An Induction Proof of the Factorization of $a^n - b^n$

Ron Persky

Editor's Note: Ron Persky is a long-time contributor to delta–K *and a strong supporter of mathematics education in the United States. This proof is included in this issue as a mathematical prompt because, in Persky's words, it is "not what you call advanced math but I just think it is nice." I agree.*

We establish that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$. (1)

To do this, the Second Principle of Finite Induction will be used as well as the following identity:
$a^{n+1} - b^{n+1} = (a+b)(a^n - b^n) - ab(a^{n-1} - b^{n-1})$.

For n = 1, (1) yields $a - b = a - b$.

Assume (1) is true. Then for n+1, we have, using the above identity:
$a^{n+1} - b^{n+1} = (a+b)(a^n - b^n) - ab(a^{n-1} - b^{n-1})$.

By the induction hypothesis and the Second Principle of Finite Induction

$$\begin{aligned} a^{n+1} - b^{n+1} &= (a+b)(a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}) + \\ &\quad -ab(a-b)(a^{n-2} + a^{n-3}b + \ldots + ab^{n-3} + b^{n-2}) \\ &= (a-b)\{(a+b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}) - ab\,(a^{n-2} + \\ &\quad a^{n-3}b + \ldots + ab^{n-3} + b^{n-2})\} \end{aligned}$$

Multiply the two terms inside the braces.

$$(a-b)\left\{\begin{matrix}[a^n + a^{n-1}b + \cdots + a^2b^{n-2} + ab^{n-1}] + [a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n] - \\ [a^{n-1}b + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1}]\end{matrix}\right\}$$

In the first bracket, [ ], the second term, $a^{n-1}b$, through the last term, ab^{n-1}, is cancelled by the terms in the third bracket. This leaves
$(a-b)\{a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n\}$ and gives formula (1) for $a^{n+1} - b^{n+1}$ which is what is required to complete the induction proof.