaining subset indicates "how many more". Hence the action suggests the sentence $17 - 8 = n$.

It may be argued that one can consider recruiting a set of children to combine with the set of 8 children so that the new set has 17 children in it. The number property of the recruited set provides the answer to "how many more". Such action suggests the sentence $8 + n = 17$. I am satisfied that one of these sentences is better than the other in reflecting the action a child would follow in comparing two groups of objects which he could move about. However, the distinction between the two analyses may be more blurred here than in the first situation.

In problems involving the combining of several sets having the same number of elements each, the multiplicative principle is involved. In general terms the mathematical sentence is

> (Number of sets) times (number of elements in each set) equals (number of elements in the combined sets).
>
> Consider the example:
>
> A man bought 4 copies of the same book. He paid $32 for them. What was the price of each book?
>
> The particular sentence based upon the general one is $4 \times n = 32$.

Arguments to the effect that this is not the only permissible sentence are based upon the mathematical understandings that multiplication is commutative and that division and multiplication are related (inverse) operations. When two factors and a product are involved, knowledge of any two will lead to the third by one of the two operations.

The "situation process" treatment of the operation of division is based directly upon the action of separation of a set of objects into a number of equivalent subsets. Again, speaking in general terms, the sentence is as follows:

> (Number of elements) divided by (number of elements in each subset) equals (number of subsets).

If I wish to distribute 84 pencils to 6 children so that each child receives the same number of pencils, I think $84 \div n = 6$. In order to justify the computation, I must think of allocating one pencil at a time to each of the 6 children so that in effect I am using sets of 6 pencils. Thus, the sentence $84 \div 6 = n$ arises in one's mind; because of its more direct connection with the actual computation, it is regarded as being equally acceptable as the other sentence.

I wish to draw to your attention several more type situations, but for the sake of brevity I shall simply describe the situation and write two or more possible mathematical sentences arising from it. I would like you to decide if the sentences are equally good or if one is better than another.

Example: A confectioner had 192 candy bars to pack into boxes, each of which held 12 bars. How many boxes did he use?

$\frac{12}{1} = \frac{192}{n}$	12 bars per 1 box, 192 bars per n boxes.
$192 \div 12 = n$	Separate 192 bars into subsets of 12 bars.
$n \times 12 = 192$	A number of boxes of 12 bars each will hold 192 bars.

Example: A teacher separated her group of children into four teams for a game in the gymnasium. She had seven children on each team. How many children were there in the class?

$\frac{7}{1} = \frac{n}{4}$	7 children per 1 team, n children per 4 teams.
$n \div 7 = 4$	Separate the group of n children into subsets of 7.
$4 \times 7 = n$	4 teams of 7 children each make up the whole class.

Example: A store advertised condensed milk at 2 cans for 35 cents. How much should a shopper pay for 8 cans?

$\frac{2}{35} = \frac{8}{n}$	2 cans of milk per 35 cents. 8 cans of milk per n cents.
$\frac{8}{2} = \frac{n}{35}$	The relationship of 8 cans to 2 cans

must be the same as the relationship of n cents to 35 cents.

$n = (8 \div 2) \times 35$ — $(8 \div 2)$ gives the number of sets of 2 cans. This is the number of groups of 35 cents to be paid.

Example: A farmer sowed 72 acres of wheat. He sowed 3 times as many acres of wheat as of oats. How many acres of oats did he sow?

$\frac{3}{1} = \frac{72}{n}$ — 3 acres of wheat to 1 acre of oats, 72 acres of wheat to n acres of oats.

$72 = 3 \times n$ — 72 is 3 times the number of acres of oats.

$72 \div n = 3$ — Separate 72 acres of wheat into 3 lots of the same size, each one to match the number of acres of oats.

Example: In a class there were half as many boys as girls. There were 22 girls in the class. How many boys were there?

$\frac{\frac{1}{2}}{1} = \frac{n}{22}$ — $\frac{1}{2}$ of a boy to 1 girl, n boys to 22 girls.

$n = \frac{1}{2} \times 22$ — Think of "half" as meaning one of two equivalent subsets.

$2 \times n = 22$ — 2 groups of n boys each are equivalent to 22 girls.

Do the foregoing examples suggest that there are several equally valid ways of writing the mathematical interpretation of the action in a situation? I have tried not to state my point of view in this presentation so far. Now I wish to summarize my ideas, most of which are based upon the presentation, some from related experiences not specifically mentioned.

1. The "situation process" technique is a valid approach to problem-solving because it provides a method of isolating the mathematical aspects of a physical situation in a meaningful manner.

2. Because there can be different points of view about the action, there may be more than one acceptable mathematical sentence. However, the criterion of choice should be the action in the situation, not mathematical convenience.

3. The writing of sentences should be used as a testing device only when the specific skill of translation is being examined and when the action is unequivocal.

4. The principles underlying mathematical operations and the relationships between them (e.g. commutativity of addition; division as the inverse of multiplication) arise out of mathematics more than out of physical situations. By illustration: $2 \times 5 = 5 \times 2$, but 2 sets of 5 books are different from 5 sets of 2 books.

5. When computational devices are being learned, their basis upon physical situations should always be clear. After they are learned, shortcuts based upon mathematical considerations may be justified.

ALGORITHMS AND COMPUTERS

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There are several areas of study in the secondary schools which may be concerned with computers. In the technical stream, students can study circuitry, testing, and repair. In the commercial stream students can study the application of computer methods to business problems (data processing). In the academic stream, students can study computer science as a separate subject or can make use of computer methods to solve problems arising in present academic classes.

This paper is concerned with a course which makes use of computer methods to further the student's knowledge of mathematics. Two possible techniques can be used. The emphasis can be placed on teaching a computer language with little training on the basic techniques or problem solving which are adaptable to computer implementation; or the emphasis can be placed on the methods of problem solving, thus dealing only incidentally with the computer language. The first type of course has a place in the education of students who are interested in acquiring a skill which has vocational value. The second type of course is one that provides a skill but also has further academic value in that it gives experience in a different type of problem solving technique.

An experimental text for such a methods-oriented course was used in a Grade XII class at University of Toronto Schools during the 1965-66 school year. The text was the preliminary edition of *Algorithms, Computation, and Mathematics* by the School Mathematics Study Group. It is composed of three separate but related sections: a main text, a FORTRAN supplement, and an ALGOL supplement. This is an attempt to influence commercial texts to provide a similar format so that new languages can be included in a basic text without a complete revision being necessary. The class involved used ALGOL since they were using a Burrough's installation (B5500) made available through the generosity of Burrough's Business Machines Ltd. and K.C.S. (a Toronto firm of consultants in operations research and computer applications). Much of the key punching for the students was done at the Ontario Institute of Studies in Education.

The course is based on the study of algorithms. An algorithm is defined as any unambiguous plan for carrying out some process in a finite number of steps.

This is a familiar concept illustrated by (1) a recipe for a cake; (2) instructions for assembling a "prefab" cottage; (3) the procedure for subtracting a polynomial; (4) the formula for the square of a binomial; (5) the formal method for determining the square root of a number; (6) Newton's method for determining the square root of a number; and (7) the procedure for obtaining the product of two matrices.

A <u>useful</u> algorithm has certain special characteristics. Besides being unambiguous and having a finite number of steps, it should be applicable to a set of problems rather than an individual problem; and if it is to be suitable for implementing on a computer, it should involve a set of repetitive steps.

A clear method for describing an algorithm is a flow chart as illustrated below.

<u>Example 1</u>

A farmer sells a cow for \$75.75. Assuming that he receives this amount in 20 dollar bills, as far as possible, calculate the number of 20 dollar bills received.

<u>Solution</u>

Number of 20 dollar bills

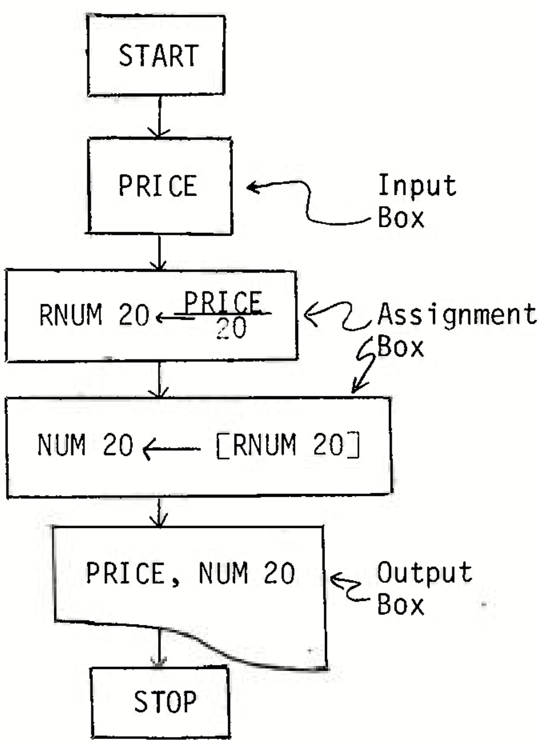
$= \text{Integral part of } \frac{75.75}{20}$

$= \left[\frac{75.75}{20}\right]$

<u>Definition</u>

Greatest integer function is the function such that $x \to [x]$, $x \in R$, where $[x]$ is the greatest integer which does not exceed x.

<u>Flow Chart</u>



The price of the cow is read in

The variable RNUM20 is assigned the value of $\frac{PRICE}{20}$

The variable NUM20 is assigned the value [RNUM20]

The value of PRICE and NUM20 is printed.

The example illustrates the use of boxes of different shapes to represent different types of instructions in a flow chart and also the symbol ← as a symbol used when assigning a particular value to a variable. From the flow chart above, with some instruction on the computer language being used, it is possible for the students to make up a program for running on a computer.

A major feature of the examples in the text is the way in which one problem is expanded upon in exercises from chapter to chapter. An illustration of this is indicated by the following example.

Example 2

A carnival wheel has 32 sections, numbered consecutively in a clockwise direction 0,1,2 ---, 31. The sections are colored blue, green, red, and yellow consecutively. The wheel is spun in a counterclockwise direction and points are awarded by the color of the section opposite the pointer when the wheel stops according to the following table.

Section color	Blue	Green	Red	Yellow
Points	-30	-10	10	30

Make up an algorithm to simulate this game.

Hint: Use S as original sector number

M as number of sectors spun through

K as a member of {0, 1, 2, 3}

where 0 ↔ Blue, 1 ↔ Green, 2 ↔ Red, 3 ↔ Yellow

P as the number of points.

Discussion

This problem requires several days to study and discuss in order to arrive at a satisfactory algorithm. A variety of interesting topics are introduced in the discussion: modular arithmetic, curve fitting; expectation. A study of particular values of S and M indicates that S, the sector number at the completion of the spin, is the remainder after obtaining the integral quotient of

$$\frac{S + M}{32}$$

This may be represented symbolically by $S' = S + M - \left[\frac{S+M}{32}\right] \times 32$

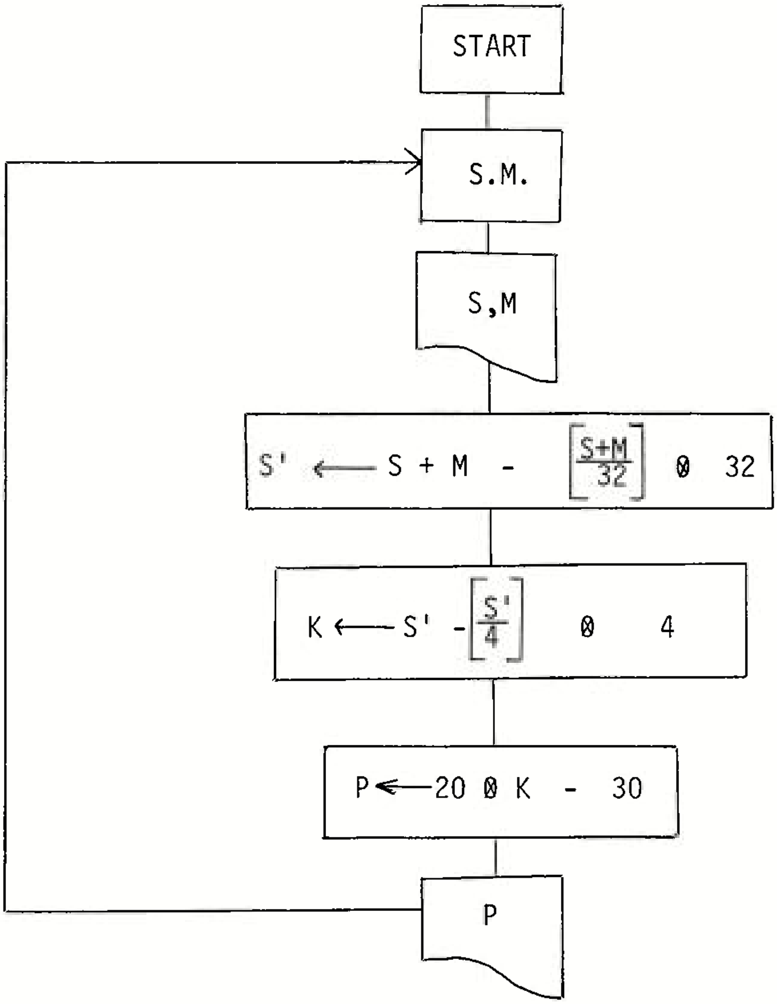
It is clear that the value of K is obtained by a similar process where 4 is the divisor. Thus

$$K = S' - \left[\frac{S'}{4}\right] \times 4.$$

To assign point values corresponding to a specific value of K, it is necessary to express P as a function of K (if a condition box is not to be used). A study of the graph of {(K, P)} suggests that the function is linear. It follows that P = 20 X K - 30.

Flow Chart

This simulation of the game provides a loop so that successive "spins" may be considered. Obvious extensions of this problem are dealt with as the students learn more programming techniques.



1. Point values are introduced which are not determined by a simple function. This requires the introduction of condition boxes.
2. Point values are changed for different sets of spins. This requires the introduction of subscripted variables.
3. A cumulative total of points scored on successive spins is kept. This requires the introduction of a loop or iteration box.
4. A count is kept of the number of spins required to reach a certain value for the total points. This requires an iteration box.

The extensions of the problem open various topics of discussion and provide for experience in the concept of the probability of an event. The students enjoyed this simulation type of problem and went far beyond the required assignments in building up better simulations of the game.[1]

In Ontario, action is already being taken at the departmental level on the question of how the curriculum should be modified to recognize the impact of computers on society: (a) A revision of the business and commerce curriculum has modernized this course, introducing a study of data processing. (b) A committee is producing a computer language for high schools. (c) A committee of the Curriculum Branch has been set up "to study the impact of computer science on the curriculum".

This committee has prepared guidelines for the type of computer suitable for use in secondary schools; sponsored an experimental use of the SMSG material in 20 schools; and entered into a study of experimental programs being carried on in the US and Canada.[2]

From our own experience with this course, we feel that for many students such a course can give an added dimension to their study of mathematics. It would be a mistake to say that all students will enjoy such a course. Even some students who enjoy mathematics find the attention to detail demand in this computer work to be discouraging.

Any teacher who contemplates introducing a course in computer methods should consider the following:

1. What computer facility will be used?

To avoid long turn around times, it is essential that the installation be powerful enough for the problems you wish to do. A small installation will limit the work to such simple problems that the students will feel the additional work involved in programming is not worth the trouble. It is doubtful if any computer which does not use punched card input and line printer output will be versatile enough for any more than an introduction to the techniques of programming. If it is expected that the students should have a chance to actually work out problems of some depth using computer methods, then a fairly powerful installation is necessary. It is not essential that students see every one of their programs processed. We found that having programs run overnight was a very feasible method.

2. What language will be used?

This will be determined by the installation used. Some variation of

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[1]For further details of the course see School Mathematics Study Group, *Algorithms, Computation, and Mathematics*, (Montreal: McGill University Press, 1961.)

[2]Further details of the Ontario studies can be obtained from G.A. Kaye, Curriculum Branch, Ontario Department of Education, 44 Eglinton Avenue, W, Toronto, Ontario.

ALGOL has the advantage over present forms of FORTRAN that the position of statements on the cards is not important. This allows corrections to be made fairly easily, an important consideration when dealing with students who are not adept at key punching.

3. What special procedures must be followed to run a program on the particular installation used?

Different installations have different procedures. It is essential that the teacher work with the installation chosen so that he can determine for himself the special procedures that must be taught the students in addition to the text material.

4. What method will be used for key punching cards?

There are four possibilities: (a) have students key punch their own cards; (b) have students use mark sensing equipment to produce their own cards; (c) have key punching done commercially; (d) have students who are receiving special training as typists or key punch operators do the key punching. A student who does not know how to type can waste hours trying to key punch his programs. Mark sensing equipment does not appear to be perfected to the stage where it can process the cards for a class of students (or a school) without breaking down just when it is needed most. Comercial key punching adds an extra cost. The best approach would be to work out a cooperative effort between students programming and students learning to type or key punch so that most of the key punching would be done by those students who will actually need this skill. The working out of the problems involved in such a team effort would be an excellent experience for the students.

The curriculum of the secondary schools is already affected by the influence of computers. We as teachers of mathematics should be aware of the effect this may have on future courses and how we can best use the computer to further our two main goals of teaching students and teaching mathematics.

TRANSFORMATION GEOMETRY IN GRADES IX AND X

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My paper on transformation geometry is based on a mixture of ideas from a new course of study in Ontario, some topics I taught to a Grade X class last year, reading of various European programs, some observation of classes overseas and discussion of school mathematics with overseas teachers during a visit to Great Britain and Denmark a year ago.

In Ontario, we have been working for several years on revisions of our program in mathematics for all grades, with possibly the greatest attention being given to algebra in the secondary schools. We have introduced what might be called a number-structure approach to mathematics. This involves careful attention to the number systems and a much earlier introduction of the concepts of relation and function.

Although geometrical topics in the Ontario curriculum were not ignored in the process of changing the program, the changes in geometry tended to be modifications of Euclid's traditional approach; the work in Grades VII and VIII stresses ruler and compasses constructions; definitions are based on sets of points; also at a time when we were introducing some of the axiomatic-deductive approach into algebra, it was assumed by some that the treatment of deduction in geometry should be made more rigorous. There was increased emphasis on the style of writing solutions of deductions and the authorities for statements. In my opinion, we were mistaken to move in that direction.

There is no doubt that by the time pupils reach Grades IX and X (ages 13 - 15) in our schools, they are ready to be shown how proof works. However, the recondite nature of mathematics can make the task difficult, particularly in geometry. There are so many special properties of geometric figures to be investigated that if we try to prove everything, by the time the pupil reaches "pons asinorum", he may be bogged down in detail and may have lost interest in discovering the main structures of the subject.

The alternative approach which I propose to describe is called transformation geometry. First, I would like to discuss briefly the background leading up to Grade X. For this I will use some details from a new Scottish program for 12-year olds to illustrate the background our 13-year olds should have for a new program. My discussion of Grades IX and X will include references to Danish and English textbooks. Finally, deductions taken from an article by Dr. Jeger, a Swiss, will illustrate how proof can work in this approach.

One reason which makes me feel qualified to say something about a new approach to geometry is that in the Ontario Five-Year Program (college preparatory) at the Grade X level we have just finished a course of study in which we are trying to change the direction of geometry by playing down the emphasis on deduction, introduing three-dimensional work, elementary transformations and vectors.

The unit on geometry will occupy about 12 weeks of the school year. Four to five weeks will be on deduction, where there is not to be a formal organization of Euclidean geometry but the development of short sequences of related theorems. One to two weeks will be spent on three-dimensional topics, three weeks on transformations and three weeks on vectors. In all, this represents a very radical change from past courses in geometry in our system.

Perhaps I should define some of the terms. A transformation may be defined as a correspondence between geometrical objects. In the simplest cases, the definition may be defined by stating that it is a 1:1 matching of sets of points. The bringing of transformations into geometry then means that we are introducing the concept of relation and function into geometry as a unifying theme. I suppose the words transformation and mapping are interchangeable. What are the advantages of the transformation approach?

In geometry we study relationships in space and the significant properties of geometric figures. "Traditional geometry lacks a methodology which is anchored to spacial reality. Logic may stretch through the whole edifice like a colored thread, but it is not satisfactory, because it is not typically geometrical." I am quoting here from an article written by Dr. Max Jeger of the Kantonschule in Lucerne, Switzerland, translated for the magazine *Mathematics Teaching* by Irene Hertz. The article is titled "The Present Conflict in the Reform of Geometry Teaching". Dr. Jeger reviews the history of the development of the traditional course in Euclid and criticizes its present state. He says: "Every generation has absorbed thousands of small details to such an extent that new features can hardly penetrate. Everything to the smallest detail has been thought out in Euclid's edifice; there is hardly any room left for the teacher's contribution in substance or method."

Such criticisms of the traditional course in Euclid are not new. Comparatively new, however, is an attempt to replace Euclid by a workable alternative, and not just to modify the old approach.

The transformation approach to geometry is due to a redefining of the subject, initiated by Felix Klein, the eminent German mathematician who lived from 1849 to 1925. Klein not only criticized Euclid but showed a method of moving away from Euclid in his famous "Erlangen" program.

Klein's method is one of sorting the properties which are important from the welter of detail in geometry and making them stand out. His definition of geometry is that it is the study of those properties of figures which are invariant (unchanged) under certain transformations. Implicit in his definition is emphasis on a more constructive approach at the basic levels, with the shifting of the axiomatic approach to a higher level.

The simplest geometrical transformations are reflections, rotations, and translations. Each of them preserves distance; they are called isometries, or rigid motions. The image is congruent to its pre-image. These three simple correspondences have, within them, all of the main structures of introductory Euclidean geometry, which is the study of rigid figures.

I have chosen to talk mainly about Grades IX and X because of my interest in the secondary field and recent work on a Grade X curriculum. I refer to ages 13-15 in the college preparatory course, in a system where mathematics is taken by nearly all of the pupils. However, it seems to me that there are some important prerequisites if this approach is to be successful. Symmetry has an important role in this kind of geometry. What sort of course should be given before Grade X so that the pupil is prepared for transformations? I hasten to add that probably some of the topics mentioned above in the new Grade X could be started much earlier; however, let us assume that we have just been transformed.

The Scottish experimental program was introduced to first-year secondary school pupils (12-year-olds) in September, 1964. A year ago I visited a few schools and talked to some of the teachers involved in this experiment. You may be interested in some details. In the geometry section of the program, the stress is on figures, beginning with the special ones: rectangles, squares, cuboids, cubes. Drawings of figures on a grid are used to clarify concepts. (Figure 1, page 57).

A question I heard asked more than once was: "In how many ways may a certain figure be fitted back into its hole in the plane, or into its hole in space?" This property of the special quadrilaterals and triangles is related to their axes and centres of symmetry and the number of ways in which they can be folded along these axes.

Throughout the Scottish work there is stress on "tiling the plane" with different figures. A rectangle is defined to be that figure which (a) can be used as a tiling agent to cover a flat surface without leaving any gaps, and (b) is such that each tile can be fitted into the shape of its own outline in four different ways.

Several concepts evolve from the study of rectangles in this manner: 1. the right angle; 2. the diagonals of a rectangle have equal lengths; 3. the diagonals of a rectangle bisect each other. If we begin with any triangle, we can develop the tiling of the plane. This brings out the sum of the angles of a triangle and the equality of alternate angles in a Z-diagram (Figure 2, page 58). Similar ideas can be developed from working with parallelograms of any shape. Note also how such a design can be used to discover intuitively the equal ratios of the lengths of the segments formed from the sides of a triangle by a line segment parallel to one side of the triangle (Figure 3, page 59).

Coordinates are introduced early to assist understanding by locating the vertices of figures. The right triangle is derived from the rectangle; their areas are related to the grids on which they are drawn. Other figures studied in the first year for their symmetries are the isosceles triangle, the equilateral triangle, the rhombus and the kite. (The use of the name "kite" for an isosceles quadrilateral seems to illustrate accidentally a pedagogical principle of the approach, namely the description of the global qualities of a geometric design as opposed to the analysis of its elementary components.

The very sketchy outline given here is based on the textbooks of the Scottish Mathematics Group *Modern Mathematics for Schools*, published by Blackie and Sons in Glasgow and London, and by Chambers. The main features of this program in geometry are emphasis on the physical manipulation of real things, apprehended globally, used before analyzed, and often special, rather than general.

Going on from the elementary level, how would you employ symmetries to introduce some of the usual geometrical ideas? Let us look at a Danish textbook for a moment. The Danish school system generally consists of three levels: elementary, ages 7-14; real skole, 14-15; and gymnasium 15-18. In a recently published textbook for the first year of real skole (age 14), the following topics are studied in the order given:

1. reflection in a straight line,
2. definition of perpendicular,
3. definition of parallel line segments in terms of a common perpendicular,
4. definition of perpendicular bisector in connection with reflection,
5. reflections of line segments and angles and use of this to define bisector of an angle.

At the end of this particular section the summary states:

A reflection in a straight line (a) carries a point over into a point, (b) carries points on the axis of reflection into themselves, (c) carries a line into a line; if a line cuts the axis of reflection in a point, the image also passes through this point; if a line is parallel to the axis, its image is also parallel to it. A line segment is carried into a line segment which is congruent to the first; the line segment which joins a point with its image is perpendicular to the axis and bisected by it. An angle is carried over into an angle which is congruent to the first; an angle is carried over into itself by a reflection in the line on which the bisector of the angle lies.

The next section discusses the circle as a locus and reviews its parts, but stresses reflections and symmetries also. For example, one question asked is, "Which circles are carried over into themselves by a reflection in a given line?" Congruent arcs and chords are developed by reflection in a diameter; the properties of intersecting circles and common chords follow very nicely from this approach. The measure of an angle is associated with a circle drawn with the vertex as centre, and the division of the circle into 360 congruent arcs. The rotation idea of angle is associated closely with transformations from the beginning; for example, the rotation of 180 degrees is equivalent to reflection in the vertex.

This is a brief sample of the discussion in a Danish textbook which includes the three rigid motions and summarizes their properties. The approach is constructive and any proofs given are informal, applying the basic assumptions for the transformations.

The congruence motions and other transformations may be used to introduce a study of the main traditional topics of introductory geometry: congruence, parallelism, area and similarity. I have tried a little of this with a Grade X class before moving into a fairly traditional axiomatic-deductive treatment. I got some of my material from *Some Lessons in Mathematics* and *School Mathematics Project, Book T* published by Cambridge University Press. Here are some samples:

Translations

A translation is movement of the plane in a particular direction without turning. Its basic properties can be illustrated very well on a grid (Figure 4).

If we use a cartesian coordinate system and represent the translations by column vectors in order to distinguish them from vertices of figures, then it is easy to show the group properties and extend the concept to three-dimensional cases.

The mention of group causes me to digress for a moment. By its simplicity and pervasiveness, group is certainly one of the best unifying themes in mathematics. Without being too formalistic about it, we can make the characteristics of a group quite clear in the case of translation vectors.

Reflection

Reflection in a line is analogous to reflection in a physical mirror. The image is on a line through the pre-image perpendicular to the axis, such that the axis bisects the line segment joining a point A to its image A' (Figure 5).

For both of these cases we stress the congruence of the image and pre-image and see the difference between congruence in the direct and opposite sense. Have you read Hermann Weyl's beautifully illustrated lectures on Symmetry? The first two are included in Newman's *The World of Mathematics*. They are well worth reading.

Rotation

This involves rotation of the plane, counter-clockwise about a given point (Figure 6). In the special case of a half turn, it is equivalent to reflection in a point.

By using coordinates we may discuss different types of reflection in the axes and the origin:

1. Reflection in O X : $(x,y) \to (s,-y)$,

2. Reflection in O Y : $(x,y) \to (-x,y)$,

3. Reflection in O : $(x,y) \to (-x,-y)$.

In my trial of this material we discussed the positions of vertices of squares, rectangles and equilateral triangles when rotated about the origin.

The case of the 45° rotation of a square made a nice little application of the Pythagorean theorem, as well as a test of the pupil's awareness of the symmetry of the figure. As with the other transformations, the congruence of the image and its pre-image were stressed.

One topic which might be discussed is the presence of invariant points. For each transformation, are there points which do not move? In the case of translation there are none; under reflection, the points on the axis are invariant; in rotation, the centre of rotation remains fixed.

The transformations discussed above can be used to give unity to the part of introductory geometry, usually called Book I. The main property exhibited is, of course, the rigidity of the figures.

We go on now to transformations which give images <u>not</u> congruent to the original figures but which do have other invariant properties.

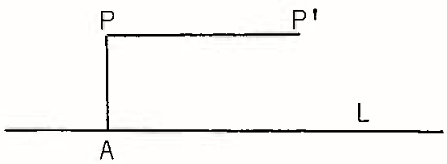
<u>Shearing</u>

A shear is a transformation with the following characteristics:

1. There is a straight line L which does not move.

2. Every other point P is carried into a point P' such that segment P P' $\parallel$ L.

3. P P' = k (P A) where P A is the distance of P from L, measured in some suitable direction, and k is a constant.



The uniform displacement of the cards in a deck gives a good illustration of the nature of this transformation. Of course, area is preserved; the principle involved is basically the one used in pre-calculus methods of developing formulae for the volume of various solids, such as the cone and pyramid.

In the case of rectangle ABCD which maps into parallelogram EBCF under a shear:

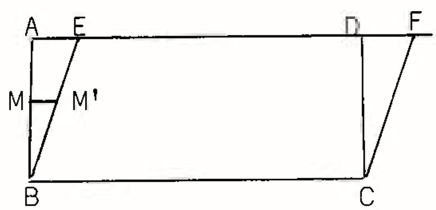
1. Suppose that AB = 6 units and AE = 2 units

2. Then AE = $\frac{1}{3}$ AB, i.e. k = $\frac{1}{3}$

3. For M, the midpoint of AB, M$\rightarrow$M'

4. $\frac{MM'}{MB} = \frac{AE}{AB}$ due to similar triangles

5. Hence MM' = $\frac{1}{3}$ MB.



Dilatations

A dilatation (or enlargement) is a transformation with the following characteristics:

1. There is a fixed point O, called the centre of dilatation;

2. If P' is the image of P, then P' lies on OP;

3. For a particular enlargement, $\frac{OP'}{OP}$ is constant, where OP and OP' are directed lengths; the constant is known as the scale factor of the enlargement.

A very good introduction to this whole topic of dilatation and similarity can be made by using a graph and coordinates. If one side of the figure is made horizontal, it is easy to see by calculation that the ratio of areas is not the same as the ratio of sides. The calculation of sides by the Pythagorean theorem gives valuable information, too (Figure 7). The invariant property here is, of course, the shape; we see this in the preservation of angle size and the equal ratios of the lengths of corresponding sides.

In summary, these are the transformations which could be best used to give life and movement to the introductory geometry. I believe that authors of textbooks using imagination could write new courses which would revolutionize our teaching of the subject. None of this material is new; it just needs reworking and simplifying for school use. For this purpose, a book I found most illuminating is *Introduction to Geometry*, by H.S.M. Coxeter, University of Toronto, published by John Wiley and Sons.

This whole question of the geometry has been bothering us in Ontario a great deal. As you may have heard in other sections, the Ontario Mathematics Commission has a committee on Geometry (K-13), which has been meeting regularly since January with the financial support of the Ontario Curriculum Institute. I believe that the line they are taking is similar to some of the ideas in my talk.

In his article in *Mathematics Teaching*, Dr. Jeger very strongly makes the point that there is not time to do geometry the old way and the new way also. On the other hand, if we are to use the transformation approach, we must be prepared to set up an axiomatic-deductive system at some stage and teach the nature of proof. Jeger states that the axioms in this sytem would be more powerful than those we have ordinarily used. I am going to take the liberty of using a couple of his proofs to illustrate deductive methods in motion geometry.

Example 1. Required to prove that $\angle$ ACB = $\angle$ ADB (Figure 8). Analysis: the main feature of this proof will be movement of the $\angle$ACB around by a rotation of the plane about the centre of the circle so that the arms of the image are parallel to the arms of $\angle$ADB.

Let us look briefly at the way arcs and chords are placed in a circle. Parallel chords are placed symmetrically with respect to a diameter through their midpoints. Now, if a chord AC is moved to a new position A'C' which is parallel to the chord AD, what can we say about the position of C'? It is easy to show that C' is at the midpoint of arc CD.

Now we have a method of proof. Let MN be the perpendicular bisector of chord AD, meeting the circle at O. Rotate $\angle ACB$ about the centre of the circle, until C', the image of C, lies at the midpoint of arc CD. Then, referring to the lengths of the arcs, we may say arc AA' = arc BB' = arc CC' = arc C'D.

Since arc AO = arc OD

and arc A'O = arc OC'

then A' is the image of C' under reflection in MN and vice versa.

Therefore, A'C' is bisected by MN and is parallel to AD.

Similarly, by using the perpendicular bisector of DB it may be proved that B'C' is parallel to BD.

Hence $\angle A'C'B'$ may be mapped onto $\angle ADB$ by a translation.

Since $\angle A'C'B' = \angle ACB$, therefore $\angle ACB = \angle ADB$.

<u>Example 2.</u> Here is a construction which may have puzzled you at one time (Figure 9).

Given: any four points A, B, C, and D.

Required: to construct a square with each side passing through one and only one of the four points.

Analysis: I believe that difficulties I experienced with this problem were caused by my failure to recognize the symmetry of the figure. The parallel sides of the square form two equidistant bands. A rotation of the square (or of the points) through 90° would give equivalently placed image points. Let M be the centre of the square. If we make a rotation of 90° about M, A, B, C, D, maps into A'B'C'D' on the adjacent sides. AC and A'C' will be perpendicular segments of equal length due to the symmetry of the "bands". Translate C' to $\bar{C}$ with a vector equal to vector A'B. Then vector $\overline{B\bar{C}}$ = vector $\overline{A'C'}$. $\overline{B\bar{C}}$ is perpendicular to segment AC and equal in length to AC.

This gives us a construction which can be drawn through the given point B, determining the side of the square through D.

<u>Conclusion</u>

The purpose of my paper has been to show how geometry may be conserved as an essential element of the teaching of mathematics by giving it new relevance, life and movement.

The use of transformation brings the function concept into geometry and, incidently, helps to clarify function and mapping by associating it with physical motion.

I am inclined to agree with Dr. Jeger that it will be successful only if we go all the way with it. Will our new crop of teachers see here a method they like so well that they use it in its full power? Or will they fail by trying to "ride two horses?"

The rigor of presentation is important. We have had difficulties over this before. Often new methods are devised by research mathematicians when they are playing the axiomatic game and being quite obtruse. Then we in the schools, mistaking "shadow for substance", condemn the new concept because we saw it first when it was couched in abstract terms. The mathematician has a responsibility to make the concept real for the schools. How well this is done will determine our teaching success. I hope that some of the examples I have used will make the possibilities of transformation geometry more real for you.

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Figure 1

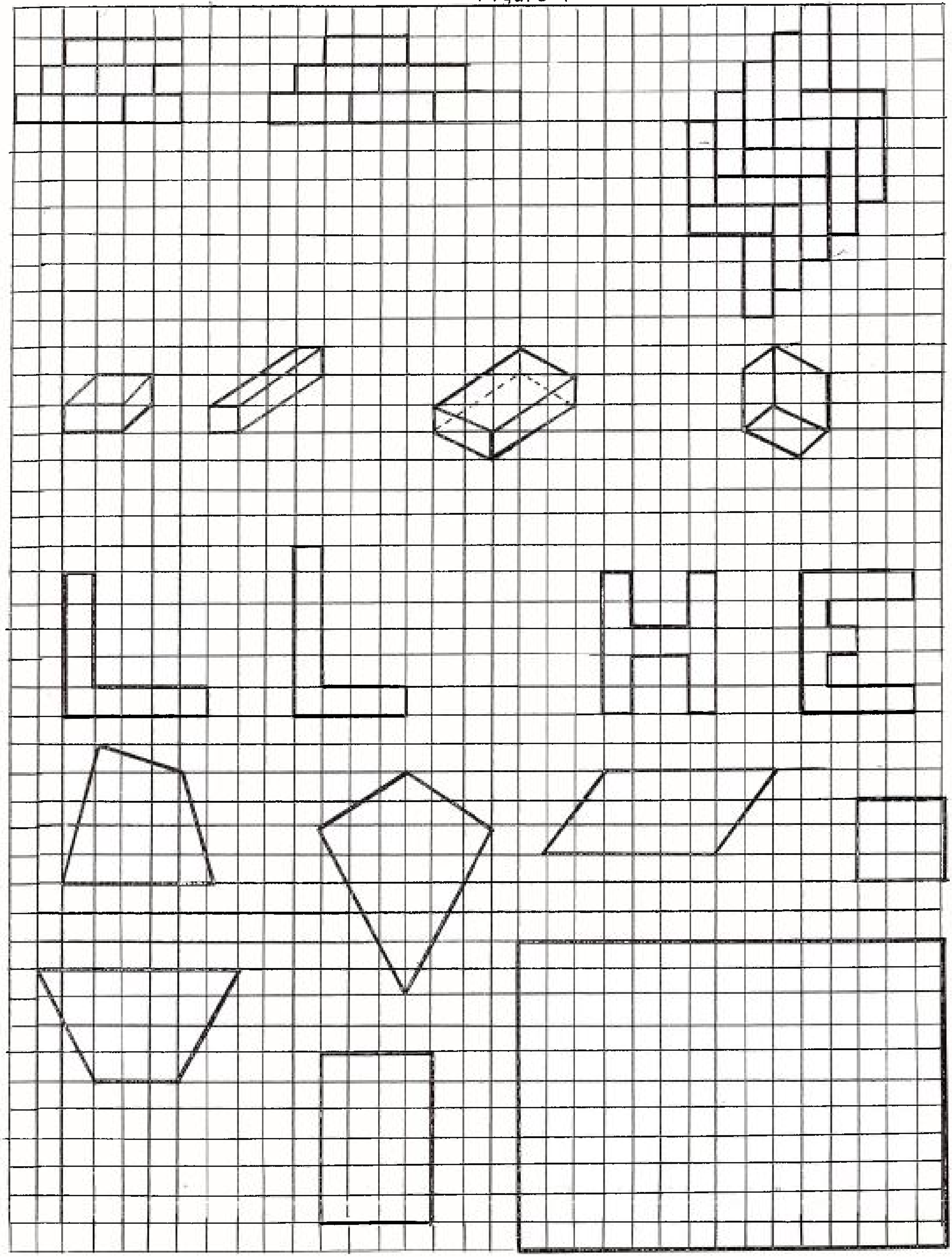


Figure 2 - Tiling the plane

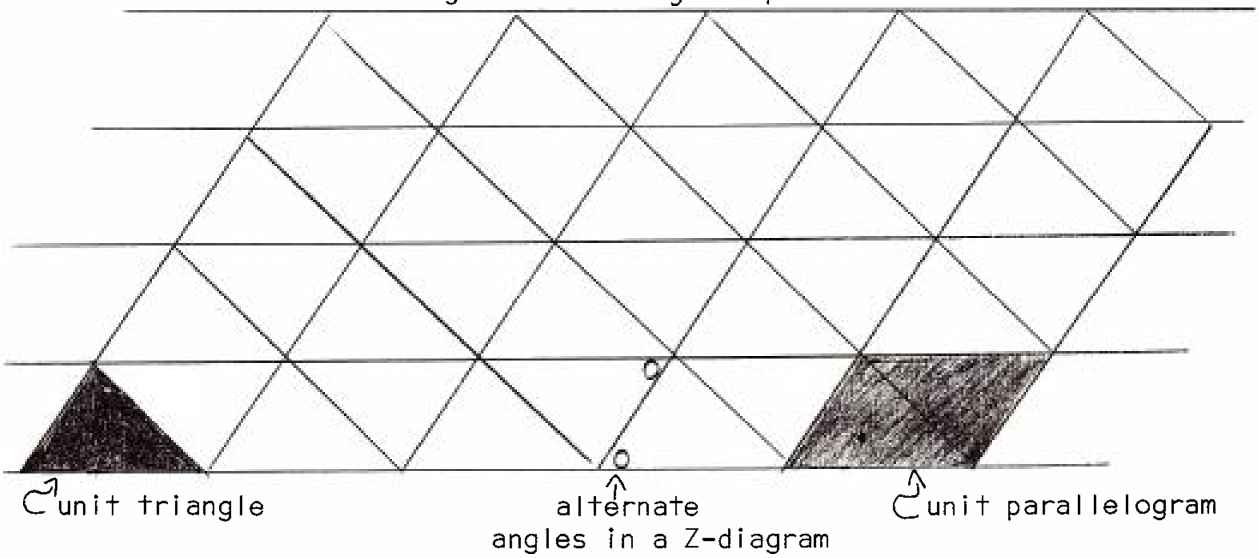


Figure 3 - Tiling the plane

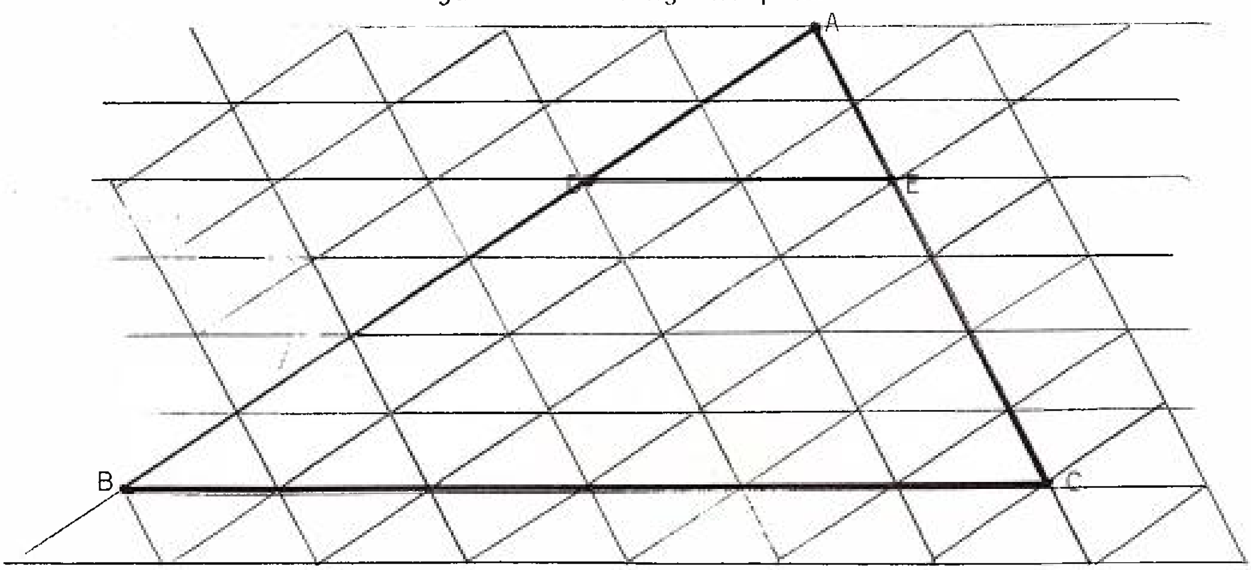
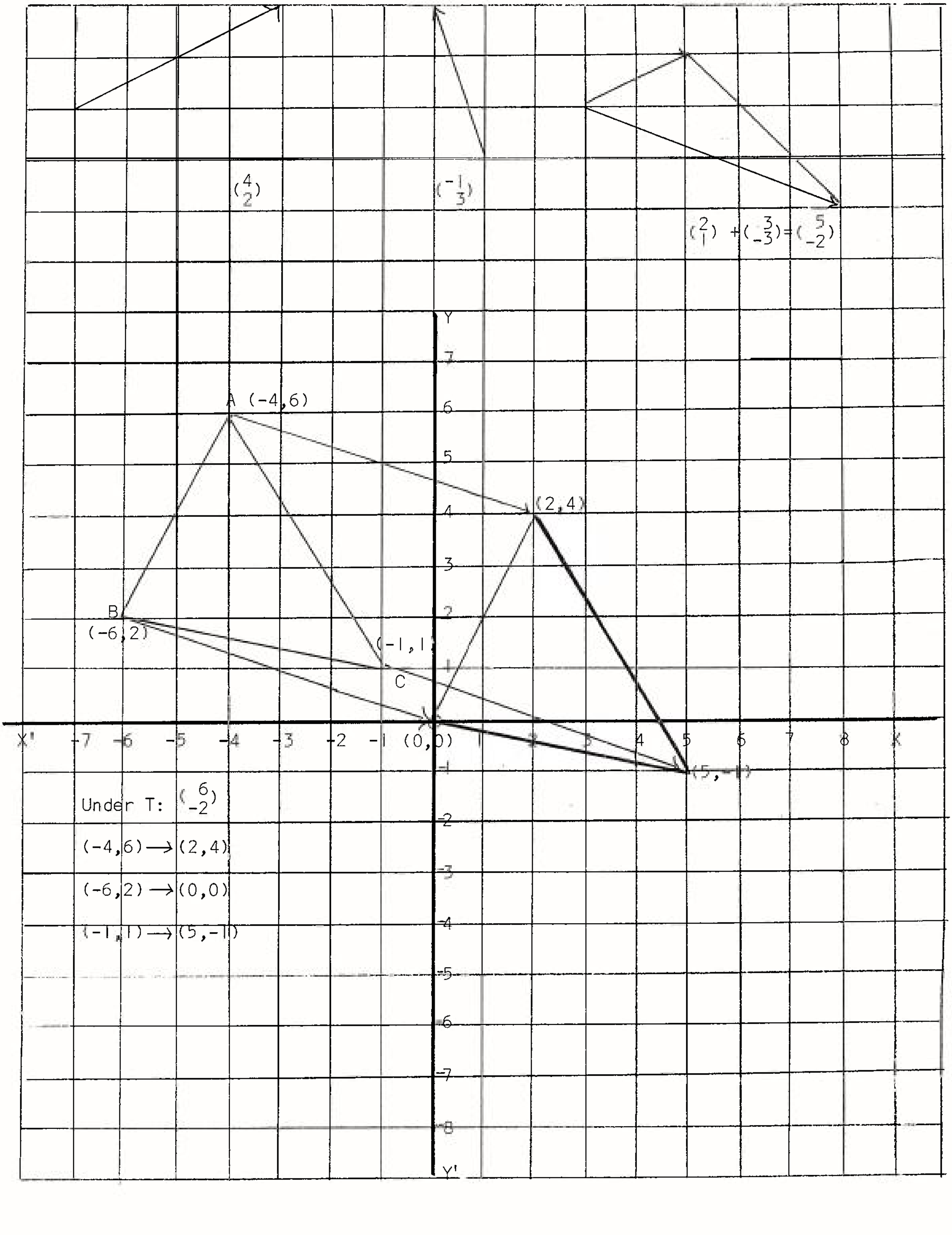


Figure 4



$\binom{4}{2}$
$\binom{-1}{3}$
$\binom{2}{1} + \binom{3}{-3} = \binom{5}{-2}$
Y
A (-4,6)
(2,4)
B
(-6,2)
(-1,1)
C
X'
(0,0)
X
(5,-1)
Under T: $\binom{6}{-2}$
(-4,6) ⟶ (2,4)
(-6,2) ⟶ (0,0)
(-1,1) ⟶ (5,-1)
Y'

Figure 5

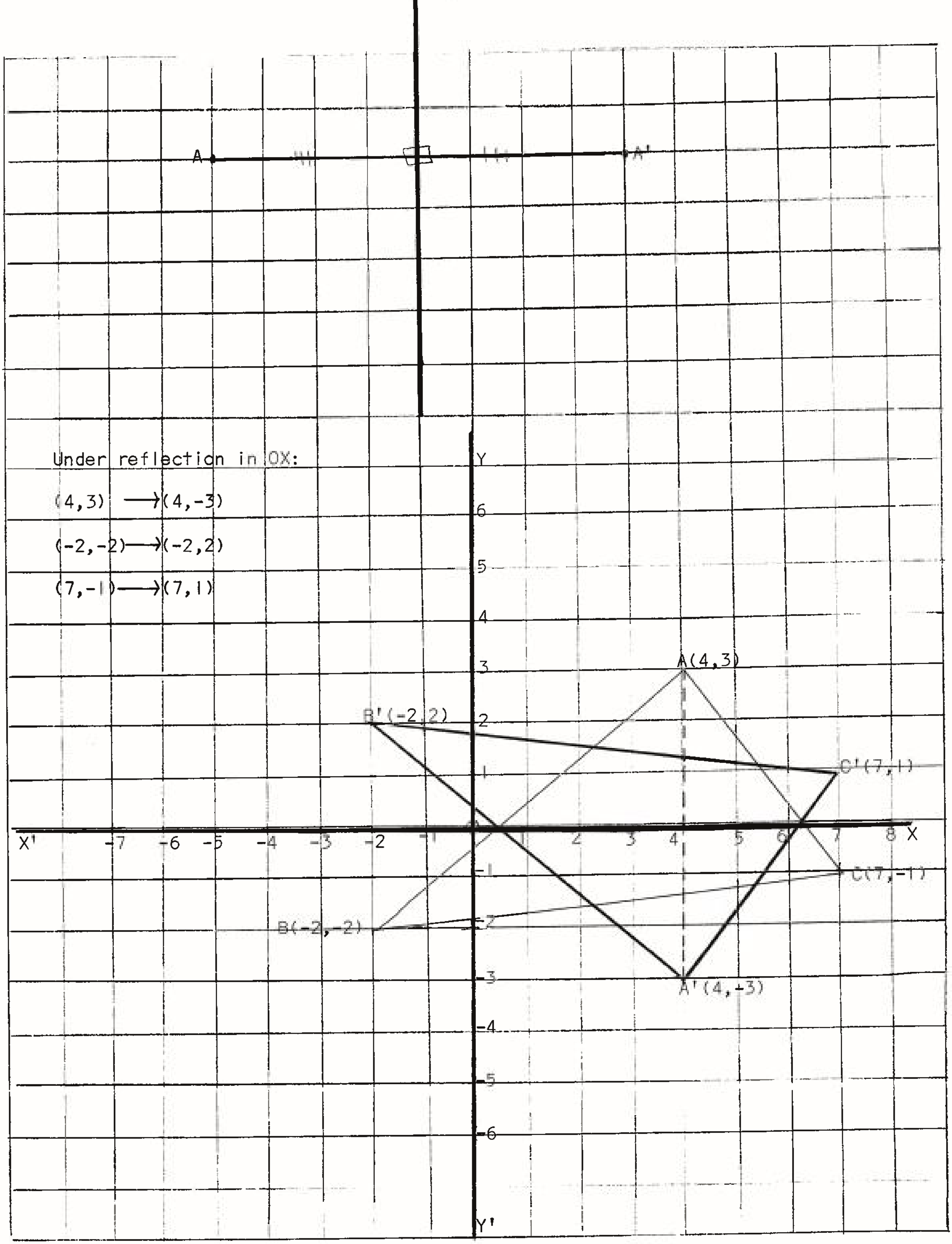

A
A'
Under reflection in OX:
(4,3) ⟶ (4,-3)
(-2,-2) ⟶ (-2,2)
(7,-1) ⟶ (7,1)
Y
X'
X
Y'
A(4,3)
B'(-2,2)
C'(7,1)
C(7,-1)
B(-2,-2)
A'(4,-3)

Figure 6

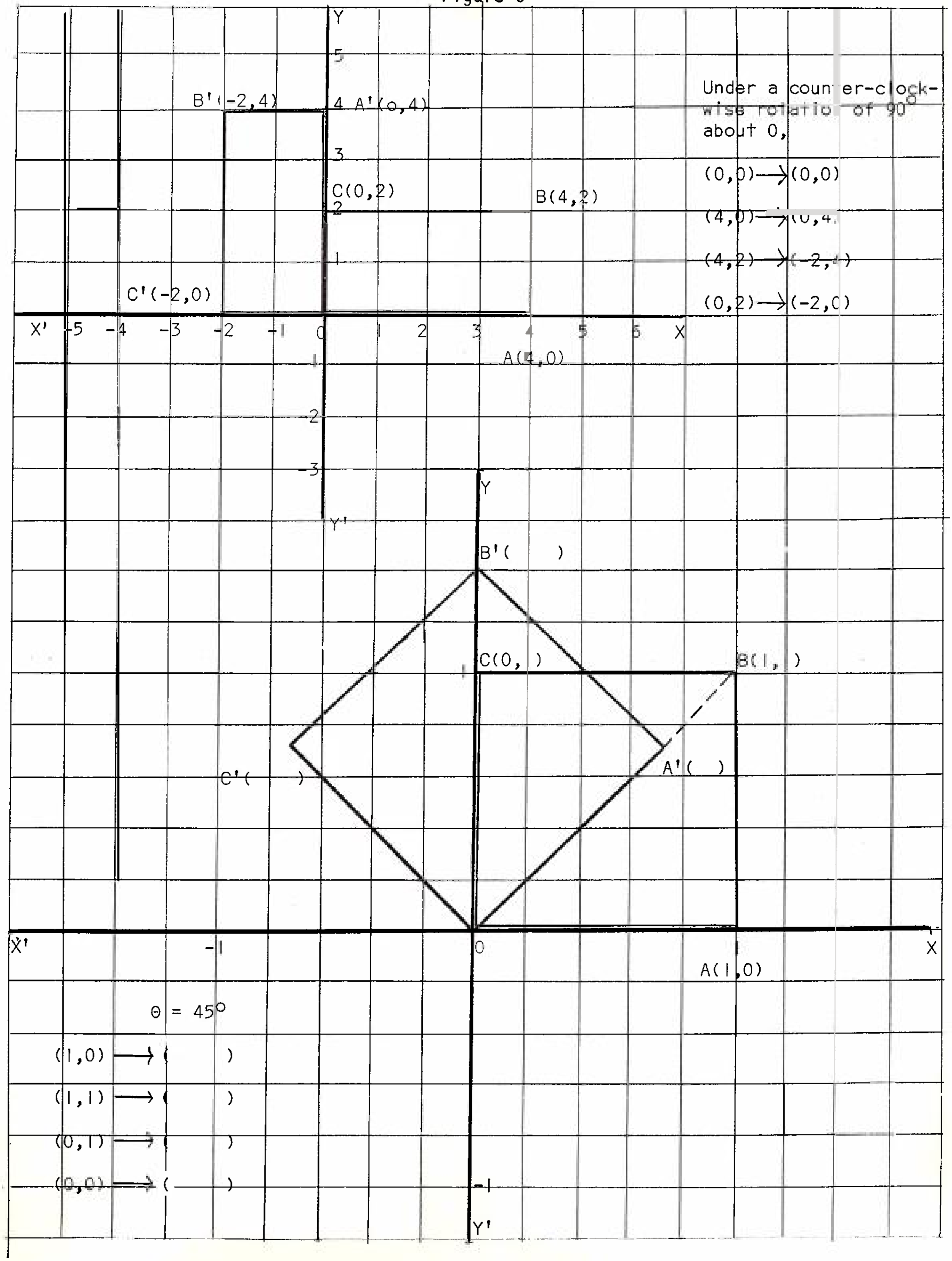

Under a counter-clockwise rotation of 90° about 0,
(0,0)⟶(0,0)
(4,0)⟶(0,4)
(4,2)⟶(-2,4)
(0,2)⟶(-2,0)
B'(-2,4)
A'(o,4)
C(0,2)
B(4,2)
C'(-2,0)
A(4,0)
X'
X
Y
Y'
B'( )
C(0,)
B(1,)
C'( )
A'( )
A(1,0)
θ = 45°
(1,0) ⟶ ( )
(1,1) ⟶ ( )
(0,1) ⟶ ( )
(0,0) ⟶ ( )

Figure 7

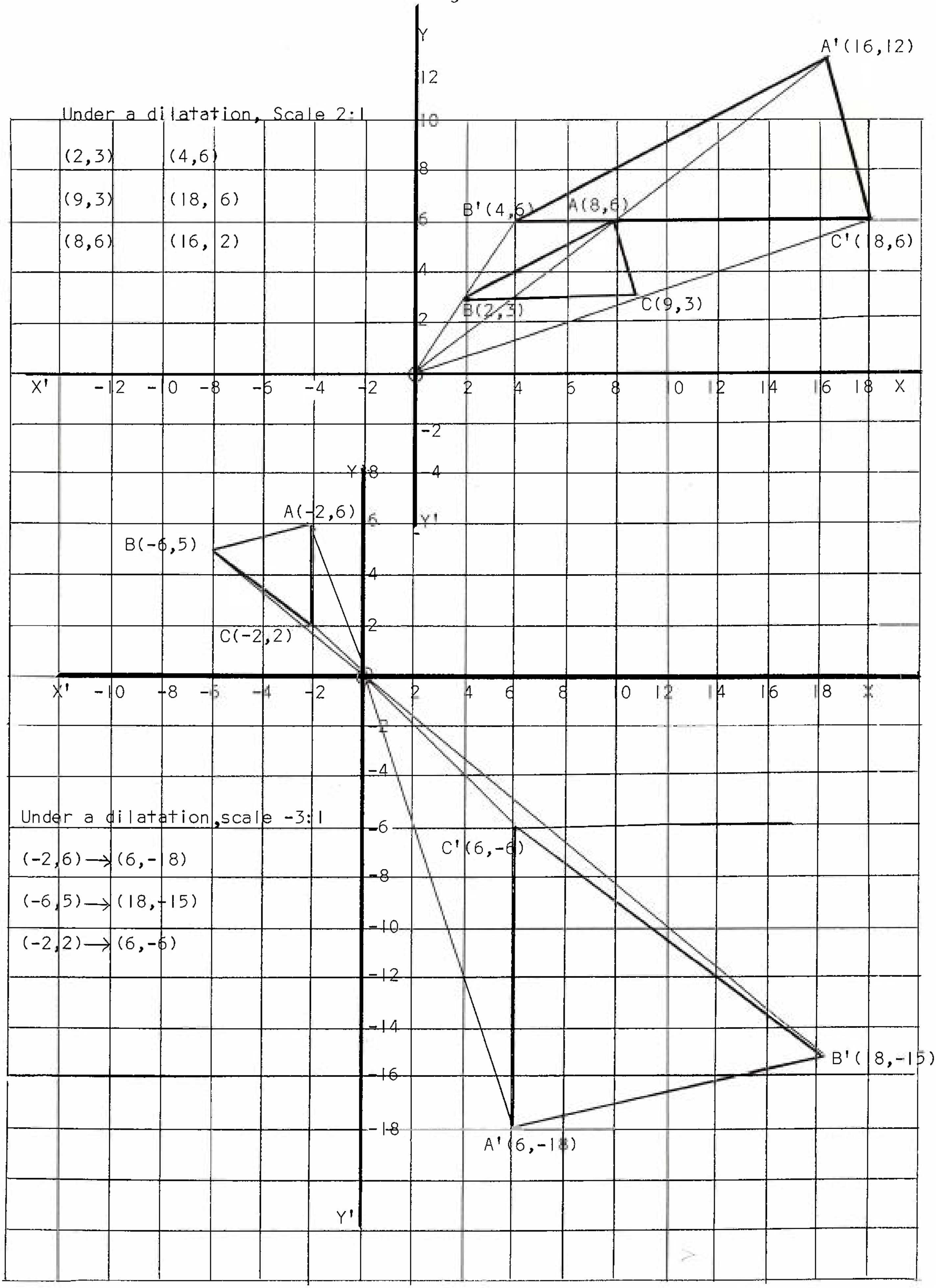

Under a dilatation, Scale 2:1
(2,3) (4,6)
(9,3) (18, 6)
(8,6) (16,12)
A'(16,12)
B'(4,6)
A(8,6)
C'(18,6)
B(2,3)
C(9,3)
Y
X
X'
Y'
A(-2,6)
B(-6,5)
C(-2,2)
Under a dilatation, scale -3:1
(-2,6) → (6,-18)
(-6,5) → (18,-15)
(-2,2) → (6,-6)
C'(6,-6)
B'(18,-15)
A'(6,-18)

Figure 8

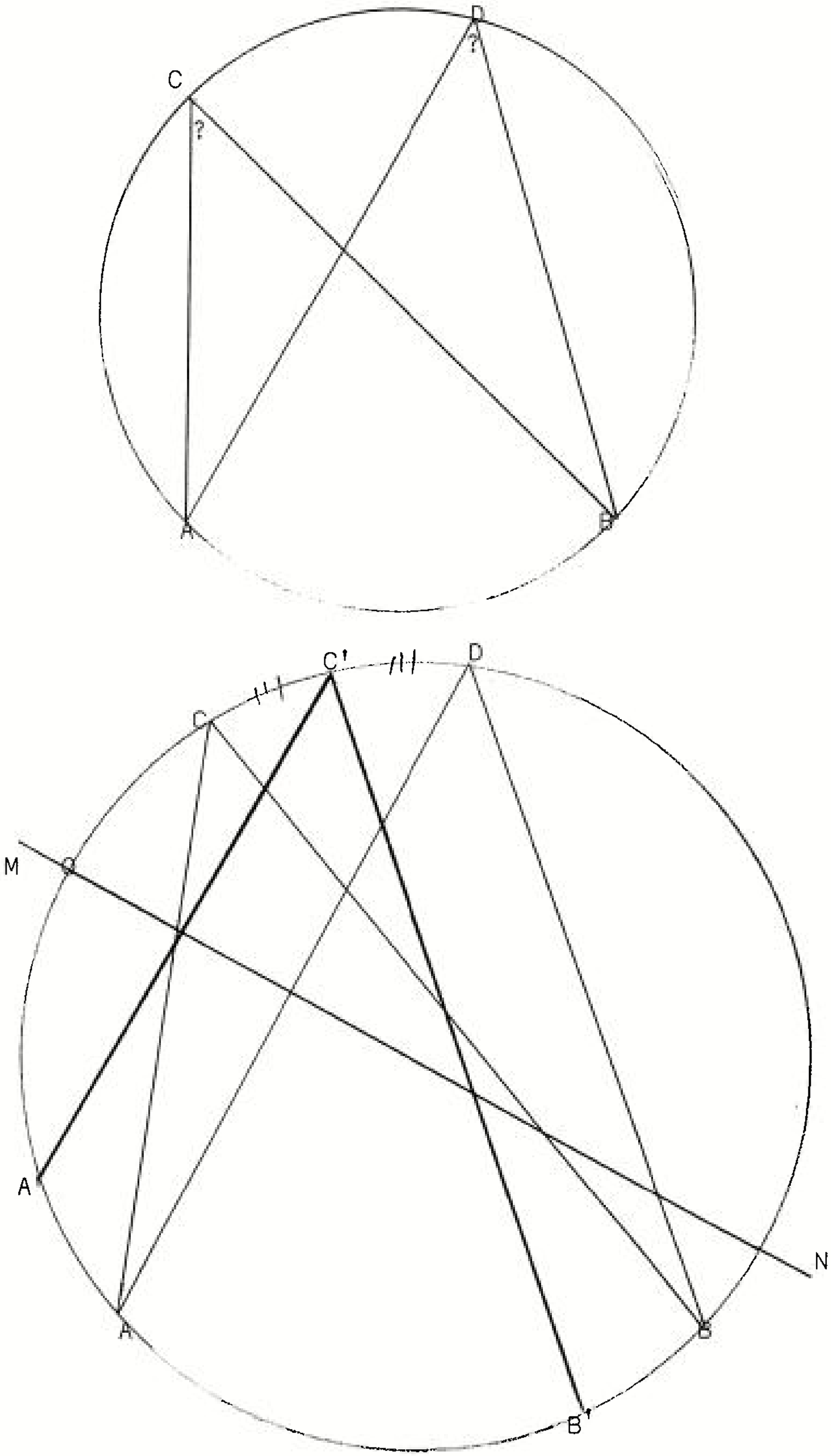
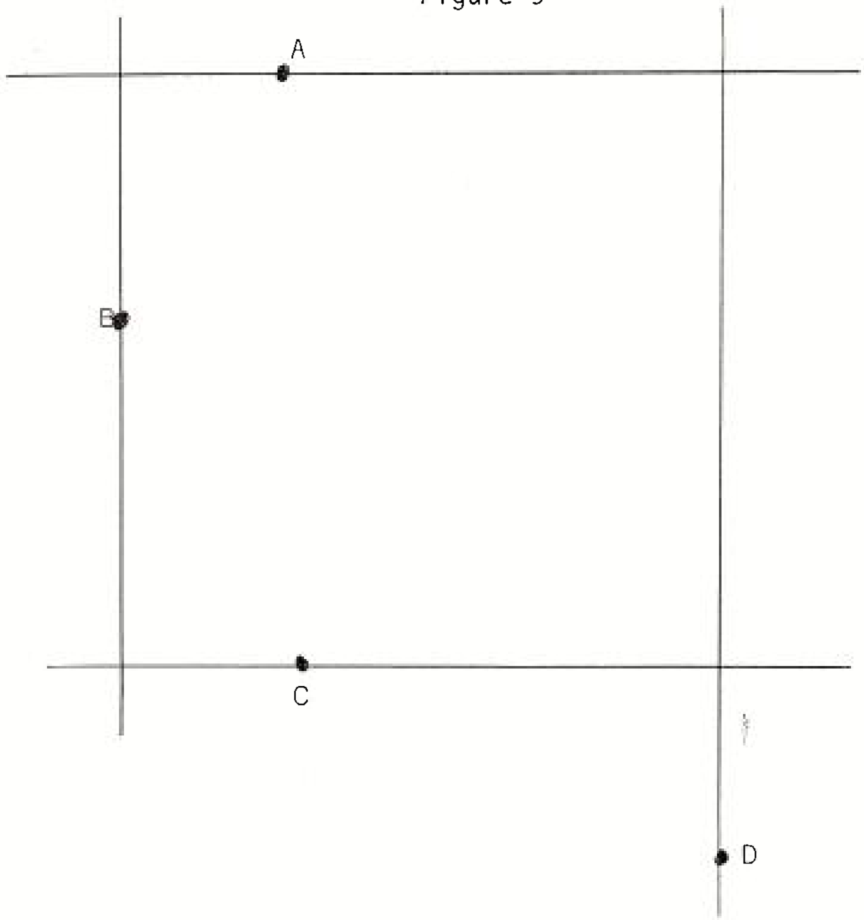
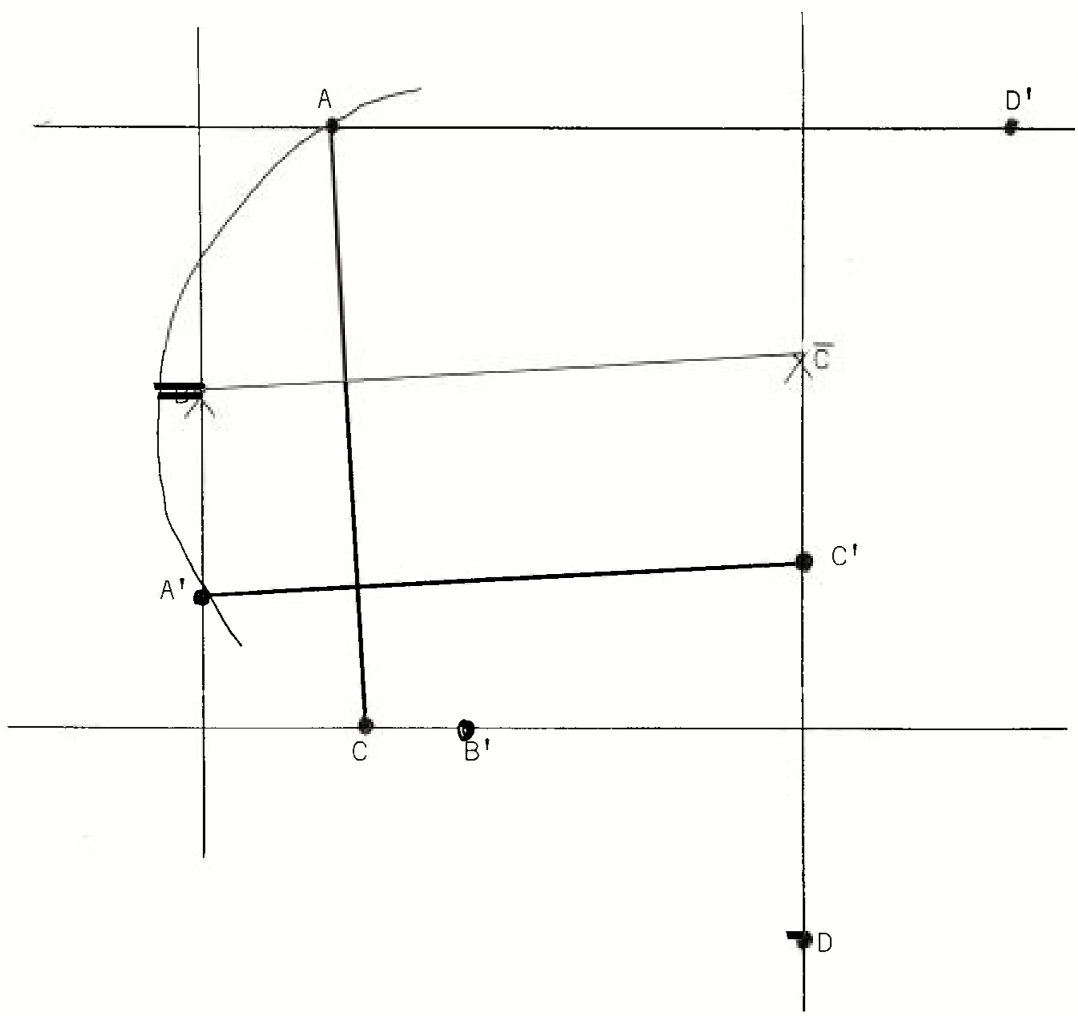


Figure 9





SOME USES OF TV ENRICHMENT TEACHING

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Calgary, Alberta

The writer and others at The University of Calgary have had recent experience with television as a medium for enrichment teaching in mathematics. This article is a report on some of these projects, a subjective appraisal of their effectiveness, together with a few suggestions for other possible applications.

Use was made of TV for the direct presentation of enrichment material to the school student, for use in teacher education classes, and inservice work for teachers.

TV in the School Classroom

Videotapes on enrichment topics were made by faculty members at the University for playback in individual classrooms. It appeared that these were well received by the junior high school students who viewed them. Although there is no reason why the classroom teacher cannot give enrichment lessons on his own - and without the use of TV - the time and effort involved suggests a pooling of resources. Certainly, as TV facilities become more easily available to classroom teachers, teams of teachers with these interests could prepare enrichment lessons and share them with others on a system-wide, or broader, basis. School systems could encourage this by making time available for such teacher participation or, alternately, by renumerating qualified teachers involved. Interested instructors at the University would be prepared to cooperate in such a venture.

During follow-up periods for the presentations referred to above, a TV camera and monitors were used successfully for demonstration in the classroom. There are classroom situations in which TV is clearly superior to other devices, such as the overhead projector, for demonstration purposes.

Although the lack of closed circuit facilities has, as yet, prevented presentation to several classrooms simultaneously, the proposed development of TV facilities in the Calgary area will make this possible in the near future.

Programs in mathematics have also reached the school classroom in Alberta over regular television channels, and some of these have contained a substantial amount of enrichment material. There are now several teachers who have participated in such programs; their experience will be useful in planning for the much heavier use of TV which is bound to come in the immediate future.

Finally, some programs on mathematics televised by commercial stations have been viewed by school students at home. Examples are credit courses offered by the University and The University of the Air series. The mathematics programs of this series consisted mainly of enrichment material.

TV in Teacher Education

TV is proving itself to be a flexible instrument in teacher training programs.

Enrichment lessons can be videotaped in the school classroom and later played back and discussed in a teacher education class. In this way, the prospective teacher may at one time be exposed to new enrichment material, techniques of its presentation, and the response of students to it.

A student teacher who has planned an enrichment lesson finds it useful to rehearse at least part of the material in front of the TV camera and then analyze his own performance when the tape is played back. In spite of the fact that an unresponsive lens replaces a responsive class, student teachers report that the playback of their presentation is very useful to them. TV does "The giftie gie us, to see oursels as others see us!", and this should be helpful to any teacher.

In Calgary, it is not yet practical to videotape a student teacher's actual performance in a classroom, but new demonstration rooms will have remote control and essentially concealed TV cameras which will make this form of self-appraisal more feasible.

The TV videotape has also been used successfully for peer-to-peer instruction. A good enrichment presentation from one year may be preserved on videotape and played back to students in subsequent years. This gives the new group of students dramatic evidence of what they, too, may accomplish.

The student teacher should also view whatever programs and videotapes are available for direct classroom use in order to broaden his background of enrichment topics and to assimilate ideas with respect to their presentation.

With the substantial closed-circuit facilities planned for the Calgary area, and with the proposed link between the new Demonstration School and classrooms in the Faculty of Education Building, TV is to play an increasing role in the preparation of teachers.

TV and Inservice Programs

Some experience has already been obtained at this level. For example, The University of Calgary has offered credit courses in mathematics by TV which were supplemented by seminars. These programs have contained some enrichment material. A major argument for such presentations is that they take the program to the teacher, rather than requiring that the teachers all come to the University with what may be a considerable expenditure of time, effort and money. So far, the programs have been televised before school hours, but other hours should soon be available. A prime viewing time for teachers would be the period from 4 to 6 p.m. The sharing of ideas by classroom teachers, supervisors, professors, and others should surely result in an enrichment of classroom teaching.

TV has already established itself as a useful medium of instruction at the three levels discussed above. Continued experimentation should go hand in hand with any increase in the use of TV for educational purposes. With respect to enrichment teaching in mathematics, it would appear to be reasonable to use TV at first in situations where material is not readily accessible to the classroom teacher or difficult and time-consuming to organize, where teachers qualified to present certain topics are in short supply, or where the visual materials required for presentation are expensive or difficult to construct.

LEARNING AND TEACHING MATHEMATICS

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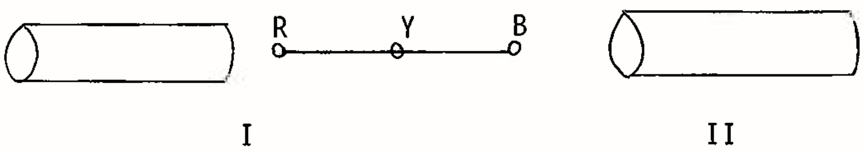
The topic of my presentation seems as old as the hills - learning and teaching mathematics. At least I have not disguised the title and made it into a discovery exercise by calling it LATM for short! My main purpose in choosing the subject is my belief that now and then the classroom teacher needs to stand back and reflect a little on what he or she is trying to do. In dealing with the subject, I want to focus on the two principal actors in the drama, the student and the teacher - in that order - for learning must (at least should) come before teaching.

The Learner

In a recent survey of the views of 25 leaders in mathematics education in the USA by Mayor, one of the most frequently mentioned needs was research in the learning of mathematics. I am not going to pretend to be an expert in this area, but I have found myself drawn more and more towards attempting to understand how children learn mathematics as I became aware of how badly we seem to teach it!

Consider, for example, this simple experiment. Put a red, yellow, and blue bead on a string in that order. Now take the string and push it into a hollow tube.



If you pull it through, which bead comes out first? If you pull it back, which comes out first? Now turn the tube with the concealed beads in it through 180°. Which comes out first now? Which will come out first after 15 turns?

Many children of age 5 or 6 cannot separate themselves from the immediate perception they are having. In this case, they would answer "yellow" quite often, unable to translate their perception into a stable conception. They are still dominated by perception.

This is one of Piaget's experiments on which he bases his theory of mental development. Very crudely, as you may know, he identifies three main stages:

A. intuitive - up to 6 or 7,

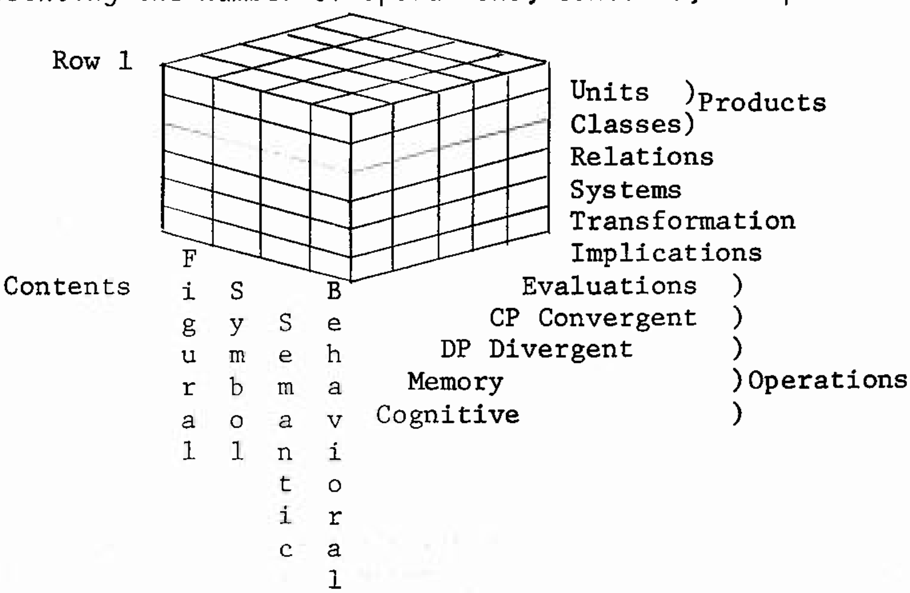
B. concrete optional - 6 or 7 or 11 or 12,

C. formal operational - from 12 up.

The point is that at stage A the child has not learned to free his thought from the world of perception, while at stage B, although he can now reason deductively and has achieved so-called reversibility of thought, he can still only abstract from his immediate personal experiences. It is only when he reaches stage C that he can think truly abstractly and be guided by form, ignoring content. Only now is he ready for if-then reasoning and consideration of the effects of hypotheses unrelated to his own experience. Piaget does not suggest that these stages are immutable, nor that they occur at these precise ages for all children. Rather, they are distinct stages through which mental development passes. They can be accelerated to some degree, but as yet to what degree is unknown. In our teaching, we must take account of these psychological facts, or we run the risk of having students memorize when they do not understand.

Our second look at the learner also concerns his intellect - from a different point of view. Work on the nature and structure of intelligence has been going on increasingly in the twentieth century. In the USA, the names L.I. Thurstone and J.P. Guilford have been prominent in recent research in this area. Guilford and others at the University of Southern California have carried out concentrated and prolonged research on cognitive and thinking abilities, paying particular attention to creativity. By applying the method of factor analysis experimentally, they have developed a unified theory of intellect. Briefly, they have identified factors of intellect which can be classed in exactly one of three ways. One of these is according to the basic operation performed. Thus, a person might be memorizing, or thinking in either a convergent way (towards one answer) or in a divergent way (searching or seeking variety). A second classification is in terms of content, which might be symbolic or behavioral, for example. When a given operation, say memorizing, is applied to a given content, say symbolic, then some kind of product is formed. Regardless of which operation and content are employed, the product is apparently one of six general kinds.

Thus, factors of intellect can be considered as falling into three classes which can be pictured as three adjacent faces of a cuboid of sides 5, 4 and 6 representing the number of operations, contents, and products factors.



If one considers the layer of the model corresponding to cognitive abilities and omits the behavioral dimension, 18 cells remain with each horizontal row of three referring to similar abilities. Fifteen out of these 18 cells do in fact represent known abilities.

What are the implications of this theory for mathematics teachers? A first and obvious one is that it should make the mathematics teacher more tolerant of students' inability to learn. It may well be that the student is weak in a particular cognitive ability. Allied to this is the existence of the possibility of obtaining an ability profile of individual children so that a teacher can teach with more understanding and knowledge of a given pupil's strength and weaknesses.

The discovery of the fact that productive thinking has two dimensions - covergent and divergent - has led to the sustained study of creativity and how it relates to education. Among the results indicated by studies are these:

1. Teachers prefer "intelligent" to "creative" children.

2. Creative students need not have markedly high IQ's. (Torrance claims 70 percent of the most creative are below the 80 percentile in IQ.)

3. Creative students exhibit different values from high IQ students.

It has become increasingly clear that economically (and otherwise) society owes much to creativity. It is, therefore, important not only to identify creative children but to teach them so that their creativity is nurtured. Creative children are liable to identify themselves by asking questions the teacher cannot answer, by producing unexpected solutions, and by seeing relationships that the teacher did not see! Teachers should not stamp on such children but try to provide scope for such thinking. Certainly modern teaching aids and more flexible curricula are strong helps to the teacher in this connection.

Let us now move to the area of motivation, and in particular to the idea of intrinsic motivation. Intrinsic motivation is the term used to account for activities which seem to be self-reinforcing. Such kinds of behavior affect the learner's psycho-physiological state and depend for their reward value on events in the central nervous system. Many avenues of research, among them the study of exploratory behavior, of personality and of attitude change, have combined to reveal the many facets of this kind of motivation. Essentially, researchers have shown that motivating disturbances can result from disharmony among central nervous system processes or conflict, as such disharmony may be called.

Conflict and curiosity have been studied intensively over a decade by Berlyne. His research shows that conflict can be generated by stimulus patterns which possess novelty, surprise, incongruity, doubt, contradiction, or complexity. These are factors which are known to evoke exploratory behavior. Thus, it seems that epistemic curiosity (i.e., curiosity for knowledge) results from conceptual conflict (conflict due to <u>discrepant</u> thoughts or beliefs or attitudes). This type of curiosity is the motivating condition for epistemic behavior which will be instrinsically reinforced and the knowledge from it retained when it resolves conceptual curiosity.

Schematically, we could represent the situation as follows:

Appropriate Stimulus Patterns (novelty, surprise, etc.)
↓
Conceptual Conflict
↓
Epistemic Curiosity
↓
Epistemic Behavior
↓
Gain of Knowledge and Intrinsic Reinforcement.

For example, the Madison Project method called "terpedeing" can produce surprise and contradiction. Consider the student who has worked with quadratic equations such as:

$$n^2 - 5n + 6 = 0$$
$$n^2 - 7n + 10 = 0$$
$$n^2 - 14n + 33 = 0$$

Where only equations with unequal prime roots are used, the student may soon think he has found the pattern by using the product rule alone. An equation such as $n^2 - 11n + 28 = 0$ may then be posed, to which the student gives as roots the set 2, 14, 4, 7. Testing reveals that he is wrong. Surprise and contradiction, and hence conceptual conflict, have been produced. This conflict can now lead to epistemic curiosity and behavior and finally to a knowledge of the need to include the "sum" rule. Intrinsic reinforcement now occurs and the conceptual conflict is relieved.

<u>Curiosity and Discovery Methods</u>. Recently, intrinsic motives and rewards have been widely used in connection with so-called discovery methods. A wide variety of forms of conceptual conflict associated with curiosity have been used including those already mentioned, to which others such as perplexity could be added. Curiosity as a powerful force in learning has come under fire from Friedlander, who points to a number of reasons why curiosity "may operate only marginally as a motivator and <u>incentive</u> for academic learning". He notes that it is often very unsystematic and noncumulative, may be satisfied in ways other than those anticipated by the teacher, may be strongest in relation to non-academic issues, and varies markedly in "degree" from one child to the next. He concludes that we should continue to use carefully the extrinsic rewards such as approval and the achievement motive, as well as the intrinsic rewards associated with discovery methods.

To what extent are discovery methods preferable to more conventional methods? The initial flush of somewhat naive enthusiasm (in retrospect) for such methods is beginning to fade to a more acceptable shade (of pink?) as researchers of renown have added their quota of qualifications. Ausubel has pointed out that there is much fuzzy thinking about the discovery method and

The Teacher

Discussion of discovery methods has led us to consider the teacher. We have seen that most of the new emphases in teaching have come from psychology and technology. Among recent prominent developments, two in particular might be singled out for attention. The second is essentially an extension of the first. I am thinking of programmed instruction and computer-assisted instruction. The name "programmed instruction" indicates what it is - a carefully worked out program or sequence of instruction. The advantages claimed for this method of learning are numerous, but three at least merit serious attention. They are:

1. the active involvement of the learner,
2. immediate reinforcement of correct responses,
3. individual pacing of learners.

These features reemphasize basic principles of sound learning, and the teacher must recognize that acceptance of these principles implies a shift in his role. I will return to this point later.

One of the more recent developments of programmed learning is the emergence of what is called non-verbal or wordless programming. M. Daniel Smith is experimenting with sequences which do not use words and, therefore, in some sense shape the ability to employ non-verbal mediating processes. I have no details of the program used other than that initially they dealt with vectors and their application to directed numbers and coordinate systems leading to matrices. However, the research has apparently produced unusual branching procedures and statistically significant advantages in learning.

I myself have employed what are essentially programming techniques in designing a series of 40 three-minute 8 mm loop films on school mathematics for the National Film Board of Canada. These films are also non-verbal and rely for their development on programmed visual sequences into which visual cues (mainly color coding and flashing) are woven when deemed advisable. Each film deals with a basic idea in mathematics and aims to present the viewer with a visual mathematical experience from which the viewer must extract the significance. The films are designed to be open-ended in the sense that they end by posing an unanswered problem or stimulate the viewer to further exploration of the ideas presented. The loop nature of the film allows for individual learning, and in many cases the material in the film is capable of being perceived at different levels of sophistication. Thus, a given film can provide an illustration of how the same mathematical relationship develops new meaning as one's knowledge of mathematics grows.

As I said earlier, all the major developments in knowledge about learning and teaching underline the need to consider the characteristics of the learner. Another requirement of major importance in mathematics teaching in my view is that mathematics teachers at all levels understand the nature of mathematics. If they do not have some reasonable grasp of its twin aspects of conjecture and proof, of exploration and formalization of results, they are not likely to be able to communicate this basic "truth" about mathematics to their students. The current reemphasis in laboratory learning which has reached Ontario via the Nuffield Mathematics Teaching Project in England is very important in this connection, for if wisely assimilated into the teacher's approach to teaching, it will greatly foster the appreciation that mathematics in the last resort draws its data from the real world and often has to define its problems before it can begin to consider how to solve them. I believe that the 8mm loop film has a significant part to play in this respect, too. The film forces the viewer to decide for himself what its meaning is and hopefully uses one or the other of the conceptual conflicts mentioned earlier to instruct the learner intrinsically. In a sense, each film may be said to depict mathematics in miniature.

We see, then, that programmed instruction need not be conceived only as words on a printed page. It is highly desirable that instruction be programmed if it is to be successful, but there is no one way or medium in which this must be done. Computer-assisted instruction can be understood as a logical development of programmed instruction and can even be designed (theoretically at least) to make provision for the student to carry out experiments in, say, geometry.

The advent of self-study and individual learning techniques I have described above mean that the teacher must face the prospect of drastic changes in his role. As Coulson says:

> His job will no longer be primarily to present information and to drill students. He will spend most of his time in diagnosing individual learning problems, remedying them in close tutorial interactions with the pupils, and leading group discussions.

Such drastic change in role cannot come overnight, nor should it even if it could. We are human beings - we need time to modify our learning and our attitudes. We need time to examine these new tools of learning free from undue pressure and anxieties. We need time to learn more mathematics and reevaluate our thinking and teaching generally. In moving to meet the changes that science and technology require teachers to make, I hope that wise planning will prevail. Above all, we must avoid situations where teachers of mathematics are teaching mathematics whose structure and value they barely understand. To teach well, the teacher should understand not only the content but its inner logic and the concepts on which it is based, together with the psychological makeup of the learner and how the content can be presented so as to interest the learner and enable him to learn actively and willingly.

To sum up, learning and teaching are complex activities. To teach well, the teacher must study children, their mental development, their intellectual and personality structure, the problem of motivation, mathematics and its content, meaning and structure, and his own role in the learning process. To teach is to learn. The teacher who stops learning soon stops teaching.

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A DISCOVERY LESSON

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Discovery teaching is mentioned, if not condoned, in every textbook and almost every article we pick up. However, it is seldom defined, the reason being that it seems very difficult to pin it down precisely. In fact, when anyone refers to discovery, he could mean a number of things. In this article I plan to explain a lesson that I taught recently. I have no particular need to call it "discovery teaching", but I think it approaches some of the ideals we have in mind when we think of discovery. I have called it a discovery lesson, although this might be a poor title.

I will now attempt to explain how the lesson was set up and then proceed to draw some educational implication from the lesson. As a precautionary note, I should mention that we conceive that the basic attack on a lesson of this kind is to set up a mathematical situation and allow students to react to it. To do this, one must be prepared to allow students to discuss mathematical ideas with very loose terminology. Eventually, it is of the utmost importance that symbols and concepts be rigidly defined, for this is the essence of mathematics learning; but accurate terminology is out of place in an introductory mathematical lesson.

The concept we dealt with was linear relations in two variables. The objective of the lesson was not to try to break down the concept into very small parts to be analyzed thoroughly, but rather to provide a broad framework in which the students were able to work. One readiness concept which the class (a Grade V class in this case) possessed was that of locating points on a rectangular coordinate system, that is, given a point (4,6) the pupils would all agree on its location. They knew that a number on the horizontal axis was represented by $\square$ and the vertical axis was represented by $\triangle$. And they further knew when an open sentence was true or false.

As we thought of presenting the lesson, we decided that the linear relation in two variables could be examined in at least three ways. It is possible to (1) picture the relationship with a graph, (2) make up a rule which the relation (linear equation) follows, or (3) make up a table of corresponding ordered pairs of numbers. Upon examining these ideas, we decided that relating the rule to the table is fairly simple. Even going from the rule to the graph or from the table to the graph or vice versa is obvious. In fact, it appeared that the most general concept of all three was the graph, and that in order to give the students the broadest perspective of this situation, we should begin working with the graph to find the rule. If the graph and the rule could be explored sufficiently, the student would have a good idea of what was involved in a linear relationship.

The question arose what particular graph should come first if the lesson were to begin with a graph on a rectangular coordinate system. One possibility was for the instructor to present a sequence of graphs to the class in a certain predetermined order to make the rule more discoverable and to enable the students to detect patterns between rules and graphs. The other alternative was for the students to make up graphs; the rule, of course, would be unknown to them. The disadvantage of this method was that the students would be presenting the class with a random selection of examples and the patterns would be less obvious. However, this second method would provide for high motivation, that is, the students would really feel that they were responsible for the class themselves, and they would have more of a chance to direct their own learning.

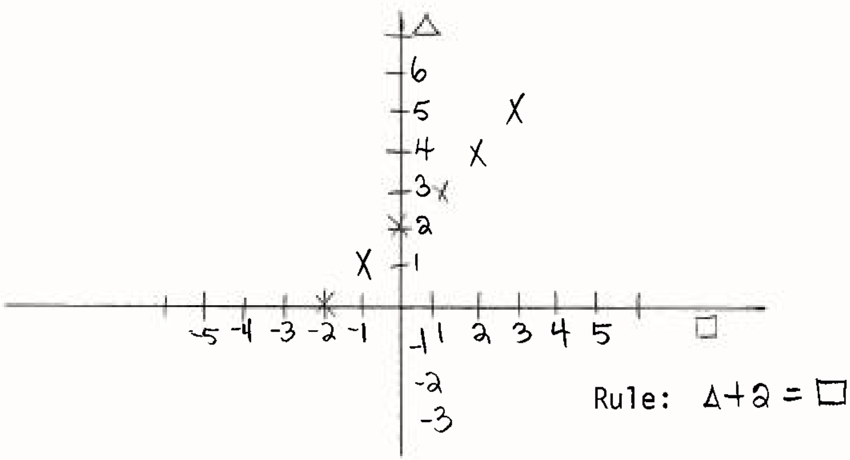
After analyzing the competence and character of the class, we decided to take the latter course of action with all its hazards. The only instructions which the children were given was that they were to make up straight line graphs in any direction they desired, but they were to try to make up what they thought would be interesting graphs. This last comment was made to try to get the children to make up graphs which were similar to the graphs previously presented or which illustrated a particular idea they had.

I will now report on the results of the activities in a group of average students just completing Grade V. I feel that these very same ideas could be used at least in Grade VII, if not in Grades VIII or IX. The class consisted of approximately 20 pupils. We limited the number in the class because of the results of other work we had done on discovery. We have found that large classes have difficulty in coping with discovery technique, especially where there is a considerable variability in mathematical ability among the class members and where the mathematical situations into which they are immersed are rather loosely structured.

I shall proceed to point out the highlights of two 40-minute periods of the discussion of graphs and rules. No mention was made of tables initially.

The first part of the activity was to have a student mark on a rectangular coordinate chart on the board a set of points which were in a straight line. The graph was put on the board:

Figure 1

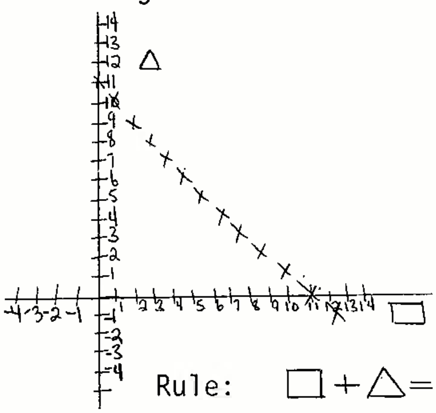


The students now understood that they were supposed to find the rule for this set of points, that is, they were supposed to find the relationship between the numbers that could go into the box (□) and the numbers that could go into the triangle (△). The first response was that they had to be equal. Five pupils spoke to this point. Someone suggested that if they were equal you would get a diagonal, although this was not clearly expressed. The discussion on the equal sign continued with a boy who said there always had to be an equal sign in the rule, but that didn't mean the two things were equal. Another comment emphasized that the box and the triangle could be equal, but then you would get something different from the thing represented by the graph. Finally, the boy who put the graph on the board noticed that his equation was two up from zero. The discussion on this problem ended here.

The most impressive thing about the comments by the students is their vagueness. The pupils had none (or little) of the terminology. They kept saying "they" have to be equal. Even Robert's comment was vague when he said his graph was two up from zero, but I feel the students all knew what he was talking about. The important aspect of this situation was that although the students knew very little of what to say and how to say it, they still reacted with a great deal of enthusiasm. The mathematical situation was structured very loosely; the students had much to learn, but still they were able to react strongly with little frustration and with a minimum amount of direction from the teacher. The problem was left when they felt (or perhaps it was when the instructor felt) they had discussed it enough.

A second graph was put on the board (see Figure 2). The instructor asked if they could extend the graph in either direction. The challenge was taken and points below the graph were found. This gave some assurance that they knew what the points of the graph were and in fact had a feeling for the relationship. Once they began discussing the rule, they suggested the statement □ + 1 = △ - 1. The instructor looked puzzled and really could not understand this rule. After a considerable discussion, the meaning of the statement was made clear. The equation meant that whenever the number in the box (□) went up one, the number in the triangle (△) went down one. They were asked to test the rule by substitution to see if it worked. Two or three examples which did not work were given. However, even after these examples many of the students knew in their hearts that this way of stating the rule made sense to them. No attempt was made by the instructor to correct this idea.

Figure 2

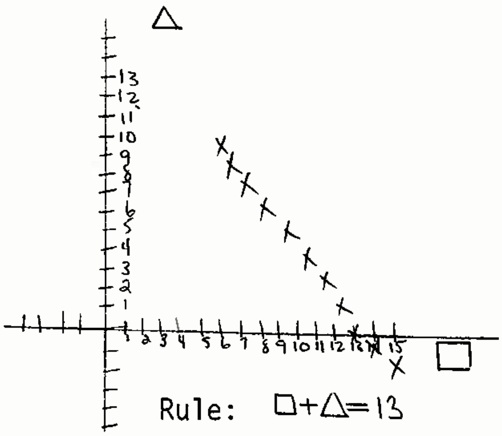


These two equations had taken 35 minutes to discuss; with five minutes left of the period, the instructor suggested trying to list the names of the points, something they had not done before on a systematic basis. After two ordered pairs were down, someone shouted "I got it!" In another minute every hand but one or two went up. The instructor asked the pupil who had his hand up first; unfortunately he did not have the correct rule. However, the rest of the class had it correct, and so the rule □ + △ = 11 was written down. The first class period ended here.

Upon discussing this lesson with an observer, the instructor was reprimanded strongly for structuring the situation to the point where he told the students to use a table. The instructor felt at this point that the students were motivated, very highly, perhaps, to the point of frustration. They were even motivated to the use of the table. He decided that the situation needed more structuring and so suggested the table. But, perhaps another kind of structuring would have been more appropriate. It is interesting to find two experienced teachers, both relatively well versed in discovery techniques, having similar mathematical backgrounds, and observing the same class, disagreeing on this point, as was the case of the instructor and the observer. It is also interesting to speculate on why the first pupil to have the answer was wrong. This is probably purely a coincidence. One conclusion was obvious: the students felt very, very cheerful and satisfied after this lesson. They had indeed discovered a mathematical relationship. I say "discovered", while the observer insisted they may as well have been told.

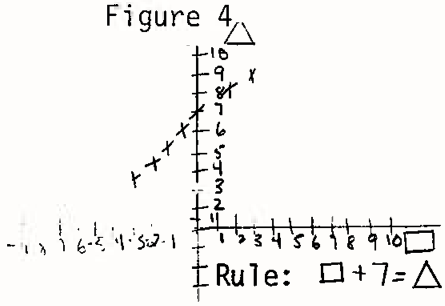
In the second period, one day later, another pupil put another graph on the board (see Figure 3). The instructor allowed a group of four students to discuss the solutions they had arrived at. He then had the students write the rule on the board since most of the pupils seem to have grasped the relationship. There was some disagreement within the group because one of the group insisted his equation was correct. The two equations presented were $\square + \triangle = 13$ and $13 - \square = \triangle$. After a lengthy discussion, it was decided that they were both right. However, at this point, a discussion ensued as to which was the better of the two. This discussion finally resolved itself into a question of which one came first and Mike insisted that $13-\square = \triangle$ came first to his mind. In opposition to Mike's stand, a student asked: "Where did you get 13 in the first place? You had to think that the numbers added up to 13." Although this latter student stated his argument very badly, Mike had to agree that he arrived at the number 13 by noticing that the box plus the triangle added up to 13. Poorly stated as it was, the argument convinced Mike that maybe his equation was an afterthought, but still correct.

Figure 3



This dicussion is interesting from two points of view. First, two different equations were presented and the children agreed that they were both correct. They were, in fact, discovering equivalent equations. It is interesting that the students at this point did not think of any other equations. It is very doubtful that they thought of these rules as being different names for the same relationship, but at least the idea that two different rules could be used to establish the same graph was brought out in the discussion. The second important point of this discussion was the idea of the students trying to decide which equation came first. They were, in fact, trying to decide which was the most natural way to solve the problem. The consensus was that you arrive at the rule by adding the number in the box ($\square$) to the number in the triangle ($\triangle$) and see what this sum is equal to. In a manner of speaking, they were discovering not only the rule but also the nature of discovering.

Figure 4



The second graph of the second period was put on the board by a student (see Figure 4). No discussion about the points was held. A group of students acting as judges were allowed to discuss the equation, and this group of five or six students ended up with the following four equations.

$\square + 7 = \triangle$, $\triangle - 7 = \square$, $7 + \square = \triangle$, and $\triangle - \square = 7$.

Each could readily see that they all applied to the set of points. However, again a debate followed as to which equation was the best. No one tried to define "best". The students seemed to have little trouble talking about these rules as representing the same idea. In fact, mention was made of one of the rules simply being the reverse of the other and, therefore, not really different. The instructor terminated the lesson at this point.

Education Outcomes

One might argue that this lesson could be presented more efficiently by telling the students that there are such things as equations for relationships. Perhaps the only way to find out which is the most efficient method is to test with control classes. However, if we wish to carry out such tests, the problem is a difficult one except for the test of mathematical information. If we admit that we are interested in something more than information, we need instruments to measure this other thing. I would now like to discuss some of the aspects and educational outcomes of the lesson other than informational aspects.

The first aspect of the lesson was the opportunity for students to discuss mathematics. More important, they were discussing mathematics without precise terminology, but they could still communicate. It would appear that the development of the ability to discuss mathematics and mathematical problems is important. Along with this would be the development of confidence, in the students, of being able to create mathematics without the assistance of an authority.

A second educational outcome is related to the notion that the student had control of the symbols and not vice versa. An example of this was when a student wanted to write the rule "$\square = \triangle$" for one of the graphs; a second student said "You may write the equation '$\square = \triangle$' but the equation will not give you the points of the graph." The feeling was clearly aroused in these classroom situations that you may do anything you want to do, but you should only do those things that are going to get you some place. Another illustration of this idea of controlling symbols was brought into the discussion which took place in the class concerning which equation was better. In that particular instance the point of the discussion was missed because "better" or "best" was never defined. "Best for what?" was never asked. Again, the matter of picking the best is an example of using symbols in the way we want to use them.

An air of self-evaluation was developed in the class. The teacher played a very minor role and the students were evaluating, on their own, their progress. I am sure that self-evaluation is a legitimate aim of school education, but even more important, it is very legitimately a behavior pattern that must be exhibited by the mathematician. As such, this outcome would be classified under knowledge of mathematics as know-how rather than information.

The fourth outcome I wish to mention is motivation, which was very high. Motivation is an educational objective. A teacher should consider his task one of making the student like mathematics as well as making the student learn it, that is, motivation in education is an end in itself, not just a means. One of the obvious outcomes, although it may not have shown up on any attitudes scale, is that students enjoy discovery-learning situations.

Another aspect of the learning situation is that a different kind of learning appears to be taking place. An example of this is the student noticing two equations can be written for the same rule. This is especially a revelation for the student who has to defend his equation, and for the others who take part in the attack. I am suggesting here that there is a difference between knowing that there are such things as equivalent equations and having a "feeling" for them. In all these discussions the word "equivalent" was never mentioned. An estimated one half of the class had a very good feeling for the notion of naming a rule in more than one way, but none of them attempted to verbalize or name this idea. The kind of learning taking place, then, is where the idea is more important than its name or description.

Another outcome relates to the discussion about which answer came first, that is, which is more natural. The students at this point were touching on the idea of how we learn. Which is the easiest way of figuring something out? "Discovering how we learn" is a different undertaking from discovering relationships. The students were trying to discover how the process of finding a relationship came about, that is, which is the easiest and least contrived way of coming up with an answer.

A final question to be asked after all this may be: "This is all very well, but did they learn any mathematics?" This is a legitimate question only after you have placed certain qualifications on it. You would, first of all, have to agree that the first six points I have mentioned all concern mathematics and, in fact, are all of vital concern to mathematics and mathematics learning. So the only questions you are asking when you say "How much mathematics did they learn?" are, for example:

1. If I gave them a graph, could they find the equation?

2. If I gave them an equation, could they find the graph?

3. If I gave them a table of ordered pairs, could they find the graph or the equation?

In other words, you would be asking how much mathematical <u>information</u> did they acquire.

Before answering this question, I would want you to concede the importance of the first six outcomes mentioned. I am sure of one thing: if they did "learn any mathematics", they could not do it with any facility. But, again, the objective of the lesson never intended to develop facility in manipulations or tabulations or rule making. Consequently, one has to be careful when asking: "How much mathematics did the students learn?" In fact, the students were not tested on mathematical information.

Summary

The seven aspects and educational outcomes of the lesson give some idea of the things this lesson was designed to do. If your views on education disagree with the emphasis that I have placed on the various objectives, then you will not use the discovery method as I have suggested. If you think another objective is more important, you will set up the mathematical situation differently. There is no "one" discovery method of teaching, but any discovery method will emphasize most of these objectives. Of one thing I am sure: it is invalid to talk about the discovery method of teaching in the light of traditional objectives of mathematics.

The greatest need in the field of mathematics education today is for a closer look at discovery. I have implied that teachers who are interested in this area must indeed be creative. They must set up situations to which students can react, not situations of special patterns or of using analogies or some very special problems in number theory, but rather of honest mathematical situations, structured highly enough so that the particular students being taught can react to them. Indeed, this would be different for every student. And finally, I have tried to impress upon you my belief that if you disregard discovery, you more than disregard a method, you disregard a whole set of objectives of present-day mathematics education.

THE TEACHER OF MATHEMATICS - FRIEND OR FOE?

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The dedicated teacher must have a real love for his pupils. No one will deny this fact. The teacher will show this love by encouraging his students, demanding their best efforts. He will exercise his knowledge and skill by assisting the students to grow in knowledge and to mature as persons.

The good teacher must know his subject; he must know how to present his material, and, perhaps even more important, he must have a personal relationship with the young men and women entrusted to him. Ultimately, the teacher is neither a teacher of mathematics nor a teacher of the textbook; he teaches people. Yet many of us realize that something or other is missing from our relationship with these young people. They tend to be distant, irresponsible, and rebellious. Teachers whose own children reach the high school age are probably more aware of this than I am. But certainly not all teachers or parents have this difficulty in relating to the young. Some teachers seem to possess the knack of appreciating the modern youngster, of understanding him. Some are able to win his confidence, to direct his energies in the pursuit of truth. What is it about some parents and some teachers that makes them able to relate to young people so well? I do not pretend to have the answer, but I will affirm that I am interested in young people and think that they recognize this interest. It is not always obvious that my interest helps them in any special way. However, there are many occasions each year that cause me to think that I am on the right track when I do try to treat each of my pupils as an individual, do accept each as a friend.

My presentation has something to do with the love that a teacher should have for his students, with my conviction that I must treat each student as an individual if I am to assist each one in developing his personality as well as his love for mathematics.

What, then, is the relevance of the title of my talk "The Teacher of Mathematics - Friend or Foe?" Young people, not unlike adults, tend to make generalizations in their contact with other people. They use expressions such as "square", "cool", "he doesn't know square root to the minus n", etc. to describe the people they know. In the title of my talk I have translated these words into the English with which you and I are more accustomed. But I, too, have been guilty of exaggeration. In a certain sense, I am maintaining that people who have a person-to-person relationship tend to classify one another into two general categories: those who are friends and those who are not friends. The latter I have called foes. I am asking which of these should be the teacher of mathematics. Let me be a little more specific. Whenever people gather to discuss the world situation, two main groups arise: one supporting the war in Vietnam, the other opposing it. At least for the duration of the discussion - and sometimes for a longer time - each person classifies the other as his friend or his foe. In the United States, there is a certain friendship among members of the Democratic party that causes them to look upon the Republicans as enemies. The same is true here in Canada among members of the Liberal and Conservative parties. Even Dodger and Giant fans in baseball, and

Leaf and Canadien fans in hockey, break up into two enemy camps. Yet there are degrees of friendship and of foeship. Ordinarily, the enmity between a Dodger fan and a Giant fan is considerably different from that between an avowed communist and a loyal citizen of a democracy. Nevertheless, no matter how slight or how strong the resentment or love may become, people in personal contact do tend to look upon one another as friends or foes. Observe that this is another generalization on my part. There are always more than two sides to any discussion, but such a division as I have made helps me to make clear the existence of the difficulty that we older men and women have in relating to modern young people.

What I am emphasizing is my original statement. The dedicated teacher must have a real love for his students. I am restating this in different terms by indicating that a teacher must be a friend to his students if he is going to be of any assistance to them in developing themselves so that they can take their place in society as responsible citizens who lead happy lives.

Before we get into the intricacies of this relationship, I think that the following may help our understanding of it. One day I was talking to a university professor who had taught in high school. I asked him about the difference between the two kinds of teaching. He made one important point that I shall always remember. "In university", he said, "the professor teaches the subject matter. In high school the teacher teaches the students." Not many people really understand this. I know that all of my students do not. In the past before the mathematics curriculum changed every year, a few students would return to school each year from college and ask the same question. "How can one who knows so much mathematics teach the same subject matter each year without becoming bored?" My answer consisted of recounting the above story and reminding the students that though the subject matter was static, the members of the class changed each year. Thus, each year was a new experience for me. There was never any chance of getting bored with the subject matter because in a certain sense it played second fiddle to my interest in the students whom I taught. I relate this story not to find fault with the university professors but to emphasize the fact that personal contact with students is one of the main characteristics of high school teaching.

It is well to say that there must be friendship between teacher and pupils, that the teacher should try to treat each of his students as an individual. How can this be done? Does not the crowded classroom that is found in so many places make this friendship extremely difficult if not impossible to achieve? Yes, we have thorny problems. Yet are not mathematics teachers noted for the eagerness with which they attempt to solve problems that do not seem to have a solution? When confronted with such a problem, we frequently employ a useful dodge. We attempt to restate the problem in a different manner or to pose another problem whose solution will help us to solve the original one. I believe that the problem we are now considering can be restated as follows: How can the teacher of mathematics help his students to attain the mathematical knowledge necessary for their intellectual well-being and still treat each pupil as an individual? A few months ago I posed this question to a number of colleagues in an Ontario Department of Education Curriculum Committee and explained that I was going to give a talk about it. One said "why not give the address the title 'How to Give a Talk on an Impossible Topic'." Another said, "this is a lecture that I should very much like to hear but not to give." Now that I am actually at the point where I must talk on the topic, I am inclined to agree with these gentlemen. But there is an ancient saying that "fools walk in where angels fear to tread", so I am not going to give up, at least not right now.

Even though the group I referred to was not really helpful, I did receive some assistance from another friend who is an inspector of mathematics teachers and mathematic teaching for the provincial Department of Education in Ontario. After I had described the problem, he said that he had never thought of the problem, at least never in those terms. He added that perhaps a really good teacher does not consciously consider the problem. Yet he conceded that one of the marks of a good teacher is the ability to relate in a personal way with each student in his class. The last remark would indicate that we may get the answer to our problem by watching and talking to those teachers whom we know do develop the personality of each pupil and at the same time lead each one to a love of mathematics.

Before we actually come to grips with the newly stated problem, I would like to employ a new method of attack. One thing I am certain most people will agree in teacher-pupil relationship is that the teacher should never do all the work. I myself am inclined to believe that the teacher should seldom, if ever, do for the pupil what the pupil can do for himself. Now, in a certain sense you and I are having a teacher-student relationship. I want you to do your own thinking before I give you my ideas.

One of my college teachers outside the field of mathematics who made the greatest impression upon me began his classes in the following way. Each class would commence with the teacher dictating a series of thought-provoking questions. These questions would never be discussed in class. Our teacher would seldom give us any answers. Indeed, many of the questions had no answer. These were just problems that this man was pondering at the time. By giving them to us, he hoped to goad us into doing some original thinking.

I have prepared a list of questions that I shall submit to you with little or no comment. It will not always be clear that a particular question will have any useful part to play in solving our larger problem. At times you will meet a question that seems to have no connection with the previous or following questions. Nevertheless, I think that much can be gained by a brief acknowledgement that certain problems do exist even though we may not actually come to solve any of them even though these special problems do not seem to have a direct bearing on our attempt to find out how we can promote a true and good relationship between ourselves and our students. After we have completed the questions, I shall give you some of my ideas on how our original problem can be solved. In the process I shall indicate my answers to some of the many questions that we shall read now. Perhaps if we have time, we will be able to arrive at other and better solutions through discussion.

1. Is it true that the modern young people will not respond to the methods of the past, e.g. the "big stick" method?

2. Is a mathematics teacher obliged to take a personal interest in his pupils?

3. Is it possible to teach an enemy? Does one's teacher have to be a friend?

4. If a teacher finds that he can either prepare his students for college or treat them as individuals but not <u>both</u>, which should he choose?

5. What academic knowledge in mathematics must a student really have to qualify for college entrance? What academic knowledge in mathematics must a non-college bound student really have to prepare him for his future life?

6. What role does the teacher play in seeing that each pupil attains this knowledge?

7. Does the teacher of mathematics ever know enough about his subject? Does he have any obligation to improve continually his knowledge of mathematics?

8. What role does an extracurricular activity play in motivating academic growth?

9. Will homogeneous grouping of students in mathematics classes lead to maturity academically or otherwise?

10. What can be done to make students understand that each is an individual and must be treated as such?

11. Is it possible to treat each pupil as an individual in a class of 40, of 35, of 30?

12. How can a teacher cultivate a respect in the students for each other, especially for the intellectual differences that exist?

13. How can a teacher help overcome the immature attitude of some students toward academic excellence? How can a teacher lead the students to respect the "egghead" rather than to deride him?

14. What is the effect on a class of recognizing that multiple solutions exist for most mathematics problems and that frequently there is no 'best' solution.

15. Can we allow adolescents to accept the responsibility for doing their homework to such an extent that we seldom check it?

16. Does a <u>broad</u> assignment of homework for a good class help the students to develop personal initiative and a love for mathematics?

17. Should the homework assignment for a non-college bound class be different from a college-bound class? Should the former type of class have any homework at all?

18. Should there be fewer formal subjects to allow pupils more time to read and think?

19. If one of my students fails, how much am I at fault if at all?

20. If one of the graduates of my school fails in college, how much blame can be attached to my school and its teachers?

21. Do my students continue to study throughout life? If they do, does this mean that they have a love for knowledge?

22. Does the fact that my school has a large number of its graduates in colleges and in graduate schools mean anything in regard to my students' love for knowledge?

23. How can I as an individual teacher encourage love of knowledge and growth in intellectual interests in my pupils?

24. What effect does the stifling of student discussion have on the growth of intellectual interest?

25. What effect does the Socratic method of teaching with its teacher-questioning and student-answering have on the growth of intellectual interest?

26. What effect does the discovery method of teaching have on the growth of intellectual interest?

27. What effect does the lecture method of teaching (in which the teacher does everything) have on the growth of intellectual interest?

28. What effect does the using of a variety of methods of teaching have on the growth of intellectual interest?

29. How can a teacher teach so that his pupils can enjoy their studies at least sometimes?

30. Do my students really appreciate the joy that comes from doing a mathematics problem correctly?

31. How can a teacher point his students towards examinations and still help them to maintain a love for mathematics in itself?

32. Is both internal and external involvement of the pupil in the classroom necessary?

33. Are pupil boardwork, seminars, talks, teaching and discussions sufficient for number 32?

34. Does praise of academic excellence through honor societies, scholarships, prizes, etc. help mature individuals?

35. How much freedom can I give my pupils in my mathematics classroom and in what spheres of activity? Should this giving of freedom be progressive?

36. What has been the effect in the past of any freedom that I have given

my students regarding discipline? Likewise, what has been the effect of any freedom I have given regarding homework?

37. What effect do other attractive things - for example, extracurricular activities, the opposite sex - have on our students' ability to use freedom properly?

38. Does a teacher need to encourage an intellectual discipline and a formal discipline in his pupils before he can give them the freedom that they want? In other words, is internal discipline a prerequisite for a mature use of freedom?

39. Can an individual teacher have any effect in these things if he stands alone in his efforts, unaided by his fellow teachers or the parents of his students?

40. Will the teaching of logical reasoning in my classes have any effect on the real life reasoning of my students?

41. How many teachers in my school actually help their students to love mathematics for itself? How many times have I visited the classes of such teachers to see how they teach? How often have I discussed teaching methods with such teachers?

42. How many teachers in my school are successful teachers of the non-college bound students? How many times have I visited the classes of such teachers or discussed their methods with them?

43. What effect do the following environmental factors have in aiding or preventing what I am trying to do for my students: the city or town, the school itself, the mathematics texts which I am using, the students' home life? Is there any way in which I can make better use of these things when they are helpful or change them when they are doing harm?

44. Which is better for the students whom I teach: a curriculum imposed from above from someone outside the school, or a curriculum determined by the teachers in my school?

45. Does the answer to the previous question depend a great deal on the competence that I have in the classroom and upon my knowledge of mathematics?

46. Is the teacher who seems to be a friend of his pupils actually a friend?

47. Is the teacher who seems to be a foe of his pupils actually a foe?

48. Should the best students in mathematics be separated from the rest?

49. Should the weakest students in mathematics be separated from the rest?

50. Should there be homogeneous grouping of students in mathematics from the best down to the weakest?

51. Can we ever expect to help the slow learner?

52. If a project or method used in a class of slow learners helps only 20 percent of the class, should this method be abandoned, or should we be happy that we have been able to help at least that many?

53. Does a class of slow learners need more personal attention, more friendship than an average class?

54. Does a class of slow learners need to be given more or less responsibility than an average class?

I shall probably not answer specifically any of the questions that have been considered. The comments I now make will, of course, indicate to some extent how I would answer the questions which have been proposed. I suspect that we shall be looking for the correct answers to questions such as these for a very long time. Indeed, I believe that there is no correct response to many of them. Any answer must take into account individual differences. Nevertheless, there is one question that I wish to answer at least partially: If a teacher considers each pupil as a friend whom he respects and loves, what effect will this have on his teaching? First, I shall indicate what I think a classroom teacher should attempt to do. Then I shall indicate some of the means which a teacher can use to accomplish the following aims.

- The mathematics teacher should treat each student as an individual person.

- The mathematics teacher must help each student to attain the mathematical knowledge necessary for the academic life that best suits him.

- The mathematics teacher must lead each student to a love of mathematics for itself. (One of the results of automation is the leisure time that it will afford the average man. This leisure time can be a good thing if a man can use it well. No man can recreate all of the time. If our students come to love mathematics for itself, they can spend many hours in worthwhile and enjoyable study of mathematics.)

- Somehow the mathematics teacher must do his part to help his students become responsible citizens. (I believe that this can be done by encouraging in each student an attitude of personal responsibility for his work, by fostering in each student an initiative that will cause each to seek out things for himself, by cultivating in each student a respect for the other students in the class - this respect can be the chief fruit of class participation. It causes the students to realize the importance of listening to the views and solutions of others and helps them to accept the truth that some will be better mathematicians than others.)

A teacher of mathematics can do the following things:

- Show a love for each student and for mathematics.

- Teach the subject of mathematics well: obtain the best possible mathematics background; prepare thoroughly for each class; use the Socratic method in which teachers question and pupils answer; make the presentation of a topic as logical as possible; allow the students to develop new ideas for themselves by careful questioning; treat mathematics with seriousness joined with joviality; make mathematics as easy as possible within certain limits - in other words, mathematics should be difficult in itself not becaue the teacher makes it difficult.

- Allow the students to participate as much as possible: never tell the students what they can tell you; allow the students to do most of the work at the blackboard or with the overhead projector; do not answer you own questions, ask questions frequently; never ask a student a question that can be answered with a "yes" or "no"; whenever a student asks a question, allow the other students to give the reply; do not act as judge and jury on pupil replies (in a sense, the teacher should become an expert who is used as a last resort and who keeps the discussion going when it lags); insist that each pupil, especially in better classes, give talks or seminars on topics outside the course of study; allow the pupils to do the teaching as often as this is feasible.

- Promote pupil responsibility: do not check homework every day; make the class find the errors in another pupil's presentation; make homework assignments broad rather than specific; have lots of reference material available in the classroom; ask a better student to teach your class when you are absent even though the principal may insist on sending someone else to keep order; if a class misbehaves when you are present and especially when you are away, do not berate them but do show you are disappointed in them.

- Encourage initiative in your students: give broad rather than specific homework assignments, especially to the better classes, and to the other classes as the year progresses; allow the students to choose their own topics for talks, seminars and projects; have lots and lots of reference material available in the classroom, especially of the pocket book variety; make homework assignments at least three or four days in advance (this encourages the students to work on their own and read ahead); advise students to participate in mathematics contests because this forces them to study mathematics that they have not taken as yet.

- Give each student as much individual attention as you possibly can: spread your questions around the class, try to get a response from <u>each</u> student <u>each</u> day; insist that the students use complete sentences in their answers and that they speak loudly enough for everyone to hear; insist that the pupils show politeness and kindness towards each other; treat discipline problems kindly but firmly and with justice; try to deal with these problems outside class time, that is do as much disciplining as possible through conversations in private; encourage all by praising good answers and saying thanks; be interested in your students outside the classroom, make a point to talk to them or at least nod to them in the corridors; have a true interest in the difficulties of your students and in the students themselves.

- Be human: laugh at yourself, and enjoy the classroom and the students in it enjoy mathematics; you will be richly rewarded - students will enjoy attending your class and learning mathematics.

- Have a healthy attitude towards examinations and tests: if a student misses a class test, give him the same one to answer, even allow him to do it at home; prepare a good answer for the students' question "Is this important for exams?"

- Be available for help or discussions outside class hours.

- Show your interest in the students by frequently attending their activities - academic, cultural and athletic.

- Show the pupils that you are interested in academic excellence.

These are noble aims. Perhaps they are unattainable in a classroom. Yet we can try to reach them. Let me now indicate to you some of the things that I do or have found others are doing which will help to bring about these goals.

I have found that students do for the most part respond in an intelligent way to the things that I have recommended. There was a time when a teacher could go into a classroom and keep order with a big stick. To me this time seems to be in the past. Freedom and responsibility is the rallying cry for so many groups now that it has penetrated even to the classroom. Yet we must avoid giving too much freedom; freedom should be aided by worthwhile guidance or it is in danger of becoming license. If we can do our part in leading our students to accept true responsibility and to have the initiative to go ahead on their own, then we will be leaving our countries in good hands.

If we, who have such a close contact with the young people of today, do not do our part in this respect, then there may be no one else left to do it. We are all very busy nowadays. There was a time when the teacher of mathematics could prepare his classes one year and then use the same preparations over and over. However, since then many new ideas have filtered into the high school. We are apt to become too preoccupied with ourselves and our own deficiencies. The student may become a face in a classroom, a name on a mark sheet. It takes a real effort to be a teacher who is a friend of his pupils, but this is the goal which, I believe, most of us should and do set up for ourselves.

Conclusion

From time to time it is necessary for us to remind ourselves what it means to be a teacher. This is what I have attempted to do. First of all, I tried to pose the problem in a way that would attract your curiosity: is a teacher a friend or foe? Then I attempted to show you the enormity of the problem by citing numerous questions whose answers had an important bearing on the problem. Finally, I tried to put the problem into real life terms by showing you how I think each one of us can begin to solve it in our classrooms. Perhaps the dimensions of this problem are beyond numbering; perhaps I have not scratched the surface of the problem. Nevertheless, I do hope that I have made clear that I think the teacher must be the student's friend and not his foe.

Yes, I believe that a dedicated teacher should have a real love for his students; a love that shows itself in the teacher's desire to teach mathematics as well as possible and at the same time to treat each student as an individual who must be helped to become a good and responsible citizen. No doubt it is easier to talk about friendship with students than to win it, but if the young men and women we teach really become our friends, our efforts will be amply rewarded.