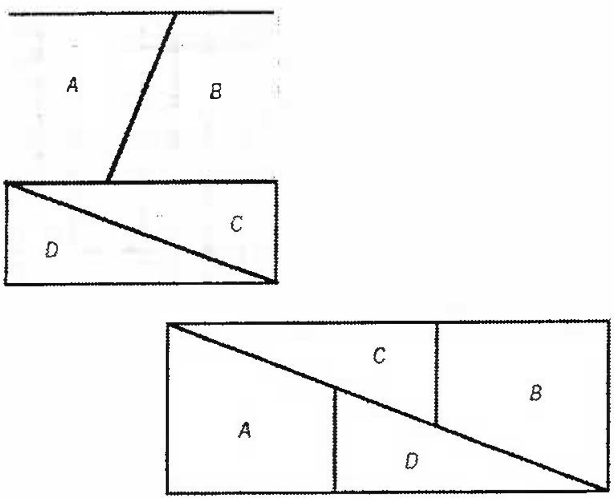
A Letter to Jim About the Other, Not-So-Magic Square

A. Craig Loewen

We are all familiar with the traditional magic square, the one in which we must fit the digits 1–9 such that all rows, columns and diagonals have the same sum. But some time ago my friend Jim introduced me to the other magic square:



1. Calculate the area of the square.
2. Cut out the four pieces and rearrange them to make the rectangle.
3, Calculate the area of the rectangle.
4. Explain the difference in the areas—or is it magic?

Jim asked, "Is there a mathematical explanation for this activity?" After exploring the problem, I wrote the following letter in response to him.

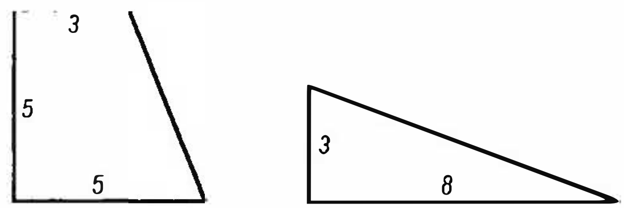
Dear Jim,

The problem seems a little familiar to me, but if I've forgotten the problem, I've also forgotten the solution! I wound up playing with it for a few hours this afternoon. Here are my thoughts and notes.

Understand the Problem

My intuition and my belief in the laws of conservation force me to doubt that the shapes grow simply by my moving them. So, let's study the square first.

We know that the square is made up of four pieces with shapes and dimensions as in the figures below (two trapezoids and two triangles):



We can calculate the areas of the trapezoids and the triangles to see if they add up to 64 or to 65. Maybe that will help us discover the trick:

$$\text{Area trapezoid} = (0.5)(\ell\text{ top} + \ell\text{ bottom})(\text{height}) = (0.5)(3 + 5)(5) = 20$$

$$\text{Area triangle} = (0.5)(\text{base})(\text{height}) = (0.5)(8)(3) = 12$$

$$\text{Area square} = (2)(\text{area trapezoid}) + (2)(\text{area triangle}) = (2)(20) + 2(12) = 64$$

Of course, this answer makes sense, because the square was 8 units on its side. We could have calculated this area a whole lot faster simply by squaring one of the sides, but I wanted to know if the areas of the pieces added up! We'd better look at the rectangle now.

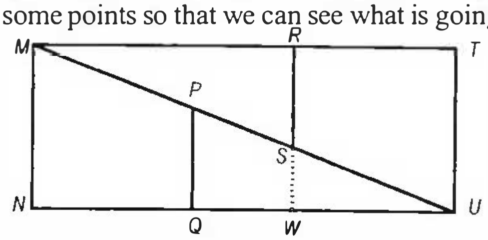
We know that the rectangle is made up of the same four parts, so it should also have an area of 64 square units. We can calculate the area of the rectangle using the length-times-width formula, but first we need to find the length and width of the rectangle. The length of the rectangle is the sum of the longest sides of pieces *A* and *D,* which is 13 units. The height of the rectangle is the same as the height of piece *A,* which is 5 units. So,

$$\text{Area rectangle} = (\text{length})(\text{width}) = (13)(5) = 65.$$

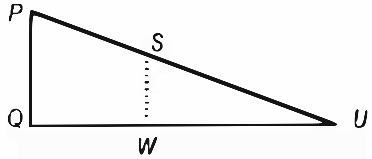
Well, for all the world, it looks like we have gained 1 square unit simply by rearranging the pieces! There is definitely something fishy about this rectangle!

Develop a Plan and Carry It Out

We had better take a closer look at the rectangle and label some points so that we can see what is going on:



First, let's extend the line *RS* until it meets the line *QU*. We'll call that point *W* (as shown in the diagram above). We know that both line *MN* and line *TU* are 5 units long. If this is a genuine rectangle, the length of line *RW* should also be 5 units. But how can we calculate the length of line *RW?* Well, we know the length of line *RS* (3 units), so all we need to do is calculate the length of line *SW.* We can do this using the properties of similar triangles. Here is our triangle:



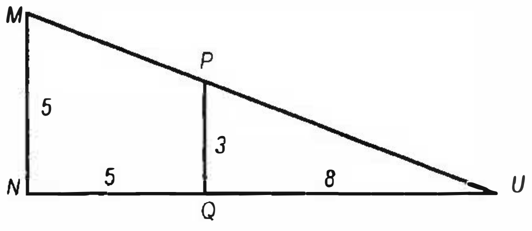
Let's make a list of what we know: $\ell PQ = 3$ units, $\ell QU = 8$ units and $\ell UW = 5$ units. Now we can construct our ratio:

$$\frac{\ell SW}{\ell UW} = \frac{\ell PQ}{\ell UQ}$$

$$\frac{\ell SW}{5} = \frac{3}{8}$$

Solving for ℓSW, we find that it is 1.875 units. Wait a minute! That means that the length of *RW* is 4.875 units . . . but it's supposed to be 5 units! In other words, our figure is not a true rectangle—at least, it's not a uniform, solid rectangle as it appears to be. Therefore, the formula for calculating the area of a rectangle (length times width) doesn't apply here.

But I still don't know exactly what is going on. It is worth investigating further. I'm wondering what happens when the two pieces (the triangle and the trapezoid) are put together. While trying to draw these shapes on the computer, I have become suspicious of their slopes. Consider this figure:



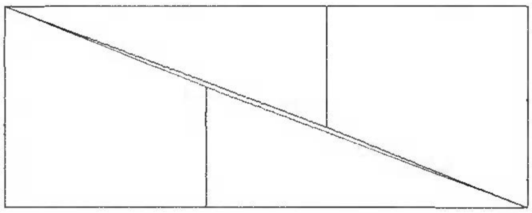
If line *MU* is indeed a straight line, then the slope of line *MP* must equal the slope of line *PU:*

$$\text{slope } PU = \frac{\text{rise}}{\text{run}} = \frac{3}{8} = 0.375 \qquad \text{slope } MP = \frac{\text{rise}}{\text{run}} = \frac{2}{5} = 0.4$$

Here it is! The slopes are dangerously similar, but they are not the same. If the slope of *PU* is less than the slope of *MP,* the two shapes do not form a true triangle. The segment from *M* to *U* bends toward *Q* at *P.* So, when we add another of these tricky triangles, putting the four pieces together, we are actually leaving a little gap! This gap is just too small to notice when we cut out the shapes and move them around. The inaccuracy of our paper model hides that 1 square unit. That is the real magic of this magic square!

Looking Back

Well, now I know why I had so much trouble drawing these shapes on my computer in the first place. Look what happens when I accurately draw the shapes with the help of my computer, rotate them and put them together to form a rectangle:



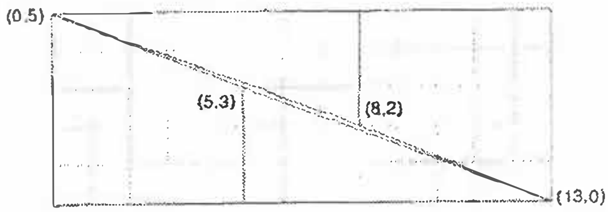
We can see a small area distributed neatly through the centre of the figure. Because I know the area of the rectangle and the areas of each of the four shapes, it makes sense that the gap has an area of 1 square unit:

$$\begin{aligned}\text{Area of gap} &= \text{area rectangle} - (2)(\text{area trapezoid}) - (2)(\text{area triangle}) \\ &= 65 - (2)(20) - 2(12) \\ &= 1.\end{aligned}$$

It would be fun to find another way to calculate (and perhaps verify) the area of the gap in the rectangle. One such way is to identify the coordinates of this tiny shape (a quadrilateral) and use the following formula:

$$\text{Area} = \frac{(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (y_1x_2 + y_2x_3 + \ldots + y_nx_1)}{2}$$

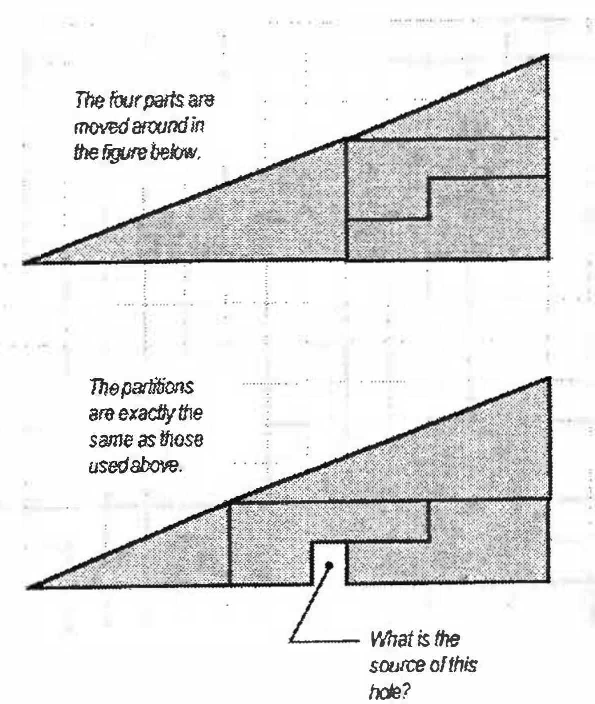
To identify the coordinates, we need only superimpose the rectangle onto a Cartesian coordinate system:



We can define the points as follows: $(x_1, y_1) = (0, 5)$, $(x_2, y_2) = (5, 3)$, $(x_3, y_3) = (13, 0)$ and $(x_4, y_4) = (8, 2)$. To use this formula, we should define the points in sequence in a counter-clockwise order. Now we can substitute the values in the formula:

$$
\begin{aligned}
\text{Area of gap} &= \frac{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (y_1x_2 + y_2x_3 + y_3x_4 + y_4x_1)}{2} \\
&= \frac{(0\cdot3 + 5\cdot0 + 13\cdot2 + 8\cdot5) - (5\cdot5 + 3\cdot13 + 0\cdot8 + 2\cdot0)}{2} \\
&= \frac{(0 + 0 + 26 + 40) - (25 + 39 + 0 + 0)}{2} \\
&= \frac{66 - 64}{2} \\
&= \frac{2}{2} \\
&= 1.
\end{aligned}
$$

Now I have a problem for you to try. I don't know the source of this problem, but I think a student gave it to me a long time ago. It has definite similarities to the problem you sent me (both in the problem itself and in its solution). Here it is:



I had a great time playing with this problem today, Jim! And the answer to your question is, yes, there is a mathematical explanation. It turns out that your square is not magical, but my afternoon really was fun!

Keep well,
Craig

A Few Notes

It is sometimes surprising that a problem can so totally capture our interest that time slips away as we play, think and explore. But, as I look back over this problem and my solution to it, I am struck by other, more important questions.

Owning a Problem

What can we do to help our students bite into a problem? When I started working on this problem, I was so sure that something was amiss—I square unit can't just disappear!—that I couldn't let it go until I could provide an adequate explanation. How can we help our students develop this sense of internal insistence? Maybe this sense is a pre cursor to that infinitely important and highly desir able sense of achievement we feel when we solve the problem!

Finding Time

How do we find time for any form of real problem solving in our classrooms? I really did spend an afternoon playing with this problem. I wanted to try several methods and see if I could come up with a good, sensible explanation. But I wonder if the volume and busyness of our curriculum (which supposedly values problem solving above all else) leaves adequate space for anything other than direct instruction based on algorithms, formulas and facts.

The Role of Technology

How can we increase the use of technology as a tool in the mathematics classroom? In at least one way, my computer was an important tool in solving this problem. Only with the aid of my computer could I construct a drawing accurate enough to reveal the small gap. The software I used was not particularly complex or expensive; it was a simple drawing package. Our students should be taught to integrate technology flexibly, using the available software to increase their problem-solving power.

Communication in the Classroom

How can we better integrate communication in the math classroom? In part, what made this problem fun for me was sharing it with a friend. What does this tell me about the need for collaboration, cooperation and communication? And what does this tell me about how my class time and, indeed, my classroom space might be better arranged?

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