# Infinite Sets and Georg Cantor

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Infinite quantities have perplexed mankind for thousands of years. To the caveman, the distance to the horizon was infinite. To the modern man, infinite is the size of the universe. Throughout man's existence, the concept of infinity has become so increasingly complex that some men have protested against the use of infinite magnitude in mathematics.

In the late 19th century, Georg Cantor shed an enormous light on the concept of infinity. His work has given the subject an acceptability in mathematics.

Georg Ferdinand Ludwig Philip Cantor was born in St. Petersburg, Russia, on March 3, 1845. By the time of his death 73 years later, Cantor had permanently changed the world of mathematics with his ideas on set theory and infinite sets. He was a revolutionary who held to his theories and the notion of the completed infinite in the face of strong opposition. Mathematicians argued against Cantor, yet in the end, Cantor prevailed. Cantor's theories were shown to be logically sound and consistent under a certain set of axioms.

The concepts Cantor introduced into set theory with his idea of transfinite sets changed the outlook of mathematicians. Students of mathematics must understand the ideas of Cantor to truly understand their discipline.

At the very heart of Cantor's theories lies the concept of cardinality or set equivalence. This concept rests on the fact that two groups need not be counted to be proven to have the same number of elements. One can attempt to establish a one-to-one correspondence between the two groups. If a one-to-one correspondence between the two groups can be set up, then they have the same number of elements. Thus Cantor defined set equivalence as follows: Two sets $M$ and $N$ are equivalent if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other. Mathematicians say that two equivalent sets have the same cardinality.

Cantor defined those sets that could be put into a one-to-one correspondence with the natural numbers as a denumerable or countably infinite set. Thus any set with a cardinality equal to that of the natural numbers was called denumerable.

In 1893, after Cantor established the different cardinality of various infinite sets, he needed a notation to represent the different cardinalities. Because of his Jewish background, he chose the Hebrew letter aleph. Cantor defined aleph-null, $\aleph_0$, as the cardinality of the natural numbers or positive integers. The aleph notation is the one used today to describe infinite sets.

It is easy to establish that the set of integers is denumerable using elementary algebra. A one-to-one correspondence with the set of integers can be shown by simply rearranging them so that there is a definite first element of a set such as 0, 1, −1, 2, −2, 3, −3, 4, −4, 5, −5, ... We see that the set of positive and negative integers is as large or has the same cardinality as the set of natural numbers.

$$\begin{array}{cccccccc} 0, & 1, & -1, & 2, & -2, & 3 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1, & 2, & 3, & 4, & 5, & 6 & \ldots \end{array}$$

Even though the set of even numbers is a proper subset of the natural numbers, it is equivalent to it in cardinality and is denumerable. We can set up a one-to-one correspondence between these two sets.

$$\begin{array}{cccccc} 1, & 2, & 3, & \ldots, & n, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2, & 4, & 6, & \ldots, & 2n, & \ldots \end{array}$$

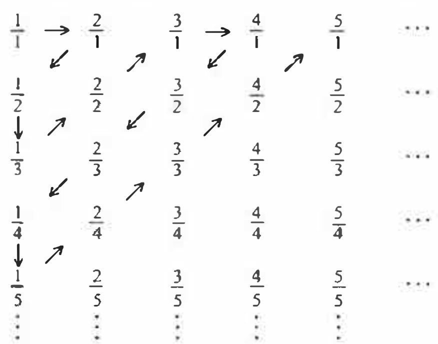
The set of odd numbers is also denumerable.

$$\begin{array}{cccccc} 1, & 2, & 3, & \ldots, & n, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1, & 3, & 5, & \ldots, & 2n-1, & \ldots \end{array}$$

Cantor used the existence of this equivalence as his definition of infinity, where he stated that an infinite set is one that can be put into a one-to-one correspondence with a subset of itself.

Finding the cardinality of the set of rational numbers required a different approach. We see that an infinity of rational numbers can be "packed in" between any two rational numbers. This means that the set of rational numbers is a dense set because no rational number has an immediate successor. The set of positive integers is discrete because every element of the set has an immediate successor. The question is whether the set of rational numbers, which is dense, has the same number of elements ($\aleph_0$) as the set of positive integers, which is discrete. How can anyone put the rational numbers in a one-to-one correspondence when an infinity of rational numbers can be "placed in" between any two?

For this proof Cantor constructed a two-dimensional array of rational numbers. When zero is placed above the array, a list of all the rational numbers is formed. Cantor then proceeded to count the rational numbers. He drew arrows up and down the diagonals of this array, effectively counting the rational numbers. This set up a one-to-one correspondence between the rationals and the natural numbers. Through this simple method Cantor showed the set of rational numbers to be denumerable.




These fractions can be written as the set of two integers and then put into a one-to-one correspondence with the natural numbers as follows:

1, 2, 3, 4, 5 ...

↕ ↕ ↕ ↕ ↕

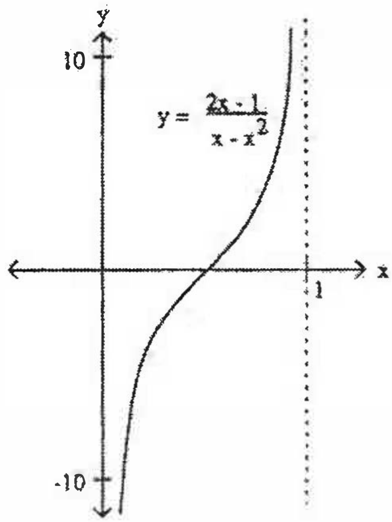
(1, 1) (2, 1) (1, 2) (1, 3) (2, 2) ...

Therefore $\aleph_0$ is also the cardinal or transfinite number for the set of rational numbers.

After these proofs, it seemed to the mathematical community that every infinite set was denumerable and that no set had a higher transfinite cardinal number than $\aleph_0$. In 1874 with his paper "On a Property of the Collection of all Algebraic Numbers," Cantor revealed that a nondenumerably infinite set existed. The paper dealt with the cardinality of the set of real numbers. Cantor proved that the set of real numbers is not denumerable.

Cantor began the proof by establishing that a one-to-one correspondence existed between the open interval (0, 1) and the set of real numbers using the function:

$$\frac{(2x-1)}{x-x^2}$$




Cantor then assumed that the set of all reals in (0, 1) was denumerably infinite. This leads to the conclusion that a list can be formed pairing each natural number with one real number. Cantor established a hypothetical list pairing the set of real numbers between zero and 1 and the set of natural numbers. He then listed a set of infinite decimals:

$1 \rightarrow 0.\ a_1\ a_2\ a_3\ a_4\ \ldots$

$2 \rightarrow 0.\ b_1\ b_2\ b_3\ b_4\ \ldots$

$3 \rightarrow 0.\ c_1\ c_2\ c_3\ c_4\ \ldots$

⋮

This set should contain all the real numbers in the given interval, (0, 1).

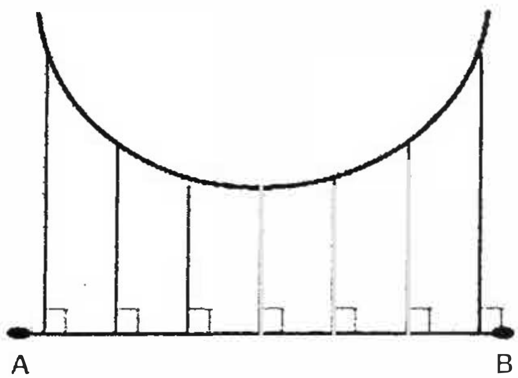
In his proof, usually known as the diagonal proof, Cantor found that he could define a real number $z$ in (0, 1) not on the list. He constructed a number that had as its $n$-th decimal place a number different from the $n$-th decimal place of the $n$-th number on the list and not equal to zero or nine,

$0.\ a'_1 b'_2 c'_3 d'_4 \ldots$

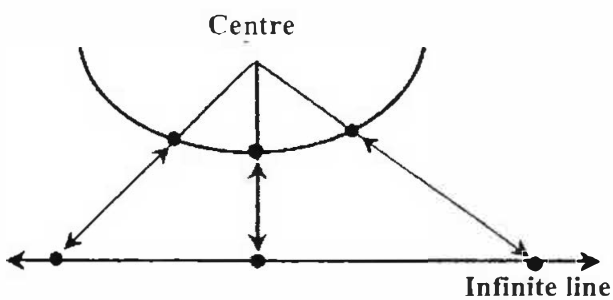
This created a number $z$ that could not be on the list. The number $z$ differed in at least one decimal place with every number on the list. The elimination of zero or nine as choices for a decimal place negated the possibility of infinitely repeating decimals equal to zero or one. Thus, the open interval (0, 1) could not be denumerable, and because the open interval (0, 1) has the same cardinality as the set of real numbers, the reals could not be denumerable.

Cantor named this cardinality aleph-one, $\aleph_1$. Modern mathematicians usually call this cardinality $C$ instead of aleph-one ($C$ for continuum, an open interval of real numbers).

To show geometrically that the reals are nondenumerable, Cantor established an astonishing fact: there are as many points along an infinite (straight) line as there are on a finite segment of it.




Each vertical line segment is perpendicular to the segment $AB$, thus ensuring not only that each vertical line segment will pass through the segment $AB$ itself, but that it will only pass through one point on the semicircle. So we can match the points of the segment $AB$ in a one-to-one correspondence with the points on the semicircle, thus proving that the segment and the semicircle have the same number of points.




Now each line segment is drawn from the centre of the same semicircle to the line, again ensuring that each segment passes through only one point on the circle and only one point on the line. Thus, the points of the semicircle are now matched one-to-one with the points of the entire line. Therefore, since the finite line segment and infinite line have both been put into a one-to-one correspondence with points on a semicircle, we can conclude that a finite line segment and an infinite line have exactly the same number of points.

But was there a set with cardinality greater than aleph-one? Where could an infinite set with a higher cardinality lie?

With his proof of the nondenumerability of the continuum, Cantor created, in effect, a hierarchy of infinites. There is no infinite set with a cardinality that is less than that of the natural numbers ($\aleph_0$), and all sets that are not denumerable (have the same cardinality as the real numbers) have a higher level of infinity than all the countable sets. At this point one may be wondering if any sets exist with a cardinality that is greater than the real numbers. It may seem reasonable to presume that C (the cardinality of the real numbers) is the greatest possible cardinality.

However, as Cantor himself soon discovered, it turns out that there are sets that are greater in cardinality than the set of real numbers.

These sets are Power Sets, the set of all subsets of a given set. For example, the power set of the set $\{a, b, c\}$ consists of the subsets $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$, to which we must add the "empty" or "null" set $\{\ \}$ and the set $\{a, b, c\}$ itself. Thus, from the original set of three elements we get a new set of eight ($= 2^3$) elements. In fact, Cantor concluded that the power set of any given set always has more elements than the original set. Cantor's theorem (as it is called today) shows that, given any set, we can always construct a set that has a greater cardinality.

Cantor's theorem was proven, once again, by contradiction, for he first assumed that there is a largest infinite set, and then demonstrated that there must be one even larger (that is, the power set).

Let $X$ be an arbitrary infinite set (of any cardinality), which we can represent as $X = \{a, b, c, d, e, \ldots\}$. Now let us assume that the members of $X$ can be put into a one-to-one correspondence with its power set, which can be represented by

$$P(X) = \{\{e\}, \{a, b\}, \{b, c, e\}, \{a, c\}, \{a\}, \ldots\}.$$

Such an arbitrary matchup would look something like:

| $X$ | | $P(X)$ |
|---|---|---|
| $a$ | ⇔ | $\{c, d\}$ |
| $b$ | ⇔ | $\{a\}$ |
| $c$ | ⇔ | $\{a, b, c, d\}$ |
| $d$ | ⇔ | $\{b, e\}$ |
| $e$ | ⇔ | $\{a, c, e\}$ |
| ⋮ | ⋮ | ⋮ |

Now let us consider the different ways that a member of $X$ could be paired with the subsets of $X$. Some of the elements of $X$ are matched with subsets that contain them. For example, here the element $e$ is matched with the subset $\{a, c, e\}$, of which it is a member. Also notice that some of the elements of $X$ are matched with subsets that do not contain them, such as the element $d$, which is matched with the subset $\{b, e\}$, of which it is not a member.

Let us consider the set of elements of $X$ that are not matched up with subsets that contain them. This set, which we'll call $S$, is clearly a subset of $X$; thus, it must appear somewhere in our matchup listing above. However, what could the element $X$ be that matches with $S$? It cannot be a member of $S$, because $S$ was specifically constructed to contain only those elements of $X$ that do not match up to the sets containing them. What happens if the element of $X$ that matches up with $S$ is not contained in $S$? Well, then it must be contained in $S$, again by the definition of $S$. Clearly, this is a contradiction. (The existence of this contradiction forces us to understand that no element of $X$ can be matched with this subset $S$.) This means that $X$ and $P(X)$ cannot be put into a one-to-one correspondence, thus indicating that they cannot have the same cardinality. Therefore, we can conclude that one set must be larger than the other. Because $X$ cannot be put into a one-to-one correspondence with a proper subset of $P(X)$, we can conclude that the cardinality of $P(X)$ must therefore be larger than that of $X$. Hence, Cantor's theorem is indeed true and, as a result, there can be no "largest infinity" and the kinds of infinity are therefore infinite.

It would seem to the observer that Cantor's set theory was an incredible success. But no one knew better than Cantor the imperfections in his theory. The problem with the theory that troubled Cantor most was his inability to prove that no aleph value existed between aleph-null and $C$. He searched his entire life for a proof, but died without ever formulating one.

It is unsurprising that Cantor never created a successful proof for his theorem. In 1938, Kurt Godel demonstrated that Cantor's "Continuum Hypothesis" (that is, that $C=\aleph_0$) could not be disproved within the confines of set theory, (that is, that the Continuum Hypothesis was relatively consistent with and did not contradict the axioms of set theory). In 1963 Paul Cohen demonstrated that the Continuum Hypothesis could not be proved from (and was independent of) the axioms of set theory. In other words, he proved that the negation of the Continuum Hypothesis ($C > \aleph_1$) would also be consistent with the axioms of set theory. This means that two different systems can be set up, both valid, using the continuum hypothesis and its negation.

Georg Cantor opened up whole new vistas in the world of mathematics. He engaged the minds of a whole generation with his concept of the infinite.

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