34th CALGARY JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

April 28, 2010

NAME: SOLUTIONS
PLEASE PRINT (First name Last name)

GENDER: ☐ M ☐ F

SCHOOL: ______________________

GRADE: (7,8,9)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART A has a total possible score of 45 points. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.

MARKERS' USE ONLY	
PART A ×5	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.
THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

PART A: SHORT ANSWER QUESTIONS

A1 The product of three different prime numbers is 42. What is the sum of the three prime numbers?

12

A2 Four athletes at the Olympic competitions are the only participants in each of eight events. For each event, three medals are awarded. Each of these four athletes wins the same number of medals. How many medals did each athlete win?

6

A3 Two sides of a triangle have lengths 5cm and 6cm. The area of the triangle is a positive integer. What is the maximum possible area of such a triangle, in cm^2?

15

A4 Rose has to write five tests for her class, where each test has a maximum possible score of 100. She averaged a score of 80 on her first four tests. What is the maximum possible average she can get on all five tests?

84

A5 Notice that $1 - 2 = -1$, $1 - (2 - 3) = 2$, and $1 - (2 - (3 - 4)) = -2$. What is

$$1 - (2 - (3 - (4 - \cdots - 100)))\ldots)?$$

-50

A6 The number

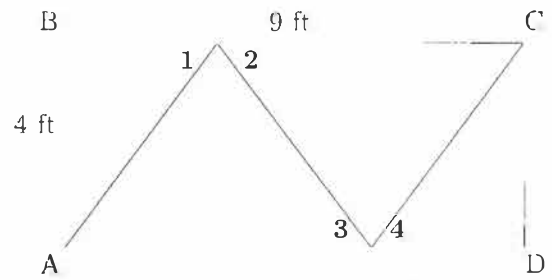
$$\frac{2^{100} + 2^{99} + 2^{98}}{14}$$

is equal to 2^n, for some positive integer n. Find n.

97

A7 A rectangular billiard table has dimensions 4 feet by 9 feet as shown. A ball is shot from A, bounces off BC so that angle **1** = angle **2**, bounces off AD so that angle **3** = angle **4** and ends up at C. What is the distance (in feet) that the ball traveled?

15



A8 Suppose that a is a certain real number so that $\frac{3x^2 - a}{x^2 + 2}$ is always the same number no matter what real number x is. What is a?

6

A9 The numbers $1, 2, 3, \cdots, 100$ are written in a row. We first remove the first number and every second number after that. With the remaining numbers, we again remove the first number and every second number after that. We repeat this process until one number remains. What is this number?

64

PART B: LONG ANSWER QUESTIONS

B1 In a video game, the goal is to collect coins and levels. A player's level is calculated by finding the number of digits of the number of coins he has collected. For example, if a player has 240 coins, then the player's level is 3, since 240 has 3 digits. Currently, Lario has 120 coins and Muigi has 9600 coins.

(a) (4 marks) What is Muigi's level? How many coins does Muigi need to collect to increase his level by 1?

Solution: Muigi has 9600 coins, which has four digits. Therefore, Muigi's level is 4.

In order for Muigi to reach level 5, Muigi must collect $10000 - 9600 = 400$ coins to increase his level by 1.

(b) (5 marks) In their next game, Lario and Muigi each collect the same number of coins, and they end up at the same level. What is the smallest number of coins that Lario and Muigi could each have collected to accomplish this feat?

Solution: Lario needs $1000 - 120 = 880$ coins to increase his level by 1 to 4. After collecting 880 coins, Muigi will have $9600 + 880 = 10480$ coins. Hence, Muigi is at level 5. Therefore, for Lario and Muigi to end up at the same level, Lario needs to collect $10000 - 1000 = 9000$ more coins. After that, Muigi will have $10480 + 9000 = 19480$. Then both Lario and Muigi will be at the same level.

Therefore, Lario and Muigi need to collect $880 + 9000 = 9880$ coins.

B2 You are preparing skewers of meatballs, where each skewer has either 4 or 6 meatballs on it. Altogether you use 32 skewers and 150 meatballs. How many skewers have only 4 meatballs on them?

Solution 1: Suppose all of the skewers contain 4 meatballs. Then the number of meatballs used is $32 \times 4 = 128$ meatballs. Therefore, we need $150 - 128 = 22$ more meatballs. Every skewer with 4 meatballs that we change to a skewer with 6 meatballs increases the number of meatballs by 2. Therefore, in order to have 150 meatballs total, there are $22/2 = 11$ skewers with 6 meatballs. Therefore, there are $32 - 11 = 21$ skewers with 4 meatballs. The answer is 21.

Solution 2: Let x be the number of skewers with 4 meatballs each. Therefore, there are $32 - x$ skewers with 6 meatballs each. Since there are 150 meatballs total,

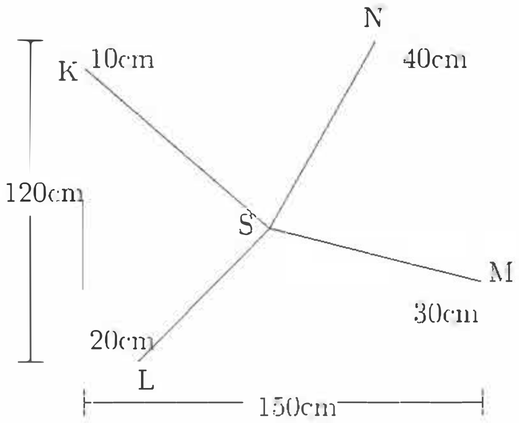
$$4x + 6(32 - x) = 150.$$

This simplifies to

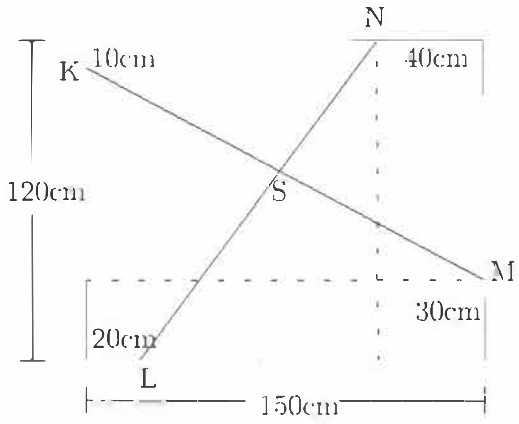
$$4x + 192 - 6x = 150 \quad \Rightarrow \quad 2x = 42.$$

Therefore, $x = 21$. The answer is 21.

B3 Khalid, Lesley, Mei and Noel are seated at 10 cm, 20 cm, 30 cm, and 40 cm, respectively, from the corners of a 120 cm by 150 cm dining table, as shown in the figure. If the salt, S, is placed so that the total distance $SK + SL + SM + SN$ is as small as possible, what is that total distance?



Solution: The sum $SK+SL+SM+SN$ is the smallest when K, S, M lie in a straight line and L, S, N lie in a straight line. In this case, $SK+SL+SM+SN = KM+LN$.



KM is the hypotenuse of a right-angled triangle with legs $120 - 10 - 30 = 80$cm and 150cm. Therefore, $KM = \sqrt{80^2 + 150^2} = 170$cm.

LN is the hypotenuse of a right-angled triangle with legs $150 - 20 - 40 = 90$cm and 120cm. Therefore, $LN = \sqrt{90^2 + 120^2} = 150$cm.

Therefore, $KM + LN = 320$cm. The answer is 320cm.

B4 ShelfCity makes shelves that hold five books each and ShelfWorld makes shelves that hold six books each.

(a) (3 marks) Jarno owns a certain number of books. It turns out that if he buys shelves from ShelfCity, he will need to buy 8 shelves to hold his books. List all of the possible numbers of books that Jarno can own.

Solution: Jarno needs more than seven shelves. Therefore, he owns at least $7 \times 5 + 1 = 36$ books. He also does not own more than $5 \times 8 = 40$ books. Therefore, the list of possible number of books that Jarno can own is $\{36, 37, 38, 39, 40\}$.

(b) (6 marks) Danny owns a certain number of books. It turns out that whether he buys shelves only from ShelfCity or buys shelves only from ShelfWorld, he will need to buy the same number of shelves. What is the largest number of books that Danny can own?

Solution 1: Note that Danny will always need at least as many shelves buying from ShelfCity than buying from ShelfWorld. If Danny owns the maximum number of books possible, then by buying books from ShelfCity, all of his shelves are full. Therefore, the answer is a multiple of 5. By trial and error, we see that if Danny owns 5, 10, 15, 20, 25, 30, 35, 40 books, then he needs to buy 1, 2, 3, 4, 5, 6, 7, 8 shelves from ShelfCity respectively and 1, 2, 3, 4, 5, 5, 6, 7 shelves from ShelfWorld, respectively. If Danny owns more than 25 books, then he will always need more shelves from ShelfCity than from ShelfWorld. Therefore, Danny owns at most 25 books.

Note: Other trial and error methods are possible and acceptable, i.e. a student does not necessarily have to considering only multiples of five.

Solution 2: Let s be the number of shelves that Danny needs from each store. Since he needs to buy s shelves from ShelfCity, the number of books he owns is from the set $\{5s-4, 5s-3, 5s-2, 5s-1, 5s\}$. He also needs to buy s shelves from ShelfWorld, so the number of books he owns is from the set $\{6s-5, 6s-4, 6s-3, 6s-2, 6s-1, 6s\}$. The number of books he owns is in both lists, and therefore is at least $6s-5$ and at most $5s$. Therefore, $6s-5 \leq 5s$. Solving this yields $s \leq 5$. Since Danny needs at most five shelves from ShelfCity, the number of books Danny can own is at most $5s \leq 25$ books. If he does own 25 books, then for both companies, he will need to buy five shelves, i.e. the same number of shelves. Therefore, the largest number of books that Danny can own is 25.

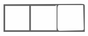
B5 Two players Seeka and Hida play a game called Hot And Cold on a row of squares. Hida starts by hiding a treasure at one of these squares. Seeka has to find out which square it is. On each of Seeka's turn, she picks a square.

• If Seeka picks the square which is where the treasure is, Hida will say "Ding!" and the game ends.

• If Seeka picks a square which is next to the square where the treasure is, Hida will say "Hot!".

• If Seeka picks a square which is not where the treasure is, and is not next to the square where the treasure is, Hida will say "Cold!".

(a) (3 marks) Suppose the game is played on three squares, as shown. Show how Seeka can pick the square with the treasure in at most two turns.



Solution: Label the squares 1, 2, 3 from left to right. Seeka first picks square 1. If it is where the treasure is, Seeka is done in one turn. If Hida says "Hot!", then Seeka knows the treasure is in square 2 and picks it on her second turn. If Hida says "Cold!", then Seeka knows the treasure is in square 3 and picks it on her second turn. In all cases, Seeka picks the square with the treasure in at most two turns.

Note: Seeka cannot accomplish this by picking the second square first.

(b) (6 marks) Suppose the game is played on nine squares, as shown. Show how Seeka can pick the square with the treasure in at most four turns.

Solution: Label the squares 1, 2, 3, 4, 5, 6, 7, 8, 9 from left to right. Seeka first picks square 5. If the square contains the treasure, Seeka is done in one turn. If Hida says "Hot!", then Seeka knows the treasure is in square 4 or 6. She picks these squares in her next two turns to guarantee finding the treasure, needing at most three turns total. If Hida says "Cold!", then the treasure is in square 1, 2, 3, 7, 8 or 9. Seeka on her second turn then picks square 2. If the square contains the treasure, Seeka found the treasure in two turns. If Hida says "Hot!", then the treasure is at square 1 or 3. Seeka picks these two squares to guarantee finding the treasure in at most four turns total. If Hida says "Cold!", then Seeka knows the treasure is at square 7, 8 or 9, with two turns remaining. We apply the strategy in (a) on these three squares to find the treasure in at most two turns. In all cases, Seeka finds the treasure in at most four turns.

Note: Other strategies are possible.

B6 Three people have identical pairs of shoes. At the end of a party, each person picks up a left and a right shoe, leaving with one shoe that is theirs and one shoe that belongs to someone else.

(a) (4 marks) In how many different ways could this happen?

Solution: Label the people 1, 2, 3 and the left and right shoes for person 1 be labeled L_1, R_1, those for person 2 be labeled L_2, R_2 and those for person 3 be labeled L_3 and R_3.

There are two choices as to which correct shoe person 1 picks, namely left or right. Person 1 then has two choices as to whose person's shoe he picks as the wrong shoe. Therefore, there are four choices of shoes for person 1, namely $(L_1, R_2), (L_1, R_3), (L_2, R_1), (L_3, R_1)$.

Suppose person 1 picks (L_1, R_2). Then person 2 must have picked L_2 (since R_2 already is in person 1's possession and person 2 must be holding one of his own shoes). If person 2 has R_1, then person 3 has L_3 and R_3, holding both of his shoes, which is not allowed. Therefore, person 2 is holding (L_2, R_3). Person 3 must then be holding (L_3, R_1). Similarly, no matter how person 1 picks the shoes, the choices for person 2 and 3 are determined.

Therefore, there are four different ways for three people to pick up a left and a right shoe leaving with one shoe that is theirs and one shoe that belongs to someone else.

(b) (5 marks) Same as part (a), except that there are four people instead of three.

Solution: We use the same labeling as in (a) and label the fourth person 4 and his left and right shoe L_4, R_4, respectively.

Consider person 1. There are two choices as to which correct shoe person 1 picks, namely left or right. There are three choices as to whose person's shoe he picks as the wrong shoe. Therefore, there are $2 \times 3 = 6$ choices of shoes for person 1, namely $(L_1, R_2), (L_1, R_3), (L_1, R_4), (L_2, R_1), (L_3, R_1), (L_4, R_1)$.

Suppose person 1 picks (L_1, R_2). Then person 2 must have picked L_2. We now consider the following cases for the right shoe that person 2 picked:

i. If person 2 is holding R_1, then either person 3 is holding (L_3, R_4) and person 4 is holding (L_4, R_3), or person 3 is holding (L_4, R_3) and person 4 is holding (L_3, R_4). Hence, there are two possibilities for this case.
ii. If person 2 is holding R_3, then person 3 must be holding L_3. Person 3 cannot be holding R_1, because then otherwise person 4 will be holding (L_4, R_4), which is both of his shoes. Therefore, person 3 is holding R_4. This leaves person 4 holding (L_4, R_1). This is the only possibility.
iii. If person 2 is holding R_4, then similar to case (ii), there is only one possibility.

Therefore, the total number of ways that all this can happen is $6 \times (2 + 1 + 1) = 24$ ways.