I SHALL TEACH ARITHMETIC

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> When called before the Tribunal of the French Revolution to state what useful thing he would do to deserve life, Legrange answered, "I shall teach arithmetic."

Some Basic Questions

Educational man has long wrestled with the age old problems of WHAT to teach, WHEN to teach it, HOW to teach it, and TO WHOM to teach it. Incidentally, too, he has grappled with the concomitant problem of WHO SHOULD DECIDE these major issues.

As examples of such problems, I draw your attention to the following highly controversial issues:

<u>What?</u> 1. Should religion be taught in the schools?

<u>When?</u> 2. At what age in a child's life should instruction in reading begin?

<u>How?</u> 3. Is the enterprise method an effective one for teaching social studies?

<u>To Whom?</u> 4. Should the children in the lowest twenty percent on a standard intelligence scale receive the same type of instruction in the same form of curriculum as the children in the highest twenty percent on this scale?

<u>Who?</u> 5. As an example of the concomitant problem of WHO SHOULD DECIDE, I urge you to consider the present issue of accreditation.

You and I have been faced with a <u>fait accompli</u> as far as the introduction of modern mathematics in the elementary school is concerned. That is, decisions have been made as follows:

<u>What?</u> 1. The STA program embodies the "what" that is necessary in arithmetic.

<u>When?</u> 2. The STA program has an approved sequence of learnings.

<u>How?</u> 3. The STA manuals outline adequate and acceptable techniques of instruction.

To Whom? 4. The STA program is for all children, with their wide range of individual differences, in our elementary schools.

Who? 5. Furthermore, the decision as to WHO DECIDES has been taken: the Minister and his Department!

Before I continue, however, I must interject a note of caution that I have neither stated nor implied a value judgment as to the rightness or wrongness of the *fait accompli* of the WHAT, WHEN, HOW, TO WHOM and DECIDED BY WHOM questions noted earlier; I simply comment that, like the recently elected government, it is now with us.

The Problem of Re-selection of Content

Emphasis Upon Basic Intellectual Aims. When one looks even casually at the STA program, one is struck by the fact that there is undoubtedly an emphasis upon intellectual aims. In essence this forms my first postulate: the STA course is essentially directed towards intellectual rather than personal, social, societal, cultural, or vocational aims; and this is in marked contrast to traditional courses.

That this is in keeping with the expressed desires of Albertans is clearly shown by the expression of such wishes as revealed in two key studies - the Andrew study, and the Downey study, both of which showed that the intellectual aims rank first in both public and professional opinion of the task of our schools in Alberta.

Application of Knowledge of the Processes of Intellectual Development in Children. My second postulate is that modern courses and their methods of instruction must be and are in keeping with what we know of the processes of intellectual development in children. The STA course does, in fact, do more than pay lip service to this fundamental principle.

The work of Jean Piaget has indicated that there are five discernible stages in perception in the growing child. They are as follows:

1. The sensory-motor stage, from birth to about 2 years of age.
2. The stage of pre-operational thought, from about 2 years to about 4 years.
3. The stage of intuitive thought at the kindergarten and primary level.
4. The concrete operations stage in the balance of the elementary grades.

5. The formal operations stage, which begins in the late elementary and continues through the junior high school grades.

As stages three to five are of major concern to us, I propose now to outline these stages in slightly more detail.

Stage Three - The child lives in a world of symbols. Through word and image symbols he has created for himself a stable internal world in sharp contrast to the shifting,changing world of perceptions previously known.

From immediate imitation of present sounds, actions and beings, he has progressed to deferred imitation of absent things. The abstraction from action to thought has been accomplished; yet but the first faltering steps have been taken in the long walk from the particular to the general. He still relies upon transductive reasoning, that is, he still holds that as he knows the results to be right, then so must the means or process of arriving at the results be right!

His notions of groups and especially of groups of operations are still hazy, and ideas of invariance are just beginning to take root. Soon, he will be thinking operationally in constructing concepts of groups of operations with invariant features.

Stage Four - In stage four he has progressed to the stage of operational groupings. He puts classes together mentally, classifies objects and actions, and forms more inclusive classes from the combination of several sub-classes. He begins to serialize asymmetrical relations of "greater than" and "less than". He begins to be capable of understanding that number systems are products of classification and seriation, or ordering.

Ideas of space, time, number, and of the material world around him flood into his mind, but these are complex ideas, and he is, so far, only capable of comprehending them in more-or-less concrete terms.

He is beginning to be less ego-centric in his attitudes and behavior and the elements of detached logical reasoning emerge slowly and often painfully.

Stage Five - By the end of the elementary school, the child's ability to perform abstract operations becomes apparent. The child of the earlier stages has been concerned with action-in-progress, with the here-and-now. At this stage of the pre-adolescent, he thinks beyond the present, back into the past and forward into the future. The historical sense emerges. Hypotheses begin to be formed and tested. The elements of postulational thinking and of logical deduction struggle for form and function.

My second postulate, then, is that the STA course is consonant with knowledge of the development of thought in children.

Perception and Perceivers. If perception is considered as the process of organizing and interpreting sensations received through the senses, then it can be shown that there is evidence for the existence of characteristic modes of perceiving and for the existence of certain types of perceivers.

Some people may have a preferred sense: the visiles, those who perceive best visually; the audiles, those who perceive best auditorily; and the tactiles, those who perceive best through kinaesthetic means.

Two main types of perceivers have been postulated, the analytic and the synthetic. The analytic tends to concentrate upon isolated detail, rarely seeing the total patterns in a situation at first, but gradually synthesizing the detail into a whole. The synthetic, on the other hand, sees the total field as an integrated whole which he later analyzes in order to perceive the details. These ideas have import for classroom teachers in that they suggest that our grouping procedures, introductory work with symbols at all levels of complexity, provision for individual differences, and our remedial work should be planned with the ideas of the different modes of perceiving and the different types of perceivers in mind. This, in essence, is my third postulate.

Scientific Method and Problem Solving. In traditional arithmetic programs, it was assumed that the fundamental processes are best learned through association and drill, and so there has been a tendency to confine thinking in this area to associative thinking alone, even in the so-called problem-solving activities within the program. This solving of problems through the use of techniques of associative thinking is in sharp contrast to the problem-solving activities of STA. Here the definition of problem-solving is closer to "the process of overcoming difficulties encountered in the attainment of objectives". The sequence of steps used is:

1. Comprehension - read the problem carefully.
2. Translation - write the equation that represents the action in the problem.
3. Computation - do the computation that accompanies the equation.
4. Interpretation - write the statement that answers the problem.

The thought processes involved then are not merely those of associative thinking but are extended to convergent thinking.

No longer do we have to rely upon key words and gimmick phrases such as: "Look at the numbers in the problem. Decide whether to add, subtract, multiply, or divide." - without ever being shown how to decide! In other words, the STA program invites a higher form of thinking in its problem-solving techniques, and makes use of two very powerful and refined aspects of mathematical technology, one of which is the equation. This is my fourth postulate.

Emphasis Upon the Contribution of Knowledge to a Recognized Subject Matter Discipline. Among the many criteria that have been used by curriculum workers in recent years - and these include "survival", "utility", "interest", "social significance", and so on - the contribution that a particular subject makes to an organized field of knowledge has tended to become the major one employed. It is not hard to see that traditional courses in elementary arithmetic have failed to meet this criterion, as they were, even in their problem-solving aspects, concerned mainly with computational speed and accuracy. Children were required to jump immediately from the statement of the problem to the computation which solved it. In the STA program, the intermediate step of translating the problem into the appropriate equation deduced from the mathematical action perceived in the problem, makes a definite contribution to the study of more advanced mathematics, and hence contributes something to a recognized subject matter discipline.

The essence of the application of this criterion in other aspects of the STA program is the emphasis upon structure: that is upon the basic principles of and the patterns of relationships within the subject. The STA program has reselected and replaced content and method in order to reveal clearly to the pupils the underlying principles and relationships that give mathematics its structure.

My fifth postulate, then, is that modern courses should - and STA does - contribute knowledge to a recognized subject matter discipline and reveal the structure of the discipline to the pupils.

The Seeing Through Arithmetic Program

From the five postulates I have just discussed with you, one could derive certain theorems which would indicate that the STA program is constructed on the following principles:

- It emphasizes the mathematical values and aims of arithmetic, rather than the social or other aims and applications, thus contributing to the basic intellectual objectives of modern education.

- The STA course recognizes and employs the most recent research findings in developmental psychology.

- In its methodology, the STA course attempts to appeal to all types of perceivers through all modes of perception.

- The STA program moves beyond associative type thinking to at least convergent thinking in its problem-solving aspects.

- The STA program emphasizes the structure of mathematics, employs several powerful aspects of mathematical technology, and thus contributes greatly to the recognized discipline of mathematics.

Deductive Nature of the STA Program

Few would deny that the Seeing Through Arithmetic and other modern approaches to elementary mathematical arithmetic are vast improvements over traditional arithmetic programs. It is my contention, however, that improved as they may be, they still suffer from a serious flaw, and strangely, this flaw arises from what I consider to be too restrictive an interpretation of a current definition of mathematics, namely that "mathematics is the search for patterns".

In the STA program, this is interpreted to mean: "Mathematicians have found certain patterns in elementary problem-solving, and have invented certain equations to describe and to reveal these patterns." Two of the equations so used can be shown generically as follows:

(1) $\boxed{a \ o \ b \ R \ c}$

(2) $\boxed{a : b \ R \ c : d}$ (rate-comparison)

Examples of the first equation are:

$$5 - 3 = 2$$
$$12 + n = 4$$
$$3 \times n = 20$$

Examples of the second are:

$$15/3 = n/1$$
$$n/20 = 35/100$$
$$3/4 = 4/n$$

As, in the first equation, the "unknown" n can appear in any one of three places, replacing either a, or b, or c, and o can be any one of "add", "subtract", "multiply", or "divide", whilst, for almost all problems, R is restricted to "equals", it then follows that there are

$3 \times 4 \times 1 = 12$ basic types of non-comparative problems.

In the case of the second, _n_ can replace, _a_, _b_, _c_, or _d_, and _R_ is again generally restricted to "equals", thus there are $4 \times 1 = 4$ basic types of rate-comparison problems.

This means that our pupils sequentially, developmentally, and systematically are taught to translate all their problems into one of the 16 basic equations, are taught a corresponding process to solve the equation, and are thus restricted to a convergent type of thought process.

The problem-solving approach, too, is necessarily _deductive_ in nature:

- Comprehend the problem.

- From the structure of the problem, determine the structure of the equation that represents it.

- Perform the standard computation for this equation-form.

- Interpret the results.

Now this is, as I have said before, a tremendous advance over traditional courses, and a most welcome one. My plea to you, however, is for the introduction of certain _inductive_ methods to supplement this deductive approach, to broaden problem-solving, to involve more than associative and convergent thinking, to involve _divergent_, _inductive_ thought in our mathematical problem-solving activities. So often, in life's problems, there is no standard, ready-made pattern of solution: one has to wrestle with the problem inductively, and the pattern only emerges after strenuous divergent thought. Let us equip our pupils with at least the readiness steps for this scientific mode of thinking.

Inductive Patterns

(Note:- the following dialogue is based upon ideas presented by Professor G. Polya in his two-volume series entitled _Mathematics and Plausible Reasoning_, published by the Princeton University Press, 1954.)

Dialogue to Illustrate the Sequence of Steps in Inductive Problem-Solving

Teacher: - (holding up before the class a regulation chess-board). How many squares are there on this chess-board?

Pupil: - (counting). Eight rows of eight . . . sixty-four.

T. - Is that all you can see?

P. - Yes, thirty-two are black and thirty-two are white.

T. - Good. What are the dimensions of each of the squares that you can see?

P. - Oh, about two inches by two inches, I guess.

T. - That seems a reasonable estimate, but - (holding up a second then a third board) - what are the dimensions of this, and this?

P. - Oh! I see. I think you can say that each of the squares is one unit in area.

T. - Good. For each board, although they are different in area, any one of the squares you can see can be said to be one unit in area.

P. - Oh! Wait! I see now! The whole board is a square and (excitedly) . . . there are some squares two units by two units . . . some three-by-three . . . There are hundreds of squares!

T. - Now you are beginning to see that the question I asked wasn't really an idle one; but <u>are</u> there actually hundreds?

P. - Well - there certainly are a lot, and (ruefully) they are very hard to see!

T. - So! They are hard for the eye to see, so let us try to "see" them in a different way - through a mathematical microscope, so to speak.

P. - How can you do that, sir?

T. - The way we have "seen" through other problems: let us <u>order</u> what we see. Let us find some pattern in the quantities before us.

P. - What sort of patterns do you mean, sir?

T. - (smiling). Look above you at the acoustic tile in the ceiling. Look beneath your feet at the tiled floor. Suppose, now, that the ceiling and the floor were like this chess-board, each a square structure of squares. Now! How could I rephrase my original question, "How many squares are there on the chess-board?" so that it is more general, so that it could refer to the chess-boards, the ceiling, the floor, in fact to any square structure of squares?

P. - (after several false starts) . . . How many squares are in an n by n square structure of unit squares?

T. - Very good. Now instead of confining our attention merely to an 8 x 8 board, we have now generalized our problem, namely, we are considering an n by n board. We have created many more problems than the very restricted one we started with. Often the solution to a set of problems is much simpler to find than is the solution to one member of the set. So let us now solve the new general problem.

P. - I know, sir! We have done this type of problem before!

T. - Good. Then what is the next step?

P. (1) - Find the simplest case . . .

P. (2) - Find the first case . . .

P. (3) - Use one . . .

T. - Good! Good! Now take it easy! Yes, you are all correct. What is the specialization, the simplest form of our new general problem?

P. - How many squares on a one by one, sir? The answer is one.

T. - Yes, of course. Now what?

P. (1) - Try a two by two, sir . . .

P. (2) - The second simplest . . .

P. (3) - Two units by two units . . .

T. - I'll need a two by two to clobber you if you shout so exitedly!

P. - There are five squares now: one big one and four little ones.

T. - Good. Now what?

P. - A three by three has ten squares.

T. - How many unit squares has it?

P. - Nine, sir . . . oh! I see, there are nine one by one's and . . four two by two's and one three by three . .

T. - Excellent, now you are beginning to see through your mathematical microscope! In fact, you are well on your way through the inductive process . . .

Now let us put some order into our observations. (writes).

n x n	No. of squares
1 x 1	1
2 x 2	1 + 4
3 x 3	1 + 4 + 9

Can you see the pattern?

P. - Yes sir, the next should be 1 + 4 + 9 + 16, 30, sir!

T. - You have formed a conjecture about an unfolding pattern. Conjectures are tricky things. Can you verify it?

P. - Yes sir! On a four by four board there will be 16 little squares, then there will be four two by two's and, let's see now . . . nine three by three's . . . Yes sir, it is verified.

T. - Yes, for one case, but is this enough?

P. - We could try it for five by five.

T. - Of course, and for other cases too, but will we then have proved our conjecture?

P. - I guess not sir, but if every one we try is true, then

T. - Well?

P. - I see sir, we cannot prove it, I guess.

T. - Later on you will find a way to prove conjectures of this nature, it is called mathematical induction. However, we will now assume our conjecture to be true if it checks out on one or two more.

P. (1) - It does for five, sir.

P. (2) - And for six . . .

T. - Very good, but let us not forget our problem! What was it?

P. (1) - How many squares are there on a chess board?

P. (2) - . . . on any square board!

P. (3) - . . . on an n by n . . .

T. - Good! Let us look again at our numbers.

n x n	Number
1 x 1	1
2 x 2	1 + 4
3 x 3	1 + 4 + 9
4 x 4	1 + 4 + 9 + 16

What do you notice in the series under "number"?

P. - They're squares, sir!

T. - Excellent! (writes).

n x n	Number	Series
1 x 1	1	1^2
2 x 2	1 + 4	$1^2 + 2^2$
3 x 3	1 + 4 + 9	$1^2 + 2^2 + 3^2$
4 x 4	1 + 4 + 9 + 16	$1^2 + 2^2 + 3^2 + 4^2$

What would be the series for an n by n?

P. $1^2 + 2^2 + 3^2$ . . . up to n^2?

T. - Excellent! Now - if our conjecture is true, we have solved our general problem, and along the way, our special problem of the chess board. Does anyone have the answer to our original problem?

P. - (chorus) - Yes, sir. Two-hundred four!

T. - Excellent . . . ah! there's the bell.

. . . .

To instruct and educate a child, therefore, does not mean to overwhelm him with bits of knowledge and with precepts, but to provide him, in proportion to his capacity and his needs, with such nourishment as he is capable of assimilating; to educate is not an effort from the outside to impose behavior or knowledge; it is to put the child in a position where he can make that effort himself ("Children are springs, not wells", a Belgian minister once wisely observed).[1]

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[1]Robert Dottrens, The Primary School Curriculum. Unesco, Place de Fontenoy, Paris - 7e, 1962. (p. 152).

Summary of Steps (At an Adult Level)

How many oranges are there in a pyramid with an equilateral triangular base if the smallest number of full dozens of oranges is used?

Generalize: How many oranges are there in a regular triangular pyramid?

Specialize to simplest cases: How many in the top layer? (1), in the next layer? (3), in the next? (6).

Conjecture the pattern: 1 + 3 + 6 + 10 + . . .

Verify for several cases: e.g., the fifth term in the series should be 10 + 5 = 15.

Verification:

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0
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000
0000
00000        (verified)
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State the general problem: What is the sum of the first n triangular numbers?

Search for patterns within the pattern:

First term		Second term		Third term	
0	1	0 00	1 + 2	0 00 000	1 + 2 + 3

n^{th} term: 1 + 2 + 3 + ... n

Now, look closer!

$1 + 1 + 1 + \ldots + 1$ Sum = $\frac{n}{1}$

$1 + 2 + 3 + \ldots + n$ Sum = $\frac{n(n+1)}{1 \times 2}$

$1 + 3 + 6 + \ldots + \frac{n(n+1)}{1 \times 2}$ Sum = ?

Conjecture:

$$\frac{n}{1} \qquad \frac{n(n+1)}{1 \times 2} \qquad \frac{n(n+1)(n+2)}{1 \times 2 \times 3} \qquad (?)$$

Verify for several cases:

One layer $\frac{n(n+1)(n+2)}{1 \times 2 \times 3} = \frac{1 \times 2 \times 3}{1 \times 2 \times 3} = 1$ (✓)

Two layers $\frac{n(n+1)(n+2)}{1 \times 2 \times 3} = \frac{2 \times 3 \times 4}{1 \times 2 \times 3} = 4$ (✓)

etc.

State the conjectured solution: The general solution is that there are $\frac{n(n+1)(n+2)}{1 \times 2 \times 3}$ oranges in a total of n layers.

Solution of special problem: The solution to the special problem is:

$$1 + 3 + 6 + \ldots + \frac{(n)(n+1)}{1 \times 2} = \frac{n(n+1)(n+2)}{1 \times 2 \times 3} = 12\ p$$

New problem! What is the smallest value of n to satisfy (in integers)

$$\frac{n(n+1)(n+2)}{1 \times 2 \times 3} = 12\ p \quad ?$$

Solution:

$$\begin{aligned} n(n+1)(n+2) &= 1 \times 2 \times 3 \times 12 \times p \\ &= 1 \times 2 \times 3 \times 2 \times 2 \times 3 \times p \\ &= (2 \times 2 \times 2) \times (3 \times 3) \times p \\ n(n+1)(n+2) &= p \times 8 \times 9 \end{aligned}$$

(intuition? or persistence?)

Therefore the smallest value of n is $n = 7$.

Check:

Layer 1 2 3 4 5 6 7
Number 1+3 +6+10+15+21+28 = 84 = 12 x 7

There are 7 dozen oranges in the structure

Recommendation

This pattern of elementary inductive problem-solving is fundamental to mathematical and scientific modes of thinking. Every attempt should be made to establish at least the readiness steps for such thinking in the elementary school. The application of this recommendation is left to the ingenuity of the arithmetic teacher.

Summary of Pattern

1. State the given, specific problem.
2. Generalize the problem.

> This procedure of heaping up new problems may seem foolish to the uninitiated. But some experience in solving problems may teach us that many problems together may be easier to solve than just one of them - if the many problems are well co-ordinated, and the one problem by itself is isolated.[2]

3. Specialize to the simplest case, or to the simplest analogous form.
4. Begin the "inductive" process.
5. Form the conjecture through seeking out the patterns, the structures of the problem.
6. Verify the conjecture for specific cases.
7. If possible, prove the conjecture, (perhaps by mathematical induction).
8. Solve the general problem.
9. Solve the specific problem.

For the Thoughtful Reader

Try the pattern on this question:

Into how many portions is space divided by 5 planes in the "general" position?

Hint: Start with points dividing lines, lines dividing areas, then to planes dividing space. Good Luck!

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[2] G. Polya, Induction and Analogy in Mathematics, Princeton University Press, Princeton, New Jersey, 1954 (p. 47).