The Alberta High School Mathematics Competition
Part I, November 17, 2009

1. If $2^x = 3^y$, then 4^x is equal to

   (a) 5^y (b) 6^y (c) 8^y (d) 9^y (e) none of these

2. Caroline bought some bones for her 7 dogs. Had she owned 8 dogs, she could have given each the same number of bones. As it was, she needed two more bones to give each dog the same number of bones. The number of bones she could have bought was

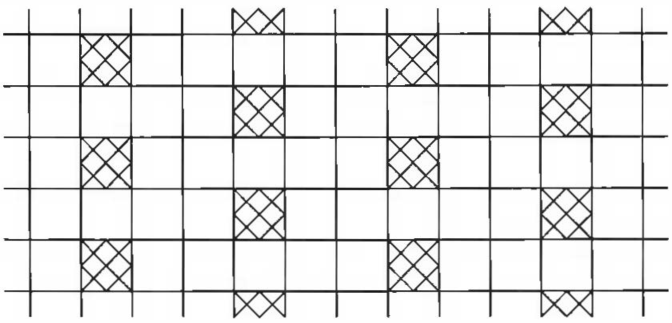
   (a) 16 (b) 24 (c) 32 (d) 40 (e) 48

3. Ace calculates the average of all the integers from 1 to 100. Bea calculates the average of all the integers from 1001 to 1100 and subtracts 1000. Cee calculates the average of all the integers from 1000001 to 1000100 and subtracts 1000000. The largest answer is given by

   (a) Ace only (b) Bea only (c) Cee only (d) exactly two of them
   (e) all three of them

4. A large rectangular gymnasium floor is covered with unit square tiles, most of them blank, in the pattern shown in the diagram below. Of the following fractions, the one nearest to the fraction of tiles which are not blank is

   (a) $\frac{1}{12}$ (b) $\frac{1}{8}$ (c) $\frac{1}{6}$ (d) $\frac{1}{5}$ (e) $\frac{1}{4}$



5. The number of integers between 1 and 2009 inclusive which can be expressed as the difference of the squares of two integers is

   (a) 1 (b) 502 (c) 1005 (d) 1507 (e) 2009

6. Among the positive integers with six digits in their base-10 representation, the number of those whose digits are strictly increasing from left to right is

   (a) between 1 and 50 (b) between 51 and 100 (c) between 101 and 500

   (d) between 501 and 1000 (e) greater than 1000

7. The number of arrangements of the letters AABBCC in a row such that no two identical letters are adjacent is

(a) 30 (b) 36 (c) 42 (d) 48 (e) none of these

8. If 2^{2009} has m digits and 5^{2009} has n digits in their base-10 representations, then the value of $m + n$ is

(a) 2007 (b) 2008 (c) 2009 (d) 2010 (e) 2011

9. An equilateral triangle has area $2\sqrt{3}$. From the midpoint of each side, perpendiculars are dropped to the other two sides. The area of the hexagon formed by these six lines is

(a) $\frac{\sqrt{3}}{2}$ (b) 1 (c) $\sqrt{3}$ (d) 2 (e) none of these

10. Two sides of an *obtuse* triangle of positive area are of length 5 and 11. The number of possible integer lengths of the third side is

(a) 3 (b) 4 (c) 6 (d) 8 (e) 9

11. $Q(x)$ is a polynomial with integer coefficients such that $Q(9) = 2009$. If p is a prime number such that $Q(p) = 392$, then p can

(a) only be 2 (b) only be 3 (c) only be 5
(d) only be 7 (e) be any of 2, 3, 5 and 7

12. A parallelogram has two opposite sides 5 centimetres apart and the other two opposite sides 8 centimetres apart. Then the area, in square centimetres, of the parallelogram

(a) must be at most 40 and can be any positive value at most 40
(b) must be at least 40 and can be any value at least 40
(c) must be 40 (d) can be any positive value (e) none of these

13. The number of positive integers n such that $\sqrt{n + \sqrt{n + \cdots + \sqrt{n}}} < 10$ for any finite number of square root signs is

(a) 10 (b) 90 (c) 91 (d) 99 (e) 100

14. A chord of a circle divides the circle into two parts such that the squares inscribed in the two parts have areas 16 and 144 square centimetres. In centimetres, the radius of the circle is

(a) $2\sqrt{10}$ (b) $6\sqrt{2}$ (c) 9 (d) $\sqrt{85}$ (e) 10

15. The number of prime numbers p such that $2^p + p^2$ is also a prime number is

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

16. Suppose that $2 - \sqrt{99}$ is a root of $x^2 + ax + b$ where b is a negative real number and a is an integer. The largest possible value of a is

(a) -4 (b) 4 (c) 7 (d) 8 (e) none of these

Alberta High School Mathematics Competition

Solution to Part I – 2009

1. If $2^x = 3^y$, then $4^x = (2^x)^2 = (3^y)^2 = 9^y$. The answer is **(d)**.

2. The number of bones Caroline bought is a multiple of 8 but 2 less than a multiple of 7. The answer is **(d)**.

3. The calculation is $\frac{(n+1)+(n+2)+\cdots+(n+100)}{100} - n = \frac{100n+(1+2+\cdots+100)}{100} - n = \frac{1+2+\cdots+100}{100}$, with $n = 0$ for Ace, $n = 1000$ for Bea and $n = 1000000$ for Cec. The answer is **(e)**.

4. Almost the entire gymnasium floor may be divided into 2×3 non overlapping rectangles each with exactly one non-blank square at the lower left corner. The answer is **(c)**.

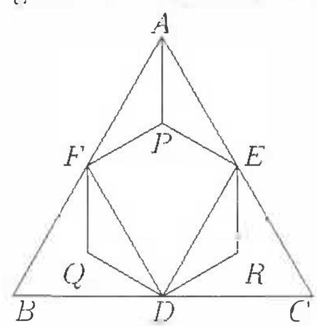
5. Observe that $x^2 - y^2 = (x-y)(x+y)$ is the product of two integers of same parity. Hence $x^2 - y^2$ is either odd or divisible by 4. Thus a number which is neither odd nor divisible by 4 cannot be expressed as a difference of two squares. On the other hand, if n is odd, then $n = 2k+1 = (k+1)^2 - k^2$. If n is divisible by 4, then $n = 4k = (k+1)^2 - (k-1)^2$. Between 1 and 2009 inclusive, there are 1005 numbers that are odd and 502 that are divisible by 4. The answer is **(d)**.

6. We can choose any six of the nine non-zero digits. The number of choices is $\binom{9}{6} = 84$. Each choice gives rise to a unique number. The answer is **(b)**.

7. Assume that the first A appears before the first B, and the first B before the first C. Then we must start with AB and continue with A or C. If we continue with A, the last three letters must be CBC. If we start with ABC, we must continue with A or B. In either case, either of the last two letters can appear before the other. So the total is $1 + 2 \times 2 = 5$. Relaxing the order of appearance, the total becomes $5 \times 3! = 30$. The answer is **(a)**.

8. Since $10^{n-1} < 2^{2009} < 10^n$ and $10^{m-1} < 5^{2009} < 10^m$, we have $10^{m+n-2} < 2^{2009}5^{2009} < 10^{m+n}$. It follows that $10^{m+n-1} = 2^{2009}5^{2009} = 10^{2009}$. Hence $m + n - 1 = 2009$. The answer is **(d)**.

9. Let ABC be the triangle and $DREPFQ$ be the hexagon, as shown in the diagram below. Triangles APE, APF, ERD and FQD are all congruent to one another. Hence $DREPFQ$ has the same area as the parallelogram $AFDE$, which is one half of $2\sqrt{3}$. The answer is **(c)**.



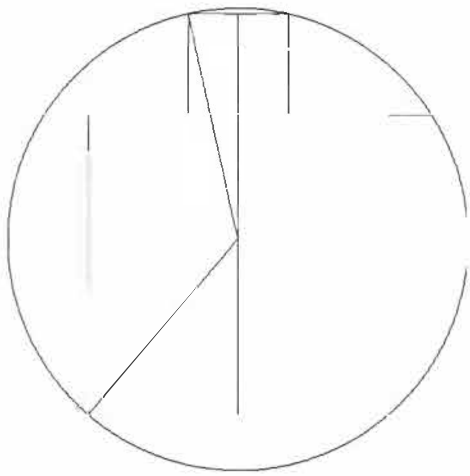
10. By the Triangle Inequality, the third side must be from 7 to 15. Now $9^2 < 11^2 - 5^2 < 10^2$, so that (5,7,11), (5,8,11) and (5,9,11) are obtuse triangles. Also, $12^2 < 11^2 + 5^2 < 13^2$, so that (5,11,13), (5,11,14) and (5,11,15) are obtuse triangles. The other three are not. The answer is **(c)**.

11. Since $Q(9) = 2009$, $x - 9$ is a factor of $Q(x) - 2009$. Since $Q(x)$ has integral coefficients, $p - 9$ divides $Q(p) - 2009 = 392 - 2009$. Since this number is odd, $p - 9$ must be odd. Since p is prime, we must have $p = 2$. Now $392 - 2009 = -1617 = -231 \times 7$, so that $Q(x)$ does exist. For instance, we may have $Q(x) = 231x - 70$. The answer is **(a)**.

12. Let 5 centimetres be the height of the parallelogram. Its base is a segment intercepted by the other pair of parallel lines 8 centimetres apart. Hence its length is at least 8 centimetres and can be arbitrarily large. The answer is **(b)**.

13. With one square root sign, $\sqrt{n} < 10$ is equivalent to $n < 100$. With two square root signs, $\sqrt{n + \sqrt{n}} < 10$ is equivalent to $n + \sqrt{n} < 100$, which is in turn equivalent to $n < 91$. With three square root signs and $n < 91$, we have $\sqrt{n + \sqrt{n + \sqrt{n}}} < \sqrt{n + 10} \leq 10$. With more square root signs, the same inequality will hold. There are 90 positive integers which satisfy $n < 91$. The answer is **(b)**.

14. Let x be the distance from the centre of the circle to the bottom edge of the larger square. The square of the radius of the circle is given by $6^2 + x^2 = 2^2 + (4 + 12 - x)^2$. This yields $x = 7$ so that the radius of the circle is $\sqrt{6^2 + 7^2} = \sqrt{85}$. The answer is **(d)**.



15. If $p = 2$, $2^p + p^2 = 8$ is not prime. If $p = 3$, $2^p + p^2 = 17$ is prime. If $p > 3$, then $p = 6k \pm 1$ for some integer k, so that $p^2 = 36k^2 \pm 12k + 1$ is 1 more than a multiple of 3. On the other hand, when divided by 3, successive powers of 2 leave remainders of 2 and 1 alternately. Since $p > 3$ is odd, 2^p is 2 more than a multiple of 3. Hence $2^p + p^2$ is divisible by 3, and cannot be prime. The answer is **(b)**.

16. Let the other root be t. Then $-a = t + 2 - \sqrt{99}$. Now $b = t(2 - \sqrt{99}) < 0$, so that $t = \sqrt{99} - 2 - a > 0$. Since a is an integer, $a \leq 9 - 2 = 7$. The answer is **(c)**.

The Alberta High School Mathematics Competition
Part II, February 3, 2010

1. Of Melissa's ducks, $x\%$ have 11 ducklings each, $y\%$ have 5 ducklings each and the rest have 3 ducklings each. The average number of ducklings per duck is 10. Determine all possible *integer* values of x and y.

2. (a) Find all real numbers $t \neq 0$ such that $tx^2 - (2t-1)x + (5t-1) \geq 0$ for all real numbers x.

   (b) Find all real numbers $t \neq 0$ such that $tx^2 - (2t-1)x + (5t-1) \geq 0$ for all $x \geq 0$.

3. Points A, B, C and D lie on a circle in that order, so that $AB = BC$ and $AD = BC + CD$. Determine $\angle BAD$.

4. Let n be a positive integer. A $2^n \times 2^n$ board, missing a 1×1 square anywhere, is to be partitioned into rectangles whose side lengths are integral powers of 2. Determine in terms of n the smallest number of rectangles among all such partitions, wherever the missing square may be.

5. Let f be a non-constant polynomial with non-negative integer coefficients.

   (a) Prove that if M and m are positive integers such that M is divisible by $f(m)$, then $f(M+m)$ is also divisible by $f(m)$.

   (b) Prove that there exists a positive integer n such that each of $f(n)$ and $f(n+1)$ is a composite number.

The Alberta High School Mathematics Competition
Solution to Part II, 2010.

1. We have $11x + 5y + 3(100 - x - y) = 1000$ or $4x + y = 350$. Since $y \geq 0$, we get $x \leq 87$. Since $x + y \leq 100$, we also have that $3x \geq 250$, so $x \geq 84$. Thus the only solutions are $(x, y) = (84, 14)$, $(85,10)$, $(86,6)$ and $(87,2)$.

2. For either (a) or (b), clearly the leading coefficient t of the quadratic must be positive.

   (a) For the inequality to hold for all real x, the discriminant must be non-positive, that is,

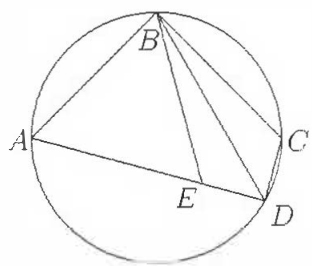
   $$0 \geq (2t-1)^2 - 4t(5t-1) = 1 - 16t^2 = (1-4t)(1+4t).$$

   Since $t > 0$, $1 + 4t > 0$, so we need $1 - 4t \leq 0$. Thus $t \geq \frac{1}{4}$.

   (b) We now have the additional possibility that the two roots of the quadratic are real and non-positive. This holds if and only if $0 < t \leq \frac{1}{4}$, $2t - 1 \leq 0$ and $5t - 1 \geq 0$. This is equivalent to $\frac{1}{5} \leq t \leq \frac{1}{4}$. Combining with the answer to (a), we have $t \geq \frac{1}{5}$.

3. **First Solution:**

   Putting $AB = BC = b$ and $CD = c$, we get $AD = b + c$. Let $\angle BAD = \alpha$. Since $ABCD$ is cyclic, $\angle BCD = 180° - \alpha$. Applying the cosine law to triangles BAD and BCD, we have $BD^2 = b^2 + (b+c)^2 - 2b(b+c)\cos\alpha$ and $BD^2 = b^2 + c^2 - 2bc\cos(180° - \alpha) = b^2 + c^2 + 2bc\cos\alpha$. Hence $b^2 + (b+c)^2 - 2b(b+c)\cos\alpha = b^2 + c^2 + 2bc\cos\alpha$, so that $b^2 + 2bc = (2b^2 + 4bc)\cos\alpha$. This yields $\cos\alpha = \frac{1}{2}$, so that $\alpha = 60°$ is the only possibility.



   Second Solution:

   Let E be the point on AD such that $DE = DC$, so that $AE = AD - DE = BC = AB$. Now $\angle BDE = \angle BDC$ since they are subtended by the equal arcs BA and BC. It follows that triangles BED and BCD are congruent, so that $BE = BC = BA = AE$, triangle BAE is equilateral and $\angle BAD = 60°$.

4. **First Solution:**

   The area of the punctured board is $2^{2n} - 1$. The base-2 representation of this number consists of $2n$ 1s. Since the area of each rectangle in the partition is a power of 2, we must have at least $2n$ rectangles. There exist such partitions with exactly $2n$ rectangles. Divide the board in halves by a horizontal grid line. Set aside the one with the missing square and cover the other with a rectangle of height 2^{n-1}. Repeating the process with the strips set aside, we obtain rectangles with decreasing heights 2^{n-2}, 2^{n-3}, ..., 2^1 and 2^0, a total of n rectangles. We now divide the resulting $2^n \times 1$ board in halves by a vertical line. Set aside the one with the missing square and cover the other with a rectangle of width 2^{n-1}. Repeating the process with the strips set aside, we obtain another n rectangles with decreasing widths, for a total of $2n$ rectangles in the overall partition.

Second Solution:

Divide the board into four congruent quadrants. Set aside the one with the missing square. Merge two of the other quadrants into one rectangle and keep the third quadrant as the second rectangle. In reducing a $2^n \times 2^n$ board down to a $2^{n-1} \times 2^{n-1}$ board, we use two rectangles. It follows that we will use exactly $2n$ rectangles in the overall partition. We now prove that we cannot get by with a smaller number. The area of a rectangle of the prescribed type is a power of 2. The smallest has area 1, and the largest has area 2^{2n-1}. Thus there are $2n$ different sizes. If we use one of each size, the total area of these $2n$ rectangles is $1+2+\cdots+2^{2n-1} = 2^{2n}-1$, exactly the size of the punctured chessboard. Consider any other collection of rectangles whose areas are powers of 2 and whose total area is $2^{2n-1}-1$. Replace any pair of rectangles of equal area by one with twice the area. Repeat until no further replacement is possible. The resulting collection consists of rectangles of distinct areas which are powers of 2, and with total area $2^{2n-1}-1$. It can only be our collection, and since mergers only reduce the number of rectangles, $2n$ is indeed minimum.

5. (a) Note that $f(M+m)-f(m)$ is a sum of terms of the form $a_k((M+m)^k - m^k)$ where a_k is the coefficient of the term x^k in $f(x)$. Since each term is divisible by $M = (M+m)-m$, so is $f(M+m)-f(m)$. Since M is divisible by $f(m)$, $f(M+m)-f(m)$ is divisible by $f(m)$. It follows that $f(M+m)$ is divisible by $f(m)$.

   (b) Since all the coefficients of f are non-negative and f is non-constant, it is strictly increasing. Let $M = f(2)f(3)$ and $n = M+2$. By (a), $f(n)$ is divisible by $f(2)$ and $f(n+1)$ is divisible by $f(3)$. Since $f(n+1) > f(n) > f(3) > f(2) > f(1) \geq 1$, both $f(n)$ and $f(n+1)$ are composite.