COMMENTARY

MATHEMATICS TEACHING AND LEARNING IN THE 1960s AS REFLECTED IN *delta-K*

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At the time I was writing this commentary, the August 2012 issue of *Scientific American* came out. Among black holes, ancient DNA, the neuroscience of joy, and gamma rays from clouds was an article entitled "Building a Better Science Teacher" (Wingert, 2012), which talks about mathematics teachers, as well.

While it is not my purpose here to review that article, I would like to note that it makes extensive reference to well-funded, large-scale projects, primarily sited in universities. These projects aim to provide better education for teachers in order to give them the rich background necessary for effective mathematics and science instruction, with special emphasis on subject matter education. President Obama has even pledged to train 100,000 highly effective mathematics and science teachers by 2020. Acronyms and catchy program names abound. Better science, technology, engineering, and mathematics (STEM) teachers and programs are needed. Projects like UTeach, UKanTeach, and Teach for America are aiming to educate

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teachers for STEM at the high level anticipated, as well as researching the effects of such programs. Deborah Ball's work on the Learning Mathematics for Teaching project (University of Michigan) and Paul Cobb and his team's work at Vanderbilt University are highlighted in the article.

As the article suggests, "Not since the Russians launched Sputnik in 1957 have American policy makers, educators and businesses been so focused on improving math and science education" (p. 62).

NEW MATH AND THE 1960s

As the Mathematics Council of the Alberta Teachers' Association (MCATA) celebrates its 50th anniversary, it may seem like we're in the 1960s again. The 1960s were a time of change in political activism; in social goals and mores; in reactions to war; in music; and, indeed, in mathematics curriculum, learning, and teaching (not to mention Tom Lehrer's "New Math" song).

As was noted in several *delta-K* articles of the time, the sixties saw large-scale projects in redeveloping and rethinking school mathematics and its teaching (see the articles by Krider and by Coulson, for example). Examples include the School Mathematics Study Group (SMSG), based at Stanford University; the University of Illinois Committee on School Mathematics (UICSM); and the bellwether report of the U.S. National Commission on Mathematics in the late 1950s. These projects—led by teams of mathematicians, educators, and teachers (particularly the SMSG)—produced curricular materials that were widely tested and researched in the United States.

And, of course, Albertans were influenced by such work. Doyal Nelson, a long-time teacher and researcher in the Department of Elementary Education, University of Alberta, did his PhD at the University of Minnesota, and his thesis was part of the effort to research the effects of SMSG curricula in schools. Many other educators (in both universities and schools) came from or had done work in the United States, which meant that many in Alberta were able to both support and evaluate ideas from such projects.

Mathematics education in Alberta was definitely part of the new mathematics phenomenon of the 1960s. But this phenomenon did not simply reflect the various large-scale curriculum projects or commercial mathematics curriculum materials. In fact, new mathematics had many facets, some of which I will discuss here.

The first of these facets (and perhaps the most talked about) reflected ideas from larger centers and projects. These ideas included an emphasis on mathematical structures and on properties of operations and of number systems themselves (natural, integer, rational, real, complex), especially when applied to computations with numbers or with algebraic expressions. The idea was that students would not simply memorize algorithms but, rather, would understand and use valid properties to solve problems and set up and work through computations, as well as being able to explain, justify, or prove the validity of their actions—hallmarks of mathematical understanding. These projects also emphasized the careful use of mathematical language, as well as looking at numerical and algebraic operations and their characterization in new ways. The emphasis at the high school level was on the elementary functions—perhaps most vividly seen in the treatment of trigonometry from a circular functional rather than only a geometric point of view. While the emphasis in these new math curricula was on properties and structures, many of the newer approaches, especially in elementary functions, could be seen as facilitating later study of calculus and linear algebra and their applications in physics, engineering, economics, and statistics.

Of course, there were many projects in new math beyond those mentioned here. All of these projects focused on big ideas in mathematics and were likely seen as less "technique-al" by practicing teachers. The emphasis was on students seeing and using patterns or relationships when engaging in mathematical activities. Another emphasis was student discovery, guided by the teacher or done independently. The 15-year Madison Project, initially directed by Robert Davis out of Syracuse University, emphasized problem solving and the discovery of patterns of operating both algebraically and numerically. Henry Van Engen and Glenadine Gibb, at the University of Northern Iowa (and later at the University of Wisconsin and the University of Texas), developed and researched pedagogies focused on alternative algorithms for operating on numbers, such as the successive subtraction algorithm for division, and alternative approaches to proportions.

These ideas were picked up and used in Alberta, which was an early venue for the use of Seeing Through Arithmetic (STA) and Seeing Through Mathematics (STM)—programs that represented an early commercialization of the ideas of new math. One author in this volume, Ray Cleveland, was closely associated with STA and, especially, the STM project. Sid Lindstad, of the University of Calgary, did early work on such developments, both in Alberta and in collaboration with U.S. colleagues. The influence of Davis and the Madison Project found its way into teacher education activities at the University of Alberta through Sol Sigurdson's connections with Davis with respect to discovery learning. In the late 1960s, the doctoral dissertation of Sigurdson's student Sandy Dawson (1969) looked at the Madison Project's teaching and learning ideas as a fallibilistic approach to mathematical reasoning and thinking. (Fallibilism was a contemporary mathematical idea from the work of Imre Lakatos, a student of the famous philosopher of science Karl Popper.)

While the term *problem solving* has been used above, another new mathematics set of ideas became available to teachers through the work of mathematician George Polya, of Stanford University, in his remarkable books of the 1950s: *Induction and Analogy in Mathematics* (1954) and *How to Solve It: A New Aspect of Mathematical Method* (1957). In these works, Polya focused on what he saw as the living edge of mathematics—plausible reasoning—which complements the more formal, demonstrative part of mathematical reasoning (best seen in the form of deductive proofs). Included here is McCall's supportive article "Polyan Mathematics"—evidence of McCall's point that this aspect of new mathematics needed to occupy a larger place in the Alberta curriculum and examination programs.

This discussion might lead one to think that new math was a U.S. phenomenon and that its influences came into Alberta school mathematics mainly from the south. Even the *Scientific American* article seems to suggest that new math, both then and now, has been a U.S. reaction to world scientific or economic events. But the 1950s and 1960s were a period rich in ideas relevant to mathematics curriculum and teaching in many places in the world.

The Soviet Union (and countries where it had influence) enlisted the working support of prominent mathematicians and educators in developing school and teacher education materials in mathematics. The famous French Bourbaki society of eminent mathematicians, who were great advocates of formal and rigorous approaches to mathematical thinking, had significant influence on French school mathematics. One need only remember the work of Jean Dieudonné, who pushed for the replacement of classical demonstrative Euclidean geometry with more algebraic approaches to the study of geometry in schools. Polya was originally from Hungary, where there was a great emphasis on problem solving, especially extracurricular contests of problem solving (which took place in many countries at this time).

Many of the new mathematics ideas that gained currency and respectability in Alberta in the 1960s had some basis in the UK. For example, mathematics education pioneer Richard Skemp was strong in his support for students actively finding meaning in mathematics, especially connecting mathematical ideas to experiences, and he developed curricular materials around this approach.

At the same time, mathematics educator Zoltan Dienes (who was born in Hungary but moved to England as a teenager) was doing his creative work in developing materials and related games whereby children could learn mathematics through experiencing the mathematical properties of number, logic, or algebra. These included Dienes blocks (logic and sets), multibase arithmetic blocks (numeration systems), and various play-like settings for invoking algebraic thinking (particularly groups). He also created activities in which children and even secondary students could be observed to be doing mathematics across several levels of abstraction, as well as experiencing the same mathematical idea sets using different physical representations. Dienes was an originator of the idea that students think in structures as they play games or work with materials, thinking that was generalized by students through the use of informal and, later, more formal writing activities in mathematics. Dienes later lived and worked in Canada, and I met him when he worked with the Minnesota Mathematics and Science Teaching (MINNEMAST) Project, with Paul Rosenbloom at the University of Minnesota in the 1960s.

Finally, Caleb Gattegno, working in England, brought to the fore the uses of purpose-designed materials (e.g., Cuisenaire rods), which enabled children and secondary students to develop their own useful ideas and patterns for working with numbers in a way that generated general patterns.

Of course, most of us who were in Alberta in the 1960s remember the use of various new math materials in preservice and in-service teacher education but also in many classrooms. Cuisenaire approaches are mentioned in the Cathcart and Liedtke article in this book, and Doyal Nelson (with Floyd Robinson) produced an extensive research report on the use of this approach and materials, which was rather widespread in Canadian schools.

Bruce Harrison, now a professor emeritus at the University of Calgary, is perhaps the leading expert on Skemp and the implications of his work for mathematics knowing and curricula in schools. Harrison's long-term relationship with Skemp began with his rigorous research, for his doctoral thesis (Harrison, 1967), on Skempian ideas related to reflective thinking and mathematical knowing in students.

Dienes's materials, and to an extent his methodologies of structural thinking, could be seen in preservice and in-service classes for teachers as providing concrete approaches to the logic and properties of mathematics in working with children.

Thus, it is clear that new mathematics ideas, of a very different character from the large-scale U.S. projects, came to Alberta from many international sources and in many forms, and they were a rich source to be taken up and, in many ways, transformed for and through their use here.

The name Richard Skemp brings to mind another source of new ideas for the learning of mathematics in the 1960s. Skemp himself would have said that his work was based on ideas drawn from the epistemological psychology of Jean Piaget. The work of Piaget and various colleagues involved studies of children working in settings related explicitly or implicitly to children's conceptions of number, space, and geometry, and Piagetian theory emphasizes the importance of "groupement" structures from mathematics itself for thinking about and exploring the mathematical actions of children. Since the article by Cathcart and Liedtke deals with these ideas, through its review of the impressively large number of Piagetian studies done at the University of Alberta in the mid-1960s, there is no need to elaborate on these ideas here. The very presence of this collection reveals the impact of Piaget on mathematics education in Alberta. But I would be remiss if I did not give credit to Doyal Nelson for making Piagetian ideas salient for the many graduate students who worked with him, and also for prompting the use of Piagetian ideas related to mathematics knowing in courses for preservice and in-service teachers, particularly at the elementary level.

The reference to Piagetian ideas relating to mathematics knowing should remind us that the new math of the 1960s didn't suddenly arise in the late 1950s. MCATA has always maintained an association with a larger body, the U.S.-based National Council of Teachers of Mathematics (NCTM). As one of its resources for teachers, NCTM has a long history of producing yearbooks using ideas from experts on various mathematical topics. Many of the grounding ideas of school mathematics of the 1960s (at least for North America) can be found in the NCTM yearbooks *The Learning of Mathematics: Its Theory and Practice* (NCTM, 1953) and *Insights into Modern Mathematics* (NCTM, 1957). The 1953 yearbook introduces mathematics educators to the roles a Piagetian constructive view of cognition might play in mathematics knowing and in the provision for such knowing. It also points to the ideas of meaning, understanding, and the role of learning mathematics in a variety of ways and at a variety of intellectual, conceptual, and maturity levels. The 1957 yearbook provides sketches of the main new ideas (e.g., functions from the set of ordered pairs point of view) and approaches to mathematics that were to become the curricular currency of the 1960s.

Once again, as noted in Galeski's article, the ideas of so-called modern or new mathematics may have had their origins in the work of mathematicians as far back as the 19th century. Thus, the mathematical ideas and approaches of modern mathematics were not totally new (one need only look at Norman Miller and Robert Rourke's Alberta-adopted *Mathematics for Canadians* text in the late 1930s to see this), but in the 1960s, school-level students were brought into regular contact with such ideas and, through rich problem-solving activities and structured learning settings, were given opportunities to take on and take in these ideas in a variety of ways. At their best, these approaches to new math align with Jerome Bruner's contention, in his famous 1960 book *The Process of Education,* that school-aged children can profitably connect with any mathematical idea if it is presented in such a way that they can engage with it using the capabilities they have.

Another facet of new math (and education in general) that was just entering the scene was the use of computers in teaching. These uses were at that time experimental but diverse. Extensive computer-based drill-and-practice schemes were developed at Stanford that allowed students to work on their own through an ordered series of computational practice sets in which criteria-referenced decision algorithms both checked their work and provided appropriate next-step activities for them. This work simultaneously focused on researching the outcomes of such instruction. But work at the University of Pittsburgh, the University of Illinois, and Pennsylvania

State University (among others) aimed to provide more elaborate computer-guided instruction involving teaching new mathematics ideas primarily in urban centres.

While not focused on mathematics education, computer-assisted instruction development and research at a world-class level of innovation occurred at the University of Alberta, under the direction of Steve Hunka. Hunka also brought computer-assisted instruction ideas from the University of Illinois that were both mathematically and technically unique.

Of course, another child of the 1960s was the Logo programming language, developed by Seymour Papert and Wally Feurzeig. Papert related Logo's development and use to the ideas of Piaget, and it allowed schoolchildren to construct procedures in real time, many of which were related to and extended the mathematical ideas of geometry. This work was at least illustrative of constructivist approaches to new geometric ideas, which Papert himself brought, at a demonstration level, to Alberta at the invitation of Hunka.

Another programming language created explicitly for use by learners was BASIC, developed by John Kemeny at Harvard. An early use of BASIC in mathematics teaching and related curriculum development was the Computer Assisted Mathematics Project (CAMP) at the University of Minnesota, which allowed secondary school students to write programs related to a wide variety of mathematical topics (such as properties of systems, prime numbers, equation solving, linear systems, and sequences and series). I came to the University of Alberta in 1967 having worked on CAMP, and in my thesis I studied the use of computer programming as a tool in grade 11 mathematics, one of the first theses studying the role of such use of computer programming in enhancing mathematics learning.

It is evident that mathematics education in the 1960s was, like the era itself, swinging with new ideas. These ideas were related to providing curriculum and methodologies for teaching big mathematical ideas in new ways (e.g., discovery learning). There was a renewed but different focus on problem solving, from Polya's pointing to the complementary values of demonstrative rigour and plausible reasoning in mathematics. There was a psychological facet of new mathematics, illustrated by European influences, such as the role of concrete experience in learning mathematics in a meaningful way through reflective thinking (Skemp) and thinking in structures with mathematically inspired materials (Dienes). There was the new influence of the constructivist thinking of Piaget and its relationship to the development of mathematical thinking. And just entering the purview of mathematics educators were the various uses of the computer in teaching, which brought new possibilities for the mathematics curriculum as well as different ways of both learning and thinking about mathematical ideas.

ARTICLES FROM DELTA-K AND THE MATHEMATICAL EDUCATION ETHOS OF THE 1960s

The articles in this volume speak well for themselves, so there is no need to review them in detail. Instead, I will try to sketch an answer to the following question: In what ways do these articles reflect the issues and facets of new mathematics in the 1960s?

Before turning to the articles themselves, I wish to assert that the articles selected for inclusion cover a wide variety of salient 1960s issues and speak to many of the facets discussed above. Thus, the Mathematics Council of the day can be seen as providing information to teachers that related to the concerns of the day. Now let me turn to a more specific analysis of how these articles address the mathematics education concerns of the 1960s.

I want to first consider McCall's article "Polyan Mathematics" and the first part of Coulson's article, "Discovery or Programming." McCall includes an extensive quotation from Polya in which he develops plausible reasoning strategies, including what he terms "guessing" and, especially, inductive thinking (not to be confused with mathematical induction), as a complement to or completion for traditional demonstrative, proof-oriented reasoning in mathematics. McCall then provides an elaborate example of finding the three dimensions of a rectangular solid of a given volume such that the surface area is a minimum. He starts by using the Polyan plausible reasoning strategy of solving a related two-dimensional problem: finding the dimensions of a rectangle of a given area with a minimal perimeter. Given his emphasis on plausible reasoning, I was surprised that he started with a general rectangle of sides x and y, giving $p = 2x + 2y$. Without discussing the nature of his plausible strategy, McCall then makes an educated guess that the solution is a square. He then does the algebraic solution to finding x and y, minimizing the difference between the area of the square of side $(x + y)/2$ and the area of the rectangle xy—in other words, his means of exploration of this minimization (when the sides are equal or $x = y$ is what would pass for a demonstrative proof). The reader is left with the question: Would there be less formal and more intuitive ways of reaching this solution more consonant with the idea of plausible reasoning, or could such a problem be solved by a grade 6 student without algebraic generalization? In his extension from three to two dimensions, McCall does provide an example of some nice plausible reasoning, but the means of his test of the plausible conclusion—of a cube as the solution—involves at least generalized if not formal reasoning. McCall has well illustrated the bi-play of plausible and demonstrative reasoning. He argues for the inclusion in schools of Polyan mathematics centered on problem solving—"mathematics for the scientific world"—and argues that the new math in Alberta schools at the time was "mathematics for a philosopher."

Coulson, on the other hand, values the contributions of programs such as SMSG and UICSM. He values mathematics as a subject worthy of study on its own terms, because its structure and relationships can be explored and discovered by students, rather than as a collection of facts, procedures, and techniques to be learned by rote. A big idea of his (which I think McCall would agree with) is that students should be given credit for and opportunities to exhibit the ability to think mathematically for themselves. For this to happen, expert teaching is needed, both in terms of knowledge of mathematics and knowledge of how students might discover the patterns and structures of mathematics. (Does this not sound like the teacher knowledge aimed at in the *Scientific American* article of 2012?) Coulson sees the following as the observable fabric of student mathematical thinking: an observation in a situation, a proof of the validity of the observation, new observations based on previous ones, another proof. Surprisingly, this fabric is not that different in pattern from the one discussed by McCall, but it too raises questions about the nature, the role, and the level of plausible reasoning by students.

In the 1960s (and even today), these two articles forced teachers to think about how they might foster and observe plausible reasoning in action at any level. Both authors prompt us to ask just how teachers might provide the setting in which students can evaluate their own mathematical actions, perhaps through explaining to peers or other methods less formal than the word proof.

Coulson also contrasts what he sees as the pattern of a discovery method with programmed learning. He sees the former as opening the path for student thinking, while the latter necessarily limits students to moving through a series of small steps and, thus, confines their thinking. Programmed learning can be seen in the work of Robert Gagné in the 1960s, which posited a carefully ordered sequence of behaviourally stated objectives describing a task, an acceptable student behaviour, and a criterion-referenced method for observing such behaviour. A question pertinent to the time but not raised by Coulson is whether a program must follow such a rigid pattern. Writing in 1965, Coulson could not have anticipated the work on using computers in mathematics. The drill-and-practice programs of the mid-1960s might have been subject to his critique, but other more complex and open computer-based individualized instructional programs in mathematics would have challenged his comparisons. In other words, Coulson's contrast of discovery-oriented mathematics instruction with programmed instruction is limited by his rather narrow view of the latter and because he could not have anticipated the technology that would, over time, change the nature of both discovery in mathematics classrooms and individualized, computer-based instruction.

Murray Falk's article does focus on a technology relatively new at the time—the overhead projector. While the means of creating projections now seem crude, Falk shows just how they might be used, especially to allow students individually and collectively to think about mathematical ideas portrayed in a visual form. This technology enhanced the study of modern mathematics concepts such as sets and operations on them, and especially thinking of functions and relations as sets of ordered pairs and the structural nature of and applications of functions of various kinds. But what struck me while reading the article was the implicit idea—not new to the 1960s but, rather, eternally relevant to mathematics teaching—that preparation is key to effective instruction, even in taking advantage of sharing student ideas that arise serendipitously.

As noted above, McCall and Coulson present contrasting views of the impact of new mathematics in Alberta schools in the 1960s. While both offer valid and well-developed views, the nature of modern mathematics was at the time elusive. This is well illustrated by Galeski's article "What Is Modern Mathematics?" The article shows the difficulty of coming up with criteria that would be useful in choosing a mathematics curriculum for the 1960s. Galeski's exploration of the nature of mathematics yields discussion of the study of form in a general sense; of formal, necessary, deductive reasoning; and of Bertrand Russell and Ludwig Wittgenstein's characterization of mathematics as a form of language game. While I was taken with Kemeny's notion that "stress should be laid on thinking mathematically and more value attached to that ability than to knowledge [of particular facts]," even this left open the meaning of such thinking, as well as the nature and possible form of plausible and demonstrative reasoning and even of the nature of "modern mathematics."

Runclius's article is a brief response to a question raised by teachers at a talk he gave on the future of school geometry. In it, he presents one view of the nature of modern mathematics in more specific terms. He provides historical criteria for teaching geometry and then describes the features and emphases of a new geometry in high school. While the new geometry would still cover important facts drawn from congruency, properties of triangles, basic constructions, and so on, the key objectives would be appreciation of the postulational structure, appreciation of different geometries with other such structures, use of algebraic and analytic methods, and appreciation of the nature of demonstrative reasoning involved in proof. To me, this does not differ much from the "old geometry." Of course, in1961, Runclius could hardly have imagined the geometries of chaos theory, or the art and mathematics of the Mandelbrot set, or geometric solutions to complex physical and statistical problems and models fostered through the use of computers. But, as Galeski might have remarked, Runclius's ideas of the modern in geometry were more 18th century than 20th. Since Donald Coxeter—a

Canadian who was considered "the man who saved geometry"—ideas such as various kinds of transformations, groups of reflections, and dimensionality and complex problems amenable to plausible reasoning at many levels of abstraction have been made available to students, providing them with a more living view of geometry. I am not trying to be critical of Runclius's off-the-cuff remarks but, rather, to point out that many times views of modern mathematics are spelled out in cautious terms.

This latter point of view relates to Krider's article. Krider points out what he sees as the cycles in school mathematics over the first 60 years of the 20th century. He views this on the basis of the psychological idea of transfer, seeing mathematics as a discipline being replaced by more specifically oriented pragmatic mathematics, and so on. This cycle was accompanied, in his view, by varying levels of conflict between mathematicians and educators in the 1950s and '60s, seeing a return of a contemporary positive view of general transfer arising from appropriate learning of the discipline (with attention also paid to citizenship), as well as Bruner's ideas on subject matter learning, which he sees as being accompanied by a wide split between mathematicians and educationalists on what should be taught in schools and how. Krider sees the curricular product of a mathematician-led project (such as SMSG) as dominating mathematics in schools, returning the emphasis to the subject matter in a way many perceive as radical. But in a call by a large group of North American mathematicians to return to a balance between content and pedagogy and a turn from mathematics that may only appeal to those who like what modern-day mathematicians like (demonstrative reason driven by more "pure" mathematics), Krider in 1963 saw the potential for turning away from what was then current.

The article by H. L. Larson in the later 1960s shows one aspect of what might be thought of as a middle-ground reaction to particular aspects of the modern mathematics curriculum. The article focuses on alternative approaches to setting up and solving proportional problems, particularly percentage problems. Larson suggests that the rate-ratio method in the new mathematics curricula favoured in Alberta was problematic for students and teachers alike, leading to the "computational bogey of cross multiply," where this technique was used blindly and was inappropriately generalized to other settings (e.g., adding fractional numbers). He proposes a return to what I remember from my school mathematics in the 1950s as an older percentage problem representational strategy in which variations of the equation $a = n\%$ of b would be matched with the various "cases" of percentage tasks. But Larson argues that the statement "50 is what percentage of 200?" leads to the equation $50 = n(1/100) \times 200$, which is nicely solvable using the properties of a field (new mathematics) and, further, is generalizable to volume problems like $V = l \times w \times h$ when three of the values are known based on "solid deductive logic" (again, new mathematics). Some reading

Larson in the 1960s might have said that he was favouring "old-tyme" computational cases, but in my reading, his thinking shows how explicit use of, for example, field laws allows students and teachers to look at computations in a non-rote, generalizable manner. And, of course, neither Larson nor the curricular proponents of the other general rate-ratio approach had the language to support or critique that which flowed from both mathematical and psychological work in the 1970s on multiplicative structures, which prompted students to think of $a{:}b = n{:}100$ in terms of using preservation of multiplicative structure relations either with respect to a to b and n to 100, or a to n and b to 100 in their solution, either of which leads to rather different, more intuitive mathematical operations.

Of course, Krider's view of the influence of new mathematics is quite different from that of Coulson. This raises the question of how well modern mathematics ideas (such as sets and number systems) were being learned and by whom. Chell and Coulson take on this question in their article "Can Students Learn Some of the Ideas of Modern Mathematics?" Of course, Larson, writing later in the 1960s, showed just how students might do such learning to their benefit at a junior high level. But Chell and Coulson looked to reports from teachers across Canada on their experiences. They started with a survey of ideas from mathematicians in Canada and the United States as to why such learning could be expected. The most concise of these arguments came from Robert Rourke, a well-known long-time teacher and curriculum developer from Kent School, in Ontario: Modern mathematics, he said, is "a means of broadening old ideas and introducing new ones to clarify, simplify, and unify our mathematical concepts [being taught in schools]."

Chell and Coulson then used a large number of Canadian sources to garner reactions as to whether using modern mathematics approaches—"modern concepts taught from a modern approach"—led to enhanced learning by students at various levels of ability. Rourke cited the Kent School experience, where set language and notation (e.g., relations as a set of ordered pairs and what follows from that logically and graphically) were at first introduced in grade 12 but were then pushed down to grades 11 and 10 and, later, to junior high because such ideas fit well with the concepts being taught there. Further, he talks about how this helped students better understand ideas such as linear functions, because the language prompted students to look at sets of pairs and solution sets. Of course, less formal language (such as the function named $2x + 3$) was used, but the set concepts allowed students to see that this expression, when used in a context, represented a set of pairs of numbers that could be explored and was not an isolated chunk of mathematical jargon. Similar positive statements about using modern mathematics were generated from a number of special summer courses and other trial projects, not surprisingly showing various positive effects in using modern mathematics with selected students at various levels. For example, in British Columbia, data from modern mathematics classes showed that students of various ability levels could learn many modern concepts while still mastering the conventional content as well as similar students in conventional classes did.

Chell and Coulson also reported on using modern mathematics with lower-performing students to good effect. For example, in Quebec, graphs (of sets of ordered pairs) were used with students who were "poor in algebra," with students showing "good comprehension." In general, Chell and Coulson found that students in various circumstances—from regular classes to special classes to math club experiences—were able to learn modern mathematics concepts under various regimes of modern instruction.

Of course, such observational reports would not satisfy the critics of new mathematics, or parents whose students were not working on computation like they used to. In my reading, I found an interesting sidebar on the effects of modern mathematics instruction with students in a brief report by Tom Atkinson in February 1965, the same issue in which Coulson's article appeared. He reported preliminary results in what seems to be a clever and rigorous study that compared 3,500 grade 7 students in Edmonton in 1961 who had not had modern mathematics instruction with 4,000 such students in 1964 who had had one or more years of instruction using Seeing Through Mathematics (STA), the Alberta modern mathematics elementary-level adoption, on a wide variety of computational tasks. The preliminary conclusion was that modern mathematics study under STA had not adversely affected students' mastery of basic computational skills.

Two other articles contain ideas that would affect teachers' approaches to teaching mathematics. The first, by Cleveland, is a brief but information-dense article on reading in mathematics, in which he presents his ideas as responses to nine questions related to mathematics as a language and to language use in mathematics in practical terms. The questions ranged from "How does a student learn the language of mathematics?" to questions about the effects of logic and semantics on such language to "What nonverbal devices (such as graphs, ordered sets of numerical data, tools such as slide rules and computers, diagrams) are used in mathematics?" His last big question—"What language implications are involved in constructing and presenting a formal mathematical system?"—clearly resonates with the other articles included here. Even in this brief article, there are many rich ideas, such as the following: "to introduce a [new] word before the concept is developed causes frustration on the part of the student and inhibits learning. The task of the teacher is to arrange student activities...dealing with concrete objects and designed to have the student discover a mathematical idea before the [formal] terminology is introduced." Students should then be given further opportunities to share their thinking with other students, addressing "discoveries in their own words," after which mathematical terminology can be introduced to give students more concise and precise ways to express themselves. Such advice resonates with ideas arising in the other 1960s articles in this collection. It might be said that the article provides a syllabus for a teacher workshop on reading and language use in mathematics learning, if not a full course.

The second article by Cathcart and Liedtke, which appeared in three parts in three issues of *delta-K* in the later 1960s, reports on a number of studies conducted by graduate students in the Department of Elementary Education, University of Alberta. These studies all involved Piaget's ideas on children's conceptual development related to mathematical ideas, particularly his stages of development (specifically, the transition from preoperational to concrete operational thinking in mathematics). This article bears reading because it represents an important early body of work on the relationship between Piagetian ideas and the mathematics knowing of children. This body of work reflects the influence of Doyal Nelson in bringing Piagetian ideas into the intellectual discussion around teaching mathematics, and it spurred continued work by graduate students both in elementary and secondary mathematics at the University of Alberta. It could also be seen as part of the genesis of a large project, led by Nelson and Sawada at the University of Alberta, that traced the thoughts and actions of children aged 3–8 as they worked with concrete-based mathematical problems. Further, reading these articles provides teachers with inputs as they think about classroom experiences in various aspects of elementary mathematics. Many of the ideas reported here resonate with Cleveland's thinking on the roles of concrete experience in children's mathematics.

I found the methodologies used in the studies very interesting and in many ways reflective of educational research in the 1960s. Although the ideas used and the reason behind the tasks developed for use in the studies were related to Piaget, for the most part these studies were not carried out as a Piagetian teaching experiment. Most of the testing involved large numbers of students using tasks individually and allowed for collecting data that could be tested statistically. These studies and the related tasks ranged across a wide variety of topics: various aspects of number with preschool students, linear measurement in grade 1, concepts of space and map reading in grades 1–6, time duration across the same grade levels, conservation of length in kindergarten and grade 1, training to conserve quantity under various transformations, relationship between linear measurement bilingualism, and many others. The findings varied. For example, Reimer found a high correlation between conservation of number and length and achievement on the STA achievement test in grade 1; however, in a brief study, Scherer found that the use of manipulative materials was no more effective than discussion in helping grade 2 students develop the ability to do word problems. One study reported in great detail was that of Sawada. He devised a clever way by which students could be tested for conservation of length by showing their responses using nonverbal means using callipers to select what they saw as the length of various objects under various transformations. Sawada found that in a group of 62 kindergarten and grade 1 students, the average age at which 50% gave an appropriate conservation response was between five years, four months and six years, two months—up to two years earlier than students who were asked to give a verbal response in the tasks. There were other interesting findings, such as the fact that both the state properties of the objects used and the transformations used were needed to explain correct conservation in this group. While age was a significant variable in performance, measured intelligence was not.

This rather extensive discussion illustrates the kind of insights into the roles that characteristics of concept formation and the nature of mental operations related to different mathematical concepts might play in the development of mathematical concepts in children. Like Cleveland's ideas on reading, a study of the ideas raised by these research works could provide part of the rich background necessary for teaching mathematics at the elementary level.

CONCLUSIONS

The articles on school mathematics published by MCATA in the 1960s make for interesting reading. They provide a broad view of mathematics education in Alberta in the 1960s. With the exception of computers and their potential use (which are mentioned only once), every facet of modern mathematics that I pointed to early in this commentary is considered in at least one article (and often in more than one). The selection of articles does justice to these facets in several ways, especially to the role of new mathematics in Alberta in the 1960s.

The reader will find contrasting views—even within one article. Thus, we can infer that there was no MCATA party line on new mathematics. I found the articles to be varied and individually quite rich. There are idea articles but also articles that provide concrete classroom-related examples. Both kinds of articles might feed the mathematics teaching thinking of the reader. The articles, while often providing support for a particular idea or practice, can rarely be read as being polemical in nature. These articles would have provided important ideas to MCATA members in its first decade and still make interesting reading today.

Of course, the ideas under discussion are from a different era and have been superseded by developments in curriculum, technology, teacher knowledge of mathematics, mathematical thinking, and methods (particularly in applications of mathematics and the use of computers in such applications), as well as knowledge of and thinking about mathematical knowing. Still, pieces of the writing here sounded very much like the writing on mathematics teaching in the 2012 *Scientific American* article. The sixties were indeed swinging with new ideas in mathematics education, and these articles show just how such ideas were taken up in Alberta and added to and transformed in rich ways.

Tom Kieren is a Professor Emeritus at the University of Alberta. He taught there from the late sixties to the late nineties teaching undergraduate preservice teachers (primarily secondary but also elementary) and worked with a large number of graduate students. At the time of his retirement he had directed or served on the supervisory committee of over 70% of the doctoral students in mathematics education in Canada. He conducted research on the use of manipulative materials in mathematics learning; using computers in mathematics learning (including work on using Logo). He did extensive, well-known research on the fractional and rational number from the middle '70s into the '90s (working with Doyal Nelson, as well as Les Steffe at U of Georgia on some of this work). He worked with Susan Pirie (Oxford and UBC) studying mathematical understanding in students from ages 10 to university level as a recursive dynamical phenomenon. He worked with many U of A graduate students (and later colleagues) including Brent Davis, David Reid, Elaine Simmt, Lynn McGarvey, Florence Glanfield, and Jerome Proulx on an enactivist view of mathematics knowing and teaching. He was widely published in the field and continues to review 20–30 articles per year for various journals. Like his research that was conducted in Alberta schools with teaching colleagues such as Susan Ludwig, Beryl Tiffen, and Bob Frizzell, Tom has continued his work with children in schools in his granddaughters' mathematics classes in Edmonton and Calgary.

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