delta-k

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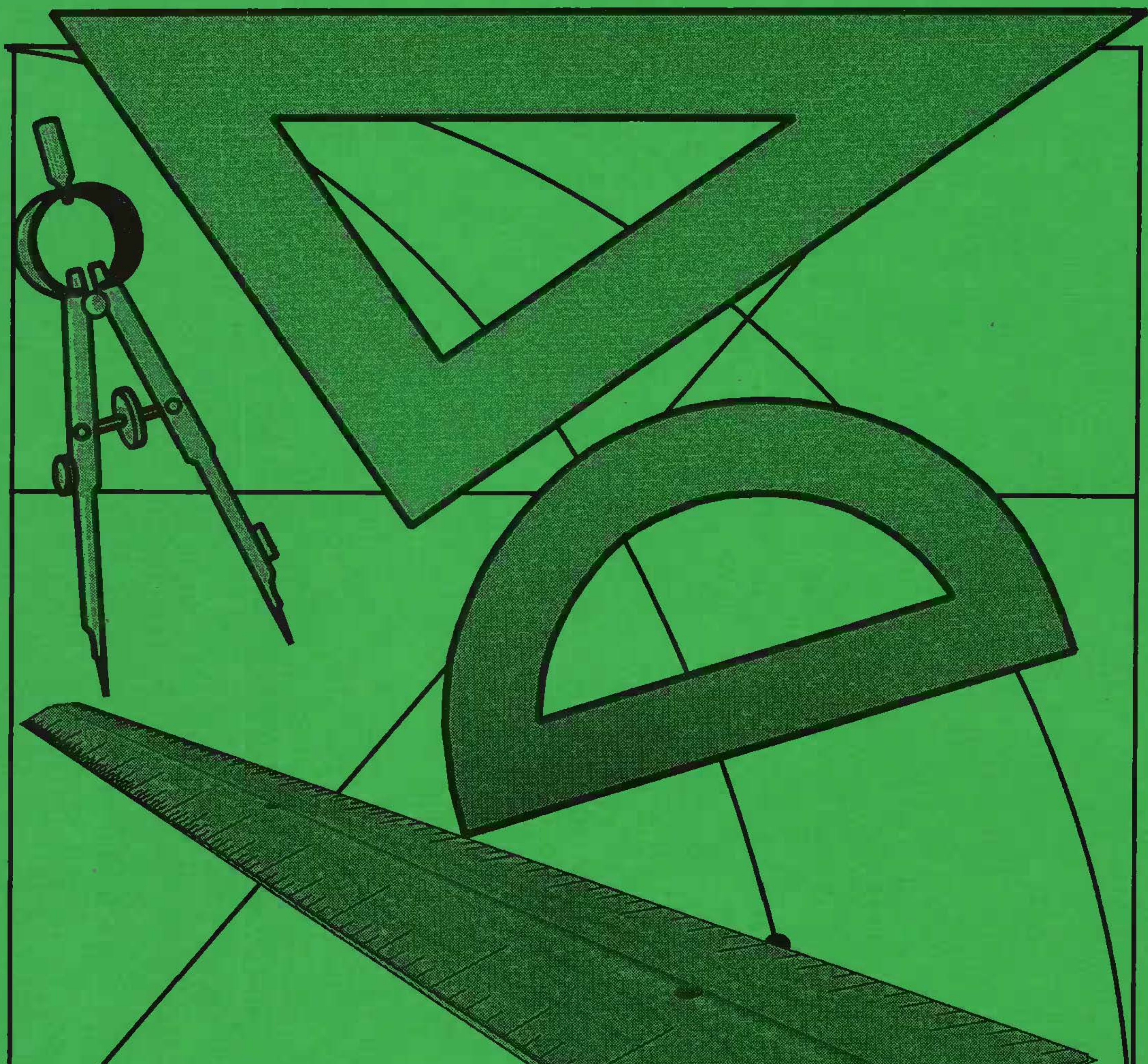

GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Submit work electronically, preferably in Microsoft Word format.
3. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. A 250–350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's website.
7. Letters to the editor or reviews of curriculum materials are welcome.
8. *delta-K* is not refereed. Contributions are reviewed by the editor(s), who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to A. Craig Loewen, Editor, 414 25 Street S, Lethbridge, AB T1J 3P3; fax (403) 329-2412, e-mail loewen@uleth.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 41, Number 2 June 2004

CONTENTS

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EDITORIAL

This is my first issue of *delta-K* since I took over as editor from Klaus Puhlmann in the fall of 2003. I want to thank the many people who have been so supportive during this transition, including ATA staff advisor Dave Jeary, ATA publications supervisor Karen Virag, MCATA publications director Shauna Boyce and all the executive members of the Mathematics Council of the Alberta Teachers' Association (MCATA). Your encouragement and support have meant a great deal!

Of course, I also want to express my thanks, appreciation and congratulations to Klaus Puhlmann as he retires as editor after so many years of superior and dedicated service. We all owe Klaus this sincere acknowledgement of his outstanding work. Thank you, Klaus!

At our last meeting, the MCATA executive voted to begin the process of making *delta-K* a refereed journal. This move will bring *delta-K* (and, by extension, MCATA) increased recognition for its publishing efforts while maintaining its traditional high standards and further enhancing its reputation.

This move will change guideline 8 in the guidelines for manuscripts on the inside front cover. Once *delta-K* becomes a refereed journal, articles will be submitted for an initial overview to the editor, who will in turn send the articles to two reviewers, each of whom has a special interest in mathematics education. Based on the reviewers' recommendations, articles will appear in future issues of *delta-K.* The final decision to publish any article remains with the editor. Guidelines for this process are not yet finalized but will become available in upcoming months.

delta-K remains committed to encouraging and publishing high-quality articles pertaining to the professional development of mathematics educators, and articles that stimulate thinking and explore new ideas and viewpoints. We continue to be interested in articles with a classroom or scholarly focus. Please feel free to submit your ideas regarding mathematics instruction—in the form of articles, activities, letters or problems. Our goal is to help teachers and other educators share their ideas and strategies with the aim of continually supporting and refining mathematics instruction across the province.

I have selected some of my favourite problems for the last page of this issue. If you have a problem you want to share, send it to me at loewen@uleth.ca. I'll try to include it in the next issue. Enjoy!

A. Craig Loewen

From the President's Pen

The previous MCATA president, Cynthia Ballheim, has moved to Columbus, Ohio, and is teaching mathematics at a Catholic high school. Because of the distance and logistical problems, Cynthia found it too difficult to continue to serve as president. She intends to stay in touch with MCATA and will be consulted as needed. I have agreed to move from vice-president to serve as president until June 2004, when the executive positions will be reviewed.

I have been a mathematics teacher for 29 years with the Edmonton Catholic School District, except for a one-and-a-half-year secondment with Alberta Learning. Currently, I am coordinator of mathematics and sciences at Holy Trinity High School in Edmonton. I have served on the MCATA executive for four years, and I hope to continue to provide the kind of leadership that Cynthia demonstrated during her many years of service.

delta-K continues to be MCATA's academic publication and is organized by our new editor, Craig Loewen of the University of Lethbridge. The executive is moving toward making *delta-K* a refereed journal, as Craig explained in his editorial. We look forward to reading the high-quality articles published in *delta-K* in the future.

Speaking of reading, if you enjoy reading books that connect mathematics to society, nature and other areas of life, I recommend the following:

- *The Millennium Problems: The Seven Greatest Unsolved Mathematical Puzzles of Our Time*, by Keith Devlin (Basic, 2002)
- *The Computational Beauty of Nature: Computer Explorations of Fractals, Chaos, Complex Systems, and Adaptation,* by Gary William Flake (MIT Press, 1998)
- *Numbers in the Dark and Other Stories,* by Italo Calvino (translated from the Italian by Tim Parks) (Knopf, 1995)

Enjoy the books, and thank you for continuing to support MCATA.

Len Bonifacio

Determining the Angles Between Two Lines

David E. Dobbs

In preparing a recent lecture for a course on non-Euclidean geometry, I needed a formula to determine the angles formed by two intersecting lines in Euclidean plane geometry. The relevant formulas in Proposition 6.2 of *The Poincaré Half-Plane: A Gateway to Modern Geometry* (Stahl 1993), the textbook for the course, depended on methods not needed again until the textbook's coverage of the hyperbolic version of the Pythagorean theorem (Theorem 8.3). I decided to seek alternative formulas with minimal prerequisites and the additional benefit of being easy to implement on modern calculators.

Because the task at hand would be meaningful for a precalculus class, I consulted a current leader in that market, *Precalculus: Mathematics for Calculus* (Stewart, Redlin and Watson 2002). The relevant method given in this textbook used the formula

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|},$$

where $\vec{u}$ and $\vec{v}$ are vectors in directions determined by the given intersecting lines, and then used the inverse cosine function and related angles (also known as reference angles) to compute the angles in question (p. 604, Example 2). The prerequisites for this approach become available rather late in a precalculus course; for instance, the above formula for $\cos(\theta)$ is proved using the law of cosines (p. 603). Therefore, I looked further for an accessible method that could be implemented with relatively few keystrokes on a calculator.

At first glance, it seemed that *A First Year of College Mathematics* (Brink 1954, 359), a textbook of 50 years ago for the precursor of today's precalculus course, contained the answer for the angles determined by intersecting nonvertical lines having slopes m_1 and m_2, by use of the formula

$$\tan(\theta) = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Unfortunately, in this formula, θ can be negative (p. 358), contrary to our natural desire to determine angles between 0 and π. (Of course, as is appropriate for precalculus and beyond, we are measuring angles in radians.)

I modernized the formula from Brink (1954) by developing some accessible, calculator-friendly formulas (see the theorem later in the article). Part (a) of the theorem concerns the case where two nonvertical lines intersect, and part (b) addresses the situation where one of the intersecting lines is vertical. This theorem can be presented as enrichment material quite early in a precalculus course because its only prerequisites are a pair of facts from geometry (equality of corresponding angles cut from parallel lines by a transversal, and relation of an external angle of a triangle to the remote interior angles of the triangle), slope, the slope-intercept equation of a nonvertical line, angle of inclination of a line, the definitions of the tangent and inverse tangent functions, and the usual expansion formula for $\tan(u - v)$. For the sake of completeness, the next section begins with a proposition that recalls the connection between the slope and the angle of inclination of a nonvertical line. The closing remark provides an example comparing the speed of applicability of the three methods mentioned above.

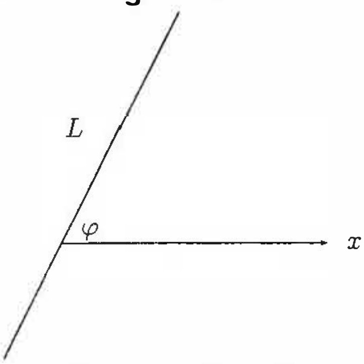
The centrality of the tangent function in trigonometry (and thus, nowadays, in precalculus) has been implicit for millennia, at least since the time of Thales. I have written a number of articles (Dobbs 1984a, 1988, 1991), suitable for use in a precalculus course, explaining how the tangent function can be used to give new proofs of various facts presented in typical high school geometry and precalculus courses. In several such notes, investigations using analytic (as opposed to synthetic) methodology have developed new results, as well (Dobbs 1984c, 1984d). This article is intended as another contribution to this program. In using it, the reader may want to consult Dobbs

(1984b) for a self-contained proof of the expansion formula for $\tan(u \pm v)$ that is more accessible than the proof in standard textbooks in that it is independent of the expansion formulas for $\sin(u \pm v)$ and $\cos(u \pm v)$. Finally, note the central role of the formula for $\tan(u \pm v)$, as it figures in a characterization of the tangent function (Dobbs 1989, Theorem 3), a result later used by the College Board and Educational Testing Service as the basis for the final question on the Advanced Placement Calculus BC examination in May 1993.

Formulas Based on the Inverse Tangent Function

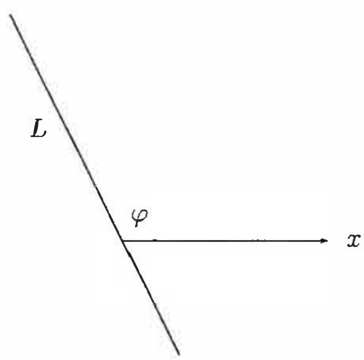
We begin by recalling the definition of what will be our key tool. If L is a line, then the angle of inclination of L is defined as the angle φ between L and the positive x-axis such that $0 \leq \varphi < \pi$. If L has positive slope, then $0 < \varphi < \pi/2$, as in Figure 1.

Figure 1
Acute Angle of Inclination



If L has negative slope, then $\pi/2 < \varphi < \pi$, as in Figure 2.

Figure 2
Obtuse Angle of Inclination



If L has slope equal to 0, then L is horizontal and it is conventional to take the angle of inclination of L to be 0. Last, if the slope of L is undefined (that is, if L is vertical), then $\varphi = \pi/2$. Part (a) of the following proposition is well known (see Brink 1954, 357); parts (b) and (c) are also essentially known and will be useful later in the proof of the theorem.

Proposition. *Let* L *be a nonvertical line having slope* m *and angle of inclination* φ. *Then,*

(a) *$\tan(\varphi) = m$.*
(b) *If φ is an acute angle, then $\varphi = \tan^{-1}(m)$.*
(c) *If φ is an obtuse angle, then $\varphi = \pi - \tan^{-1}(-m) = \pi + \tan^{-1}(m)$.*

Proof. (a) If φ is acute, then $\tan(\varphi)$ and m are the same ratio of two sides of a right triangle having φ as one of its angles. Suppose next that φ is obtuse, with related angle θ. Then, the preceding reasoning gives that $\tan(\theta) = -m$. Moreover, $\tan(\varphi) = -\tan(\theta)$ by the definition of the tangent function, as given in Brink (1954, 200, 233). The assertion follows easily. (The preceding argument was tailored for classes whose definition of the trigonometric functions is, like that in Brink [1954], based on angles in standard position and related angles. An alternative proof should be given to classes whose definition of the trigonometric functions is based on the unit circle.)

(b) The assertion in (b) follows from (a) and the definition of the inverse tangent function.

(c) Suppose that φ is obtuse. Let θ be the related angle of φ. Since $\varphi + \theta = \pi$, it follows that θ is an acute angle. Also, as noted in the proof of (a), $\tan(\theta) = -m$. Then $\theta = \tan^{-1}(-m) = -\tan^{-1}(m)$, the first equality holding by the definition of the inverse tangent function and the second equality holding because $\tan^{-1}$ is an odd function. Substituting these facts into the equation $\varphi = \pi - \theta$ leads to the assertions in (c), to complete the proof. □

The formulation of our main result ignores the case of perpendicular lines because this case can be handled directly. Indeed, if L_1 and L_2 are coplanar lines having slopes m_1 and m_2, respectively, every precalculus course covers the fact that L_1 and L_2 are perpendicular if and only if $1 + m_1m_2 = 0$. Moreover, a vertical line L_1 is perpendicular to a coplanar nonvertical line L_2 having slope m_2 (at their point of intersection) if and only if $m_2 = 0$.

Theorem. *Let* L_1 *and* L_2 *be two intersecting nonperpendicular lines in the Euclidean plane. Then,* (a) *suppose that* L_1 *and* L_2 *are each nonvertical, having slopes* m_1 *and* m_2, *respectively. Then, the two acute angles formed by* L_1 *and* L_2 *at their point of intersection are each given by*

$$\tan^{-1}\left(\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|\right),$$

and the two obtuse angles formed by L_1 *and* L_2 *at their point of intersection are each given by*

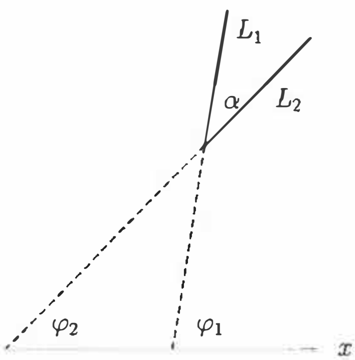
$$\pi - tan^{-1}\left(\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right).$$

(b) *Suppose that* L_1 *is vertical and that* L_2 *has slope* m_2. *If* $m_2 > 0$, *then the two acute angles formed by* L_1 *and* L_2 *at their point of intersection are each given by* $\pi/2 - tan^{-1}(m_2)$, *and the two obtuse angles formed by* L_1 *and* L_2 *at their point of intersection are each given by* $\pi/2 + tan^{-1}(m_2)$. *If* $m_2 < 0$, *then the two acute angles formed by* L_1 *and* L_2 *at their point of intersection are each given by* $\pi/2 + tan^{-1}(m_2)$, *and the two obtuse angles formed by* L_1 *and* L_2 *at their point of intersection are each given by* $\pi/2 - tan^{-1}(m_2)$.

Proof. (a) Four angles are formed at the intersection of L_1 and L_2. Since vertically opposite angles are congruent, it suffices to determine one of these angles, say α. The other three angles of intersection are then α, $\pi - \alpha$ and $\pi - \alpha$. It is convenient to distinguish six cases.

Case 1. $0 < m_2 < m_1$, with α acute. The data are depicted in Figure 3.

Figure 3
Case 1



By a basic fact about triangles in Euclidean geometry, the exterior angle φ_1 is the sum of the two remote interior angles, α and φ_2, and so $\alpha = \varphi_1 - \varphi_2$. Moreover, by part (a) of the proposition, $\tan(\varphi_1) = m_1$ and $\tan(\varphi_2) = m_2$. Therefore, by the expansion formula for $\tan(u - v)$, we have that

$$\tan(\alpha) = \tan(\varphi_1 - \varphi_2) = \frac{\tan(\varphi_1) - \tan(\varphi_2)}{1 + \tan(\varphi_1)\tan(\varphi_2)}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

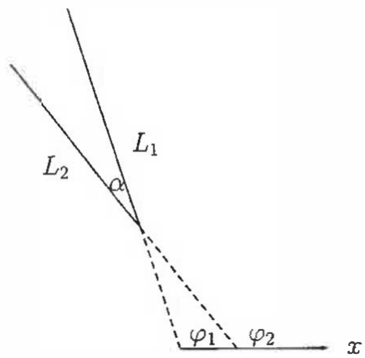
Since α is an acute angle, the definition of the inverse tangent function ensures that

$\alpha = \tan^{-1}(\tan(\alpha))$

in this case, so the asserted formula for α has been established.

Case 2. $m_1 < m_2 < 0$, with α acute. The data are depicted in Figure 4.

Figure 4
Case 2



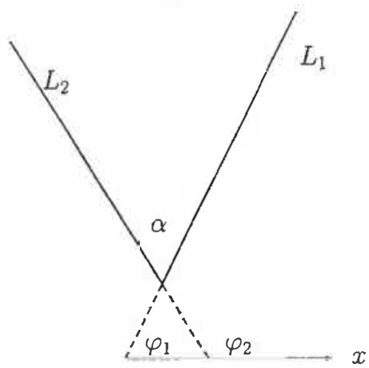
The exterior angle φ_2 is the sum of the two remote interior angles, α and φ_1, and so $\alpha = \varphi_2 - \varphi_1$. Combining part (a) of the proposition, the expansion formula for $\tan(u - v)$ and the definition of the inverse tangent function as above, we see that

$$\tan(\alpha) = \frac{m_2 - m_1}{1 + m_2 m_1} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

and $\alpha = \tan^{-1}(\tan(\alpha))$. The asserted formula for α follows.

Case 3. $m_2 < 0 < m_1$, with α acute. The data are depicted in Figure 5.

Figure 5
Case 3



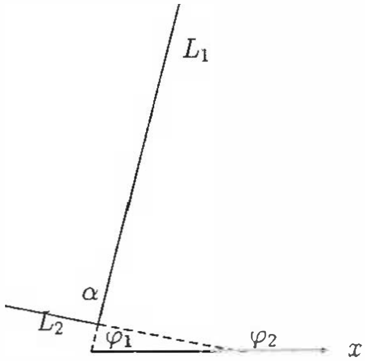
As in the analysis for Case 2, we infer that

$$\tan(\alpha) = \frac{m_2 - m_1}{1 + m_2 m_1} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

(The last equality holds since $m_2 - m_1$ and $1 + m_2 m_1$ are both negative, but it is not really necessary to observe this, because we need only appeal to the fact that any acute angle has a positive tangent.) Case 3 can now be completed in the earlier cases by appealing to the definition of the inverse tangent function.

Case 4. $m_2 < 0 < m_1$, with α obtuse. The data are depicted in Figure 6.

Figure 6
Case 4



As in the analyses for Cases 2 and 3, we infer that

$$\tan(\alpha) = \frac{m_2 - m_1}{1 + m_2 m_1} = -\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|,$$

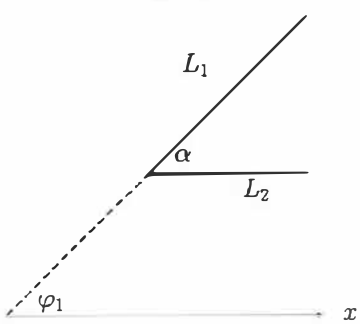
the last equality holding since obtuse angles have negative tangents. Next, note what was effectively established in part (c) of the proposition—that any obtuse angle α satisfies

$$\alpha = \pi - \tan^{-1}(|\tan(\alpha)|);$$

this can also be seen as a consequence of the basic facts about related angles (see Brink 1954, 233, Rule). Combining the assembled information leads to the asserted description of α.

Case 5. $0 = m_2 < m_1$, with α acute. The data are depicted in Figure 7.

Figure 7
Case 5



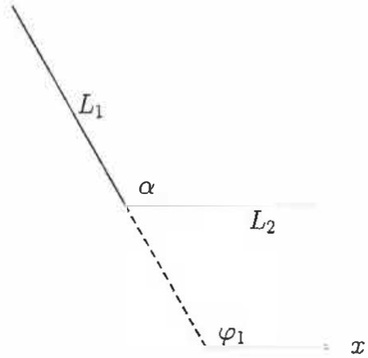
Relative to the transversal L_1, the parallel lines L_2 and the x-axis cut off exterior corresponding angles α and φ_1. Therefore, by a fundamental result in Euclidean geometry, $\alpha = \varphi_1$. Hence, by part (b) of the proposition,

$$\alpha = \tan^{-1}(m_1) = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right),$$

and the asserted description of α follows easily.

Case 6. $m_1 < m_2 = 0$, with α obtuse. The data are depicted in Figure 8.

Figure 8
Case 6



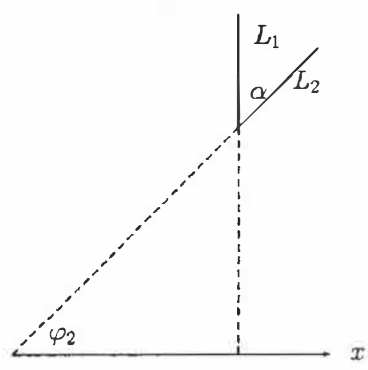
As in the analysis of Case 5, we see that α and φ_1 are exterior corresponding angles and, hence, equal (in measure). In particular, part (a) of the proposition yields that $\tan(\alpha) = m_1$. Then, since α is obtuse, a fact recalled in the analysis of Case 4—or an application of part (c) of the proposition—yields that

$$\alpha = \pi - \tan^{-1}(|\tan(\alpha)|) = \pi - \tan^{-1}(|m_1|).$$

Since $m_2 = 0$, the asserted description of α now follows easily.

(b) The opening comments in the proof of (a) are enough to prove the assertions concerning the acute angles of intersection α. Let φ_2 denote the angle of inclination of L_2. Suppose first that $m_2 > 0$, as in Figure 9.

Figure 9
L_1 Vertical and L_2 with Positive Slope

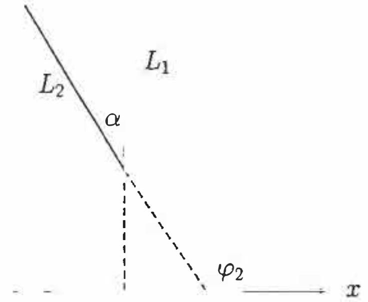


Since $m_2 > 0$, we have that φ_2 is an acute angle, so part (b) of the proposition gives $\varphi_2 = \tan^{-1}(m_2)$. Therefore, since α and φ_2 are complementary, we have $\alpha = \pi/2 - \varphi_2 = \pi/2 - \tan^{-1}(m_2)$, as asserted.

Finally, suppose that $m_2 < 0$, as in Figure 10.

Figure 10

L_1 Vertical and L_2 with Negative Slope



We have the exterior angle φ_2 equal to the sum of the two interior remote angles, α and $\pi/2$. Since φ_2 is obtuse, we could now complete the proof by using the fact recalled in the analyses of Cases 4 and 6. For variety, we argue instead through part (c) of the proposition. This gives that $\varphi_2 = \pi + \tan^{-1}(m_2)$. Therefore, $\alpha = \varphi_2 - \pi/2 = (\pi + \tan^{-1}(m_2)) - \pi/2 = \pi/2 + \tan^{-1}(m_2)$, to complete the proof. □

Remark. Consider the lines L_1 and L_2 with Cartesian equations $2x - 3y + 4 = 0$ and $5x + 6y + 7 = 0$, respectively. Solving for y, we obtain the equations in slope-intercept form as $y = \frac{2}{3}x + \frac{4}{3}$ and $y = -\frac{5}{6}x - \frac{7}{6}$, respectively. Therefore, the slopes of the given lines are the coefficients of x in slope-intercept form: $m_1 = \frac{2}{3}$ and $m_2 = -\frac{5}{6}$. Implementing part (a) of the theorem (with the aid of a TI-86 graphing calculator), we see that the radian measure of an acute angle formed by L_1 and L_2 at their point of intersection is

$$\tan^{-1}\left(\left|\frac{\frac{2}{3} - \frac{-5}{6}}{1 + \frac{2}{3}\cdot\frac{-5}{6}}\right|\right) \approx 1.28274087974,$$

and so an obtuse angle formed by L_1 and L_2 is given approximately by the supplement of the preceding value:

$$\pi - 1.28274087974 \approx 1.85885177385.$$

Notice that, in implementing the theorem, we need no diagram and there are no ambiguities. In particular, because of the absolute value symbol appearing in the formulas in part (a) of the theorem, it does not matter which line we called L_1 and which L_2. In addition, the above narrative displays the relatively few calculations and keystrokes needed in this routine application of the theorem.

Let us compare the above work with how the methods in Brink (1954, 359) and Stewart, Redlin and Watson (2002, 604) would handle the same problem.

First, we consider the method from Brink (1954, 359), using the formula

$$\tan(\theta) = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

With the above values of m_1 and m_2, we find that $\tan(\theta) = \frac{27}{8} = 3.375$. One such θ is $\tan^{-1}(3.375) \approx 1.28274087974$, the acute angle that we found using the theorem; thus, by calculating the supplement of θ, one would also find the obtuse angles formed by L_1 and L_2.

However, one should not conclude that Brink's (1954, 359) method is as useful in general as that in part (a) of the theorem. What if we interchanged the labels on the lines L_1 and L_2? We would then be led to consider an angle θ such that $\tan(\theta) = -3.375$. This θ is neither the acute nor the obtuse angle that we are seeking! Moreover, a calculator cannot come to the immediate rescue, because the inverse tangent of this θ is negative. Granted, with a careful diagram and some thought about related angles, a skilled user of this method could eventually find the desired acute and obtuse angles. In contrast, a user of the theorem need never worry about such matters, because the case analyses in the proof of the theorem took care of them once and for all.

Next, we consider the currently popular vectorial method in Stewart, Redlin and Watson (2002, 604). To apply this method, we first need to find a vector $\vec{u}$ in a direction determined by L_1 and a vector $\vec{v}$ in a direction determined by L_2. To find $\vec{u}$, we first find two points on L_1, say the intercepts of L_1 on the x- and y-axes. Setting one variable equal to 0 in an equation for L_1 and solving for the other variable, we are thus led to the points $P_1(0, \frac{4}{3})$ and $P_2(-2, 0)$ on L_1. A suitable $\vec{u}$ is then the vector $\overrightarrow{P_1P_2} = \langle -2, -\frac{4}{3} \rangle$; some notational conventions would write this vector as $(-2, -\frac{4}{3})$ or $-2\vec{i} - \frac{4}{3}\vec{j}$. A similar amount of work with an equation for L_2 would find a suitable $\vec{v}$ to be the vector $\langle -\frac{7}{5}, \frac{7}{6} \rangle$. Applying the formula on page 604, we are then led to an angle θ such that

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{-2\left(-\frac{7}{5}\right) - \frac{4}{3}\cdot\frac{7}{6}}{\sqrt{(-2)^2 + \left(-\frac{4}{3}\right)^2}\sqrt{\left(-\frac{7}{5}\right)^2 + \left(\frac{7}{6}\right)^2}}$$

$$\approx 0.284088329691.$$

One such θ is $\cos^{-1}(0.284088329691) \approx 1.28274087974$, the acute angle that we found using the theorem; thus, by calculating the supplement of θ, one would also find the obtuse angles formed by L_1 and L_2.

As one might suspect from the above discussion, this method can be made as useful in general as that in the theorem, but one would need the following additional provisos. If the calculated value of $\cos(\theta)$ is positive (resp., negative), then taking the inverse cosine of this number produces the acute (resp., obtuse) angle(s) formed at the point of intersection of L_1 and L_2.

The method in Stewart, Redlin and Watson (2002, 604) does have an advantage: it does not need to consider separately the case in which one of the intersecting lines is vertical, as we did in part (b) of the theorem. However, as the above example illustrates, the number of calculations and keystrokes needed to implement this method is considerably greater than the corresponding effort in applying the theorem.

Last, I indicate another aspect, which I view as a drawback, of this method. Notice that if we interchange the labels on the points P_1 and P_2 considered above, then $\vec{u}$ is replaced with $-\vec{u}$ and the calculated value of $\cos(\theta)$ changes to the negative of the previous value. Thus, this method cannot guarantee a priori whether the first angle θ that it finds is going to be acute or obtuse. As explained above, this ambiguity can be removed, at the possible cost of calculating a supplementary angle, after observing the sign of the calculated value of $\cos(\theta)$. By way of contrast, no such thought or supplementary calculation (pun intended) is needed in applying the theorem; once again, the point is that the case analyses in the proof of the theorem took care of such issues once and for all.

If time is short, a classroom presentation covering the main points given above could be based on the proposition; the statement of the theorem; cases 1, 4 and 6 from the proof of the theorem; and the part of the remark in which the theorem is applied.

References

Brink, R. W. *A First Year of College Mathematics*. 2nd ed. New York: Appleton-Century-Crofts, 1954.

Dobbs, D. E. "Proving Heron's Formula Tangentially." *College Mathematics Journal* 15, no. 3 (1984a): 252–53.

———. "A Classroom Note on the Tangent Expansion Formula." *Computer and Mathematics Education Journal* 18 (1984b): 119–20.

———. "A Trigonometry-Based Method for Constructing Square Roots by Straight-Edge and Compass." *International Journal of Mathematics Education in Science and Technology* 15 (1984c): 127–28.

———. "A Tangential Approach to Central Angles." *Illinois Mathematics Teacher* 36, no. 4 (1984d): 18–23.

———. "The Law of Tangents: A New Proof and an Application." *International Journal of Mathematics Education in Science and Technology* 19 (1988): 759–63.

———. "A Characterization of the Tangent Function." *Elemente der Mathematik* 44 (1989): 101–3.

———. "A Tangential Approach to Angle-Bisectors." *Illinois Mathematics Teacher* 42, no. 2 (1991): 23–25.

Stahl, S. *The Poincaré Half-Plane: A Gateway to Modern Geometry*. Boston: Jones and Bartlett, 1993.

Stewart, J., L. Redlin and S. Watson. *Precalculus: Mathematics for Calculus*. 4th ed. Pacific Grove, Calif.: Brooks/Cole, 2002.

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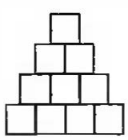
Sums of Arithmetic Sequences: Several Problems and a Manipulative

A. Craig Loewen

An extremely powerful and important link exists between manipulatives and problem solving. Through the use of manipulatives, we come to understand mathematics; through problem solving, we are challenged to apply what we have learned.

Consider the following four problems:

1. There are 50 people at a party. Each person shakes hands once with each of the other people. How many handshakes occur in total?
2. How many diagonals are in a regular hectagon (a polygon with 100 sides)?
3. Bricks are stacked to create a pyramid like the one shown below. How many bricks would be required to build a pyramid 75 rows tall?



4. As in the well-known Christmas song, on the first day of Christmas, my true love gives me one partridge in a pear tree. On the second day, my true love gives me two turtledoves. If my true love continues for a full year (365 days) to give me one gift more each day than the previous day, how many gifts will I receive in all?

Though these problems look unlike on the surface, they share at least one important quality: the solution to each requires summing a series of consecutive whole numbers.

A Historical Note

A story about a famous mathematician, Carl Friedrich Gauss, tells how Gauss, while still a schoolboy, was required to sum the numbers 1–100 as a punishment. He finished the task far ahead of his classmates. It seems that Gauss realized that by grouping the numbers he could identify a pattern and thus simplify his work:

$$\begin{aligned}\text{Sum} &= 1 + 2 + 3 + \ldots + 98 + 99 + 100\\ &= (1 + 100) + (2 + 99) + \ldots + (50 + 51)\\ &= 101 + 101 + \ldots + 101.\end{aligned}$$

There are 50 pairs, each totalling 101. Thus,

$$\begin{aligned}\text{Sum} &= 50 \times 101\\ &= 5{,}050.\end{aligned}$$

There is a similar way to understand and attack this problem. Below the first equation, write the equation in reverse order, then add the two equations:

$$\begin{aligned}\text{Sum} &= 1 + 2 + \ldots + 99 + 100\\ \text{Sum} &= 100 + 99 + \ldots + 2 + 1\\ 2 \cdot \text{Sum} &= 101 + 101 + \ldots + 101 + 101\end{aligned}$$

Because we know that the equation has a hundred 101s, we can write

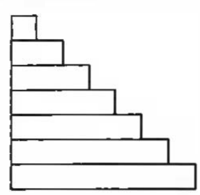
$$\begin{aligned}2 \cdot \text{Sum} &= 100 \times 101\\ &= 50 \times 101\\ &= 5{,}050.\end{aligned}$$

Starting with a simpler problem, let's look at a manipulative that shows why this might work.

Solve a Simpler Problem

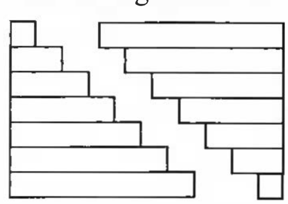
Consider the following task: Find the sum of the numbers 1–7.

A familiar manipulative such as Cuisenaire Rods can be used to model this task. Take one each of the seven shortest rods and arrange them in a staircase as shown:



The total length of all the stairs is equal to the sum we are trying to find. Also, because each rod has a width of 1 unit, the area of the staircase is equal to that sum.

Now arrange a second staircase as shown:



Slide the two staircases together to create a rectangle.

- What are the length and width of this rectangle?
- How does the area of the rectangle compare with the sum we are trying to find?
- What is the largest such staircase you could build?
- Using graph paper, draw a staircase showing the sum of the numbers 1–20. Draw the related rectangle.
- Could the same process be used regardless of the number of stairs? Why?
- Describe how you could find the sum of the counting numbers up to any given value.
- Write a formula to show how this sum could be easily calculated.

Through this manipulative and exploration, we come to see that the sum of n counting numbers starting at 1 is

$$\text{Sum} = \frac{n(n+1)}{2}.$$

Applying What We Know

Now we can easily apply what we have learned to the original four problems.

In the first problem, the first person at the party will shake hands with 49 people, the second person will shake 48 hands and so on. Thus, the solution to the first problem is

$$\text{Sum} = 49 + 48 + 47 + \ldots + 2 + 1$$
$$= \frac{49(49+1)}{2}.$$

Likewise, the solution to the third problem is

$$\text{Sum} = \frac{75(75+1)}{2}.$$

The solution to the fourth problem is

$$\text{Sum} = \frac{365(365+1)}{2}.$$

But be careful! The solution to the second problem is a bit trickier. We know that a hectagon has 100 sides, so it is tempting to think that we can just use Gauss's answer for the second problem. But the correct answer is not 5,050!

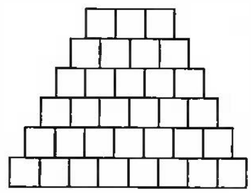
Imagine the vertices of a hectagon spread evenly around a large circle. We can begin to connect these vertices two at a time to create our diagonals. There will be 99 lines from the first vertex to each of the other 99 vertices, 98 lines from the second vertex and so on. This implies that the final answer is half of 99×100. But again, it is not! In drawing these lines, we have included the edges of the hectagon itself, and these edges are *not* diagonals (by definition, diagonals must pass through the interior of the figure). So, we must remember to subtract the 100 edges of the hectagon to reach our final answer. Thus, the number of diagonals in a regular hectagon is

$$\text{Diagonals} = \frac{99(99+1)}{2} - 100.$$

Note that, even when we have a useful formula, we must still think carefully to apply it appropriately.

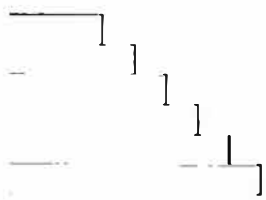
One Step Further: Extending What We Know

Now, let's return to the third problem and imagine a pyramid of bricks like the one shown:

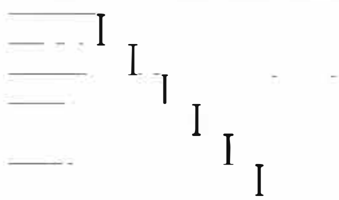


This is the same type of pyramid except the first row has something other than a single brick.

Once again, we can apply our manipulative to this sequence. First, build a staircase starting with the Cuisenaire Rod 3 units in length, and use each of the rods to a length of 8 units.



Like before, add a second staircase to form a rectangle.



- What are the length and width of this rectangle? How are the length and width determined?
- How does the area of the rectangle compare to the sum we are seeking?
- If you knew only the length of the first stair and the length of the last stair, could you predict how many stairs there are in all?
- How could we use our first formula,

  $$\frac{n(n+1)}{2},$$

  to generate a formula for this problem?

- Rewrite the Christmas problem so that my true love gives me something other than one gift on the first day. Apply your revised formula to solve this problem.

Other Variations

We can vary these problems in many other ways to increase the challenge. For example, let's assume that my true love gives me two gifts on the first day, four gifts on the second day, six gifts on the third day and so on. How many gifts would I receive in 12 days?

This problem differs from the others in that the successive values in our sequence increase by two rather than one:

Sum = $2 + 4 + 6 + \ldots + 24$.

In general terms, the first sequences we added were of the form $1, 2, 3, \ldots, n$.

Next, we considered sequences like $c, c + 1, c + 2, \ldots, c + n$.

This last problem introduces another sequence: $a, 2a, 3a, \ldots, na$.

The next logical step is to consider the sequence $c, c + a, c + 2a, c + 3a, \ldots, c + na$. In this sequence, we can begin at a value other than 1, and the difference between successive elements in the sequence can be a value other than 1. For example, in the sequence 6, 10, 14, 18, 22, . . ., $c = 6$ and $a = 4$.

- What would a pyramid of bricks that followed the fourth sequence above ($a \neq 1$ and $c \neq 1$) look like? Generate several examples.
- How are the four types of sequences related?
- Develop a formula for finding the sum of a sequence such as 7, 12, 17, . . ., 717.

Conclusion

Our most powerful learning experiences are those in which we explore and experiment in a meaningful context. Manipulatives help us to see not only how but also why something works. Also, students need opportunities to apply mathematics through problem solving. It is not necessary or even desirable to treat manipulatives and problem solving separately. When problem solving is incorporated in a manipulative activity, we can provide many dynamic learning opportunities for our students.

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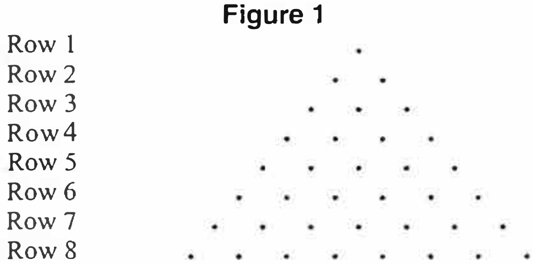
Developing Three Distinct Number Patterns from a Single Diagram

David R. Duncan and Bonnie H. Litwiller

Mathematics is in large part a study of relationships and patterns. Teachers are always on the lookout for settings in which these relationships and patterns can be discovered and considered. Finding several patterns in a single setting is a serendipitous occurrence.

Consider the following set of points:

Figure 1



We will discuss three problems in this setting.

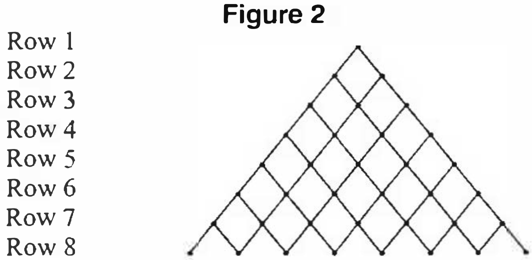
Problem 1

How many points does Figure 1 contain? Rows 1, 2, 3, . . ., 8 contain, respectively, 1, 2, 3, . . ., 8 points. Consequently, the eight rows together contain 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 points. Your students may recognize this to be the eighth triangular number.

Problem 2

Figure 2 displays the same set of points as Figure 1 but with connecting pathways superimposed.

Figure 2



In how many ways can you proceed downward from Row 1 to Row 8, following only the indicated paths? We can break this task into a series of seven consecutive tasks:

- Task 1: You can proceed from Row 1 to Row 2 using either of two paths.
- Task 2: From any point in Row 2, you can proceed to Row 3 using either of two paths.
- Task 3: From any point in Row 3, you can proceed to Row 4 using either of two paths.
- Tasks 4–7: From any point in each of Rows 4–7, you can proceed to the next row using either of two paths.

Using the fundamental principle of counting, we find that the number of ways to perform Tasks 1–7 is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$. Thus, there are 2^7, or 128, distinct paths from Row 1 to Row 8.

Problem 3

Let us again consider Figure 1. In Problem 2, we were restricted to only two pathways from each point to the following row. Suppose that we abandon that condition and proceed directly from a point in one row to any point in the next row. In how many ways can you proceed from Row 1 to Row 8 under these more flexible rules?

- Task 1: You can proceed from Row 1 to Row 2 using either of two paths.
- Task 2: From any point in Row 2, you can proceed to Row 3 using any one of three paths (remember that you can go directly to any of the three points in Row 3).
- Task 3: From any point in Row 3, you can proceed to Row 4 using any one of four paths.
- Task 4: From any point in Row 4, you can proceed to Row 5 using any one of five paths.
- Task 5: From any point in Row 5, you can proceed to Row 6 using any one of six paths.
- Task 6: From any point in Row 6, you can proceed to Row 7 using any one of seven paths.
- Task 7: From any point in Row 7, you can proceed to Row 8 using any one of eight paths.

Again using the fundamental principle of counting, we find that the number of ways to perform Tasks 1–7 is $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$, or $8!$. Thus, there are exactly $8!$, or 40,320, distinct ways of proceeding from Row 1 to Row 8.

We have been using a set of eight rows of points for Problems 1–3. If Figure 1 were extended to n rows of points formed in the same way, the answers to the three problems would be as follows.

Problem 1

How many points does the figure contain? There are $1 + 2 + 3 + 4 + \ldots + n$ (the nth triangular number). Recall that the nth triangular number is $\frac{n(n+1)}{2}$.

Problem 2

How many paths can you take from Row 1 to Row n when each point is connected to two points in the following row? You can take $\underbrace{2 \cdot 2 \cdot 2 \ldots 2}_{n-1 \text{ factors}}$, or 2^{n-1}, such paths.

Problem 3

In how many ways can you proceed from Row 1 to Row n if the paths of Problem 2 need not be followed? You can proceed in $2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$, or $n!$, such ways.

These three results represent three fundamentally different categories of mathematical formulations: summations, exponentials and factorials.

Have your students check out these formulas for specific values of n by drawing the figures and counting whatever the problem calls for. Can you and your students find other problems arising from Figure 1?

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A Focus on Fostering Number Sense Makes a Lot of Sense

Werner Liedtke

Number sense is the most important pre- and corequisite for numeracy. Number sense contributes to flexible thinking in numerical situations and the ability to solve problems. Without a conscientious focus on fostering number sense through key aspects of the curriculum (the teacher, resources, materials and so on), it is unlikely that most students will develop this sense. Consideration of number sense must go beyond unit topics and should, whenever possible, be part of the ongoing program. When students are not able to make sense of the numbers being manipulated, mathematics learning becomes rote or, as research shows, overwhelming and anxiety inducing.

In *A Handbook on Rich Learning Tasks,* Flewelling and Higginson (2001) label the term *rote learning* an oxymoron (p. 24), identify rote learning as a major source of anxiety (p. 28) and suggest that it is an impediment to problem solving (p. 26). The authors state, "Rote-learning-plus-practice techniques train problem solvers as well as paint-by-numbers techniques train artists" (p. 27).

In this article, I identify the important components of number sense and illustrate them through examples using two-digit numbers. Most of these ideas can easily be adapted to other whole numbers and to fractions, decimals and integers.

Visualization

When students hear or see a two-digit number, they should be able to visualize the number. In responses to interview questions about numbers, students uttered phrases such as "I see it in my brain" and "I see it in my mind." For example, when they hear the words *thirty-four* or see the numeral *34,* students should be able to "see" the smallest number of base-10 blocks, dimes and pennies, or $10 and $1 bills needed to represent that number.

To develop this sense, students should learn to make as many groups of 10 as possible from the number, recording the result in a box titled "Tens." What is left goes in a box titled "Leftover Ones." To enhance the association of digits with the appropriate place values, the students record the digits again below and beside these boxes.

Some resources suggest that the main reason for grouping by tens and ones is that it is faster and easier. Nothing could be further from the truth. For children in the early grades, it is much faster to count by ones and easier, or less work, not to group the objects. In fact, the main reason for adopting this procedure is that it allows us to use only 10 digits (an accident of nature?) to record an infinite number of number names.

Flexible Thinking

Students learn that, using only tens and ones, they can show two-digit numbers in at least two ways. For example, students can be given the following problem:

> What are the different ways to show 42 using tens and ones? How do you know that you have found them all?

Students can solve and even create riddles such as the following:

> I have only dimes and pennies. I have six coins. How much money do I have? How do you know that you have recorded all the possible answers?

Connecting

Students should be shown that two-digit numbers connect to many aspects of their experience—money, games, books on shelves, book orders, children in classrooms and so on.

Relating

When we talk about or compare two-digit numbers, we use terms such as *greater than, less than, close to, between, far apart, ones place, tens place, odd, even, sum of the digits* and so on. Teachers can solve and create riddles for missing, hidden or secret numbers on a 99 chart (see Figure 1).

Figure 1
99 Chart

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Over the years, I have collected many excellent secret-number riddles created by Grades 1–7 students. Samples provided by the same group of students over a period of time indicate that sophistication and accuracy increase as students are given more opportunities to create riddles.

The following is an example of a secret-number hunt:

The number . . .
. . . is less than 54.
. . . is not between 36 and 54.
. . . does not have a 3 in the tens place.
. . . is greater than one dozen.
. . . has an odd number as the sum of its digits.

As hints of this type are presented, one at a time, students look at the 99 chart and are invited to respond to the questions, "Which number(s) do you think it could not be?" and "Which number do you think it could be?"

Estimating

A key strategy for learning how to estimate involves using a referent—in this case, a group of 10 or 10 fingers. For example, when students are asked to look at a group of objects and estimate how many objects there are, they use their fingers as a referent and pretend to put the objects in groups of 10. They then report their estimates to the nearest ten (that is, "About __ tens"). For a variation, rather than recording an estimate, students could be given three choices, asked to select the best estimate and be ready to justify their choice. Should any estimates be deemed unreasonable or illogical? Teachers should take great care in assessing estimates. Number sense develops slowly, and estimates may differ greatly from student to student. Labelling an estimate, if it is an estimate, as illogical seems illogical to me.

Resources for teachers include ideas for estimation tasks that pose a question such as "How many marbles are in the jar?" The greatest value of these types of tasks lies not in the numerical responses provided by the students but, rather, in the strategies students use to arrive at their responses and in the follow-up discussion comparing the strategies.

Subitizing

The term *subitizing* refers to the ability to recognize the numerousness of small sets and to attach the appropriate name to the sets without having to count each member. I would like to think that a type of subitizing is possible for two-digit numbers. After students have had many opportunities to represent two-digit numbers in different ways, the teacher could try the following task:

> Ask three children to come to the front of the room. Ask two of them to hold up 10 fingers each and one of them to hold up four fingers (whispering in their ears so the rest of the students can't hear). Then, ask the remaining students, "How many fingers do you see?"

Depending on their previous experiences, many students will be able to identify the number without having to count each finger. As they look at the numbers displayed, students will also reach a stage at which they can state numbers that are one more, one less, 10 more, 10 less, or even double or half of the numbers shown.

Mental Calculation

The task described under the category of subitizing involved visualization or aspects of mental calculation.

It is discouraging to encounter students who, when asked to describe how they might proceed to find an answer without using pencil and paper, explain something that is the same as a recorded algorithm. That is not the intent or the goal of mental calculation. Opportunities for flexibility exist, and different levels of number sense can be accommodated. For example, the teacher can give students the following task:

> Let's pretend that we have 24 books and we order 13 more. How many books will we have?

After two groups of students have been asked to represent these two numbers at the front of the room, different ways to find the answer without using pencil and paper can be illustrated. An interesting discussion can revolve around the question, "Which starting point did you like best? Why?"

Practice

As I pointed out earlier, rote practice is of little value. In *Future Basics: Developing Numerical Power*, Charles and Lobato (1998, 17) state that "appropriate practice can promote the development of numerical power." Number sense is an important aspect of this numerical power. The authors define *appropriate,* in this case, as something involving reasoning, communicating, connecting and problem solving. One of their examples asks students to explain through words and pictures two different ways to find the solution to 43 – 18 without using a calculator.

The calculator can be used in fostering the development of number sense—if it is used in ways other than to check answers (which leads students to ask, "Why did we not use the calculator in the first place?"). Students have to think about the numbers when, for example, a question asks them to think of four different ways to enter the numbers to get the answer and to record how these numbers were entered. Pointing out incorrect answers and determining what went wrong and why can make students think about the numbers (for example, "Someone got an answer of 4,318. What do you think happened?").

Conclusion

With a little imagination, teachers can generate high-order questions that will provoke students to think about the numbers they are working with and, thus, further develop a sense of number. The time and effort required to select and modify activities to make them more appropriate or effective in developing number sense will pay great dividends, because such tasks also foster self-confidence and a positive attitude.

References

Charles, R., and J. Lobato. *Future Basics: Developing Numerical Power.* Golden, Colo.: National Council of Supervisors of Mathematics, 1998.

Flewelling, G., and W. Higginson. *A Handbook on Rich Learning Tasks: Realizing a Vision of Tomorrow's Mathematics Classroom.* Kingston, Ont.: MSTE Group, Faculty of Education, Queen's University, 2001.

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Working Toward Mathematical Literacy

Anne MacQuarrie

As a former math illiterate, I knew there had to be a way to present math so that everybody, even people like me, could understand it. I was determined to find a way to teach *all* children to love math and appreciate it as a real and living thing.

I came to my love of math rather late in life—in my 30s, if I remember correctly. Before that, math seemed a giant ogre in the classroom, lying in wait for me with its convoluted operations that occurred for no discernable reason and its numerals that were supposed to be numbers (I thought the word *two* referred only to two items, not to the numeral *2*). I thought I was being punished for not being very smart. It turned out I was being tricked.

When it finally occurred to me that math is all about patterns, I became a zealot. I started seeing math everywhere, and it was beautiful and exciting. I needed to pass the message on, and my classroom was where I began trying to convert mathophobes to mathophiles. Some children will get math and love it no matter how it is presented. I was more concerned about those children whose math flowers were still shaded buds in the gardens of their minds. Those buds needed sunshine in order to open.

For several years, MCATA has been working on developing mathematical literacy—promoting understanding of big ideas and recognizing that arithmetic is the grammar and spelling of mathematical thinking. I try hard to develop math literacy in the children I teach. Here, I share a story about my Grades 2/3 class.

A *Calgary Herald* story about the results of the Grade 9 provincial achievement test in mathematics included a sample question that went like this:

> A farmer and his son left the barn to do a fence check on their property. They walked in opposite directions—the farmer at 4 km/h and his son at 3.5 km/h. How long did it take until they were 2.5 km apart?

The possible answers were 1/5 hour, 1/3 hour, 3 hours and 5 hours. Could it be any more fun? I decided to do the question with my Grades 2/3 class—not using algebra (we don't usually get that tricky in the lower grades) but, instead, considering the question, What is a *reasonable* answer?

First, we converted 3.5 and 2.5 into fractions (3½ and 2½). I emphasized that a decimal is just another way of writing a fraction and demonstrated that 5/10 is the same as 1/2.

We then considered the clock. I drew a clock on the board and broke it into quarters, and the children counted off the minutes for each quarter. We considered how fast the farmer and his son were walking. By labelling each section of the clock with "1 km," I helped the children easily recognize that 4 km/h (the farmer's pace) means 1 km in 15 minutes. I asked the children if the son, who was walking slower than his father, could have gone 1 km in 15 minutes, as his father had. As a test, I had two students walk away from each other in the classroom. The longer the students walked, the farther they got from each other, and the faster person got farther from the start point in the same time period. The children agreed that the son wouldn't have walked as far as his father in 15 minutes. On the board, I drew a diagram showing two stick figures walking in opposite directions. Knowing that in 15 minutes the farmer had walked 1 km and his son had walked less than 1 km, the children concluded that after 15 minutes the farmer and his son were less than 2 km from each other.

Referring to the clock again, I drew five circles on the board to represent each hour in the answer choices. I asked the children how I could show that the farmer was going 4 km/h and was instructed to put "4 km" in each circle. Some children multiplied and some added, but they all agreed that if the farmer travelled 4 km in 1 hour, he went 12 km in 3 hours and 20 km in 5 hours. The children told me that 3 hours and 5 hours were much too long because the farmer and his son only had to be 2½ km, not 12 or 20 km, from each other. I erased the choices of 3 hours and 5 hours from the board and, in so doing, demonstrated to the children a test-taking strategy that they can use in the future.

We were left with the answer choices of 1/5 hour and 1/3 hour and, therefore, had to undertake the job of comparing the size of fractions. We talked about what the denominator means, and the children chanted our fraction mantra: "The number on the bottom is the number of equal-sized pieces the whole thing is broken into." We reviewed what happens to the size of the pieces when the denominator gets bigger. We talked about what the fraction is called when you break something into five equal pieces. We looked at the clock again to see what it could tell us about the relationship between 60 (the number of minutes in an hour) and 5 (the number of pieces we had to break the hour into to get fifths). We could see that, counting by fives starting at the top of the clock, 12 sets of 5 are 60. Because I always have the children give me four math sentences for every fact, we also determined that 5 sets of 12 are 60, 60 broken into sets of 12 is 5, and 60 broken into sets of 5 is 12. I wrote the sentences on the board in standard algorithmic form, explaining again that this is how the words look in the language of mathematics. We determined that each fifth was worth 12 minutes because that was the size of each piece when you broke 60 into 5 equal parts. We looked at the quarters on the first clock and found the difference between the 15-minute quarters and the 12-minute fifths. The fifth was three minutes less than the quarter—not enough time for the farmer to go 1 km. If 12 minutes wasn't enough time for the farmer to travel 1 km, it certainly wasn't enough time for the son, and together they would have travelled less than 2 km. Thus, we had to reject 1/5 hour as a possible answer. That left us with 1/3 hour as the only possible answer.

Obviously, the procedure did not go that smoothly or quickly, so, please, all you elementary generalists out there who say, "I could never do that," believe me when I say, "Yes, you can!" I did not give the children the answers. I got excited. I jumped up and down. I got the children moving around, and they got excited, too. I asked and probed and made them construct meaning, drawing on their own experiences and actions. I talked about patterns—the basis of all mathematics—and urged the children to find the patterns in the problem. By drawing on their knowledge, I was able to stimulate the children to put things together. Not all of them got all parts of the problem, but they all got some of it. In this Grade 9 test question, I was able to find something for even the least developed math mind in Grades 2/3 while addressing the needs of the other students, including the most capable.

The children were thrilled that they had solved a Grade 9 test question and that most of them had understood it. They all understood that the answer had to be reasonable and that 3 hours and 5 hours were not reasonable answers. Comparing the fractions was a far more rigorous task, but most of the children were able to make some meaning out of it.

Math should be something that excites, not terrifies, children. The more broadly based a problem is, the more willing children are to try it. I always stress to them that, although they may not be able to do everything, there will always be something in a problem that they can grasp. It is important for children to see that math is not the ogre in the corner or the demon in the red notebook who is out to get them. They also need to see math as numbers and shapes and forms, not just numerals arranged in algorithms. They need lots of practice estimating so that they can learn to discern a reasonable solution.

Math is everywhere, and it is beautiful!

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An E-Conversation with a Math Olympian

Robert Wong

Recently, MCATA asked me to do a write-up on Robert Barrington Leigh, a medal winner in both the 2003 International Mathematical Olympiad (IMO) and the 2003 International Physics Olympiad. Two questions came to mind: Where is Robert Barrington Leigh? and, What can I ask him that various news media haven't already asked? With Andy Liu's help, I was able to find the whereabouts of the famous Edmontonian: Robert is now in his first year of study of mathematics at the University of Toronto. A face-to-face conversation with him would have been difficult, so e-mail it was. I thought it would be best if I asked questions from a student perspective. After brainstorming with my students and collaborating with Shauna Boyce, MCATA publications director, I came up with a set of questions. What follows is my e-mail conversation with Robert Barrington Leigh.

Robert Wong: What math topics did you have to work with in the IMO?

Robert Barrington Leigh: The problems I solved during this year's competition—two out of six—concerned geometry (the usual: circles, lines and angles) and number theory (properties of whole numbers). There were also one algebra question and two more number theories. The hardest problem concerned powers of prime numbers. Significantly, the IMO considers calculus postsecondary material; thus, I didn't feel obliged to study it! The problems chosen for the Olympiad are generally more difficult when tackled with calculus than without.

RW: How competitive was the Olympiad? Was it stressful?

RBL: The Olympiads are organized to be as friendly as possible: we write the contest within a few days of our arrival and then we try to concentrate on enjoying ourselves and making friends for the rest of the event. I particularly recall playing a card game with members of the Chinese team, who seemed just as relaxed as the Canadians. Understandably, though, some of us were anxious while the questions were being marked and right before the results were posted. In addition to the opportunity to meet other students, the International Olympiads provide the opportunity to explore a new country and culture. I was fortunate this year to visit Tokyo, because I do not often travel outside Canada and certainly had never left the Western Hemisphere. I think it's essential to have a sense of the scale and diversity of our planet, and, although the abstract field of mathematics might not seem the best field in which to develop such an understanding, the IMO lets students engage in both mathematics and cultural exploration.

In general, the Olympiad was not stressful—even the 270-minute exams were more relaxed than one might expect. If I had panicked, I would have made more mistakes than usual, and I would certainly have had no room in my thought processes to discover solutions. Thus, the exam is a mixed bag of exploration, insight and occasionally frantic writing.

RW: What kind of recognition have you received as a result of the competition?

RBL: Articles about the Olympiads were published in Edmonton, Calgary and Banff newspapers, and I was heard on an Edmonton radio show last spring.

RW: Did you spend much time outside of school working on or studying math topics?

RBL: Certainly. I have never been content with learning only what I am being taught officially—and there are always contests to study for. One useful tool has been a correspondence program for high school students organized by Ed Barbeau of the University of Toronto: every month he sends out a problem set and marks everyone's solutions. Moreover, over the past few years I have been invited to many math camps, which are, perhaps surprisingly, almost as much fun as they are instructive. These camps include the Alberta summer camp held alternately in Edmonton and Calgary, a corresponding national camp at the University of Western Ontario, a spring camp at the University of Waterloo, a January camp for IMO contenders at York University and a July training camp for the IMO itself that was held this year in Calgary and Banff. I am deeply grateful to the professors and university departments involved and to the sponsors of these camps: ESSO, the Canadian Mathematical Society and the Pacific Institute for the Mathematical Sciences.

RW: What are your career plans? Is math a part of those plans?

RBL: Most likely math will be my principal focus for decades to come. I am not yet certain how I will manage to incorporate this focus in a career.

RW: What are you studying now?

RBL: I am enrolled in a math–physics double major (with greater emphasis on math), but I think I'll also pursue the odd computer science course. Right now, I'm taking second-year math and physics, a first-year computer science course and a sociology seminar for first-year students.

RW: What have been your accomplishments in the last three years?

RBL:

- Placed first for Alberta in the 2001 Canadian Open Math Challenge
- Placed first for Alberta in the 2003 Alberta High School Math Competition
- Received an honourable mention in the 2002 Canadian Mathematical Olympiad and placed third in 2003
- Placed third in the 2002 Leonardo da Vinci Competition
- Placed eighth in the 2002 Canadian Association of Physicists High School Prize Exam and first in 2003
- Placed third for Alberta in the 2003 Chemical Institute of Canada National High School Chemistry Examination
- Placed ninth for Alberta in the 2003 University of Toronto National Biology Competition
- Won the bronze award at the 2002 International Mathematical Olympiad in Glasgow, U.K. (22 points out of 42; tied 113th–132nd of 480 participants) and the bronze award in 2003 in Tokyo (18 points out of 42; tied 107th–123rd of 457 participants)
- Won the silver award at the 2003 International Physics Olympiad in Taipei, Taiwan (28.7 points out of 50; 38th place out of 239)
- Participated, as a member of the three-person University of Toronto team, in the 2003 William Lowell Putnam Mathematical Competition (a North American math competition for university students) on December 6, 2003

RW: How many articles or books have you written and published?

RBL: A few years ago, I published two articles with Richard Ng, with help from Andy Liu of the University of Alberta—"Zigzag" and "Minimizing Aroma Loss" (which was reprinted in *delta-K,* Volume 38, Number 1, December 2000). I am in the process of helping Professor Liu with a book of translated problems and solutions from a Hungarian math competition (*Hungarian Problem Book IV*).

RW: Were your parents good at math?

RBL: Yes, I would say that interest in math runs in the family. My father was my first math teacher and the one who could teach me best for many years.

RW: When did you start working on math problems?

RBL: Sadly, I don't remember that far back, but in Grade 1 I was asking my teacher for enrichment to the math curriculum. In Grade 5, I wrote my first multiple-choice math contest, and in Grade 6 I joined a math club organized by Professor Liu. There I discovered a long-answer math contest called the International Tournament of the Towns, which I enjoyed immensely despite its being slightly above my level.

RW: How many hours of math did you do in a week in elementary, junior high and senior high?

RBL: About three—I certainly don't remember in elementary. Also, sometimes I'm not expressly working on math but am just thinking about a problem—in the shower or what have you. But, naturally, I wished I had more time for math than was available.

RW: What do you enjoy doing when you have free time? Hobbies?

RBL: Apart from math and chatting with friends, I enjoy music, in particular playing the piano, as well as cross-country skiing and running. Edmonton has a supportive Nordic ski club that I've belonged to since elementary school. Also, I have a casual interest in computer programming—my brother-in-law is a software developer.

RW: What kinds of books do you read?

RBL: Like many people, I must confess that I ought to be reading a much greater variety and volume of books than I do. Currently my reading agenda consists of math books—recreational and otherwise—and other science-related nonfiction, fiction and biography.

RW: Who's your mentor in math?

RBL: Without a doubt my mentor is Professor Andy Liu, an award-winning mathematics educator at the University of Alberta, who even tutored me privately in Grade 11. I have been in contact with him since Grade 6 through his math club, and most recently he was the leader for the 2003 IMO team. Professor Liu not only has been a dedicated mathematics tutor and friend but also has introduced me to many other young mathematicians and, crucially, shown me how rewarding a career in math can be.

RW: How do you prepare for math tests and contests?

RBL: The same way as anyone else does: I familiarize myself with the standard problem-solving techniques and then I attempt to solve sample problems. If I get stuck on a problem, I either move on to the next or look up the solution in case the same approach can be applied elsewhere. In a contest, the range of insights needed for different problems is much broader; therefore, knowing the solution to a particular problem is less valuable than it might be on a school test. Seeing the solutions to many sample problems is still helpful, but solving them myself is key.

RW: What are your educational and career goals?

RBL: Learn a great deal of math and physics. I have no idea—save the world . . . .

RW: Do you have any advice for others who want to excel in math?

RBL: Find others with the same goal; it's more exciting when you challenge each other to excel. There are so many good math problems on the Internet and in the library that I'll never run out of them. The Canadian Mathematical Society also has some great resources, such as Dr. Barbeau's Mathematical Olympiads Correspondence Program (www.cms.math.ca/Competitions/MOCP/info.html), especially if you need someone to mark your solutions—an important part of training.

I must stress that, without practice, speed and creativity in math diminish over time.

RW: What would you say to people who do not like or who struggle with math?

RBL: Well, I certainly don't hold it against them: though I pick up mathematical ideas faster than average, I find other skills difficult and, therefore, unpleasant. On the other hand, mathematics is diverse enough that someone may abhor one branch but appreciate the charm of another. Puzzles created by Binary Arts Inc.—for example, Rush Hour—tend to be friendly tools for exercising the mathematical parts of one's brain without even noticing.

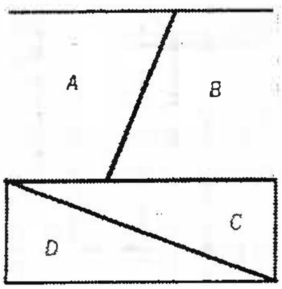
Robert Wong, MCATA webmaster, teaches at Vernon Barford Junior High School in Edmonton, Alberta.

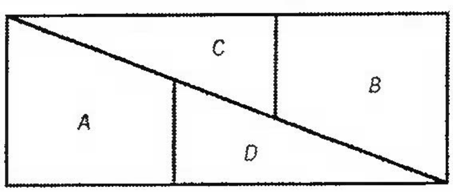
This interview was previously posted on the MCATA website (www.mathteachers.ab.ca). Minor editorial changes have been made in this version.

A Letter to Jim About the Other, Not-So-Magic Square

A. Craig Loewen

We are all familiar with the traditional magic square, the one in which we must fit the digits 1–9 such that all rows, columns and diagonals have the same sum. But some time ago my friend Jim introduced me to the other magic square:





1. Calculate the area of the square.
2. Cut out the four pieces and rearrange them to make the rectangle.
3, Calculate the area of the rectangle.
4. Explain the difference in the areas—or is it magic?

Jim asked, "Is there a mathematical explanation for this activity?" After exploring the problem, I wrote the following letter in response to him.

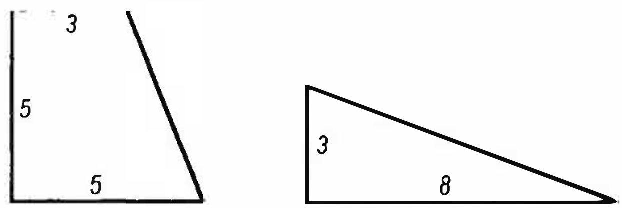
Dear Jim,

The problem seems a little familiar to me, but if I've forgotten the problem, I've also forgotten the solution! I wound up playing with it for a few hours this afternoon. Here are my thoughts and notes.

Understand the Problem

My intuition and my belief in the laws of conservation force me to doubt that the shapes grow simply by my moving them. So, let's study the square first.

We know that the square is made up of four pieces with shapes and dimensions as in the figures below (two trapezoids and two triangles):



We can calculate the areas of the trapezoids and the triangles to see if they add up to 64 or to 65. Maybe that will help us discover the trick:

$$\begin{aligned} \text{Area trapezoid} &= (0.5)(\ell\text{ top} + \ell\text{ bottom})(\text{height}) \\ &= (0.5)(3 + 5)(5) \\ &= 20 \end{aligned}$$

$$\begin{aligned} \text{Area triangle} &= (0.5)(\text{base})(\text{height}) \\ &= (0.5)(8)(3) \\ &= 12 \end{aligned}$$

$$\begin{aligned} \text{Area square} &= (2)(\text{area trapezoid}) + (2)(\text{area triangle}) \\ &= (2)(20) + 2(12) \\ &= 64 \end{aligned}$$

Of course, this answer makes sense, because the square was 8 units on its side. We could have calculated this area a whole lot faster simply by squaring one of the sides, but I wanted to know if the areas of the pieces added up! We'd better look at the rectangle now.

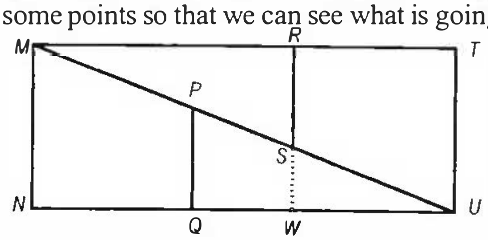
We know that the rectangle is made up of the same four parts, so it should also have an area of 64 square units. We can calculate the area of the rectangle using the length-times-width formula, but first we need to find the length and width of the rectangle. The length of the rectangle is the sum of the longest sides of pieces A and D, which is 13 units. The height of the rectangle is the same as the height of piece A, which is 5 units. So,

$$\begin{aligned} \text{Area rectangle} &= (\text{length})(\text{width}) \\ &= (13)(5) \\ &= 65. \end{aligned}$$

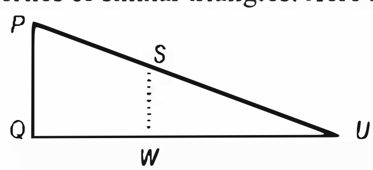
Well, for all the world, it looks like we have gained 1 square unit simply by rearranging the pieces! There is definitely something fishy about this rectangle!

Develop a Plan and Carry It Out

We had better take a closer look at the rectangle and label some points so that we can see what is going on:



First, let's extend the line *RS* until it meets the line *QU*. We'll call that point *W* (as shown in the diagram above). We know that both line *MN* and line *TU* are 5 units long. If this is a genuine rectangle, the length of line *RW* should also be 5 units. But how can we calculate the length of line *RW?* Well, we know the length of line *RS* (3 units), so all we need to do is calculate the length of line *SW.* We can do this using the properties of similar triangles. Here is our triangle:



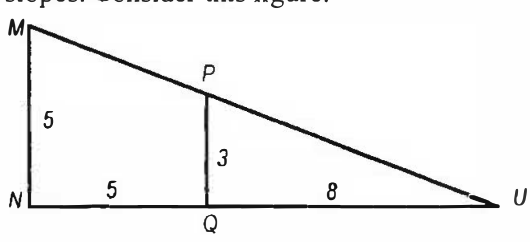
Let's make a list of what we know: $\ell\, PQ = 3$ units, $\ell\, QU = 8$ units and $\ell\, UW = 5$ units. Now we can construct our ratio:

$$\frac{\ell\, SW}{\ell\, UW} = \frac{\ell\, PQ}{\ell\, UQ}$$

$$\frac{\ell\, SW}{5} = \frac{3}{8}$$

Solving for $\ell\, SW$, we find that it is 1.875 units. Wait a minute! That means that the length of *RW* is 4.875 units . . . but it's supposed to be 5 units! In other words, our figure is not a true rectangle—at least, it's not a uniform, solid rectangle as it appears to be. Therefore, the formula for calculating the area of a rectangle (length times width) doesn't apply here.

But I still don't know exactly what is going on. It is worth investigating further. I'm wondering what happens when the two pieces (the triangle and the trapezoid) are put together. While trying to draw these shapes on the computer, I have become suspicious of their slopes. Consider this figure:



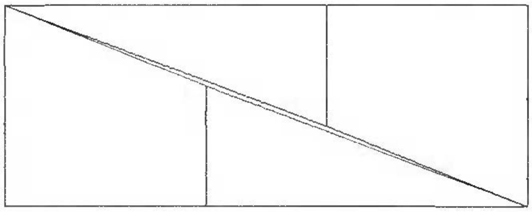
If line *MU* is indeed a straight line, then the slope of line *MP* must equal the slope of line *PU:*

$$\text{slope } PU = \frac{\text{rise}}{\text{run}} = \frac{3}{8} = 0.375 \qquad \text{slope } MP = \frac{\text{rise}}{\text{run}} = \frac{2}{5} = 0.4$$

Here it is! The slopes are dangerously similar, but they are not the same. If the slope of *PU* is less than the slope of *MP,* the two shapes do not form a true triangle. The segment from *M* to *U* bends toward *Q* at *P.* So, when we add another of these tricky triangles, putting the four pieces together, we are actually leaving a little gap! This gap is just too small to notice when we cut out the shapes and move them around. The inaccuracy of our paper model hides that 1 square unit. That is the real magic of this magic square!

Looking Back

Well, now I know why I had so much trouble drawing these shapes on my computer in the first place. Look what happens when I accurately draw the shapes with the help of my computer, rotate them and put them together to form a rectangle:



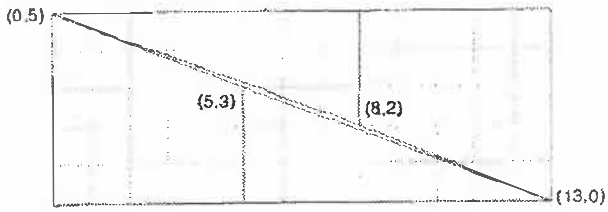
We can see a small area distributed neatly through the centre of the figure. Because I know the area of the rectangle and the areas of each of the four shapes, it makes sense that the gap has an area of 1 square unit:

$$\begin{aligned}\text{Area of gap} &= \text{area rectangle} - (2)(\text{area trapezoid}) - (2)(\text{area triangle})\\ &= 65 - (2)(20) - 2(12)\\ &= 1.\end{aligned}$$

It would be fun to find another way to calculate (and perhaps verify) the area of the gap in the rectangle. One such way is to identify the coordinates of this tiny shape (a quadrilateral) and use the following formula:

$$\text{Area} = \frac{(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (y_1x_2 + y_2x_3 + \ldots + y_nx_1)}{2}$$

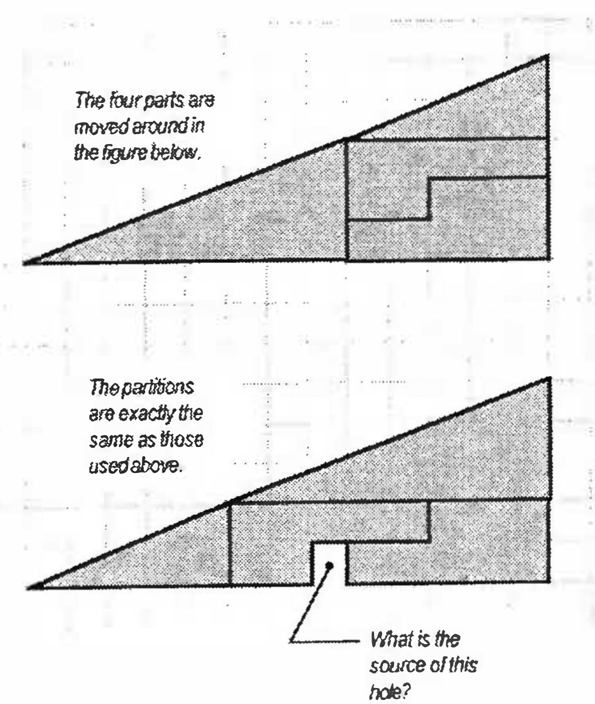
To identify the coordinates, we need only superimpose the rectangle onto a Cartesian coordinate system:



We can define the points as follows: $(x_1, y_1) = (0, 5)$, $(x_2, y_2) = (5, 3)$, $(x_3, y_3) = (13, 0)$ and $(x_4, y_4) = (8, 2)$. To use this formula, we should define the points in sequence in a counter-clockwise order. Now we can substitute the values in the formula:

$$\begin{aligned}\text{Area of gap} &= \frac{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (y_1x_2 + y_2x_3 + y_3x_4 + y_4x_1)}{2} \\ &= \frac{(0\cdot3 + 5\cdot0 + 13\cdot2 + 8\cdot5) - (5\cdot5 + 3\cdot13 + 0\cdot8 + 2\cdot0)}{2} \\ &= \frac{(0 + 0 + 26 + 40) - (25 + 39 + 0 + 0)}{2} \\ &= \frac{66 - 64}{2} \\ &= \frac{2}{2} \\ &= 1.\end{aligned}$$

Now I have a problem for you to try. I don't know the source of this problem, but I think a student gave it to me a long time ago. It has definite similarities to the problem you sent me (both in the problem itself and in its solution). Here it is:



I had a great time playing with this problem today, Jim! And the answer to your question is, yes, there is a mathematical explanation. It turns out that your square is not magical, but my afternoon really was fun!

Keep well,
Craig

A Few Notes

It is sometimes surprising that a problem can so totally capture our interest that time slips away as we play, think and explore. But, as I look back over this problem and my solution to it, I am struck by other, more important questions.

Owning a Problem

What can we do to help our students bite into a problem? When I started working on this problem, I was so sure that something was amiss—I square unit can't just disappear!—that I couldn't let it go until I could provide an adequate explanation. How can we help our students develop this sense of internal insistence? Maybe this sense is a pre cursor to that infinitely important and highly desir able sense of achievement we feel when we solve the problem!

Finding Time

How do we find time for any form of real problem solving in our classrooms? I really did spend an afternoon playing with this problem. I wanted to try several methods and see if I could come up with a good, sensible explanation. But I wonder if the volume and busyness of our curriculum (which supposedly values problem solving above all else) leaves adequate space for anything other than direct instruction based on algorithms, formulas and facts.

The Role of Technology

How can we increase the use of technology as a tool in the mathematics classroom? In at least one way, my computer was an important tool in solving this problem. Only with the aid of my computer could I construct a drawing accurate enough to reveal the small gap. The software I used was not particularly complex or expensive; it was a simple drawing package. Our students should be taught to integrate technology flexibly, using the available software to increase their problem-solving power.

Communication in the Classroom

How can we better integrate communication in the math classroom? In part, what made this problem fun for me was sharing it with a friend. What does this tell me about the need for collaboration, cooperation and communication? And what does this tell me about how my class time and, indeed, my classroom space might be better arranged?

A. Craig Loewen, editor of delta-K, *is an associate professor of mathematics education and assistant dean of Student Program Services in the Faculty of Education at the University of Lethbridge in Lethbridge, Alberta.*

Activities for the Middle School Math Classroom: Games Using Manipulatives

A. Craig Loewen

We have all taught students who can do mathematics yet who do not really understand that mathematics. Such students succeed with one problem or task, but if given a similar but slightly altered or camouflaged task, they are lost. They have probably focused on the routines or procedures (algorithms) associated with concepts but have not really learned the concepts themselves. It is entirely too easy for students to see mathematics as a disconnected collection of algorithms rather than a meaningful body of knowledge overflowing with connections and integrally related to life outside the classroom. Often our students are happy to learn only the algorithms (until their memories are taxed), and we lapse into the contentment of teaching to this inclination. The students' focus on algorithms is probably related to a desire to quickly finish or avoid homework.

But if there is so much more to mathematics than algorithms, what more is there to know? Or, in other words, what does it mean to understand mathematics? The answer to this question is neither short nor simple. However, it is fair to say that incorporating manipulatives in our mathematics instruction usually represents a genuine intent to teach for meaning and to go beyond the first level of instruction, the algorithm.

When we teach with manipulatives, we allow students to learn through more senses and to literally see mathematics in action. Manipulatives lend themselves to exploration, conversation and investigation. In general, we use manipulatives to provide a representation of a concept that enables learning at a greater depth. The use of manipulatives enables conceptual learning as opposed to procedural or algorithmic learning.

However, like any other useful instructional method, manipulatives are not meant to stand alone. Let's remember that we want our students to know the algorithms in addition to the concepts underpinning those algorithms. A well-balanced program includes opportunities to explore ideas; draw conclusions; and formulate, test and practise algorithms. Manipulative-based games provide motivation and a wonderful learning context in which students can work together to explore ideas, generate new ideas and practise what they have learned.

Here are some considerations for integrating games in your math classroom:

- Where possible and appropriate, include manipulatives in your games. Students must learn how to work with representations and tools, and games provide a motivating and nonthreatening environment in which to do so.
- Consider collecting student game sheets, cards and notes as a way to catch student errors. Be sure to watch the students play the games, and note where misconceptions become evident.
- Be prepared for a more active, noisier classroom. Motivating activities such as games are often noisier than traditional activities.

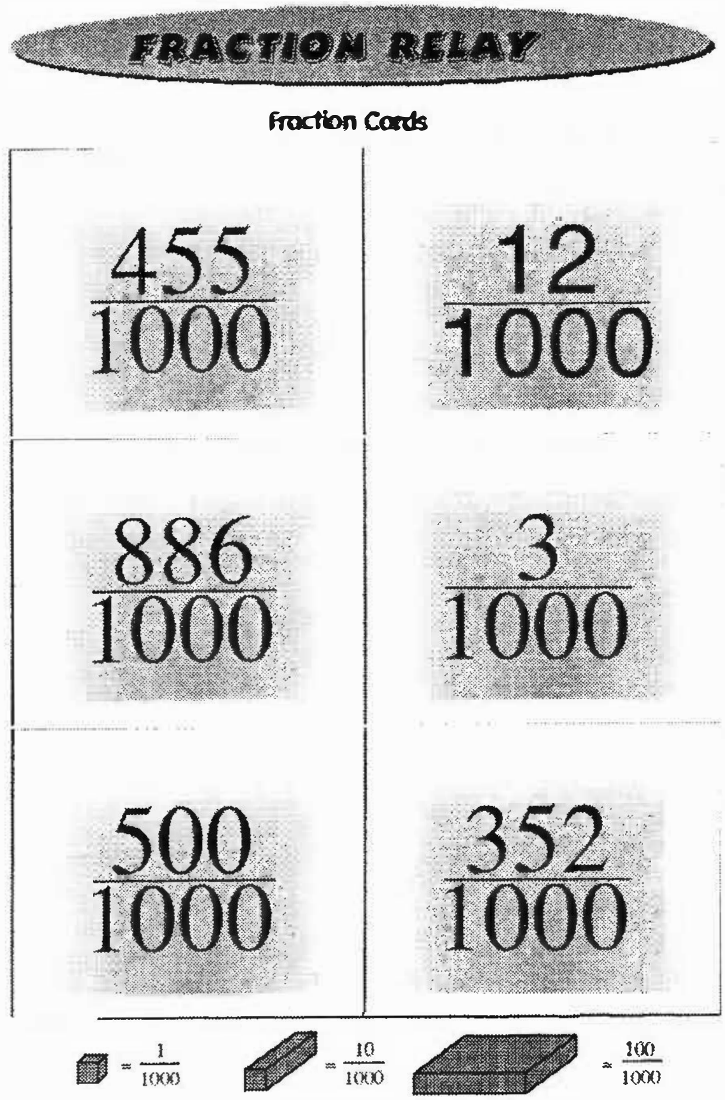
Note: The objectives in the following games are based on outcomes from Alberta Learning's (1996, 1997) mathematics program of studies.

Fraction Relay

Objective: Represent and describe proper fractions (Number [Number Concepts], Grade 5, Outcome 7)

Materials: Base-10 blocks (small set for each team), hi-lo cards, 6–10 fraction cards

Players: Two or more teams



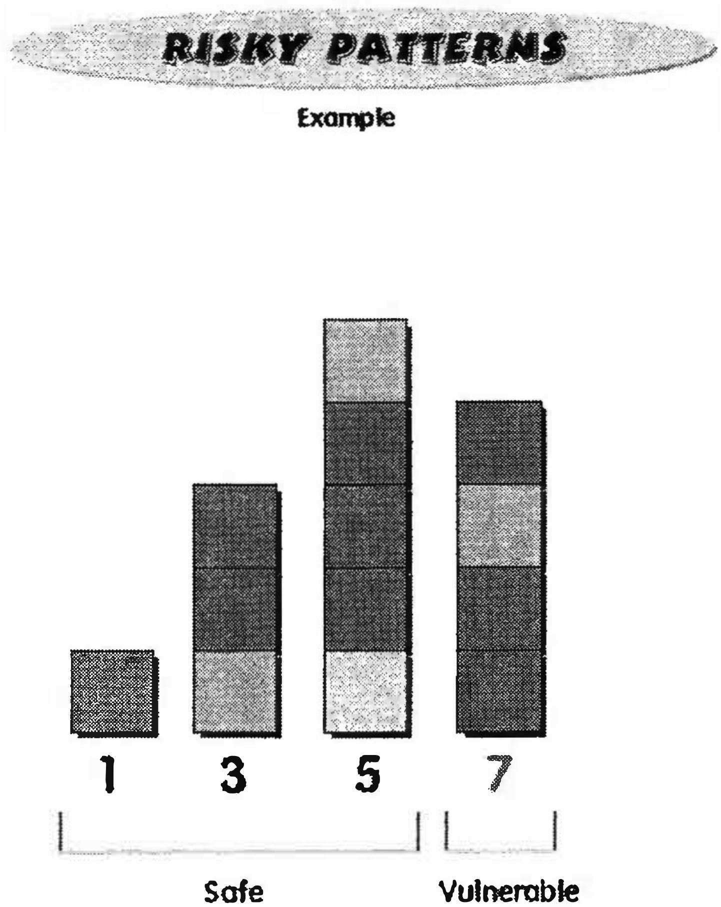
Rules

1. Before introducing the game, the teacher must make a hi-lo card for each team and 6–10 fraction cards. Hi-lo cards are index cards labelled "Too high" on one side and "Too low" on the other. Fraction cards are index cards with a fraction on one side. The fractions should be of the form $\frac{x}{1,000}$—for example, $\frac{156}{1,000}$ or $\frac{24}{1,000}$.
2. Each team selects one member to start. This person comes to the front of the classroom with the starters from the other teams. The teacher shows the starters the first fraction card.
3. The starters return to their teams to help their teammates build the fraction using the base-10 blocks. The starter may not talk but may give his or her teammates clues by showing them the appropriate side of the hi-lo card.
4. Once the team has built the fraction, a team member other than the starter comes to the front of the room and tells the teacher the fraction. If the fraction is correct, the teacher shows the player the next fraction to be built. This player now uses the hi-lo card to provide clues to his or her teammates as they try to build the fraction (as in Step 3).
5. The team that works through the whole set of fraction cards first wins.

Adaptations

1. Instead of limiting the fractions to the form $\frac{x}{1,000}$, use fractions of the form $\frac{x}{10^n}$, such as $\frac{7}{10}$, $\frac{56}{100}$ or $\frac{854}{1,000}$.
2. Play several rounds with every team working on the same fraction at the same time. The team that wins the most rounds wins the game.

Risky Patterns



Objective: Construct and expand patterns in two and three dimensions, concretely and pictorially (Patterns and Relations [Patterns], Grade 5, Outcome 3)

Materials: Tiles, a six-sided die

Players: Two or more

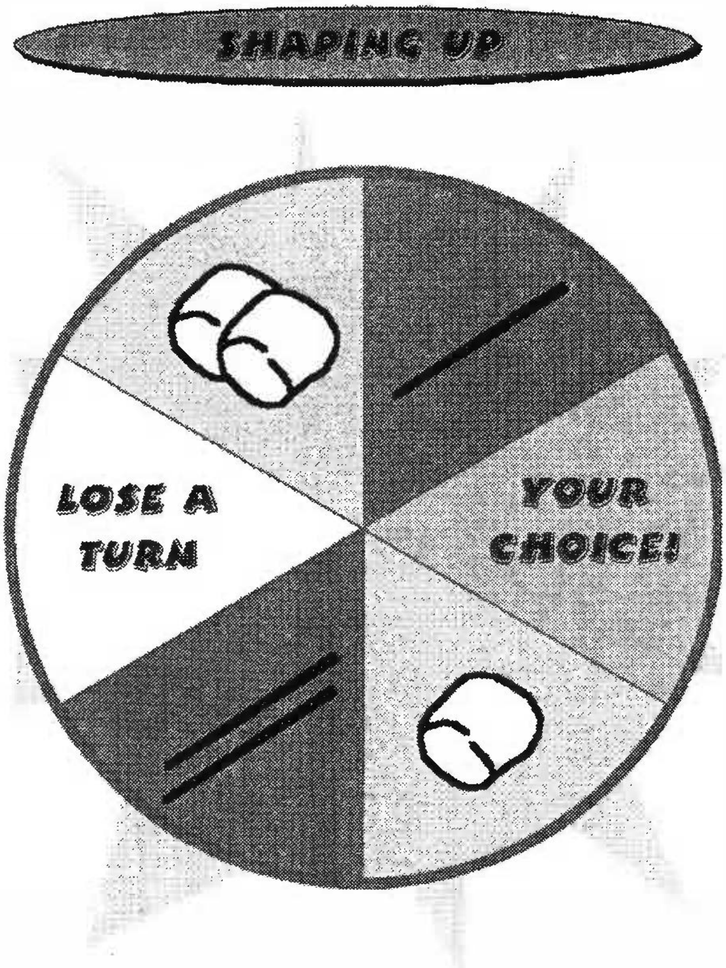
Rules

1. Players will attempt to construct the first six elements in the sequence 1, 3, 5, 7, ...
2. On a turn, a player rolls the die and then adds the specified number of tiles to columns representing the elements in the sequence (see the figure at right). He or she may complete a column and start a new column on the same turn. Alternatively, the player may remove the specified number of tiles from an incomplete column belonging to an opponent. The player may not both add tiles to his or her own columns and remove tiles from an opponent's column on the same turn.
3. The first player to construct all six columns wins.

Adaptations

1. Have students construct other sequences, such as 2, 4, 6, 8, . . . (six columns); 1, 1, 2, 3, 5, . . . (seven columns); 1, 2, 4, 7, 11, . . . (six columns); 5, 2, 6, 3, 7, 4, . . . (nine columns).
2. Change the rules so that a player can add to only one column on a turn and must complete one column before starting another (constructing the columns in sequence).

Shaping Up



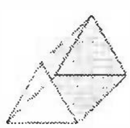
Objective: Build, represent and describe geometric objects and shapes (Shape and Space [3-D Objects and 2-D Shapes], Grade 5, Outcome 15)

Materials: Toothpicks, mini-marshmallows, the Shaping Up spinner mat, an overhead spinner

Players: Two or more

Rules

1. In the first round of this game, players will race to build a cube out of eight marshmallows and 12 toothpicks.
2. On a turn, a player spins the spinner and adds toothpicks or marshmallows according to the result of the spin. If the spinner lands on Lose a Turn, the player adds nothing to the shape. If it lands on Your Choice!, the player may choose to add either one or two marshmallows or one or two toothpicks.
3. If the spinner lands on something the player does not need, play passes to the left.
4. After the player has built the cube, he or she starts building the triangular prism out of six marshmallows and nine toothpicks.



5. After the player has built the triangular prism, he or she starts building the square pyramid.

6. The first player to build all three geometric shapes wins.

Adaptations

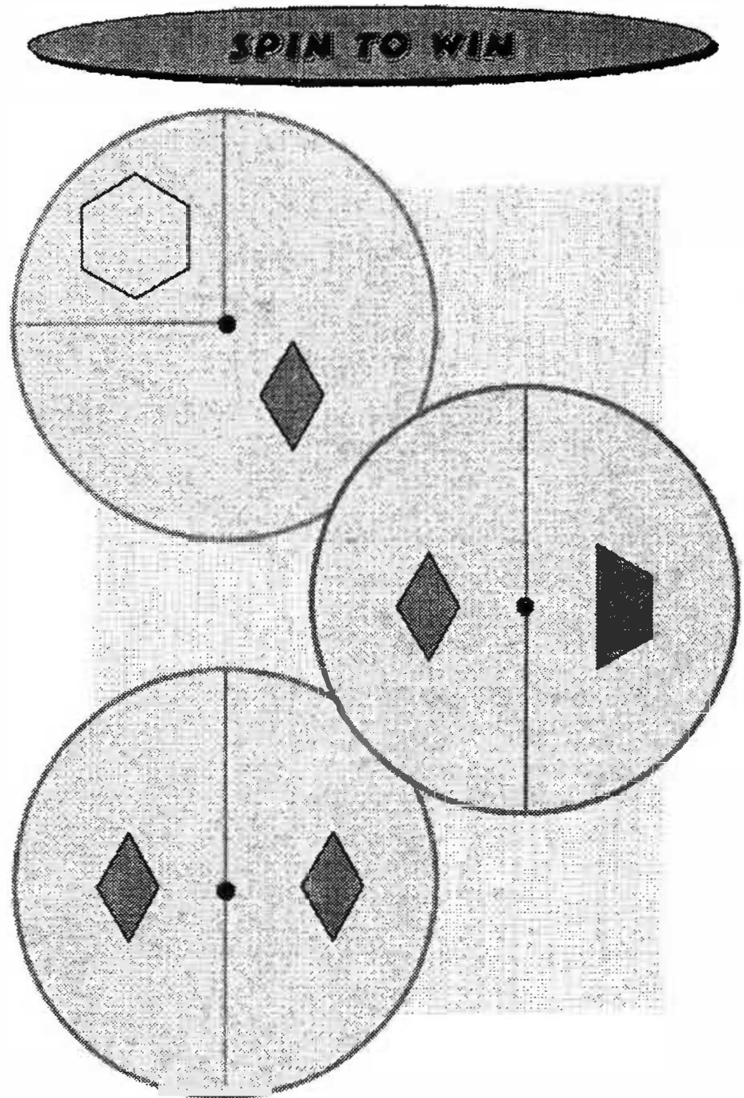
1. Change the rules so that, if the player does not need the part spun, his or her opponent (or the player to the left) may use it instead.
2. Build a spinner that changes the odds of landing on certain elements.
3. Have the students build compound shapes (for example, a clock tower comprising a cube with a square pyramid on top).

Spin to Win

Objective: Describe events using the vocabulary of probability: *always, more likely, equally likely, less likely, never* and so on (Statistics and Probability [Chance and Uncertainty], Grade 5, Outcome 10)

Materials: Spin to Win spinner mat, an overhead spinner, pattern blocks

Players: Two or more

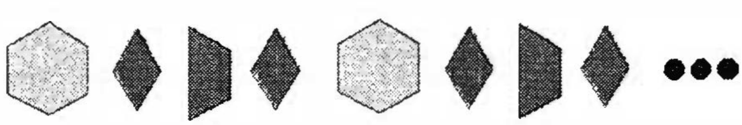


Rules

1. On a turn, a player selects one of the three spinner mats (shown at right) and spins the spinner. The player then adds the block specified by the spinner to his or her set. (Here, the diamond represents the blue block, the quadrilateral represents the red block and the hexagon represents the yellow block.) If the player rejects the block, any other opponent can claim it.
2. When a player has three blue blocks or two red blocks, he or she may trade them for one yellow block.
3. The first player to collect four yellow blocks wins. (The player must collect exactly four yellow blocks—that is, he or she may not accept any block that would build a collection greater than four yellow blocks).

Adaptations

1. Change the rules so that a player must trade up to a yellow block before starting to gather blocks for another yellow block. In other words, if a player starts collecting blue blocks, he or she must complete the collection (three blue blocks) before accepting a red block.
2. Change the spinners to include green blocks.
3. Change the game such that a player tries to build (in sequence) the first 15 elements in the following pattern:



So Very Close!

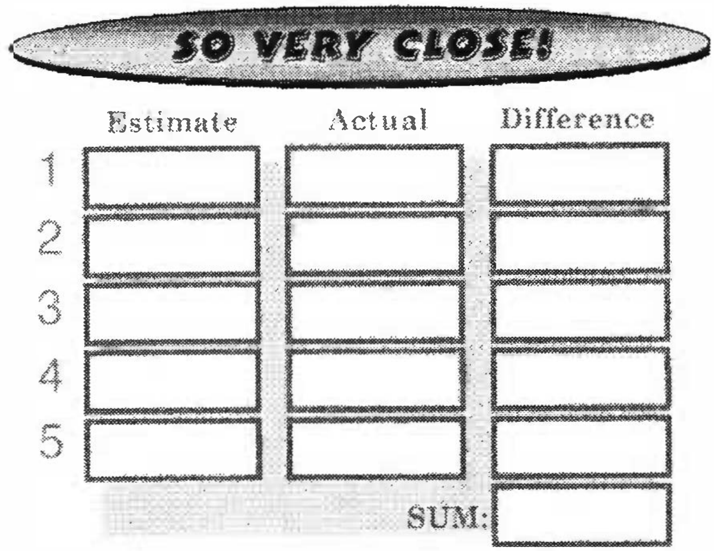
Objective: Determine the volume of an object by measuring the displacement of a liquid by that object (in cubic centimetres or in millilitres) (Shape and Space [Measurement], Grade 6, Outcome 8)

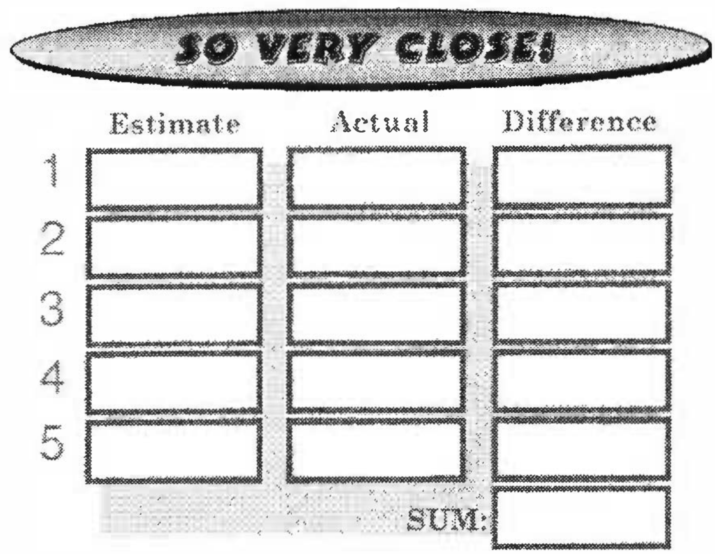
Materials: So Very Close! game board, a graduated cylinder or beaker, water, five objects of various sizes

Players: Any number

Rules

1. To begin the game, the players will need to agree on five objects that can be immersed in water and that fit inside the graduated cylinder or beaker.
2. Each player estimates the amount of water (in cubic centimetres or in millilitres) that will be displaced by the object.
3. After players have recorded their estimates, each object is immersed in water and the displacement is determined.
4. Each player records the displacement and calculates the positive difference between his or her estimate and the actual displacement.
5. After the displacements for all five objects have been measured, each player sums the differences from his or her five estimates. The player with the lowest sum wins.





Adaptations

1. Change the rules so that a player scores one point if his or her estimate is within 10 mL of the actual measurement. After five rounds, the player with the most points wins.
2. To increase the challenge of estimating, use some regular objects (such as cubes), some irregular objects, some objects that float and some objects that don't float.
3. Change the scoring so that a player scores a point when his or her estimate is closer than the previous estimate.

Tangled Angles

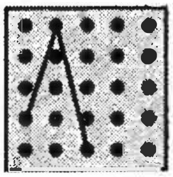
Objective: Classify given angles as acute, right, obtuse, straight and reflex angles (Shape and Space [Measurement], Grade 6, Outcome 12)

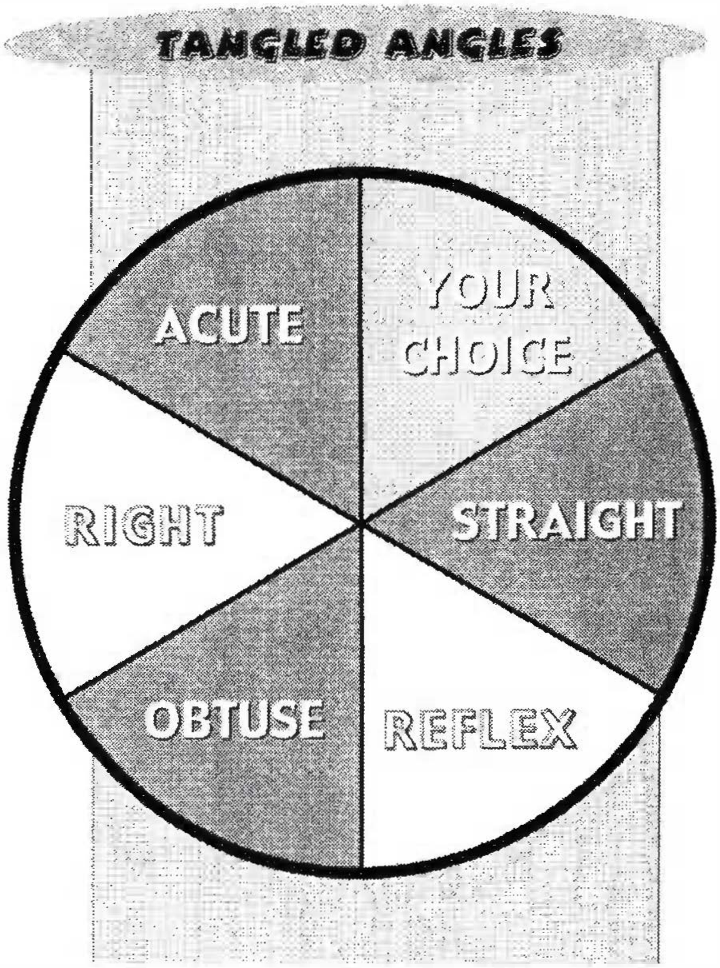
Materials: A geoboard, elastic bands, an overhead spinner, the Tangled Angles spinner mat

Players: Two or more

Rules

1. To play this game, players take turns spinning the spinner on the spinner mat (shown at right).
2. On a turn, a player takes an elastic band and, using any three pegs on the geoboard, builds the type of angle specified by the spinner. For example, if the spinner lands on Acute, the player might build an angle like the following:





   Players must follow these rules when building angles:
   - The arms of any two angles must not cross each other.
   - A peg must not be used to build more than one angle (that is, once a peg is used, it is no longer available for use in another angle).
   - The arm of any angle may span more than two pegs, but all such pegs are considered used and, therefore, no longer available.
3. Play continues until a player does not have pegs in the appropriate orientation to build the specified angle. This player must then drop out of the game. The last remaining player in the game wins.

Adaptations

1. Replace the angle types on the spinner with specific angles for players to build and measure.
2. Have students play cooperatively to see how many angles they can build before the requisite collection of pegs is no longer available.

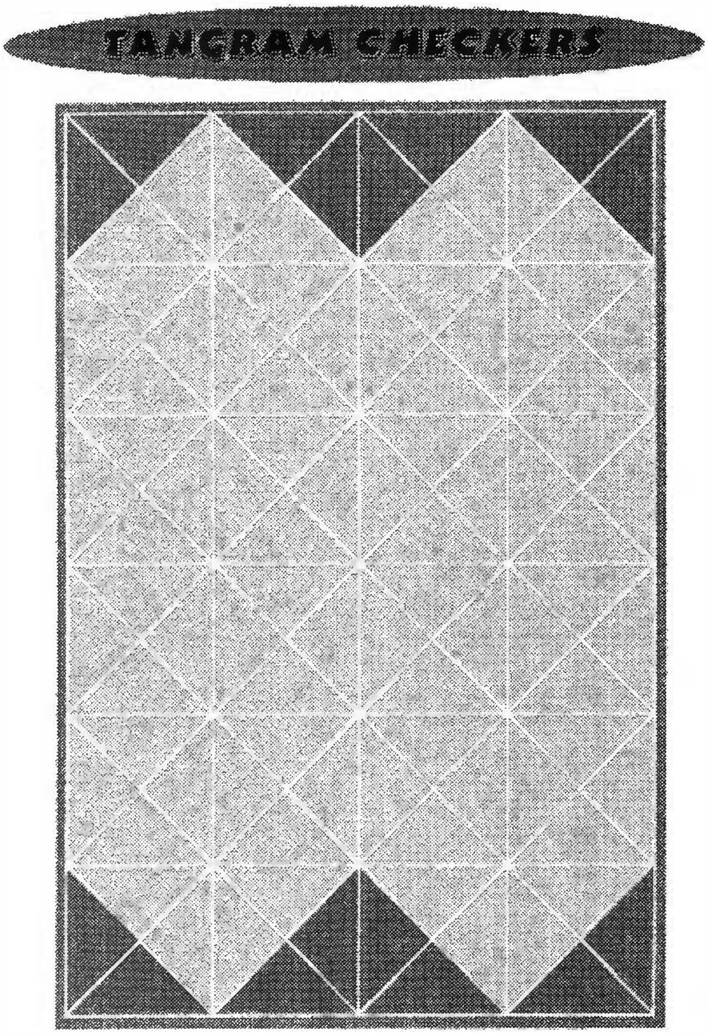
Tangram Checkers

Note: The idea for the original version of this game belongs to James Reynolds.

Objective: Create, analyze and describe designs using translations (slides), rotations (turns) and reflections (flips) (Shape and Space [Transformations], Grade 7, Outcome 11)

Materials: The Tangram Checkers checkerboard, tangrams (two sets, different colours)

Players: Two



Rules

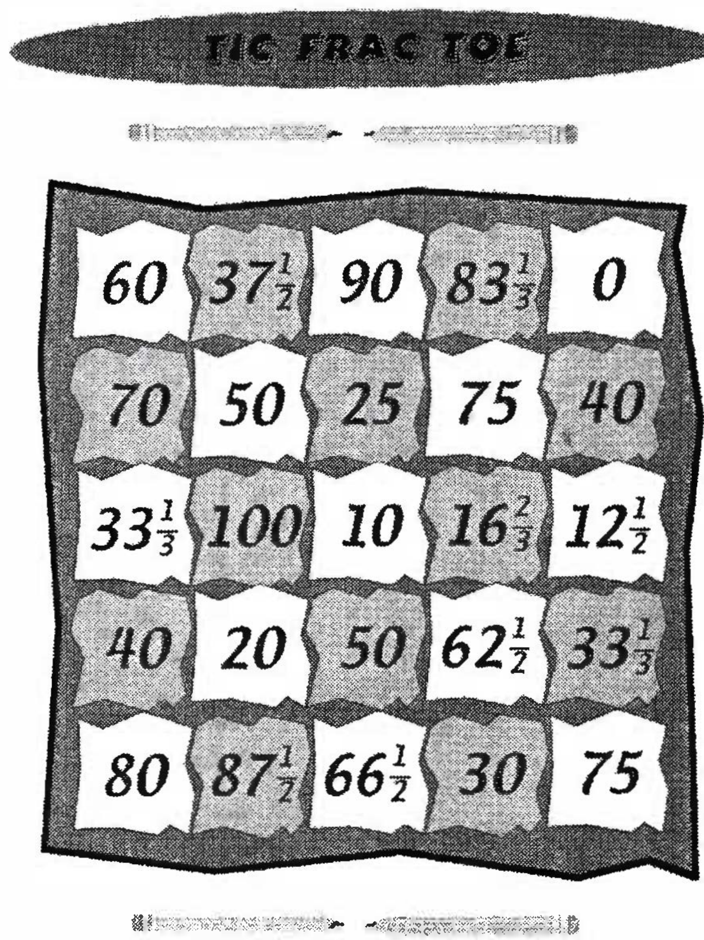
1. To start the game, each player places the appropriate triangular tangram pieces (one medium-sized triangle and two small-sized triangles) on the dark spaces on his or her side of the checkerboard.
2. On a turn, a player may make one of the following moves:
   - Slide one piece one space horizontally, vertically or diagonally.
   - Flip one piece over an identified line of reflection (which must be an edge of the piece to be moved).
   - Rotate one piece a quarter-turn clockwise or counter-clockwise around any corner of that tangram piece.
3. Before a player may move a piece, the player must identify the move he or she will be making, including identifying the corner used in the rotation or the edge used in the reflection.
4. Two tangram pieces must never overlap, and no part of a tangram piece may rest off the checkerboard.
5. The first player to reposition his or her triangles in the home positions originally occupied by the opponent wins.

Adaptations

1. Substitute or include shapes other than triangles.
2. Place a tangram silhouette in the centre of the checkerboard and ask players to build it together (following the same rules).
3. Modify the board by adding home positions on the right and left sides of the checkerboard. Play the same game with four players.
4. Darken some spaces on the checkerboard to represent obstacles around which players must navigate.

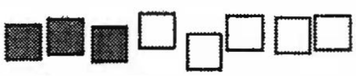
Tic-Frac-Toe

Objective: Convert, mentally, among fractions, decimals and percentages to facilitate the solution of problems (Number [Number Operations], Grade 7, Outcome 21)
Materials: Tic-Frac-Toe game board, two-colour chips, bingo chips, a four-sided die, a six-sided die, a calculator
Players: Two



Rules

1. In this game, players will build models of fractions using the two-colour chips. For example, the fraction shown below represents 3/8:

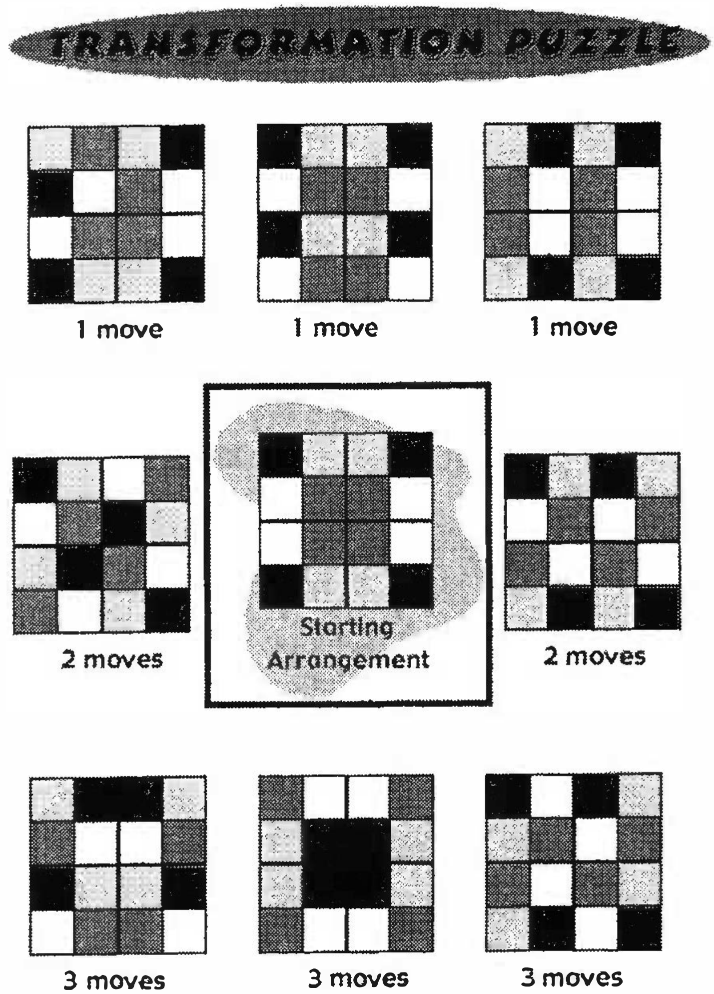


   The players will then convert the fractions to percentages (mentally or with the aid of a calculator).
2. Players start with no chips. On a turn, a player rolls either the four-sided die or the six-sided die. The player then adds to or removes from his or her set the same number of chips as the value rolled. For example, if the player rolls a 2, he or she may add two chips with the red side up or two chips with the white side up. Alternatively, the player may remove two red chips or two white chips. If the value rolled is not favourable, the player may pass the turn. A player may not add or remove a combination of red and white chips.
3. When the player has built a new fraction, he or she converts the fraction to its equivalent percentage and places a bingo chip on the corresponding value on the game board (shown at right). Play passes to the left.
4. If the appropriate percentage is already occupied, the player passes his or her turn.
5. If a player is shown (with the calculator) to have converted incorrectly, his or her chip is removed from the game board and the player passes the turn.
6. The first player to get three chips in a row—vertically, horizontally or diagonally—wins.

Adaptations

1. Replace the percentages on the game board with their decimal equivalents.
2. Change the rules such that, if a player creates a fraction for a percentage that is already occupied, the player can steal the space, replacing the chip with his or her own. Four chips in a row wins.

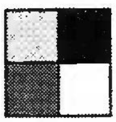
Transformation Puzzle



Objective: Create, analyze and describe designs using translations (slides), rotations (turns) and reflections (flips) (Shape and Space [Transformations], Grade 7, Outcome 11)
Materials: 16 gram cubes (four each of four colours)
Players: One

Rules

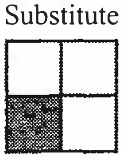
1. To begin the game, the player takes the 16 cubes and builds four squares, each comprising four blocks (one block of each colour) as shown below:



2. The player now arranges the four sets in a larger square like the starting arrangement shown in the figure at right.
3. Each of the following counts as a single move:
   - Turning a set of four blocks a quarter-turn clockwise or counter-clockwise
   - Flipping a set of four blocks vertically or horizontally
   - Flipping two adjoining sets of four blocks vertically or horizontally
   - Turning the entire puzzle (all four sets of four blocks) a quarter-turn clockwise or counter-clockwise
   - Flipping the entire puzzle (all four sets of four blocks) vertically or horizontally
   - Switching two adjoining sets of four blocks without changing their orientation
4. Using the prescribed number of moves (and returning the blocks to the starting arrangement before beginning each puzzle), the player tries to re-create each of the patterns shown at right.

Adaptations

1. Substitute different combinations of blocks, as below:



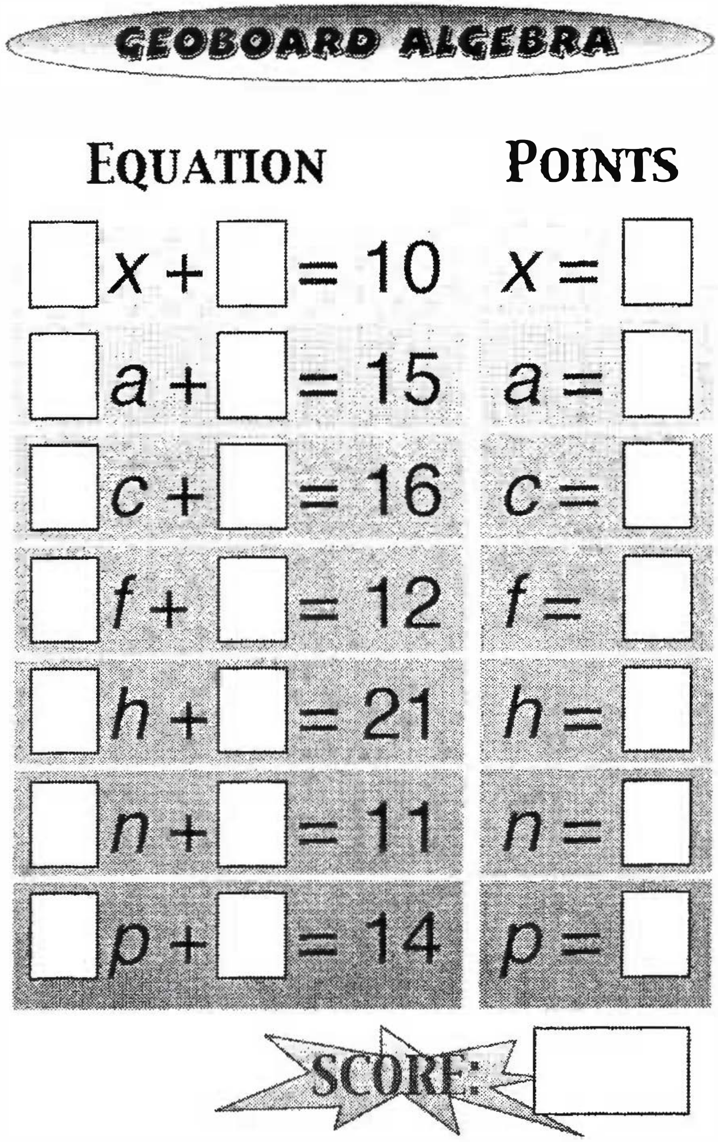
2. Increase the number of moves required.
3. Have students make cards showing patterns requiring up to three moves. The students then exchange decks of cards with other players and race to see who can solve each other's pattern puzzles first, working through the decks in sequence.

Geoboard Algebra

Objective: Solve and verify one- and two-step, first-degree equations (Patterns and Relations [Variables and Equations], Grade 8, Outcome 5)

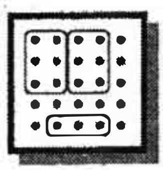
Materials: Geoboard Algebra game board, geoboards, elastic bands, a four-sided die, a six-sided die, an eight-sided die, a pencil

Players: Two or more



Rules

1. On a turn, a player selects a die and rolls it. The player may now enter the value rolled in any of the blanks in the Equation column of the game board (shown at right). A value can only be entered into an equation if it constructs a whole number root (for example, $3x + 1 = 10$ is acceptable whereas $3x + 2 = 10$ is not). Players will have to think carefully before entering any value.
2. Play continues until an equation has both missing values entered.
3. When both values for an equation are determined, the player models his or her equation using the geoboard, thus demonstrating the value of the variable in the completed equation. The player scores points equal to the value of the variable. The example shows $2a + 3 = 15$, and the player scores six points for determining that $a = 6$.



4. Players continue completing and solving equations until one player has solved all seven equations. Players now total their points. The player with the most points wins.
5. If a player cannot (or chooses not to) enter a rolled value, the player passes that turn.

Adaptations

1. Change the rules such that a player must enter a value on every turn. If an equation does not have a whole-number root, then the player scores no points.
2. Omit the geoboard model and allow roots that are not whole numbers.

References

Alberta Learning. *Mathematics Grades 7–8–9.* Edmonton, Alta.: Author, 1996. Available at www.learning.gov.ab.ca/k_12/curriculum/bySubject/math/jhmath.pdf (accessed April 19, 2004).

———. *Mathematics Kindergarten to Grade 6.* Edmonton, Alta.: Author, 1997. Available at www.learning.gov.ab.ca/k_12/curriculum/bySubject/math/elemmath.pdf (accessed April 19, 2004).

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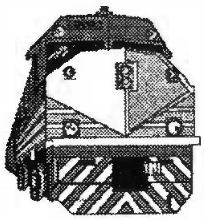
A Page of Problems

A. Craig Loewen

High School

Herb the Hobo is 3/7 of the way across a railroad bridge when he hears a train whistle behind him. If he runs forward, he can jump off the bridge just in front of the train. If he turns around and runs back toward the train, he can also jump off the bridge just in front of the train. If the train is travelling at 35 km/h, how fast must Herb run?



Source: C. Kantecki and L. E. Yunker, "Problem Solving for the High School Mathematics Student," in *Problem Solving in the Mathematics Classroom,* Math Monograph No. 7 (Edmonton, Alta.: MCATA, 1982), edited by S. Rachlin, 49–60.

Junior High

Substitute a different digit for each letter to make the following statement true:

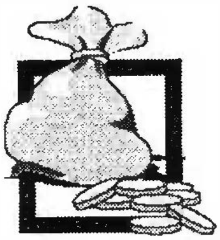
MATH
- IS
FUN

How many solutions can you find? How many solutions are there?

Elementary

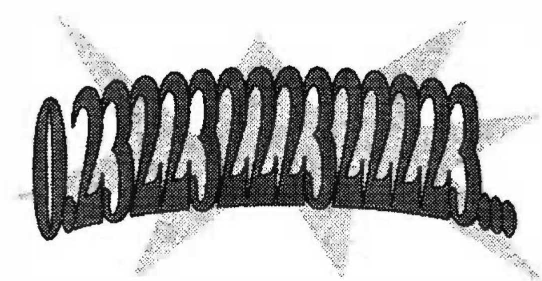
How many ways can you make up 55¢ using only nickels and dimes?

How many ways can you do so if you can use quarters, too?



High School

In the following decimal, how many 2s are there before the 100th 3?



Source: M. A. Sobel and E. M. Maletsky, *Teaching Mathematics: A Sourcebook of Aids, Activities, and Strategies,* 2nd ed. (Englewood Cliffs, N.J.: Prentice Hall, 1988).

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