TRANSFORMATION GEOMETRY IN GRADES IX AND X

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My paper on transformation geometry is based on a mixture of ideas from a new course of study in Ontario, some topics I taught to a Grade X class last year, reading of various European programs, some observation of classes overseas and discussion of school mathematics with overseas teachers during a visit to Great Britain and Denmark a year ago.

In Ontario, we have been working for several years on revisions of our program in mathematics for all grades, with possibly the greatest attention being given to algebra in the secondary schools. We have introduced what might be called a number-structure approach to mathematics. This involves careful attention to the number systems and a much earlier introduction of the concepts of relation and function.

Although geometrical topics in the Ontario curriculum were not ignored in the process of changing the program, the changes in geometry tended to be modifications of Euclid's traditional approach; the work in Grades VII and VIII stresses ruler and compasses constructions; definitions are based on sets of points; also at a time when we were introducing some of the axiomatic-deductive approach into algebra, it was assumed by some that the treatment of deduction in geometry should be made more rigorous. There was increased emphasis on the style of writing solutions of deductions and the authorities for statements. In my opinion, we were mistaken to move in that direction.

There is no doubt that by the time pupils reach Grades IX and X (ages 13 - 15) in our schools, they are ready to be shown how proof works. However, the recondite nature of mathematics can make the task difficult, particularly in geometry. There are so many special properties of geometric figures to be investigated that if we try to prove everything, by the time the pupil reaches "pons asinorum", he may be bogged down in detail and may have lost interest in discovering the main structures of the subject.

The alternative approach which I propose to describe is called transformation geometry. First, I would like to discuss briefly the background leading up to Grade X. For this I will use some details from a new Scottish program for 12-year olds to illustrate the background our 13-year olds should have for a new program. My discussion of Grades IX and X will include references to Danish and English textbooks. Finally, deductions taken from an article by Dr. Jeger, a Swiss, will illustrate how proof can work in this approach.

One reason which makes me feel qualified to say something about a new approach to geometry is that in the Ontario Five-Year Program (college preparatory) at the Grade X level we have just finished a course of study in which we are trying to change the direction of geometry by playing down the emphasis on deduction, introduing three-dimensional work, elementary transformations and vectors.

The unit on geometry will occupy about 12 weeks of the school year. Four to five weeks will be on deduction, where there is not to be a formal organization of Euclidean geometry but the development of short sequences of related theorems. One to two weeks will be spent on three-dimensional topics, three weeks on transformations and three weeks on vectors. In all, this represents a very radical change from past courses in geometry in our system.

Perhaps I should define some of the terms. A transformation may be defined as a correspondence between geometrical objects. In the simplest cases, the definition may be defined by stating that it is a 1:1 matching of sets of points. The bringing of transformations into geometry then means that we are introducing the concept of relation and function into geometry as a unifying theme. I suppose the words transformation and mapping are interchangeable. What are the advantages of the transformation approach?

In geometry we study relationships in space and the significant properties of geometric figures. "Traditional geometry lacks a methodology which is anchored to spacial reality. Logic may stretch through the whole edifice like a colored thread, but it is not satisfactory, because it is not typically geometrical." I am quoting here from an article written by Dr. Max Jeger of the Kantonschule in Lucerne, Switzerland, translated for the magazine *Mathematics Teaching* by Irene Hertz. The article is titled "The Present Conflict in the Reform of Geometry Teaching". Dr. Jeger reviews the history of the development of the traditional course in Euclid and criticizes its present state. He says: "Every generation has absorbed thousands of small details to such an extent that new features can hardly penetrate. Everything to the smallest detail has been thought out in Euclid's edifice; there is hardly any room left for the teacher's contribution in substance or method."

Such criticisms of the traditional course in Euclid are not new. Comparatively new, however, is an attempt to replace Euclid by a workable alternative, and not just to modify the old approach.

The transformation approach to geometry is due to a redefining of the subject, initiated by Felix Klein, the eminent German mathematician who lived from 1849 to 1925. Klein not only criticized Euclid but showed a method of moving away from Euclid in his famous "Erlangen" program.

Klein's method is one of sorting the properties which are important from the welter of detail in geometry and making them stand out. His definition of geometry is that it is the study of those properties of figures which are invariant (unchanged) under certain transformations. Implicit in his definition is emphasis on a more constructive approach at the basic levels, with the shifting of the axiomatic approach to a higher level.

The simplest geometrical transformations are reflections, rotations, and translations. Each of them preserves distance; they are called isometries, or rigid motions. The image is congruent to its pre-image. These three simple correspondences have, within them, all of the main structures of introductory Euclidean geometry, which is the study of rigid figures.

I have chosen to talk mainly about Grades IX and X because of my interest in the secondary field and recent work on a Grade X curriculum. I refer to ages 13-15 in the college preparatory course, in a system where mathematics is taken by nearly all of the pupils. However, it seems to me that there are some important prerequisites if this approach is to be successful. Symmetry has an important role in this kind of geometry. What sort of course should be given before Grade X so that the pupil is prepared for transformations? I hasten to add that probably some of the topics mentioned above in the new Grade X could be started much earlier; however, let us assume that we have just been transformed.

The Scottish experimental program was introduced to first-year secondary school pupils (12-year-olds) in September, 1964. A year ago I visited a few schools and talked to some of the teachers involved in this experiment. You may be interested in some details. In the geometry section of the program, the stress is on figures, beginning with the special ones: rectangles, squares, cuboids, cubes. Drawings of figures on a grid are used to clarify concepts. (Figure 1, page 57).

A question I heard asked more than once was: "In how many ways may a certain figure be fitted back into its hole in the plane, or into its hole in space?" This property of the special quadrilaterals and triangles is related to their axes and centres of symmetry and the number of ways in which they can be folded along these axes.

Throughout the Scottish work there is stress on "tiling the plane" with different figures. A rectangle is defined to be that figure which (a) can be used as a tiling agent to cover a flat surface without leaving any gaps, and (b) is such that each tile can be fitted into the shape of its own outline in four different ways.

Several concepts evolve from the study of rectangles in this manner: 1. the right angle; 2. the diagonals of a rectangle have equal lengths; 3. the diagonals of a rectangle bisect each other. If we begin with any triangle, we can develop the tiling of the plane. This brings out the sum of the angles of a triangle and the equality of alternate angles in a Z-diagram (Figure 2, page 58). Similar ideas can be developed from working with parallelograms of any shape. Note also how such a design can be used to discover intuitively the equal ratios of the lengths of the segments formed from the sides of a triangle by a line segment parallel to one side of the triangle (Figure 3, page 59).

Coordinates are introduced early to assist understanding by locating the vertices of figures. The right triangle is derived from the rectangle; their areas are related to the grids on which they are drawn. Other figures studied in the first year for their symmetries are the isosceles triangle, the equilateral triangle, the rhombus and the kite. (The use of the name "kite" for an isosceles quadrilateral seems to illustrate accidentally a pedagogical principle of the approach, namely the description of the global qualities of a geometric design as opposed to the analysis of its elementary components.

The very sketchy outline given here is based on the textbooks of the Scottish Mathematics Group *Modern Mathematics for Schools*, published by Blackie and Sons in Glasgow and London, and by Chambers. The main features of this program in geometry are emphasis on the physical manipulation of real things, apprehended globally, used before analyzed, and often special, rather than general.

Going on from the elementary level, how would you employ symmetries to introduce some of the usual geometrical ideas? Let us look at a Danish textbook for a moment. The Danish school system generally consists of three levels: elementary, ages 7-14; real skole, 14-15; and gymnasium 15-18. In a recently published textbook for the first year of real skole (age 14), the following topics are studied in the order given:

1. reflection in a straight line,
2. definition of perpendicular,
3. definition of parallel line segments in terms of a common perpendicular,
4. definition of perpendicular bisector in connection with reflection,
5. reflections of line segments and angles and use of this to define bisector of an angle.

At the end of this particular section the summary states:

A reflection in a straight line (a) carries a point over into a point, (b) carries points on the axis of reflection into themselves, (c) carries a line into a line; if a line cuts the axis of reflection in a point, the image also passes through this point; if a line is parallel to the axis, its image is also parallel to it. A line segment is carried into a line segment which is congruent to the first; the line segment which joins a point with its image is perpendicular to the axis and bisected by it. An angle is carried over into an angle which is congruent to the first; an angle is carried over into itself by a reflection in the line on which the bisector of the angle lies.

The next section discusses the circle as a locus and reviews its parts, but stresses reflections and symmetries also. For example, one question asked is, "Which circles are carried over into themselves by a reflection in a given line?" Congruent arcs and chords are developed by reflection in a diameter; the properties of intersecting circles and common chords follow very nicely from this approach. The measure of an angle is associated with a circle drawn with the vertex as centre, and the division of the circle into 360 congruent arcs. The rotation idea of angle is associated closely with transformations from the beginning; for example, the rotation of 180 degrees is equivalent to reflection in the vertex.

This is a brief sample of the discussion in a Danish textbook which includes the three rigid motions and summarizes their properties. The approach is constructive and any proofs given are informal, applying the basic assumptions for the transformations.

The congruence motions and other transformations may be used to introduce a study of the main traditional topics of introductory geometry: congruence, parallelism, area and similarity. I have tried a little of this with a Grade X class before moving into a fairly traditional axiomatic-deductive treatment. I got some of my material from *Some Lessons in Mathematics* and *School Mathematics Project, Book T* published by Cambridge University Press. Here are some samples:

Translations

A translation is movement of the plane in a particular direction without turning. Its basic properties can be illustrated very well on a grid (Figure 4).

If we use a cartesian coordinate system and represent the translations by column vectors in order to distinguish them from vertices of figures, then it is easy to show the group properties and extend the concept to three-dimensional cases.

The mention of group causes me to digress for a moment. By its simplicity and pervasiveness, group is certainly one of the best unifying themes in mathematics. Without being too formalistic about it, we can make the characteristics of a group quite clear in the case of translation vectors.

Reflection

Reflection in a line is analogous to reflection in a physical mirror. The image is on a line through the pre-image perpendicular to the axis, such that the axis bisects the line segment joining a point A to its image A' (Figure 5).

For both of these cases we stress the congruence of the image and pre-image and see the difference between congruence in the direct and opposite sense. Have you read Hermann Weyl's beautifully illustrated lectures on Symmetry? The first two are included in Newman's *The World of Mathematics*. They are well worth reading.

Rotation

This involves rotation of the plane, counter-clockwise about a given point (Figure 6). In the special case of a half turn, it is equivalent to reflection in a point.

By using coordinates we may discuss different types of reflection in the axes and the origin:

1. Reflection in O X : $(x,y) \to (s,-y)$,

2. Reflection in O Y : $(x,y) \to (-x,y)$,

3. Reflection in O : $(x,y) \to (-x,-y)$.

In my trial of this material we discussed the positions of vertices of squares, rectangles and equilateral triangles when rotated about the origin.

The case of the 45^o rotation of a square made a nice little application of the Pythagorean theorem, as well as a test of the pupil's awareness of the symmetry of the figure. As with the other transformations, the congruence of the image and its pre-image were stressed.

One topic which might be discussed is the presence of invariant points. For each transformation, are there points which do not move? In the case of translation there are none; under reflection, the points on the axis are invariant; in rotation, the centre of rotation remains fixed.

The transformations discussed above can be used to give unity to the part of introductory geometry, usually called Book I. The main property exhibited is, of course, the rigidity of the figures.

We go on now to transformations which give images not congruent to the original figures but which do have other invariant properties.

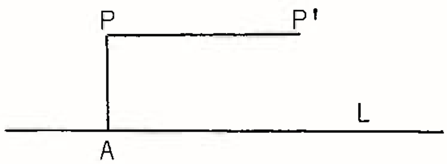
Shearing

A shear is a transformation with the following characteristics:

1. There is a straight line L which does not move.

2. Every other point P is carried into a point P' such that segment P P' $\parallel$ L.

3. P P' = k (P A) where P A is the distance of P from L, measured in some suitable direction, and k is a constant.



The uniform displacement of the cards in a deck gives a good illustration of the nature of this transformation. Of course, area is preserved; the principle involved is basically the one used in pre-calculus methods of developing formulae for the volume of various solids, such as the cone and pyramid.

In the case of rectangle ABCD which maps into parallelogram EBCF under a shear:

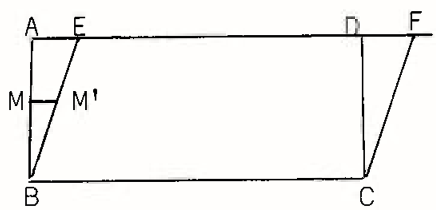
1. Suppose that AB = 6 units and AE = 2 units

2. Then AE = $\frac{1}{3}$ AB, i.e. k = $\frac{1}{3}$

3. For M, the midpoint of AB, M$\rightarrow$M'

4. $\frac{MM'}{MB} = \frac{AE}{AB}$ due to similar triangles

5. Hence MM' = $\frac{1}{3}$ MB.



Dilatations

A dilatation (or enlargement) is a transformation with the following characteristics:

1. There is a fixed point O, called the centre of dilatation;

2. If P' is the image of P, then P' lies on OP;

3. For a particular enlargement, $\frac{OP'}{OP}$ is constant, where OP and OP' are directed lengths; the constant is known as the scale factor of the enlargement.

A very good introduction to this whole topic of dilatation and similarity can be made by using a graph and coordinates. If one side of the figure is made horizontal, it is easy to see by calculation that the ratio of areas is not the same as the ratio of sides. The calculation of sides by the Pythagorean theorem gives valuable information, too (Figure 7). The invariant property here is, of course, the shape; we see this in the preservation of angle size and the equal ratios of the lengths of corresponding sides.

In summary, these are the transformations which could be best used to give life and movement to the introductory geometry. I believe that authors of textbooks using imagination could write new courses which would revolutionize our teaching of the subject. None of this material is new; it just needs reworking and simplifying for school use. For this purpose, a book I found most illuminating is *Introduction to Geometry*, by H.S.M. Coxeter, University of Toronto, published by John Wiley and Sons.

This whole question of the geometry has been bothering us in Ontario a great deal. As you may have heard in other sections, the Ontario Mathematics Commission has a committee on Geometry (K-13), which has been meeting regularly since January with the financial support of the Ontario Curriculum Institute. I believe that the line they are taking is similar to some of the ideas in my talk.

In his article in *Mathematics Teaching*, Dr. Jeger very strongly makes the point that there is not time to do geometry the old way and the new way also. On the other hand, if we are to use the transformation approach, we must be prepared to set up an axiomatic-deductive system at some stage and teach the nature of proof. Jeger states that the axioms in this sytem would be more powerful than those we have ordinarily used. I am going to take the liberty of using a couple of his proofs to illustrate deductive methods in motion geometry.

Example 1. Required to prove that $\angle$ ACB = $\angle$ ADB (Figure 8). Analysis: the main feature of this proof will be movement of the $\angle$ACB around by a rotation of the plane about the centre of the circle so that the arms of the image are parallel to the arms of $\angle$ADB.

Let us look briefly at the way arcs and chords are placed in a circle. Parallel chords are placed symmetrically with respect to a diameter through their midpoints. Now, if a chord AC is moved to a new position A'C' which is parallel to the chord AD, what can we say about the position of C'? It is easy to show that C' is at the midpoint of arc CD.

Now we have a method of proof. Let MN be the perpendicular bisector of chord AD, meeting the circle at O. Rotate /ACB about the centre of the circle, until C', the image of C, lies at the midpoint of arc CD. Then, referring to the lengths of the arcs, we may say arc AA' = arc BB' = arc CC' = arc C'D.

Since arc AO = arc OD

and arc A'O = arc OC'

then A' is the image of C' under reflection in MN and vice versa.

Therefore, A'C' is bisected by MN and is parallel to AD.

Similarly, by using the perpendicular bisector of DB it may be proved that B'C' is parallel to BD.

Hence /A'C'B' may be mapped onto /ADB by a translation.

Since /A'C'B' = /ACB, therefore /ACB = /ADB.

Example 2. Here is a construction which may have puzzled you at one time (Figure 9).

Given: any four points A, B, C, and D.

Required: to construct a square with each side passing through one and only one of the four points.

Analysis: I believe that difficulties I experienced with this problem were caused by my failure to recognize the symmetry of the figure. The parallel sides of the square form two equidistant bands. A rotation of the square (or of the points) through 90° would give equivalently placed image points. Let M be the centre of the square. If we make a rotation of 90° about M, A, B, C, D, maps into A'B'C'D' on the adjacent sides. AC and A'C' will be perpendicular segments of equal length due to the symmetry of the "bands". Translate C' to $\bar{C}$ with a vector equal to vector A'B. Then vector $\overline{BC}$ = vector $\overline{A'C'}$. $\overline{BC}$ is perpendicular to segment AC and equal in length to AC.

This gives us a construction which can be drawn through the given point B, determining the side of the square through D.

Conclusion

The purpose of my paper has been to show how geometry may be conserved as an essential element of the teaching of mathematics by giving it new relevance, life and movement.

The use of transformation brings the function concept into geometry and, incidently, helps to clarify function and mapping by associating it with physical motion.

I am inclined to agree with Dr. Jeger that it will be successful only if we go all the way with it. Will our new crop of teachers see here a method they like so well that they use it in its full power? Or will they fail by trying to "ride two horses?"

The rigor of presentation is important. We have had difficulties over this before. Often new methods are devised by research mathematicians when they are playing the axiomatic game and being quite obtruse. Then we in the schools, mistaking "shadow for substance", condemn the new concept because we saw it first when it was couched in abstract terms. The mathematician has a responsibility to make the concept real for the schools. How well this is done will determine our teaching success. I hope that some of the examples I have used will make the possibilities of transformation geometry more real for you.

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Figure 1

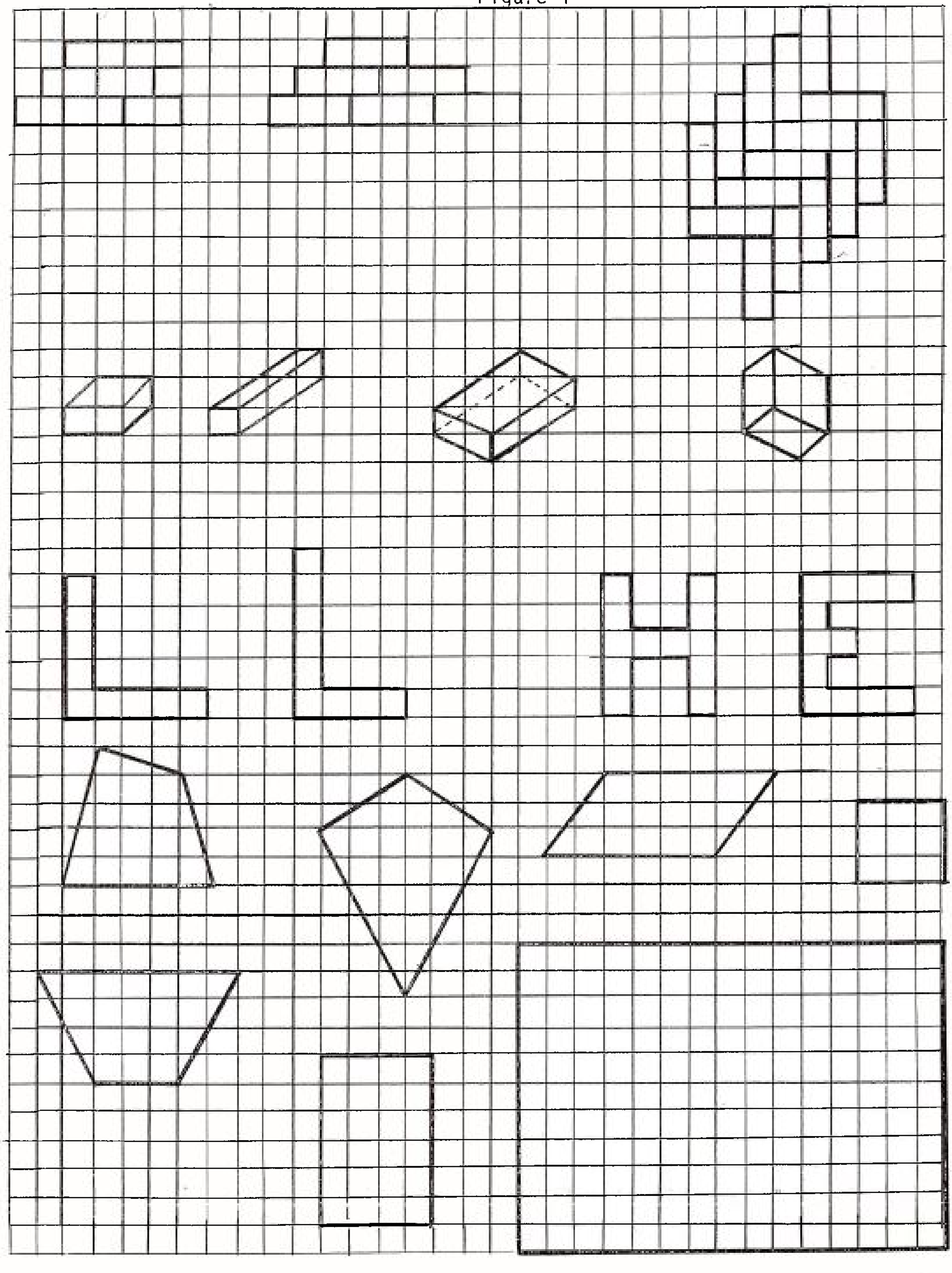


Figure 2 - Tiling the plane

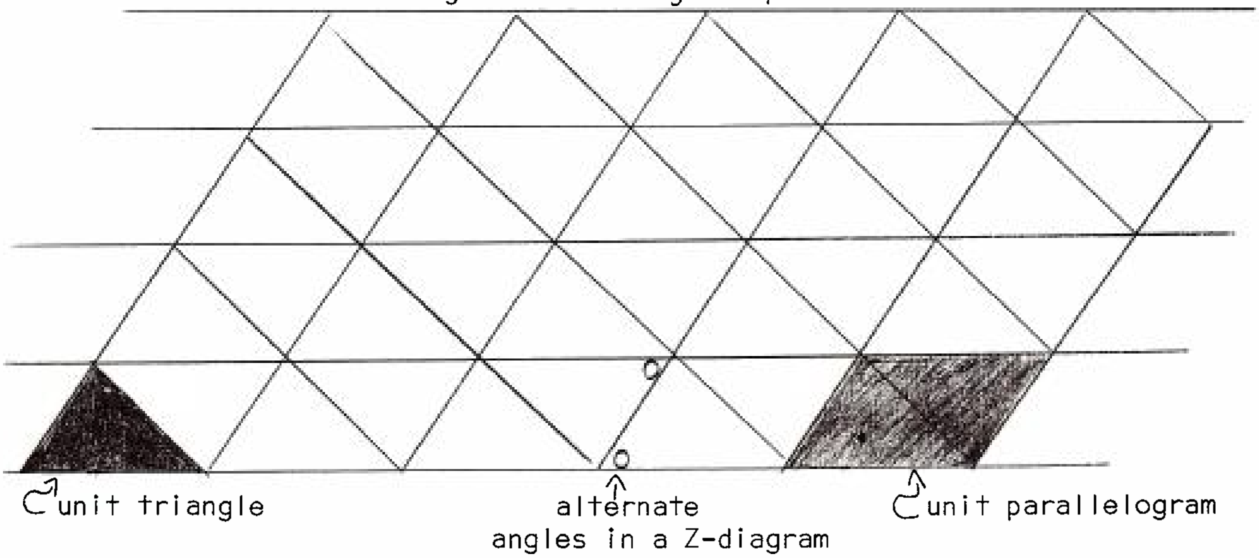


Figure 3 - Tiling the plane

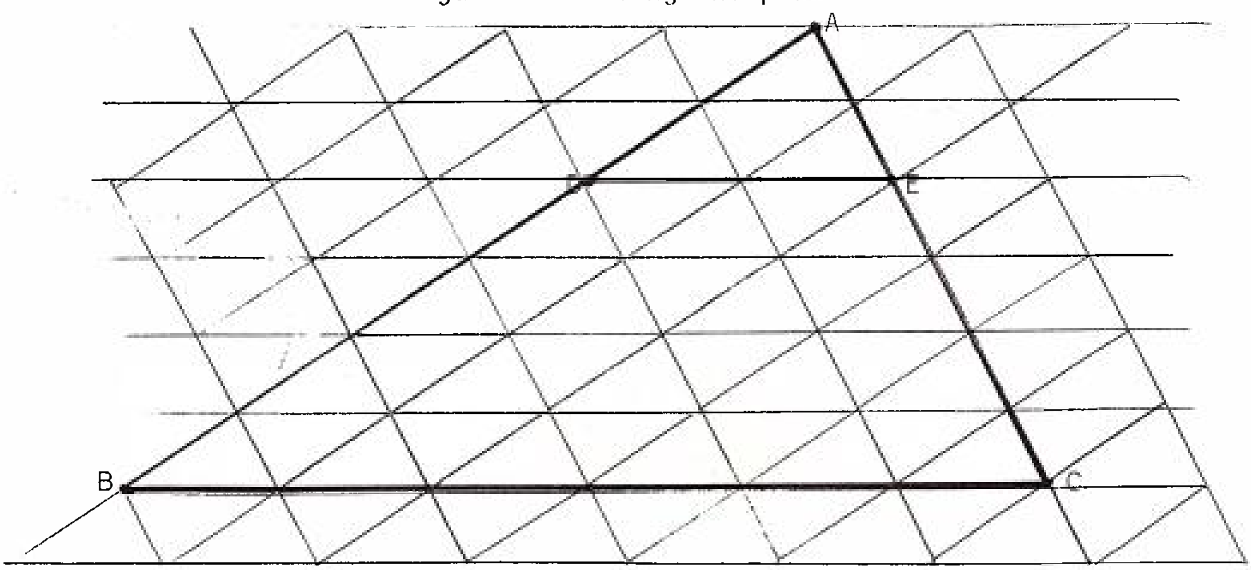


Figure 4

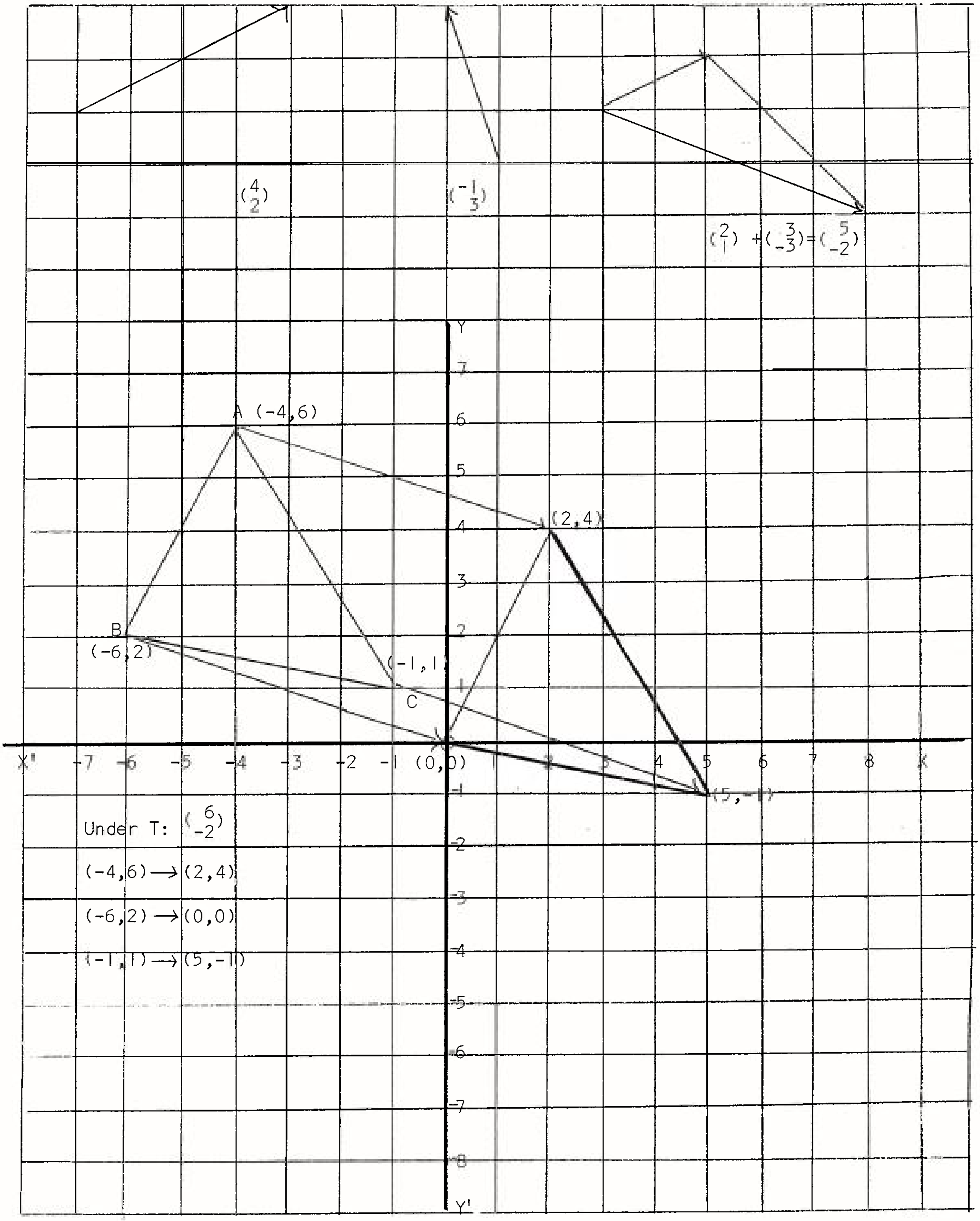


Figure 5

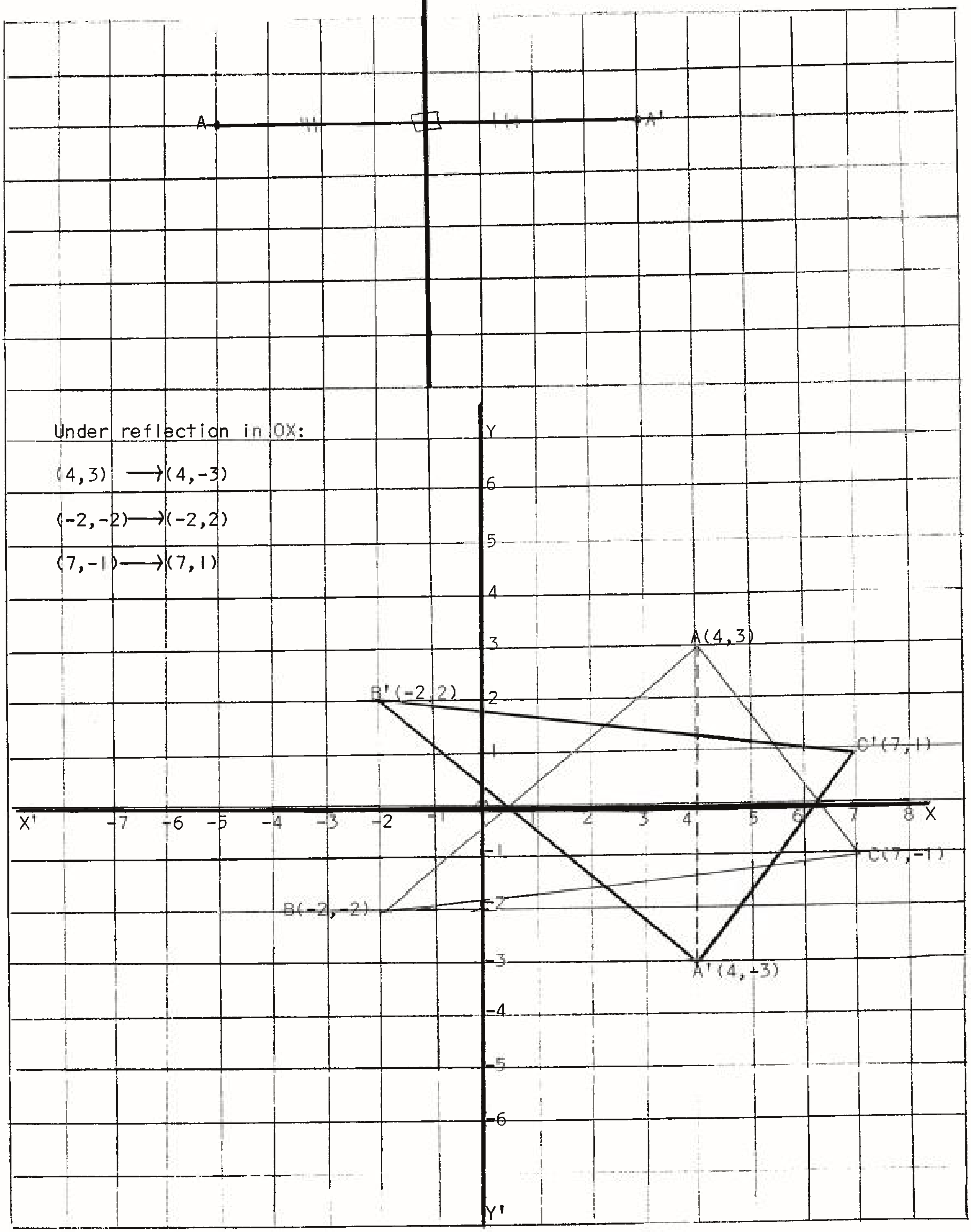
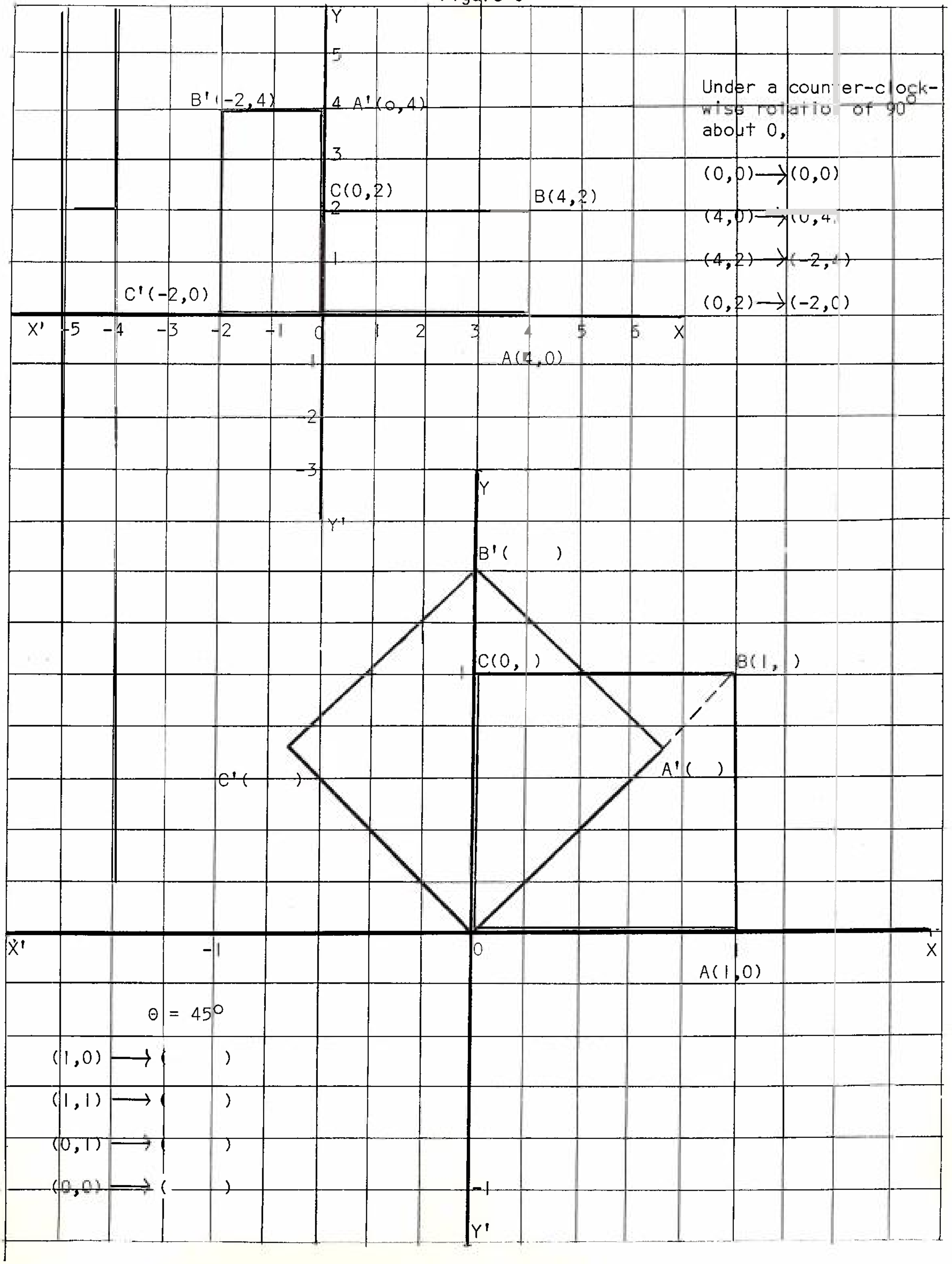
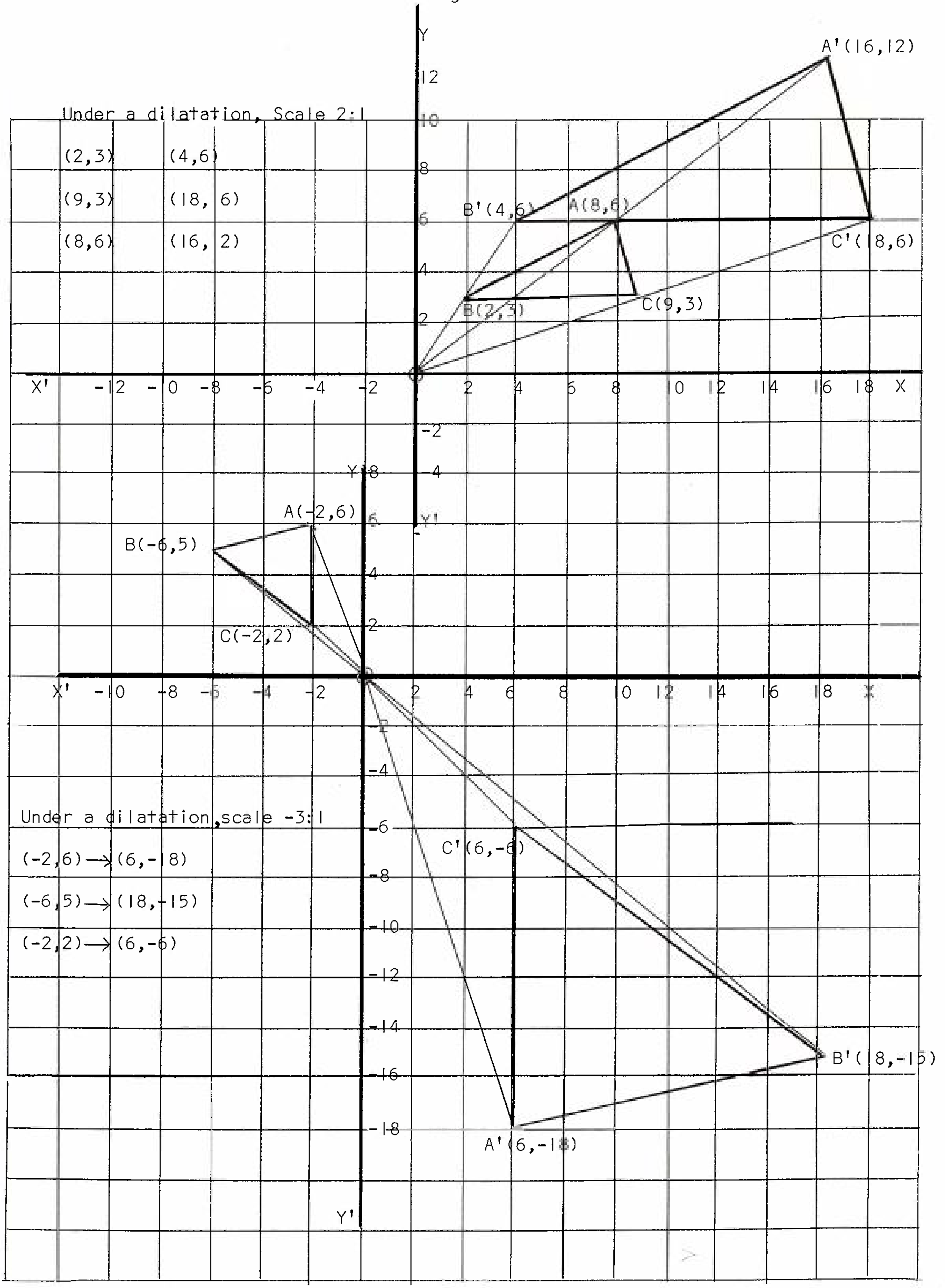

A
A'
Under reflection in OX:
(4,3) ⟶ (4,-3)
(-2,-2) ⟶ (-2,2)
(7,-1) ⟶ (7,1)
Y
Y'
X
X'
A(4,3)
B'(-2,2)
C'(7,1)
C(7,-1)
B(-2,-2)
A'(4,-3)

Figure 6



Under a counter-clockwise rotation of 90° about 0,
(0,0) → (0,0)
(4,0) → (0,4)
(4,2) → (-2,4)
(0,2) → (-2,0)
B'(-2,4)
A'(o,4)
C(0,2)
B(4,2)
C'(-2,0)
A(4,0)
X'
X
Y
Y'
B'( )
C(0,)
B(1,)
A'( )
C'( )
A(1,0)
θ = 45°
(1,0) → ( )
(1,1) → ( )
(0,1) → ( )
(0,0) → ( )

Figure 7



Under a dilatation, Scale 2:1
(2,3) (4,6)
(9,3) (18, 6)
(8,6) (16,12)
A'(16,12)
B'(4,6)
A(8,6)
C'(18,6)
B(2,3)
C(9,3)
X' X Y Y'
A(-2,6)
B(-6,5)
C(-2,2)
C'(6,-6)
B'(18,-15)
A'(6,-18)
Under a dilatation, scale -3:1
(-2,6) → (6,-18)
(-6,5) → (18,-15)
(-2,2) → (6,-6)

Figure 8

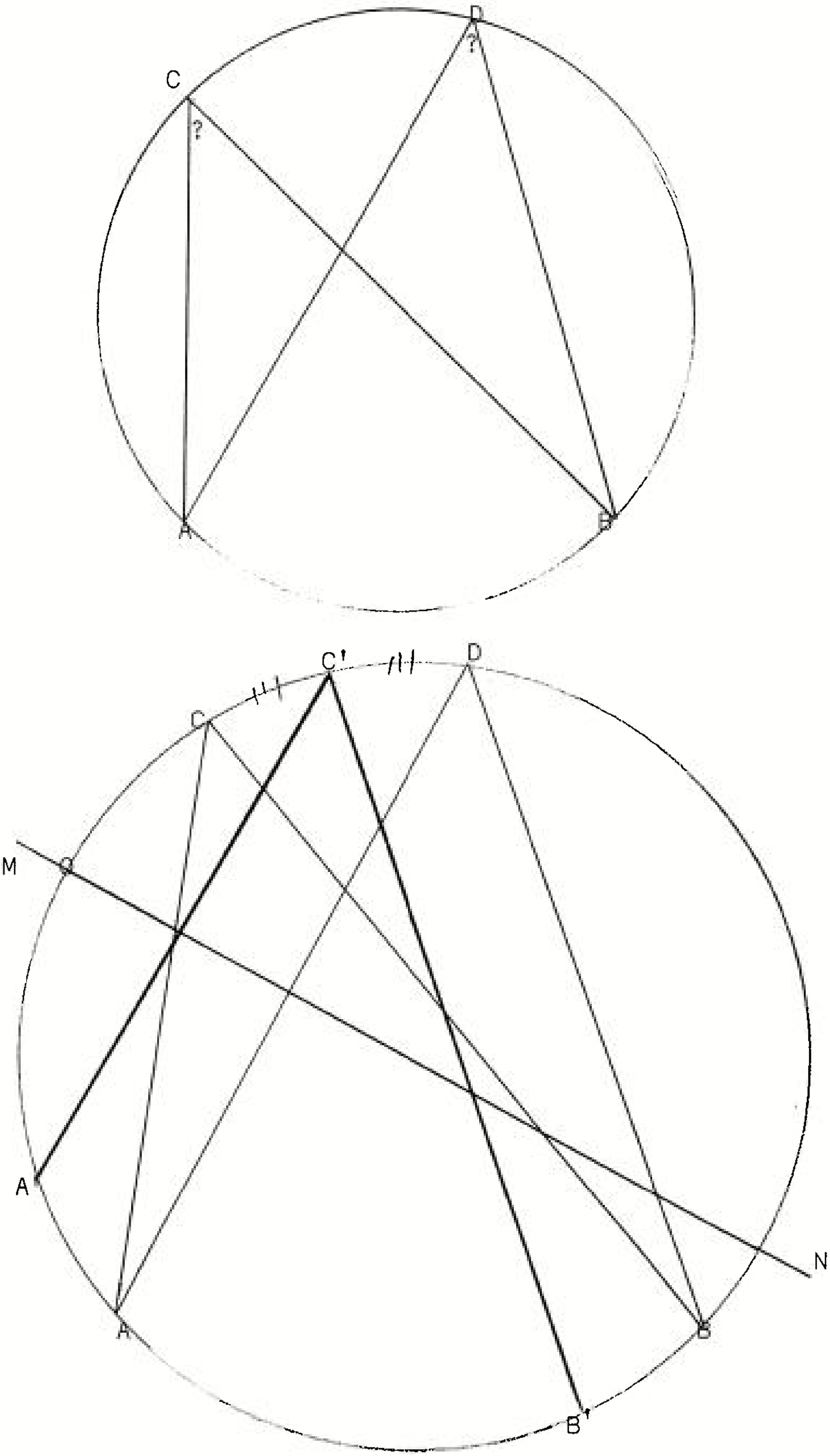


Figure 9