SOME K-6 GEOMETRY
MODERN MATHEMATICS

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What is modern mathematics?

We ask this question often to initiate a discussion wherever there is interest. Repeating it should not bore us, for, as yet, no one has found a concluding answer. And, as we get more insight, we tend to revise and improve our answers. To date, in the light of the work of the National Council of Teachers of Mathematics during the past 47 years, 'modern mathematics' means improved mathematics.

Continuing Concepts

In order to help the phrase 'modern mathematics' to signify improved mathematics, our comments will emphasize five geometrical concepts. These properly begin in early school days and continue through later school days. They include sets, forms, measures, illustrations, and proofs. They permeate geometry, and geometry permeates life.

Readiness

Indeed, the proper subject to study by all of us is geometry. It is everywhere. Forms, just as well as numbers, interest people of all ages. Even a pre-school child is shape-conscious. Interest in shapes, we believe, continues through the years.

Geometry not only appears in mathematical applications; it also underlies much of further work in mathematical theory. Yet, all too often in the past we have let children go on to secondary schools painfully innocent of geometry. This we now deplore. To defer geometry to the secondary school is sometimes to eliminate it entirely. For, being grossly ignorant of the subject, many pupils do not elect it. Instead, when they enter high school, our pupils should possess a rich knowledge of geometric facts, a thriving readiness for reasoning, and a strong desire to continue to learn geometry.

Sets

Mathematics treats sets - sets of numbers and sets of points. These, and the things that people can do with them, comprise the subject. Whereas the set of all numbers has no one-word name, the set of all points is called 'space'. Both the set of all numbers and the set of all points have infinitely many members, and both have numerous subsets. Through the years new numbers have produced an abundance of new algebras and new arithmetics. The idea of adding new points to the collection of all points may, at first thought, seem less likely than the idea of creating new numbers. But there is at least an abundance of new ways of looking at points, and the creation of new geometries has become commonplace.

So it is that studying sets of numbers and sets of points provides an excellent approach for pupils of all ages. Indeed, to summarize briefly what modern mathematics is all about, to impress your friends, to convince people that you are in the know, and to mystify those who query, you can simply say that mathematics today deals with "sets and stuff".

Subsets

Seldom do students of geometry investigate the set of all points. Rather, they look at interesting subsets. Hence the remainder of our discussion will focus on special subsets. These collections of points underlie the geometry of form, an especially exciting study for pupils in the grades. Note how simply the ideas evolve and how familiar the materials illustrating the ideas are!

We begin with a sheet of paper. What is its shape? What name does this shape have? If the sides all had the same measure, what would we call the shape? Is a square a rectangle? Sometimes this provokes a lively discussion. Squares and oblongs as special cases of rectangles and, possibly, rhombi and rhomboids as special cases of parallelograms might well ensue.

Definitions

Another outcome stresses the outline of the paper, be it an oblong (a rectangle with not all four sides having the same measure) or a square (a rectangle with all sides having the same measure). The set of points under consideration, not the interior of the figure, makes up the figure. This implies that, from kindergarten on, we try to use precise language, while the children are developing correct concepts. Of course, numerous activities to help children create clear conceptions should precede their memorizing of technical words. Once they know how squares differ from oblongs, and once they recognize how oblongs and squares are alike, the pupils then appreciate that the words 'square', 'rectangle', and 'oblong' may be useful additions to their vocabularies.

Definitions, just as well as technical terms, should grow from experiences rather than start the experiences. The idea of square corners, fundamental to rectangles, outweighs the word 'rectangle' in importance. Much of our discussion to follow in this article stems from the assumptions: (a) experience, concept, name; (b) activity, idea, definition.

Let us reassert, then, that geometry in 1967 emphasizes sets of points (outlines in such figures as squares, circles, triangles, et cetera) rather than portions of planes. Therefore, circles, ovals, triangles, rectangles, and other configurations can hardly be colored. To "color the triangles red" would mean literally to retrace the line segments that the pictures represent. To fill the interiors of triangles with a red crayon detracts from the concept we seek, namely, the border set of points.

More valuable than the tasks of "Color the triangles red", "Color the squares green", and "Put a cross in each circle" would be simply to "Change the paper with oblong shape to a paper with square shape", "How many parts with shapes like triangles can you make by folding a square paper once?" and "If you fold an oblong paper twice, how many oblong shapes can you get?" Similarly, the experience of "Fold a paper once in any way; then fold the paper again so that part of the folded edge fits along the rest of the folded edge. Then open the paper and tell about the folds" packs a greater potential than cutting and pasting disks and other parts of planes.

Creases

Let us return to our sheet of paper. It carries us on. Crease the sheet from one corner to the opposite corner. With squares this comes easily; with oblongs, not so easily. We urge the pupils to achieve a sharp crease. Indeed, we call for a perfectly sharp crease. Hopefully, some pupil(s) will demur at this. How can a crease be perfect?

Then we appeal to pupils' imagination. Suppose that the crease were perfect. How wide would the crease be? Further creases, such as the crease joining the other two corners, all illustrate the idea of length without width. While this is percolating for slow learners using a square paper, the sharpies can be working with an oblong paper. To name the idea which we have uncovered, we might suggest joining opposite corners of another paper with a mark made along a ruler. What can pupils call such a mark? A line. A ruler, incidentally, is a device for ruling (marking) lines. Since many rulers contain linear scales, we can also measure lengths with them. The key idea, however, is that of marking lines. A ruler primarily helps us to represent lines and segments of lines.

Lines

As pupils develop readiness for lines, we extend the idea that lines have length without limit. Our creases show parts (segments) of lines. Really, lines are neither creases, nor marks, nor anything physical. They are mental constructs - things with length, but no width. A bit of incidental learning involved, once pupils have the idea of opposite corners, arises in the word 'diagonal'. Possibly some have already learned "catty-cornered", "kitty-cornered", "cater-cornered", or some such expression, which possibly not only lacks elegance but also, etymologically, means four-cornered (French 'quatre'), rather than opposite.

Crosses

From sharp creases we focus attention on the place where two sharp creases cross. A pair of diagonals will do it, so will any pair of creases that cross. Then, - and please note the deduction and the imagination - if two creases, each perfectly sharp, cross, what has to be the size of the place where they cross? This leads to the idea of place without size. Pupils learn to call this idea 'point'. They learn further that a point, exacting as a line and as a number, stems strictly from the mind. Number, point, and line are ideas - not symbols, marks, or creases. With this in mind, if we move a paper containing a specific point, do we move the point? To answer this, pupils only need to consider the sole property that a point has position.

Points

Repeated foldings reveal that lines contain numerous points. There can be many crossings - in fact, an infinity. So a line is a set of points. Similarly, a paper represents another set of points, a plane. Furthermore, all points together make up space. So planes and lines are subsets of space. Occasionally, a mathematically mature youngster may think of lines as sets of points, planes as sets of lines, and solids as sets of planes. Rarely a pupil may inquire, "What does a set of solids make up?" This pushes the imaginations of most of us. We can use objects to represent point, line, plane, and solid. Theoretically, we can go on. However, we do not have models to represent four-space, five-space, et cetera.

Separators

This bears on various objects as separators also. A point separates a line into two half lines. A line separates a plane into two half planes. A plane separates space into two half spaces. What does a space separate into what?

Angles

The idea of sets of points goes farther. An angle is a set of points - two rays not on the same line, with a common endpoint. (A ray is a half line with a specific endpoint.) As you probably read in *The Mathematics Teacher* for January, 1967, not all mathematicians accept this definition for an angle. Such a definition does indeed raise questions about straight angles, angles of a triangle, and angles increasing or decreasing via rotation of one or both sides of the angles. As with the case of moving a specific point, can we rotate a set of points about a fixed endpoint?

The concept of an angle as a set of points, composed of two half lines and one common endpoint, probably suffices for K-6 geometry.

Measures

Except for the discussion about rectangle, all that we have thus far considered requires no measurement; much of geometry does not depend on size. Unfortunately, this very important idea in mathematics suffers from want of attention. Ironically, on the other hand, the idea of measurement also suffers from neglect. All of us know that quantity as well as number means much in life. All of us can recall in a moment numerous applications of mathematics that involve measurements. Yet the daily use of geometric forms and units of measure in traditional programs in elementary schools concentrates on computation, as if computation were the sole goal in mathematics for children!

Accordingly, we hope that all pupils have frequent opportunities to make rough, or unstandardized, measurements. How many steps from the throwing place should the target be? How many spans is your marble from Jack's marble? How many hands high is your horse? How long is your desk in pencil lengths? How many cupfuls does this jar contain? How many papers do we need to cover a desk? How many newspapers to cover the floor? How many wooden blocks to fill the box? Questions such as these can get at the much needed understandings.

Comparing an object with another object used repeatedly should happen early and often. It prepares the way for a sticky sort of question. How can a set of points, each having zero size, make up a line that has length other than zero length? The answer goes back to Archimedes, circa 250 B.C., who stated that a small segment laid down sufficiently often end to end can exceed a large segment. This assumption, which shifts from points to segments, is the key to measurement.

Number Lines

After pupils have had some experience with measuring lengths with understandardized units, they become ready to apply Archimedes' postulate in the making of number lines. (Strictly, they are number-line segments). On segments which pupils represent on papers or chalkboard they choose a starting point. To the right of this point they mark other points each at some chosen distance from the marked point nearest it. Then, using numerals, they assign numbers to each of the points marked. Helping children early to set up such a correspondence between points and numbers amounts to a tremendous accomplishment. This idea rates highly in importance geometrically. The same idea, by the way, promotes understanding in arithmetic. Order is basic. Here, early in the pupils' lives, arithmetic and geometry reinforce each other.

Paths

Unions of line segments become paths that interest most pupils of all ages. Making the transition from straight-line segments to curves leads the way to closed curves, which include polygons, such triangles, quadrilaterals, pentagons, hexagons, et cetera. The study of the properties of closed curves provides much interesting geometry. Even the girl who referred to "clothed curves" made a contribution to the day's work.

Points inside closed curves have significance, too. They constitute space of two dimensions, later to be measured and reported via a number, called 'area'.

Drawings

Along with our traditional neglect of geometry for the sake of geometry (even where some classes study geometry for the sake of drill in arithmetic via formulas not often comprehended) we tend also to neglect drawing-board geometry. Apparently we do not know for sure what the least age for ability for properly handling ruler and compasses may be. In some quarters people fear that pupils will use compasses as weapons of offense and defense. However, we doubt that we shall ever learn this minimum age for proper use by keeping compasses away from pupils until they reach high school. Numerous worthwhile projects on the drawing board conduce to enhanced geometrical insights. They contribute to the store of knowledge that pupils need for general education and for mathematical education. Besides numerous standard exercises such as bisections, parallels, and perpendiculars, teachers and pupils can improvise some.

Illusions

Drawing, folding, measuring, cutting, and observing provide numerous opportunities for pupils to learn fundamental lessons of life: Things are not always what they seem to be. No better way exists to get at this principle. When the teacher holds up a white cylindrical stick, asks the pupils what it is, and calmly unrolls the paper which they will take to be a piece of chalk, the lesson endures. Illustrations of numerous optical illusions should be in the teacher's supply of materials.

Op Art

Recently museums of art have become ever more fascinating. The collections of geometrics interest us especially. The popularity of this kind of art has widely spread to encompass illustrations in books, journals, and newspapers. Advertising matter of all kinds includes geometrics. Blurbs for books now contain geometric designs, whether the books contain mathematics or not. Illustrations of a special variety, Op Art, seem to predominate. They appear not only in galleries of art but also in shops and stores, especially in materials for women's clothing, wall hangings, upholstery, and floor coverings. The implications for interesting projects in art and mathematics have almost no limit. This aspect of modern mathematics helps us to emphasize the contemporary spirit, the feeling for discovery, the creative urge.

Proof

The gradual development of ideas of abstract geometry, such as point, line, plane, closed curve, interior point, exterior point, half line, half plane, angle, et cetera, encourage pupils to go beyond things visible to things invisible. We refer to mathematical constructs, ideas, and objects of thought which become what we make them. Physical objects help us to learn, but the elements of geometry are mental objects.

An aim of present-day mathematics emphasizes the importance of correct language. 'Number' differs from 'numeral'. 'Line' differs from 'line segment'. We represent length, area, and volume by using numerals, because we can count (directly or by rule of computation) the units. Yet the segments, surfaces, and solids that we draw, measure, and calculate contain points which have infinite numerosity.

We have seen that mathematics stems from well-taught imaginations. Points have position only; we do not actually use them; we use imagination to understand them; what we see only crudely represents them. Similarly, 'line', 'plane' and 'space' denote subsets of points that we can imagine perfectly but which physical objects represent imperfectly.

To communicate one's ideas (imaginations) to one's fellows, one needs to use language correctly. To convince one's fellows that one's conclusions merit acceptance, one needs to use proof correctly. As the child matures from pre-schooler to teenager, he adopts progressively more logical forms of proof. To expect him to produce full-blown proofs at some arbitrary age, say 15, is unrealistic. Rather, we might appeal first to authority; these two rectangles have the same size because Dad says so. Next, the pupil may depend on measuring; these two triangles measure the same along three pairs of sides. Next, he may depend on computing, which entails counting; these two oblongs contain equal numbers of square units employed as measuring sticks.

Reasoning appears in all such developments. Gradually, pure reasoning renders dependence on physical objects less necessary. The child reasons that, if sticks represented the idea of 40, then 40 means four tens of sticks. To answer the question "How many eights are in 40?" one needs only to imagine that two sticks removed from each 10 makes four twos, or one eight. This plus the four eights, made from the tens, produces five eights. One reason that there are five eights in 40.

Similarly, questions in geometry receive convincing answers via reasoning, with configurations needed only for more and more complicated questions. Indeed, sketches often suffice. One reasons correctly from incorrect figures.

Pure reason serves also to produce answers not entirely possible to achieve via physical objects. If one perfectly sharp crease crosses another perfectly sharp crease, the size of their intersection has to be zero. That is, a zero width (abstracted from a less-than-perfectly-sharp crease) crossing another zero width has to be a crossing of zero width.

Numerous reasonings such as these prepare pupils not only for more formal and more rigorous mathematical deductions but also for less formal and less rigorous everyday decisions. The upshot emerges as the transfer of some training for which teachers have quite specifically planned and taught. Proof of the sort appropriate to the maturity of the children belongs to every program in mathematics. To defer it to the period as late as high school is to rob pupils of their fair and proper intellectual birthright.

<u>Summary</u>

Proof lies at the heart of mathematics; if our endeavors do not lead to proofs, then they hardly merit the name 'mathematics'. Traditionally, in grade schools we have emphasized the computations of arithmetic. And we have shunned geometry. In so doing, we have erred - twice:

1. Geometry could not be separated from life; it is everywhere. To defer it was to delay an important part of general education. To postpone it was, in some cases, to eliminate it, for the students later elected not to take it.

2. Geometry could not be learned effectively in the traditional Grade X crash program. Pupils had to learn numerous facts and compose proofs at the same time. As a result, many resorted almost entirely to memorizing the facts and the proofs.

The approach in improved mathematical programs encourages pupils to learn geometric facts and geometric ways of thinking from kindergarten on. This does not mean moving the traditional, concentrated course to a lower grade. Rather, it signifies a gradual approach - spiral learning from simple essentials to more and more mature understandings. Intuition lies at the heart of the matter. Objects, pictures, and drawings illustrate geometric ideas. What could be simpler than sheets of paper to illustrate planes, or creases in paper to illustrate lines, or crossings of creases in paper to illustrate points?

Along the line teachers emphasize the importance of experimenting - drawing, observing, guessing, counting, measuring, and reasoning. Insight increases because the subject fascinates pupils, and interest underlies learning.

Outlook

The best time to study geometry reaches from the cradle to the grave. This makes sense to the teacher who, being a victim of a by-gone sink-or-swim course in high school, promptly sought other electives. This heartens the teacher who, despite the shock he had in Grade X, now confronts the suggestion (or, possibly the command) that he begin to teach geometry to pupils in the lower grades. This emboldens the teacher who, aware of the fact that his memory fails on some of the details, can recapture them in the bright line of good understanding. This light prevails as his pupils learn geometry in a gradual, intuitive program that emphasizes observations, inductions, and deductions.

Are you willing to embark or to enlarge your endeavors on such an important journey in the education of all children?