delta-k

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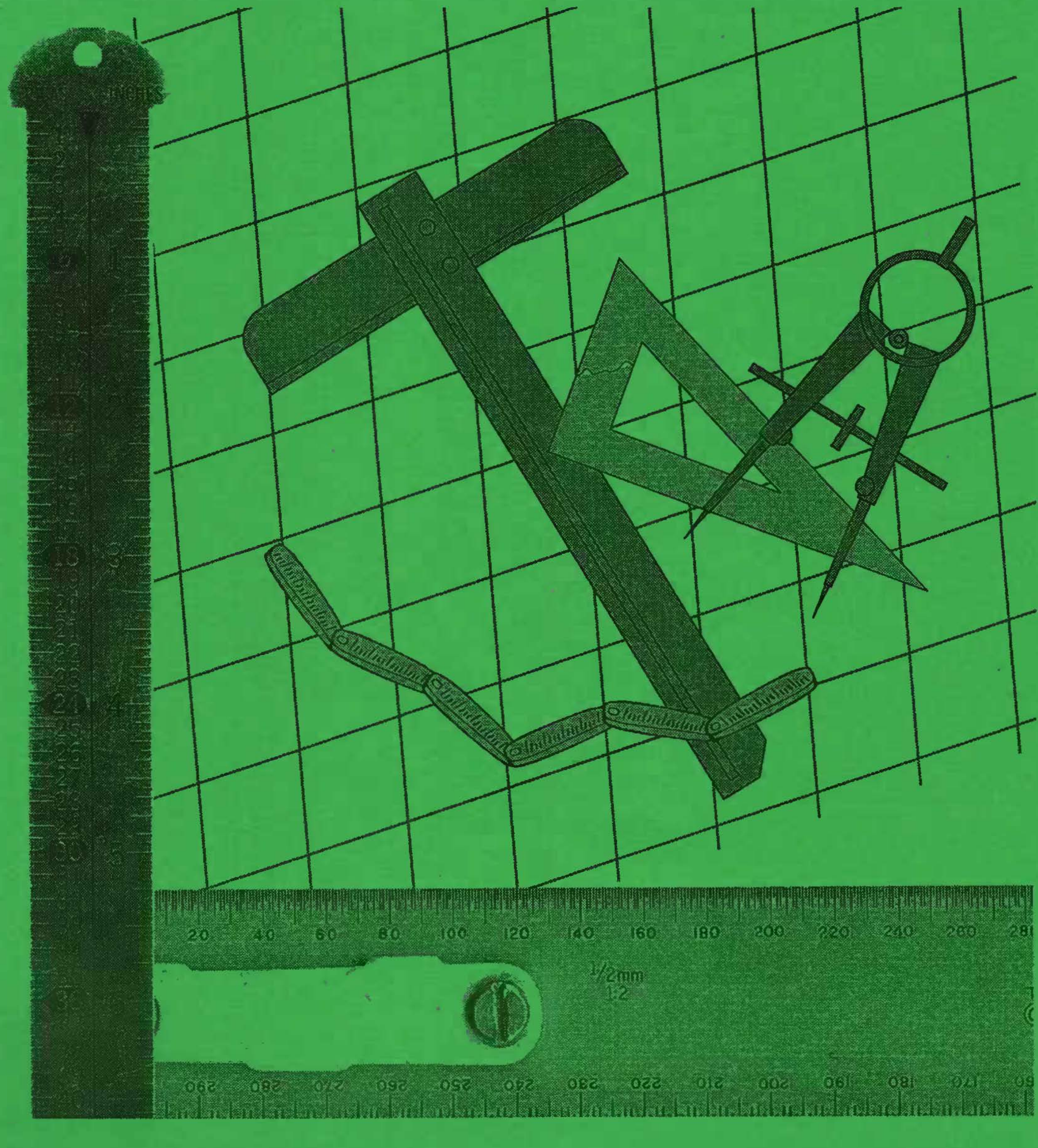

GUIDELINES FOR MANUSCRIPTS

delta-K is a professional journal for mathematics teachers in Alberta. It is published to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; and
- a focus on the curriculum, professional and assessment standards of the NCTM.

Manuscript Guidelines

1. All manuscripts should be typewritten, double-spaced and properly referenced.
2. Preference will be given to manuscripts submitted on 3.5-inch disks using WordPerfect 5.1 or 6.0 or a generic ASCII file. Microsoft Word and AmiPro are also acceptable formats.
3. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
4. If any student sample work is included, please provide a release letter from the student's parent allowing publication in the journal.
5. Limit your manuscripts to no more than eight pages double-spaced.
6. A 250–350-word abstract should accompany your manuscript for inclusion on the Mathematics Council's website.
7. Letters to the editor or reviews of curriculum materials are welcome.
8. *delta-K* is not refereed. Contributions are reviewed by the editor(s), who reserve the right to edit for clarity and space. **The editor shall have the final decision to publish any article.** Send manuscripts to Klaus Puhlmann, Editor, PO Box 6482, Edson, Alberta T7E 1T9; fax 723-2414, e-mail klaupuhl@gyrd.ab.ca.

Submission Deadlines

delta-K is published twice a year. Submissions must be received by August 31 for the fall issue and December 15 for the spring issue.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 40, Number 2 September 2003

CONTENTS

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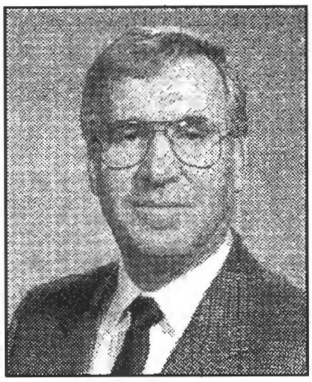
The Image of Mathematics and Mathematics Teaching



Real teachers in real schools in Alberta face real challenges as they implement the intended mandatory program of studies at the various grade levels. Although many teachers are excited about the possibilities for new kinds of instruction, they feel bogged down by the overwhelming expectations about how they should provide that instruction and by the public image of mathematics.

The public image of mathematics is not the best. The work of mathematicians is not considered an important contribution to our civilization. Part of the difficulty lies with mathematics itself, because it presents abstract concepts and images to a public inundated with pictures that are concrete and real. As a result, mathematicians have the tough job of putting mathematics in the right light. Another part of the difficulty lies with the mathematicians, because many of them are unwilling to demonstrate the importance of their work to the general population out of fear that the flawless image of mathematics will be damaged. Yet another part of the difficulty lies with the curriculum content and instructional strategies in the mathematics classroom. Such mathematical content and instructional strategies have little hope of emerging unless there is an educational environment in which teachers teach against external assessment structures completely aligned with the intended curriculum. Schools must become places where students have access to mathematical knowledge and where student engagement in learning is merited by the mathematical content and the instructional strategies.

The public, including the students, generally believes that mathematics is uninteresting, too abstract, difficult to digest and devoid of exciting new developments. Although our program of studies—its rationale, emphasis and goals—reflects the rich cultural heritage and dimensions agreed upon by curriculum developers, the cultural heritage is in a form that students can readily consume to succeed on the external exam.

Although mathematics is one of the oldest scientific fields and the most widely taught subject in schools worldwide, it suffers from an image problem. Mathematicians and mathematics educators have a great responsibility for the mathematics education of future generations. What educators teach in the classroom or present in lecture theatres not only is important for the immediate mathematics needs of the students but also has, in the larger perspective, a significant impact on the image of mathematics in society. Mathematics meets a real need in modern societies and individuals. All teachers of mathematics—from the elementary to the postsecondary level—must think about these issues and, above all, be ready to act to improve mathematics teaching at all levels.

The world over, mathematics is taught as an important subject. Why is mathematics so important? Mathematics teaches children to think—much sooner than in the other subjects and requiring often very little knowledge. Mathematics provides endless possibilities for creative thinking at a high level, unrestricted by authoritarian principles, using common sense and the flight of the imagination. It allows students to unravel the beautiful patterns and the connections to other branches of human activity and thought.

But are we really achieving these ends? Is mathematics, as taught today, really imparting to our students the freedom and pleasure of thinking and generating knowledge? Or are the goals so eloquently stated in our program of studies just empty slogans, mere declarations of importance?

The everyday reality is that in some places of learning mathematics is boring and empty, and in others it is too difficult and a source of constant frustration. Mathematics is seldom what it should be. Why is this the case? Here are only two possible causes: (1) we have not gone beyond the traditional use of the textbook, and (2) we feel that the usual subject matter of mathematics is not suited to original thinking and learning through exploration and discovery.

What needs to be done? What is the role of teachers in engaging students in learning mathematics?

- Teachers must establish an atmosphere in which mathematics and learning are important and in which students feel safe to take risks and share ideas.
- Teachers must create activities and tasks in which students will engage their intellect, stretch their thinking, increase their mathematical understanding and solve relevant problems.
- Teachers must decide on the discourse in their class, which includes deciding how students will interact with each other.
- Teachers must reflect constantly on the teaching and learning that take place in the classroom.
- Teachers must constantly assess the direction in which the students are going and then make adjustments to ensure mathematics learning.
- Teachers must assess the needs and talents of the students, giving the students responsibility for their own learning while knowing when more guidance is needed.
- Teachers must ensure that students make the correct mathematical connections to other concepts and subjects.
- Teachers must constantly upgrade and develop professionally to remain current.
- Teachers must encourage students to become mathematics teachers and not just counsel them to pursue careers in other mathematics-related fields.
- Teachers must have patience and perseverance to develop students to their fullest potential.

The image and the teaching of mathematics are very much influenced by the beliefs and values of the mathematics teachers. To change this image, we need to address the characteristics that prevent teachers from focusing on the intended curriculum.

Klaus Puhlmann

From the President's Pen

In the previous issue of *delta-K,* I brought forward the idea that we are all mathematics leaders and concentrated on John C. Maxwell's (1998) law of reproduction, from his book *The 21 Irrefutable Laws of Leadership.* As I talk with teachers in Alberta, especially those on the Mathematics Council of the Alberta Teachers' Association (MCATA) executive and the U.S.-based National Council of Teachers of Mathematics (NCTM) regional conference planning committee, I continue to be struck by their dedication, commitment, work ethic and boundless energy. Again I refer to Maxwell's work, specifically to the law of magnetism and the law of the Big Mo.

The law of magnetism says that who you are is whom you attract. No wonder study groups are popping up all over our province, attendance at our symposia is increasing, our annual conference fills up early, MCATA has created a new executive position with the sole mandate of working on special projects and the NCTM has selected Alberta as the site of its next leadership conference. Teaching, motivating and learning are second nature to Alberta mathematics teachers, and that is being recognized by individuals and organizations from far and wide.

The Big Mo is the law of momentum, which helps us perform better than we ever thought we could. Against all odds, Alberta's mathematics teachers are inspired and inspiring. The mathematics horizon looks bright, trouble seems temporary and our obstacles are surmountable. Alberta mathematics leaders do not sit back and give up but, instead, look for ways to move forward and ensure success.

Thanks, Alberta mathematics leaders, for the following:

- Attending MCATA's annual conference in record numbers
- Working so thoughtfully at the symposium and on many special projects
- Trying to implement collaborative-learning environments in classrooms and on committees
- Believing in a constructivist philosophy of teaching and learning
- Volunteering on countless committees
- Planning an infinite number of meetings and inservices
- Burning the midnight oil for the sake of mathematical understanding
- "Growing leaders" wherever and whenever possible
- Never forgetting, even in the hectic pace of everyday life, why we became mathematics teachers
- Always putting students first
- Helping the "unmathematically inclined" to understand what the teaching and learning of mathematics is all about
- Although sometimes becoming discouraged, never giving up what we believe in
- Sticking together, through thick and through thin, and making MCATA stronger than ever

I hope that you enjoyed your summer and a well-deserved break!

Reference

Maxwell, J. C. *The 21 Irrefutable Laws of Leadership: Follow Them and People Will Follow You.* Nashville, Tenn.: Thomas Nelson, 1998.

Cynthia Ballheim

The Right Angle

Deanna Shostak

Learner Assessment Branch

Diploma Examinations

Diploma examinations for Pure Mathematics 30, Applied Mathematics 30 and Mathematics 33 will be administered in June and August 2003. August will be the last administration of the Mathematics 33 diploma examination.

Information Bulletins

Information bulletins provide teachers and students with information about the diploma examination in each subject. The bulletins for Pure Mathematics 30 and Applied Mathematics 30 are updated annually and are available at www.learning.gov.ab.ca/k_12/testing/diploma/bulletins/default.asp. Although teachers are encouraged to review a bulletin in its entirety, for quick reference a summary of changes is included at the beginning of each bulletin when major changes have been made.

Calculator Policy

The list of approved calculators and instructions for clearing calculators can be found at the back of the information bulletins or at www.learning.gov.ab.ca/k_12/testing/diploma/dip_gib/sec11_policy_2.asp. The list of approved calculators is updated annually. Note that the HP39 calculator will be removed from the approved list at the end of the 2002/03 school year. The Alberta Learning calculator policy applies to all mathematics and science diploma exams.

Learning Technologies Branch

The following mathematics resources have been developed by the Learning Technologies Branch (LTB) and are available from the Learning Resources Centre (LRC), 12360 142 Street NW, Edmonton T5L4X9; phone (780) 427-2767, fax (780) 422-9750, e-mail lrccustserv@gov.ab.ca; website www.lrc.learning.gov.ab.ca.

Secondary Resources

- *Applied Mathematics 30 Module Pack* (2002) (Product #474982)
- *Applied Mathematics 10: Companion CD* (CD-ROM v.1.0, Windows/Mac) (2000) (Product #431164)

The LTB is currently developing print resources for Mathematics 14, projected to be available in September 2003.

Elementary Resources

- Grade 3 Mathematics and Grade 6 Mathematics

With the completion of the Grade 3 Mathematics and the Grade 6 Mathematics, all the elementary mathematics courses are now completed.

A support resource containing interactive lessons for the key concepts of Grade 6 mathematics will soon be available on CD-ROM from the LRC. In addition to the concept lessons, this resource contains a glossary of mathematical terms, strategies for learning basic math facts, problem-solving strategies, notes to parents and printable activity sheets. This resource can also be accessed at www.learnalberta.ca/Math6/.

Curriculum Branch

The Learning Equation (TLE) 11 is a multimedia resource that addresses the outcomes of Pure Mathematics 20. TLE 11 has three components (courseware on CD-ROM, a teachers' manual and a student refresher) and has been jointly developed by Alberta Learning, CogniScience and Nelson. TLE 11 will soon be available in both French and English through the LRC and also through Nelson Thomson Learning.

Direct questions on the development of TLE 11 to Hugh Sanders, Project Management Coordinator, TLE Mathematics, Learning Technologies Branch, Alberta Learning; phone (780) 415-4504, e-mail hugh.sanders@gov.ab.ca.

Direct questions on the delivery and pricing of TLE 11 to Barbara Morrison, Senior Director and Lead Educator, TLE Math, Nelson; phone (403) 938-1945, voice mail 1-800-914-7776 (ext. 506), e-mail barbara.morrison@nelson.com.

In this section, we will share your points of view on teaching and learning mathematics and your responses to anything contained in delta-K. *We appreciate your interest and value the views of those who write.*

What Is a Good Mathematics Teacher? How Do We Find One?

John C. Egsgard

The following article is a summary of presentations and discussions I conducted in North America and Europe when I was president of the National Council of Teachers of Mathematics (NCTM). It seems to be appropriate for me to recall some of these ideas for our many young teachers of mathematics. More details can be found in the Proceedings of the Fourth International Congress on Mathematical Education *(Boston: Birkhäuser, 1983), pages 144–52.*

Good teachers of mathematics use their knowledge and love of mathematics, as well as their love and respect for their students, to lead these students to enjoy the study of mathematics.

I have said nothing about students being successful, for I believe that students who enjoy the study of mathematics will learn mathematics and will be successful. My definition also makes it very simple to determine whether or not someone is a good teacher of mathematics. It is only necessary to discover the attitude toward mathematics of the students of that person. My definition of good mathematics teachers indicates that there are two key elements that will lead students to enjoy the study of mathematics. One is the love and knowledge teachers have of mathematics. The other is the love and respect teachers show their students. In other words, a personal relationship must exist between teacher and student, founded on the mutual respect they show each other.

How do good teachers of mathematics show their love and respect for their students?

My experience is with high school students. In order to show these students that I loved and respected them, I tried to give each one as much individual attention as possible. To solve the dilemma of doing this in classes of varying size, I used the Socratic method of teaching, which involves questions and answers—the questions being asked by both teacher and students and answered wherever possible by the students. This enabled me to ask most, if not all, of the students one question each day. This question provided daily personal contact with each student. Ordinarily, half of the class time was given over to students working alone on problems. Again this allowed me the opportunity of personal contact as I went from desk to desk answering questions, praising good work, encouraging students and offering them hints. In addition, I was available before and after school to help students who had been away or who had difficulties that had not been cleared up during class time. In all of this, I found that praise and encouragement tended to help students try harder. Students want us to think that they are more than just students of mathematics, so I found that I could increase my personal relationship with students if I showed a genuine interest in their other activities—academic, cultural and athletic.

In the classroom, especially, the key to my effort to show respect was kindness—kindness to all students. Kindness is easy to give to someone who shows us respect. But kindness demands a true self-discipline on the part of teachers as they try to encourage students who appear to be unaffected by their efforts. Meanness, lack of work and rowdiness by the students must all be treated firmly but kindly. It is a slow and sometimes discouraging process trying to change students who dislike mathematics into being happy to be doing mathematics in one's class. Each year, I found students in my classes who only began to enjoy mathematics as the year was drawing to a close. The fact that these students did change gave me the confidence that I needed to continue working with all students no matter how hopeless it might have seemed.

Students Must Learn Mathematics, Too

Yet, all of the kindness and respect for students will not help unless the students find that they are learning mathematics, sometimes in spite of themselves. Indeed, successful learning and doing mathematics are absolute necessities for any student who is going to enjoy mathematics. For every time a student solves a mathematics problem successfully, especially if success seemed impossible to the student, that student has made a giant step on the road to the enjoyment of mathematics. For this to happen, the teacher has to teach so that students can learn. There are many mathematical ideas that are difficult in themselves, but good teachers of mathematics are able to peel away the unnecessary complications and present the core ideas so that these ideas can be understood. (This is why a solid mathematics background is so important.) Good mathematics teachers, through their questions and explanations, are able to help all of their students to understand and successfully do mathematics.

The previous discussion of good mathematics teachers leading students to enjoy the study of mathematics makes two assumptions: teachers have the necessary knowledge of mathematics, and teachers enjoy mathematics themselves.

What Is the Necessary Academic Knowledge for Good Teachers of Mathematics?

In general, as a minimum, teachers of mathematics must know all of the material that they will need to teach as well as the place of this material in the spectrum of the mathematics curriculum. This process of learning mathematics is never finished for good mathematics teachers because they must continually learn—for example, through reading the national and provincial mathematics education journals such as the Ontario Association for Mathematics Education's (OAME) *Ontario Mathematics Gazette* and the NCTM's *Mathematics Teacher, Teaching Children Mathematics* and *Mathematics Teaching in the Middle School;* by attending conferences of the OAME and the NCTM; and by taking courses to keep their knowledge up to date.

How Do We Find Good Mathematics Teachers?

Much is demanded of good mathematics teachers. It is the obligation of the profession itself to see that mathematics teachers receive as much assistance as possible in their growth to become better mathematics teachers. If organizations such as OAME and NCTM are to be helpful to teachers at all levels (elementary, secondary and postsecondary), teachers must lead the organizations from all levels. One of the gravest mistakes that organizations like these can make is to think that such representation is not needed. Mathematics education associations will be most helpful if the articles in their journals and the talks at their conferences are given by teacher experts who are communicating with people at their own level—namely, elementary, secondary or postsecondary. I believe that teachers learn best from teachers who are experts at the same level. All of the goodwill in the world cannot make up for the fundamental error made in assuming that elementary teachers learn best about teaching elementary mathematics from secondary or postsecondary teachers, or that postsecondary teachers can help secondary teachers in their classroom practice. Teachers at the postsecondary level can and must assist elementary and secondary teachers to increase their knowledge of mathematics. It is only rarely that postsecondary school teachers are able to give worthwhile practical help, useful in the classroom, to elementary and secondary school teachers. Peers must aid peers. The good elementary school teacher must become an apostle assisting other elementary school teachers. The good secondary school teacher must be willing to share with other secondary school teachers. The good postsecondary school teacher must find ways to show other postsecondary teachers that good teaching is important and possible at that level.

The question now remains—can one help a given individual become a good teacher of mathematics? First, let me make some statements that follow logically from my definition and my discussion on it:

- Only a person with the required knowledge of mathematics for level X should become a teacher at level X.
- Only people who are happy should become teachers of mathematics.
- Only people who enjoy other people and want to help other people should become teachers of mathematics.
- Only teachers who want to become mathematics teachers should become mathematics teachers.
- Only teachers who love mathematics should become teachers of mathematics.

Note that this is a list of necessary conditions and not of sufficient conditions for determining whether or not an individual could become a good teacher of mathematics. In other words, no one lacking one of the five qualities should become a teacher of mathematics. But it does not mean that everyone who has these five qualities will become a good teacher of mathematics.

I believe that the best way to help someone with the necessary conditions to become a good mathematics teacher is to assign that person as an apprentice to an outstanding teacher. Many students who graduate from teacher-training institutions in mathematics are not ready to become good teachers of mathematics. Indeed, many of these teachers give up after a short time in the classroom. Those who do go on to become good teachers of mathematics usually have had much assistance from their peers who are good teachers of mathematics.

The real learning about teaching and about whether or not one should become a teacher of mathematics comes in the classroom. I firmly believe that one learns to teach by doing, not by listening. An apprenticeship type of training for beginning teachers would require "experts" to whom they would be assigned in elementary or secondary schools. These "experts" would need to be successful teachers who are currently teaching in their own classroom each day. A maximum of three teacher trainees would be apprenticed to each of these teachers for a minimum of one year. As in other apprenticeship programs, the trainees would be paid a nominal salary.

As I envision the program, the expert teacher and the trainee would be responsible for the same classes. Ample time would be set aside in the school day for them to prepare classes together, to decide what is to be taught, how it is to be taught, the questions that would be asked of the pupils and so forth. Some days the teacher would teach all the classes, and on other days, the teaching would be shared with the trainee. Time would be spent daily to discuss both the teacher's and the trainee's classes so that the trainee could understand the why as well as the how.

Conclusion

I was a teacher of mathematics for 47 years. During that time, I had many students ask me why I had been content to teach the "simple" mathematics of the high school level when I could be challenged more by university-level mathematics. They openly wondered why it was not boring to teach the same mathematics content year in and year out. My reply has been a simple one: "I teach students rather than mathematics." Even though the mathematics did not change from year to year, my students did. Each year I had a new group of individuals to be with in my classroom. My years in the classroom were happy ones. I treated my students with respect, and they reciprocated. We all had much fun in my classroom, and we all learned. But my respect for the students was not enough for them. They needed to be successful. Somehow, in each of my classes, I was able to find something that every student could do well. I was fortunate in being able to do this in classes with four students and in classes with 45 students. The fact that my students learned, and that they enjoyed learning, brought me great joy.

Teaching students is difficult. Many people try it and cannot handle the one-to-one and many-to-one relationships that are part of teaching. Yet, too much emphasis is given to the difficulties of teaching. We must encourage our young teachers to look for more joy in their teaching. We must assure them that it is there. One of the reasons I think that apprenticeship is an essential part of teacher preparation is that it gives the teacher trainees an opportunity to share in the joy of teaching—a joy that some may believe does not exist.

As you proceed in your own teaching career, think about my definition, adjust it, change it, but think about it. Always remember the following: good teachers of mathematics use their knowledge and love of mathematics, as well as their love and respect for their students, to lead these students to enjoy the study of mathematics.

Reprinted with permission from Ontario Mathematics Gazette, *Volume 40, Number 3, March 2002, pages 9–13. Minor changes have been made to spelling and punctuation to fit ATA style.*

STUDENT CORNER

Communication is an important process standard in school mathematics; hence, the mathematics curriculum emphasizes the continued development of language and symbolism to communicate mathematical ideas. Communication includes regular opportunities to discuss mathematical ideas and to explain strategies and solutions using words, mathematical symbols, diagrams and graphs. Although all students need extensive experience in expressing mathematical ideas orally and in writing, some students may have the desire or should be encouraged by teachers to publish their work in journals.

delta-K *invites students to share their work with others beyond their classroom. Submissions could include papers on a mathematical topic, a mathematics project, an elegant solution to a mathematical problem, an interesting problem, an interesting discovery, a mathematical proof, a mathematical challenge, an alternative solution to a familiar problem, poetry about mathematics, a poster or anything of mathematical interest.*

Teachers are encouraged to review students' work prior to submission. Please attach a dated statement that permission is granted to the Mathematics Council of the Alberta Teachers' Association to publish the work in delta-K. *The student (or the parents if the student is under 18 years of age) must sign this statement, indicate the student's grade level and provide an address and telephone number.*

MCATA invites teachers of Pure Mathematics 30 or Applied Mathematics 30 to submit their best student projects. An independent panel will judge the projects. The students who submit the best two projects will each receive $50, and their work will be published in delta-K. *The students will also be acknowledged in the* Mathematics Council Newsletter. *Submissions must include both the project questions and answers. Students must submit both a hard copy of the project and a disk containing an electronic version. Entries must be accompanied with an application form that includes the student's name, home address and phone number; teacher's name; and school name and address. Students must sign a release form allowing the project to be published in* delta-K *and making it the property of MCATA. If the student is under 18 years of age, the parents must sign the release form. The project and the disk will be returned to the student along with complimentary copies of* delta-K. *The submission deadlines are February 15 for first-semester projects and July 15 for second-semester projects. Submission/release forms are available on the MCATA website (www.mathteachers.ab.ca) under Grants and Awards. Send submissions to Lorraine Taylor, Director of Awards and Grants, Mathematics Council of the Alberta Teachers' Association, 10 Heather Place, Lethbridge T1H 4L5; fax (403) 329-4572, e-mail lorraine.taylor@lethsd.ab.ca.*

No submissions were received for this issue. We look forward to receiving your submissions for the next issue.

NCTM Standards in Action
The Measurement Standard

Klaus Puhlmann

An underlying view of mathematics education expressed in the National Council of Teachers of Mathematics (NCTM 2000) *Principles and Standards for School Mathematics* is that students should be actively involved both mentally and physically in constructing their own mathematical knowledge.

Principles and Standards for School Mathematics identifies 10 standards that form an essential and comprehensive foundation for students from Kindergarten to Grade 12 as they learn mathematics. The standards are divided into five content standards (number and operations, algebra, geometry, measurement, and data analysis and probability) and five process standards (problem solving, reasoning and proof, communication, connections and representation). This article focuses on one content standard: measurement.

The Alberta program of studies for K–12 mathematics contains a measurement strand that emphasizes describing and comparing everyday phenomena using direct or indirect measurement. At the Kindergarten level, the general outcome focuses on demonstrating awareness of measurement, whereas the general outcomes at the elementary level include estimating, measuring and comparing in different units of measure, and solving problems using measurement concepts and various tools. At the secondary level, the measurement strand includes problem solving involving properties of circles and their connections to angles and time zones. At the start of the secondary level, students are introduced to indirect-measurement procedures and use of trigonometric ratios to solve problems involving right triangles. In the final years of the measurement strand, the general outcomes include demonstrating and understanding scale factors and their interrelationship with dimensions of similar shapes and objects. Furthermore, students are exposed to problems involving triangles (including three- and two-dimensional applications); the use of measuring devices to perform calculations in solving problems and analyzing objects and shapes; and processes to solve cost-design problems, coordinate-geometry problems, polygons and vectors, and conic-section problems.

The measurement standard is of central importance to the teaching and learning of mathematics in Alberta. It helps students to see that mathematics is useful in everyday life, and it is the basis for developing many mathematical concepts and skills. The measurement strand also leads students to naturally recognize the need for learning about fractions and decimals. Because measurement is a building block in the learning of mathematics, students must understand the attributes to be measured as well as what it means to measure. For such understanding to occur, students must first experience a variety of activities that focus on comparing objects directly, covering them with various units and counting the units. In short, students must have a variety of qualitative-measurement experiences. Teachers must guard against the premature introduction and use of instruments and formulas when teaching students the measurement strand. The qualitative-measurement experiences should include awareness of the size of units as well as situations that require estimates. Students should become aware that measurements, rather than being exact, are approximations, which later leads students to discuss and estimate the error of a measurement. Eventually, students encounter measurement ideas both in and out of school, in such areas as geometry, architecture, art, building and design, cooking, sports and map-reading. As students mature and become more conscious about the world in which they live, they develop a natural desire to explore the world and engage in measuring objects.

Geometry is exceptionally rich in opportunities for students to be involved not only in measurement activities but also in activities related to other content and process standards. For example, tangram activities reinforce the idea of conservation of area. Not only young students find it difficult to accept that a square and a triangle can have the same area. A geoboard or geopaper problem can be used to explore this idea. Students may want to cut apart each figure created on geopaper and rearrange the pieces to form a rectangular area to verify their solutions. Pentominoes with areas of five squares can be used in enjoyable and worthwhile activities linking visualization and measurement. Students in higher grades can explore the Pythagorean relationship using triangles on geopaper. Drawing the triangles on geopaper allows students to investigate quickly and accurately the squares on the sides of several triangles. This investigation, in turn, allows them to draw conclusions and make generalizations based on several examples rather than just the two or three that traditionally appear in textbooks. Nonright triangles can also be explored to determine if the property holds. Students should be encouraged to ponder such questions as, Is there another way? and, What would happen if . . . ? For example, What would happen if we used isosceles triangles instead of right triangles? What if we drew semicircles on the sides instead of squares? What if we drew equilateral triangles (or other regular polygons) instead of squares?

Geometry and measurement are interconnected and support each other in many ways. The concept of similarity, for example, can be used in indirect measurement, and the perimeter and area of irregular figures can be determined using line segments and squares, respectively.

Measurement activities involve the handling of concrete objects and materials, which is helpful in gaining a deeper understanding of the concepts and skills related to measurement. As students advance through the grades, measurement concepts become more sophisticated and complex. Whatever the grade level, students should have many informal experiences in understanding measurable attributes before using tools or formulas to measure them. Measurable attributes, of course, increase in complexity as the grade level increases, as does the relationship between attributes. For example, cutting up a shape and rearranging its pieces may change the perimeter, but it will not affect the area.

The selection of different units to measure different attributes is sometimes difficult for students in the lower grades to understand. Hence, learning to choose an appropriate unit is an important part of understanding measurement. For example, the perimeter of a geometric shape is a linear measure, but the area of that shape is measured in square units and the volume is measured in three-dimensional or cubic units. Knowing how to select appropriate linear measures is also important. For example, measuring the distance from Edmonton to Calgary in centimetres would not be appropriate.

The application of appropriate techniques, tools and formulas to determine measurements is an important part of the measurement strand. Students must become proficient in the use of tapes, rulers, scales and clocks. Using formulas to measure an attribute is introduced as early as the elementary level and further developed in the later grades. Estimating as a legitimate measurement technique is also developed throughout the school years.

Measurement is one of the most widely used applications of mathematics. It helps to connect ideas within mathematics and ideas between mathematics and other disciplines. Measurement concepts and skills can be developed in the context of other strands and need not be taught as a separate unit of study. Other mathematics strands, science, art and technical courses provide natural opportunities to use and understand measurement in real-life and realistic contexts.

The four articles that follow deal with measurement in various ways and contexts and offer many practical suggestions for teachers. "Teaching Geometry and Measurement Through Literature" identifies texts that can support mathematical thinking. "How Big Is Your Foot?" is a hands-on discovery project that aims at improving students' understanding of measurement and showing that measurement is meaningful in the real world. "A Dynamic Way to Teach Angle and Angle Measure" addresses elementary students' lack of understanding of and attention to the concept of angle. "Let's Do It: Measurement for the Times" discusses measurement and multiplication using many practical applications.

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Teaching Geometry and Measurement Through Literature

Sara Delano Moore and William P. Bintz

As teacher educators, our work involves helping novice and experienced middle school teachers integrate literature in the mathematics classroom. We have noticed a recurring dilemma that can best be described through the voice of a composite middle school mathematics teacher:

> I'm really excited about the possibilities of using literature to teach math in my class. And yet, I'd be less than honest if I didn't say that the idea makes me feel a little uncomfortable. After all, I'm a math teacher. I haven't had any training or experience in this area, so I don't really know what literature is out there to teach math and where to find it.

Simply stated, mathematics teachers know mathematics but often have little knowledge of and experience with using literature to teach mathematics.

This article helps resolve this dilemma by making connections between literature and mathematics. We have three purposes: (1) to provide examples of learning experiences that use literature to teach aspects of geometry and measurement, (2) to discuss instructional implications of these experiences in the contexts of Van Hiele theory and differentiated instruction and (3) to identify and organize a variety of literature that has "text potential" (Burke 1995) for supporting mathematical thinking.

Looking Inside a Classroom

Teaching geometry in the middle grades traditionally includes many manipulative activities. Students use geoboards, pattern blocks, tangrams and other tools to explore such geometric concepts as properties of figures, area, perimeter and tessellations. In a literature-based geometry unit, stories can provide meaningful contexts for these experiences. For example, a common activity is to ask students to fit various shapes together in tiled patterns. The book *A Cloak for the Dreamer* (Friedman 1994) provides a context for this exploration.

This story involves a tailor and his three sons. The two older sons enjoyed working with their father; the youngest son dreamed of travelling the world. One day, the archduke ordered the tailor to make him three new cloaks for an important journey. The tailor in turn asked each son to sew a different cloak and to make sure that each garment would protect the archduke from the wind and rain. One son used the pattern of bricks on the floor to make a cloak by sewing rectangles together. Another son made a cloak by stitching triangles together. The youngest, however, cut circles to represent places he would love to visit and sewed his circles together into a cloak. When the sons showed the cloaks to their father, the youngest realized that his cloak was useless—it was full of open space. That night, the tailor and his two older sons snipped the circles into hexagons and sewed the hexagons together into a beautiful cloak for the youngest son, who used it to protect himself from the wind and rain as he set off to travel the world. This story illustrates that the successful completion of a cloak depends on the properties of the shape chosen to construct the cloak. The shape chosen for the cloak must fit together precisely; it must not contain open space, or the cloak will not protect its wearer. Students can use tools such as pattern blocks or attribute pieces to explore which shapes can be used to make a cloak and explain why some shapes tile and others do not.

The teacher could also extend this study of geometric figures and their properties by using the process of cutting circles into regular hexagons as a link to compass and straightedge constructions. After reading the story, ask students, "What mathematical skills do the father and his sons need to create accurate regular hexagons from the circles? What other shapes could they have made?"

Using Literature to Support Mathematical Thinking

As we read, we construct meaning from the text. The meaning that we create depends on many factors: our background knowledge, our purpose for reading and the social context in which the reading occurs. As a family story, *A Cloak for the Dreamer* emphasizes supporting each child in his or her chosen path in life. When this book is read in the context of a mathematics class, it can become a launching point for investigating geometric figures and their properties. We encourage teachers to look for important mathematics in each text they explore, because the mathematics will not always be obvious.

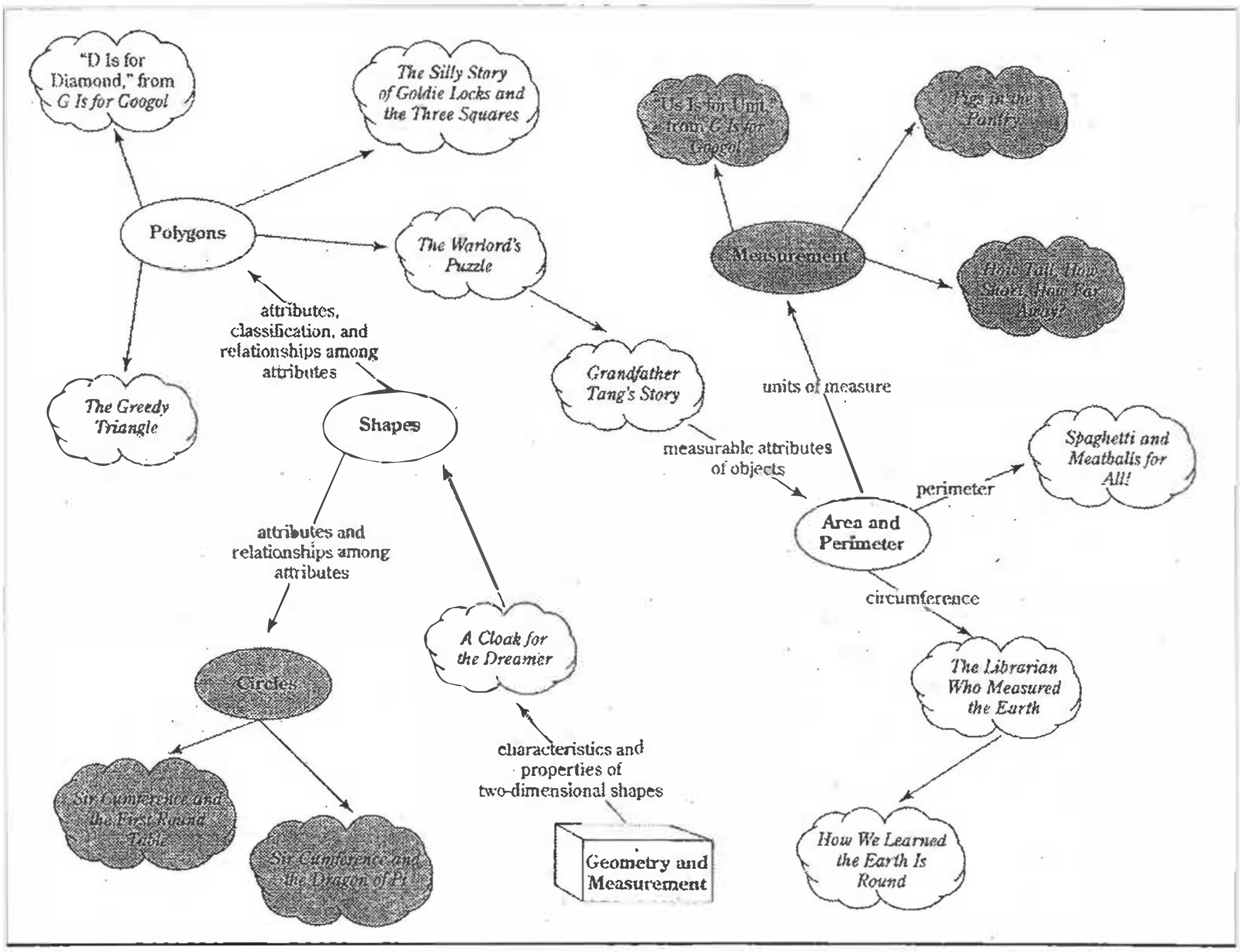
Figure 1 highlights clusters of literature that have potential for teaching concepts in geometry and measurement: circles, polygons, area and perimeter, and measurement. For simplicity's sake, we describe these clusters in a linear way; later, however, we discuss these clusters in terms of their possibilities for teachers to differentiate instruction.

The clusters in Figure 1 provide a framework for studying this facet of the National Council of Teachers of Mathematics (NCTM 2000, 233) geometry standard: "Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships." *A Cloak for the Dreamer* is a good starting point to introduce these concepts.

Circles and Polygons

In studying shapes, Figure 1 includes two clusters, one focusing on circles and the other on polygons. The cluster dealing with circles includes two interrelated texts. *Sir Cumference and the First Round Table: A Math Adventure* (Neuschwander and Geehan 1997) introduces basic concepts: parts of a circle and the terms used to label them. *Sir Cumference and the Dragon of Pi: A Math Adventure* (Neuschwander 1997) extends further the study of the properties of circles by using pi to show relationships between the radius, diameter and circumference of circles.

Figure 1
Literature Related to Shapes and Measurement

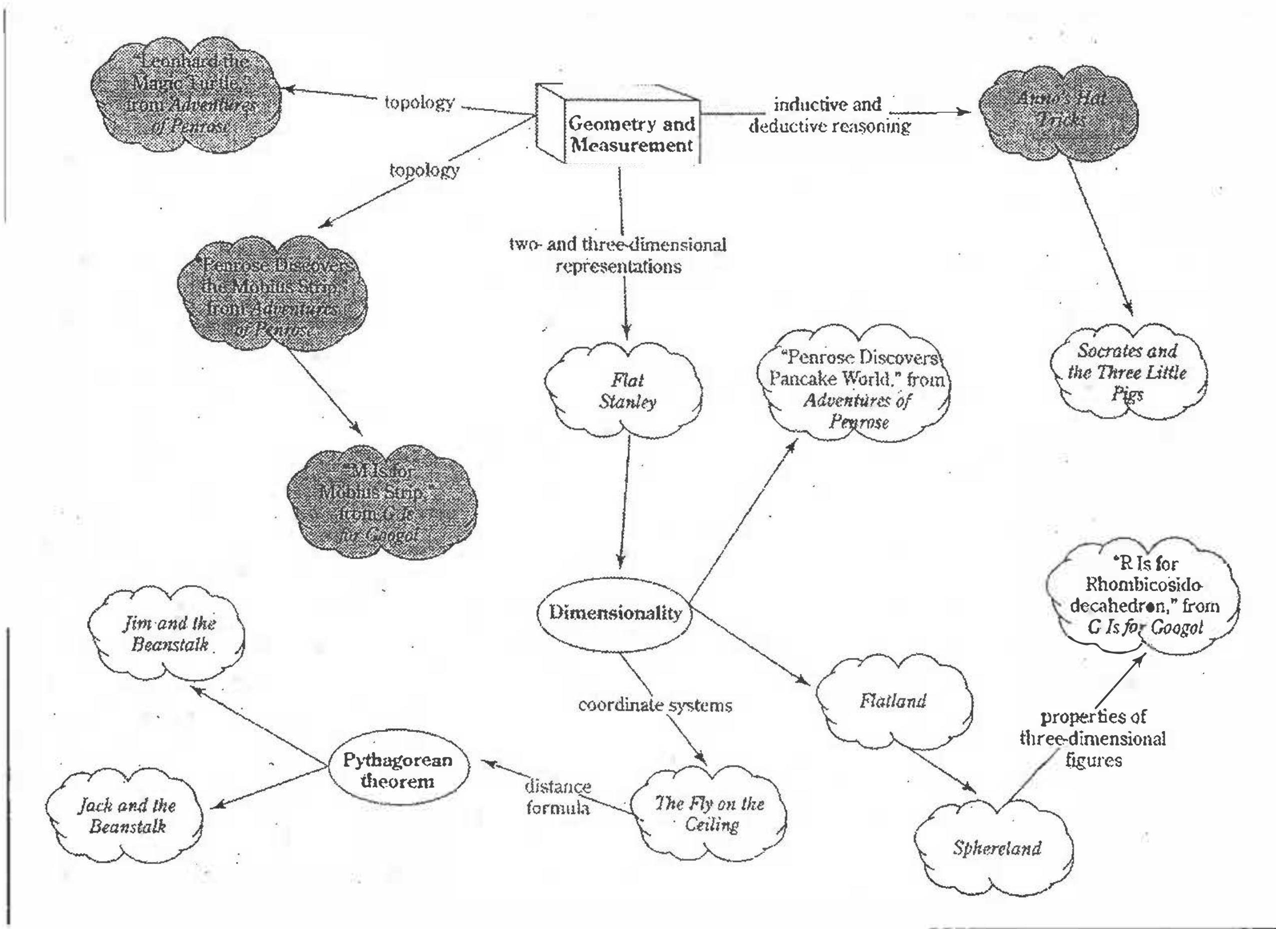


In addition, studying shapes can lead to the study of polygons. The cluster dealing with polygons relates to *The Greedy Triangle* (Burns 1994). This delightful book identifies and names a progression of polygons (triangle, quadrilateral, pentagon, hexagon and so on). If students have geoboards at their desks, they can follow along with the story. Defining properties of shapes can be discussed as students compare and describe the various triangles, quadrilaterals or other shapes that they make. "D Is for Diamond" (Schwartz 1998a), from the book *G Is for Googol,* identifies different properties of polygons (for example, one attribute of a square is that all four sides are congruent) and reinforces distinctions between colloquial names for figures (for example, *diamond*) and proper mathematical terms (for example, *rhombus*). *The Silly Story of Goldie Locks and the Three Squares* (MacCarone 1996) addresses relationships between properties (for example, the more sides to a regular polygon, the larger each interior angle).

Of course, some shapes tile a plane and some do not. *A Cloak for the Dreamer* describes shapes that do tessellate, or tile. *The Warlord's Puzzle* (Pilegard 2000) creates similar patterns (tiling with no gaps and no overlaps) but uses multiple shapes rather than a single, repeated shape. In this story, the challenge is to find out how to combine a variety of different pieces (specifically tangram pieces) to re-create a broken square tile. This book can be followed by *Grandfather Tang's Story* (Tompert 1997), in which the same pieces of a tangram set are used to create different figures. One relationship that we can explore is the relative areas of the pieces of the tangram set. If the small triangle represents one unit of area, students can describe the areas of the other shapes in the set and the figure that they create with some or all of the shapes in terms of this unit. This explanation provides a smooth transition to measurement.

Figure 2

Literature Related to Reasoning, Topology and Dimensionality



Area and Perimeter

In this cluster, we consider three concepts related to measurement: area, perimeter and circumference, and units of measure. Units of measure are particularly important because middle grades students do not always choose appropriate units for a particular measurement task. Readings such as "U Is for Unit" (Schwartz 1998d), from *G is for Googol, Pigs in the Pantry* (Axelrod 1997) and *How Tall, How Short, How Far Away?* (Adler 1999) help students explore units of measure—mass, capacity, length—appropriate for various contexts. These texts can also connect to science through measurement tools, such as a balance scale or a graduated cylinder, used in the science lab.

Spaghetti and Meatballs for All! (Burns 1997) extends students' mathematical thinking about the topics of area and perimeter by illustrating and describing how square units can be used to measure a shape's surface area. It also describes how to use units of length to determine the distance around tables of different shapes to seat varying numbers of dinner guests.

Finally, *The Librarian Who Measured the Earth* (Lasky 1994) and *How We Learned the Earth Is Round* (Lauber 1994) explore circumference—in particular, the challenge of finding the distance around the earth. These stories connect to science, as the earth's circumference is represented as the equator and prime meridian. They also connect to social studies, since the contributions of the ancient Greeks are described in part as we learn how Eratosthenes determined the circumference of the earth with astonishing accuracy.

Still More Possibilities in Geometry and Measurement

Figure 1 illustrates the literature that can tie the study of geometry and measurement to shapes and measurement. Figure 2 illustrates other texts that can help explore other facets of geometry and measurement: inductive and deductive reasoning, topology, two- and three-dimensional representations, and properties of figures.

Inductive and Deductive Reasoning

The NCTM's (2000, 232) geometry standard states that instructional programs should enable students to "develop mathematical arguments about geometric relationships." Specifically, middle grades students should "create and critique inductive and deductive arguments concerning geometric ideas and relationships, such as congruence, similarity, and the Pythagorean relationship" (p. 232). *Anno's Hat Tricks* (Anno and Nozaki 1985) can start a discussion on reasoning by introducing students to binary logic as they analyze their hat colour by looking at others' hats. *Socrates and the Three Little Pigs* (Anno 1986) extends this discussion of logical reasoning with more complex problem-solving situations. Students can critique the reasoning of the wolf, Socrates, as he attempts to find the three little pigs hiding in various houses. Although the fundamental mathematics involves combinations and permutations, teachers can frame questions so that students develop skills associated with "If . . . , then . . ." deductive reasoning and explore all possible cases of a given situation. These activities will be useful in developing formal proof skills in high school mathematics classes.

Topology

Although topology is not explicitly part of the middle school curricula, it is an engaging facet of mathematics for middle school students. A good place to start discussions about topology is "Leonhard the Magic Turtle" (Pappas 1993a). In this story, Leonhard attempts to traverse the seven bridges of Königsberg. Presented in story form, this classic problem is an effective introduction to network theory.

Similarly, the selection "Penrose Discovers the Möbius Strip" (Pappas 1993b) introduces the exploration of a single-surface Möbius strip. "M Is for Möbius Strip" (Schwartz 1998b) provides a more formal mathematical discussion and activity suggestions.

Two- and Three-Dimensional Representations

A number of texts explore properties of two- and three-dimensional figures and make connections to the coordinate plane. In *Flat Stanley* (Brown 1992), a little boy wakes up one morning to discover that his bulletin board fell on his bed during the night and that he is flat. The story describes the effect of Stanley's being two-dimensional on various events of his day. For example, Stanley poses unnoticed inside a picture frame to help catch an art thief. This story introduces students to the differences between two- and three-dimensional representations. For students who might need a simpler introduction, "Penrose Discovers Pancake World" (Pappas 1997) covers comparable ground. For students who are ready for a more sophisticated discussion, *Flatland* (Abbott 1992) and its "cousin" *Sphereland* (Abbott, Burger and Asimov 1994) not only provide a good mathematical foundation but also can serve as an opportunity to connect to social studies through a discussion of Victorian society, particularly the social status of men and women. If a teacher wishes to extend *Sphereland* into a discussion of properties of three-dimensional figures, "R Is for Rhombicosidodecahedron" (Schwartz 1998c) introduces Euler's formula, which describes the relationship between faces, edges and vertices of a solid figure.

Finally, a two-dimensional world is explored further in *The Fly on the Ceiling* (Glass 1998). This entertaining piece of historical fiction hypothesizes about the circumstances under which Descartes developed his system of coordinates. If a teacher wishes to make connections to measurement, the distance formula serves as a link between the Pythagorean theorem and the coordinate plane. Both *Jack and the Beanstalk* (Kellogg 1991) and *Jim and the Beanstalk* (Briggs 1970) discuss situations in which these concepts can be applied. For example, after retelling *Jack and the Beanstalk,* students may determine the length of ladder needed to reach a golden egg resting at a particular height on the beanstalk. Although these explorations happen in mathematics class, the language arts teacher could talk about folklore, exaggeration in folktales and the balance necessary between reality and fantasy inherent in an effective story.

Using Literature to Differentiate Instruction

We do not expect teachers to use these texts exclusively or to work through these texts in the linear fashion in which we have described them. Rather, we offer this model as a way to organize a collection of texts about related mathematical ideas so that teachers can use literature to make mathematics more engaging and relevant to their students.

Van Hiele Theory

The work of Pierre van Hiele and Dina van Hiele-Geldof describes a sequential and hierarchical series of stages through which students progress as they learn about shapes, properties and relationships between properties. One aspect of the theory is that progression through the stages relates more closely to instruction than age or cognitive maturation (Van Hiele 1984, 1986; Van Hiele-Geldof 1984). The first cluster of books described in Figure 1, particularly the polygons group, is an excellent starter set to help students move from an early level where shapes are recognized holistically ("I know that's a rectangle because it is the same shape as the [rectangular] door") to recognition of properties ("A square has four right angles, four congruent sides, two pairs of parallel sides and so on"). These experiences can also support students as they learn to recognize relationships between properties ("If both pairs of opposite sides in a quadrilateral are parallel, then opposite sides are congruent"), the level that best supports success in high school geometry classes (Usiskin and Senk 1990).

Differentiated Instruction

A perennial challenge to middle school teachers is the fact that "one size fits all" instruction is not effective. Students come to mathematics classes perhaps grouped by skill in computation or algebra, but they range widely in their reading ability and knowledge of geometry and measurement. Differentiated instruction is the practice of creating related learning experiences in which students can be taught content matched to their instructional level (Tomlinson 1999). The texts about polygons mentioned earlier can create related learning experiences that are responsive to students at varying Van Hiele levels. For example, *The Greedy Triangle* can be discussed in terms of naming shapes, identifying properties of shapes or exploring relationships between properties by comparing shapes. The teacher could group students for these discussions depending on their levels of geometric understanding. The texts about two-dimensional worlds ("Penrose Discovers Pancake World," *Flat Stanley* and *Flatland*) can be used so that students of varying reading levels can all explore two-dimensional reality through literature. Each student would read the text that is most closely related to his or her reading level, then the students could work in groups across texts to complete tasks related to two- and three-dimensional representations. For example, teams could be given various two-dimensional drawings of a figure seen from different sides and use unit cubes to construct the three-dimensional figure that is consistent with the drawings.

Conclusion

In this article, our goal was to identify texts that can potentially support mathematical thinking. To this end, we illustrated and described two sets of literature that can be used to teach various aspects of geometry and measurement. We also discussed the connection between these texts and Van Hiele theory and shared examples of implementing these experiences in a differentiated classroom. Our hope is that this article will start new conversations and create new possibilities for using literature to teach mathematics.

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Twenty men did ¼ of a job in eight days. Then, because of an emergency, it became necessary to complete the job in the next five days. How many additional men were added to the crew of 20 to accomplish this?

How Big Is Your Foot?

Suzanne Levin Weinberg

Concepts relating to fractions and measurement are difficult for students in the upper elementary and middle school grades to grasp (Bright and Heoffner 1993; Coburn and Shulte 1986; Levin 1998; Thompson and Van de Walle 1985; Thompson 1994; Witherspoon 1993). As a first-year teacher, I learned the value of relating difficult concepts, especially abstract concepts, to students' real-world experiences. The "How Big Is Your Foot?" project grew out of a question that I asked my Grade 8 students during my first year of teaching. We had just finished studying conversions in the metric system and had begun working with conversions in the customary system. As a warm-up question, I asked my students to describe the distance from my desk to the door of the classroom. I wrote their responses on the chalkboard as they called out estimates: 1 m, 60 m, 25 ft., 300 in., 300 cm. The students did not seem to have any grasp of the length of a metre or an inch. One of my more boisterous students got out of her desk and walked the distance heel to toe; the class counted with her and agreed that 9 "feet" was a fair estimate. I then asked the class whether the distance was greater or less than 9 ft. When no clear consensus was reached, I postponed my planned lesson and launched this activity involving comparisons of standard and nonstandard measurements.

Measure Your Foot

I asked each student to take out a pencil and two pieces of notebook paper. I then instructed the students to take off their shoes and determine which of their two feet was longer. Students traced the outlines of their longer feet, then cut their traces out carefully. My students quickly commented on how big or small their classmates' feet were. I asked students to form pairs. Each person was to determine distances using his or her own traced foot as the unit of measure. Partners were instructed to do the measuring. In other words, to determine the height of Person A, Person B would use Person A's traced foot to measure Person A's height. Then the roles would switch. I asked for the following measurements in the students' own "feet":

- Vertical height without shoes
- Arm span: tip of the third finger to tip of the opposite third finger when the arms are held out as far as possible and parallel to the floor
- Fingertip to shoulder: tip of the third finger to the shoulder when the arm is held out as far as possible and parallel to the floor
- Length of head: distance from chin to top of head
- Wrist to elbow
- Hand span: distance from pinky to thumb when the hand is stretched out as far as possible

The measurements that I requested were not random. The ratios of distances in the human body are well documented in anatomy books and work for most students as young as Kindergarten age. These anatomical relationships were examined and painstakingly sketched by Leonardo da Vinci in the late 1400s. I remembered these relationships from a design class that I had taken in college.

All but the hand-span measurement have the following relationships to the length of the foot: vertical height is 7 "feet"; arm span is 7 "feet"; fingertip to shoulder is 2.5–2.75 "feet," depending on the build of the body; length of head is 1 "foot"; and wrist to elbow on the inside of the arm is 1 "foot." The hand-span measurement has no predetermined relationship to the foot length because the span is affected by stretching of the fingers, such as when students play the piano. I included the hand-span measure because I wanted students to recognize that not every measurement on the human body is genetically determined. For middle school students, who are extremely concerned about their appearance, this knowledge of body proportions can lead to important discussions.

Measure Using Your Foot as the Unit

After giving the assignment, I watched as students groped with issues relating to the measurement tasks: "Should we round up or down? May we use fractions? Where do I start measuring the head? Do we measure the inside or outside of the arm?" One creative pair, who had made dots on the chalkboard to mark Person B's fingertips, complained, "It doesn't match!" The measurement on the chalkboard did not exactly match the measured distance when Person A measured Person B's arms and back. When everyone had finished measuring, we discussed some of these issues. I explained that, in measurement, mathematicians use certain conventions, such as starting at 0 rather than 1, and that the units we use help to eliminate some of the confusion. For this activity, we might label a measurement as a Beth-foot or a Robert-foot, and we would understand that 1 Beth-foot might not be equal in length to 1 Robert-foot.

With other issues for which no rule seemed to exist, the students explained and defended their choices. For example, one student argued that measuring the height of the head was not the same as measuring the distance from the top of the head to the chin because the first was an "up and down" measurement, whereas the other was a "perimeter" measurement. A quick consensus was reached to use the "up and down" measurement, and several students measured their partners' heads again. A more contentious problem, the question of how to measure the length between the wrist and the elbow, provoked heated discussion. Some students claimed that the task was "understood" to mean the inside of the arm; others countered that they did not understand this constraint at all. One student explained that, because we always want a large measurement, we should measure the outside of the arm. Another noticed that the distance changes when the wrist is bent. Still another student thought that averaging the length of the inside and outside of the arm would produce a fair result. Finally, a student who almost never participated suggested labelling the measurement "the inside of the arm from wrist to elbow." The class applauded, and we moved on.

As the discussion continued, I asked what the difference was between estimated and exact measurements. Some of the students said that a rounded measurement was an estimate. Others claimed that no measurement could be perfectly exact because humans are not exact. The class listed reasons for error when we cut out our "feet": (1) a person might have moved his or her foot during tracing, (2) the fingers that were used to hold the pencil and mark the paper had width, (3) the pencil point might not be sharp when drawing and a person might slant the pencil instead of holding it in perfect vertical alignment, and (4) cutting exactly on a line is difficult with scissors.

Unit Conversion

The next part of the activity asked students to compare their measurements to see if they could find any common answers or answers that were very nearly the same. Some students already knew that the arm span is nearly the same length as a person's height. Many did not know, however, that other relationships exist involving the foot as a unit of measure. As the students continued to discuss their results, they noticed that even though nonstandard measurements had been used, the ratios appeared to be about the same for everyone! With my help, they were able to express the relationships in algebraic terms. For example, 7 × head = height and 2.5 × wrist to elbow = fingertip to shoulder.

I then extended the lesson by asking students to predict the length of their own feet in both inches and centimetres. After students made their predictions, I handed out rulers with both centimetre and inch markings and instructed students to measure the length of their feet. At this point, several students, referring to the original question asked at the beginning of the lesson, called out, "It [the distance to the door] is less than 9 ft.!" When I asked students to tell me how they could determine, without measuring, the length in centimetres from fingertip to shoulder, nearly every hand went up. One student spoke for her classmates: "If 2.5 'feet' are needed and if my foot is 22 cm, I just multiply 2.5 by 22; 22 cm for each of my feet and 11 cm for half my foot." Her partner then used the ruler to determine that the first student's fingertip-to-shoulder measurement was 65 cm. When I asked whether the girl had overestimated or underestimated the length using her own foot, the class answered immediately that she had slightly underestimated.

I finally asked students to look at the lengths of their feet and describe objects in the classroom that were approximately 1 "foot" in length, 2 "feet" in length and so on. We then compared the longest "foot" and the shortest "foot" measurements with the measurements using a standard foot. I closed the lesson by asking students to estimate the length of the chalkboard in standard feet. Students wrote their estimates, then measured using rulers. This time, the answers ranged from 9 ft. to 15 ft. The actual length of the chalkboard was 12 ft.

That day, my students discovered that measurement is inexact. By paying close attention to the size of the unit, however, my students learned that making reasonable estimates is not difficult. Nonstandard units can help us estimate the lengths of objects in inches, centimetres, feet and so on, by giving a frame of reference. The students enjoyed this lesson and mentioned it frequently in subsequent weeks, especially when we revisited this idea by comparing 1 kg with the weight of two medium-sized apples, 1 g with the weight of a paper clip, a small bowl of cereal with the weight of about 30 g and the liquid in four large coffee mugs with the approximate capacity of 1 L.

Adapting the Project for Different Audiences

The first time that I presented this project, I had not carefully planned out what I would say or have the students do. As I wrote out the project for students in successive years, I changed the language that I used to reflect the difference between the student's foot length and the length of a standard foot. To emphasize the fact that every student's foot was different, I asked students to refer to the person's name followed by the words *shoe-unit.* Hence, Brian's body would measure about 7 Brian-shoe-units in length, and Tanisha's body would measure about 7 Tanisha-shoe-units in length.

Renaming the units was a cosmetic change to eliminate confusion. As I thought about this project, I wanted to make some conceptual additions that would connect with the mathematics that students had seen earlier that year and in previous classes. I wanted to connect work on ratios and proportions with measurements through the application of scale. I also wanted the students to do more of the activity on their own. Figure 1 shows the final form of the project.

Figure 1
Final Format of Project for Middle School Students

How Big Is Your Foot?

On a piece of paper, trace the outline of your largest foot *without your shoe on.* The heel should just touch the bottom edge of the paper. Now, cut out this outline of your foot. The length of this foot will now be used as a nonstandard measure. Name your nonstandard unit in the following way: <your name>-shoe-unit. For example, the length of Chad's foot will now be called a Chad-shoe-unit. Use your shoe-unit to measure yourself. For example, Chad would measure himself in Chad-shoe-units. Chad's partner would *not* use Chad-shoe-units. In pairs, complete the following information:

Name of Person A ______________	Name of Person B ______________
Person A will be measured in ______________-shoe-units.	Person B will be measured in ______________-shoe-units.
➢ I am ______________-shoe-units tall.	➢ I am ______________-shoe-units tall.
➢ I measure______________-shoe-units from the tip of my longest finger to my shoulder.	➢ I measure______________-shoe-units from the tip of my longest finger to my shoulder.
➢ I measure______________-shoe-units from the top to the bottom of my head.	➢ I measure______________-shoe-units from the top to the bottom of my head.
➢ I measure ______________-shoe-units from the tips of my fingers on my left hand to the tips of my fingers on my right hand when I hold my arms out parallel to the floor.	➢ I measure ______________-shoe-units from the tips of my fingers on my left hand to the tips of my fingers on my right hand when I hold my arms out parallel to the floor.
➢ I measure ______________-shoe-units from the tip of my thumb to the tip of my pinky when my hand is opened as far as possible.	➢ I measure ______________-shoe-units from the tip of my thumb to the tip of my pinky when my hand is opened as far as possible.
➢ I measure ______________-shoe-units from my wrist to my elbow.	➢ I measure ______________-shoe-units from my wrist to my elbow.

Complete the following:

- Write at least three observations that you made while measuring.
- Write any questions that you faced while doing this activity.
- If you came up with an answer to your question, write it and say how you came up with this answer.
- Compare answers with several other pairs. Do you notice any similarities or any differences for a specific measurement?

Estimate the length of your shoe-unit in inches and centimetres, then use a ruler to measure your foot. Describe how you can determine the lengths in the chart above in inches and centimetres without actually measuring.

Another change that I made was to incorporate the use of calculators to verify the estimates and measurement conversions. Although the computation is not difficult in this project, having calculators available allowed many students to complete the basic questions quickly and gave the class more time to examine the conceptual questions. Because of time constraints, I do not currently include spreadsheets with this project; however, one natural extension might be to have middle school students use spreadsheets to compare the average length of different measurements on the body in terms of "feet." These averages could then be displayed using bar graphs.

I have had the opportunity to do this project with more than 400 middle school students, more than 200 undergraduate elementary education students and more than 30 inservice teachers. Although the initial activity for preservice and inservice teachers is essentially the same as the student activity, additional questions are included to examine the prerequisite knowledge needed for successful completion of the project, and suggestions are given for adapting this activity for students who have special needs.

The preservice and inservice teachers recognized that all students need to go through a process of learning measurement. Teachers need to guide their students through this process by addressing the following prerequisite skills for this project, which involve

- a basic understanding of how to measure (that is, a single unit is placed end to end with no space and no overlap);
- gross- and fine-motor skills, including the abilities to trace and cut;
- an understanding of how to record data in a table;
- an understanding of how to find the arithmetic mean, or average, of a set of data, including knowing how to add and divide whole numbers;
- the ability to identify a ratio as a comparison of two related quantities or numbers;
- the ability to write and solve proportions, including in whole-number multiplication and division;
- an understanding of how to measure to the nearest inch and centimetre;
- an ability to make a reasonable estimate for linear measurement; and
- an awareness of how to interpret a scale measure.

If students have not attained these prerequisite skills, then this activity can easily be modified to include instruction in any of the mathematical topics listed above. Additionally, this activity can be used as a preassessment to gauge how well students understand these topics.

When asked how they would adapt this activity for students who have special needs, the preservice and inservice teachers made the following suggestions:

- For students who have vision impairments, use a rough piece of paper, such as sandpaper, to enable students to measure the length.
- For students who have vision impairments, let the partner say "hot" or "cold" to refer to how close to the end of the "foot" the student's finger is.
- Allow students who have hearing impairments to measure after their partners do, so that they can follow the example of the first partner. This same suggestion was made for students who do not speak English.
- Also for students who have hearing impairments, write the findings on the chalkboard as the discussion continues.
- For students who tend to need more time, make one or more of the measurements that are one "foot" long (such as the wrist to elbow or the height of the face) optional. Have students who tend to finish quickly estimate then measure other body parts, such as the distance from knee to ankle or from neck to waist.

Preservice and inservice teachers suggested tying this project to reading map scales in geography, because the notion of converting between one measurement system and another is related to this concept. Another adaptation suggested by the inservice teachers is to read *How Big Is a Foot?* (Myller 1962). This story illustrates why we need standard measurements. The teachers agreed that more emphasis should be placed on helping students write algebraic equivalents involving two lengths, such as $7 \times$ foot length $\cong$ arm span. Finally, the teachers suggested that students find out more about Leonardo da Vinci, who identified these and other anatomical relationships over 500 years ago. Students might be interested to learn that Da Vinci wrote his findings backward so that no one would steal them and that he drew more than 1,000 sketches relating to architecture, anatomy, maps, nature and art. In his sketches and artwork, he successfully used perspective, which requires the application of proportional reasoning and ratios.

What Have I Learned?

This project began in response to students' inability to reason about standard and nonstandard measurement and has grown to include the concepts of measurement conversion, ratio, proportion and scale. During my first year of teaching, I shied away from projects like this one precisely because I knew they would take extra time, which I felt I did not have, and would lead to difficult questions for which I did not always have answers. Such projects call for students to make and validate conjectures, neither of which is an easy task for middle school students. I worried about how students would feel if they could not do the assigned task quickly and accurately.

After doing this project year after year, I realize that students need to struggle with mathematical concepts. Memorizing conversion formulas, such as 1 yd. = 3 ft. and 1 m = 100 cm, is important, but if a student does not know how long a centimetre is, then the conversion is just another piece of mathematics trivia to be learned for the test and quickly forgotten. I now strive in my teaching to make the concepts relevant by connecting new mathematics to previously learned mathematics and, whenever possible, by connecting mathematics to other disciplines and the real world. Furthermore, I try to avoid reciting mathematics facts by incorporating these types of projects into my teaching as often as possible.

By watching my students work on "How Big Is Your Foot?" I discovered that they did not really mind the struggle when they understood what was being asked. For some of my students, the struggle required two or more periods. Although I had assumed that my students would rebel if they did not get the answer quickly, many of them begged me not to give away the answer!

My students have enjoyed the hands-on discoveries of this project, and I am convinced by their discussions in class, as well as their test scores, that their understanding of measurement and ratio is more complete. Additionally, this project forces students to acknowledge that mathematics topics can be meaningful in the real world, as well as in class.

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A salesman travelled at 110 km/h while making a 220 km trip to a client. He then returned home at 90 km/h. What was the average speed for the round trip?

A Dynamic Way to Teach Angle and Angle Measure

Patricia S. Wilson and Verna M. Adams

What is missing from the following definitions of a rectangle?

Grade 6 Students' Definitions of a Rectangle

- A four-sided quadrilateral
- An object with four sides
- A figure with two sets of parallel, equal lines
- Has four long sides

Grade 8 Students' Definitions of a Rectangle

- A figure with four sides that are parallel. Two sides are equal to each other, and the other two are equal to each other.
- A figure that has four straight lines. Both pairs are parallel.
- A four-sided figure having two sets of parallel lines.
- A closed figure with four sides.
- A four-sided figure with two sides the same length.

These quotations reflect the properties of rectangles that students thought were important when they were directed, "In your own words, define a rectangle." Although most students stipulated that a rectangle needed four sides and some struggled with the concept of parallel opposite sides, no mention was made of angles! The foregoing definitions are from a study involving 145 Grade 8 students and 143 Grade 6 students. Of these students, only 2 per cent of the Grade 8 students and 1 per cent of the Grade 6 students mentioned angles or square corners in their definitions of a rectangle (Wilson 1988). These findings may reflect a lack of attention to the concept of angle until middle school, where it is generally given an abstract treatment.

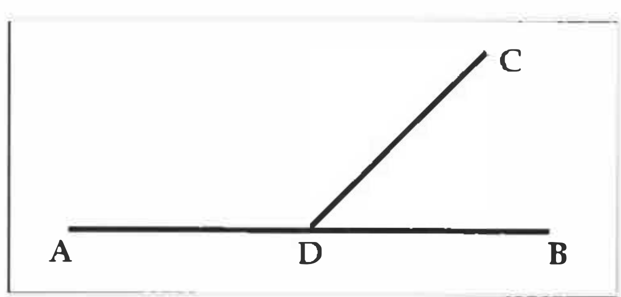
Getting students to think about angles and angle measure long before they are handed protractors and asked to verify a given theorem was addressed by classroom teachers at an elementary school in Georgia. In this article, we draw from their experiences to show that the development of the concept of angle through a dynamic interpretation as a turn can begin in Kindergarten and be part of the mathematics curriculum at every grade level. First, we discuss how students learn angle concepts and outline a basic teaching strategy. The main part of the article offers a sequence of sample activities for developing the concept of angle in K–6.

What We Know About Children's Concepts of Angle

Piagetian research indicates that children's concepts of angle develop slowly (Piaget, Inhelder and Szeminska 1960, 411). Students were shown two supplementary angles (Figure 1) and asked to make another drawing exactly like it without looking at the model while drawing. (The figure shown to the students did not have arrowheads to indicate rays.) The student could suspend drawing and refer to the model at any time. Preschool and primary-level students (4–7 years) used only visual estimates to record the slant of $\overline{CD}$ and did not attempt to devise any way to measure the slant. Middle grades students (7–9 years) tried to copy the slope of the slanted segment but could not devise a measure that would help them. Upper-level elementary school students (9–11 years) began to identify the angle and devised some ways to compare their angles. Some older students used the lengths of $\overline{AC}$, $\overline{CB}$ and $\overline{AD}$ to locate points D and C, creating an accurate slope for $\overline{DC}$. Others drew various lines perpendicular to $\overline{AB}$ to help them approximate the slope of $\overline{DC}$.

Figure 1

Piagetian Task: Make a Drawing Like This One



Piagetian research raises two important points:

1. The concept of angle develops over an extended period of time.
2. A static view of an angle (line segments on a piece of paper) is difficult for young students to perceive.

We also know that students progress through different levels of understanding about geometric figures (Van Hiele 1986). At first, students look at angles holistically. As they begin to recognize angles, students may notice that a triangle always has three angles, or corners, but they do not focus on any particular properties of those angles. Later, they understand that the measure of an angle may be smaller than the measure of a right angle (acute) or larger than the measure of a right angle (obtuse) and begin to identify properties and relationships of angles. The next step of development is to operate with such relationships as "a triangle cannot have more than one obtuse angle because the three sides must form a closed figure." From a developmental perspective, these ideas are not rules; they are understandings that students have developed from working with angles and triangles. Van Hiele's research leads to a third important point:

3. Students need good activities designed to help them explore angles and their properties and relationships.

These three ideas lend insight into how students learn about angles and have implications for curriculum and instruction. Among the many ways to perceive of the concept, angle as a rotation, or turn, seems to be especially appropriate for instruction at the elementary school level. This way of thinking about angle allows the student to anchor the concept of angle on the concrete experience of turning her or his own body. An advantage of perceiving angle as a turn is that it counteracts students' common misconceptions that the size of the angle is determined by the length of the pencil marks used to represent the angle and that one side of the angle must be horizontal.

A Basic Strategy for Teaching Measurement

One of the most important influences on students' achievement is the opportunity to learn. Affording the opportunity to develop conceptual understanding requires a greater depth of coverage than is accorded many topics in the elementary school classroom. Geometry and measurement often receive attention only when extra time is available. To add to the bleak picture, the topic of angle and angle measurement is not usually addressed directly. If it is addressed at all, angle measurement is usually a part of a discussion focusing on triangles or polygons. We suggest that the following basic strategy for teaching measurement be used as a guideline for planning instruction and increasing the opportunities for elementary school students to learn about angle.

Several authors (Hiebert 1984; Wilson and Osborne 1988) have suggested strategies for teaching measurement that share the common elements of exploring, comparing, developing a unit and creating formulas. These elements form the basis for the following teaching strategy, with each step extending the previous step and introducing the next step. Together the steps help students build useful concepts. It is important to cycle through the sequence several times using practical applications and activities that actively involve the students. Each cycle should give students an opportunity to learn at a deeper level. The following list summarizes the four steps as they relate to measuring angles:

1. Explore the concept of angle. What do we want to measure when we measure an angle? Measuring length, measuring volume and measuring angles are extremely different tasks. Students are most familiar with length and often overgeneralize ideas they have learned in measuring length. Students commonly think that they can measure an angle using a ruler; they may try to measure the length of a ray or a distance between rays. We suggest that the amount of turning can be used as the attribute being measured for angles.
2. Compare angles. How can we tell if one angle measure is greater than another angle measure? The act of comparing helps students focus on the attribute being measured and should initially be done without using any particular units. This activity promotes a new perspective on the work done in step 1. When students want to know how much larger one angle measure is than another, they are motivated to move to the next step.
3. Develop a unit that can be used to measure angles. How can we compare angles using units and instruments? Develop a nonstandard unit and a tool, such as a wedge, to help make comparisons. Using several wedges, create an instrument that can be used to count units (a protractor). After the pros and cons of various units are discussed, move to degrees (the standard unit) and the traditional protractor. Students are motivated to move to the fourth step so as to make the process more efficient.

4. Observe relationships and invent rules. What rules help us count the units and apply our knowledge of angles quickly? Formulas or rules should grow out of counting strategies that students develop for themselves. Students might develop such rules as "The sum of the angle measures of a triangle is 180°" or "Supplementary angle measures total 180°."

These four steps were the basis for the development of activities in a three-year project in which researchers at the University of Georgia and classroom teachers at South Jackson (Georgia) Elementary School worked on a K–6 geometry-and-measurement curriculum that would take advantage of what we know about students' learning and effective classroom teaching. Several cycles of writing, piloting and revising furnished insight into what is helpful in teaching about angles. The following activities focus on helping elementary school students progress through the first three steps of the teaching strategy outlined previously.

Activities for Learning About Angle

The activities described in this section illustrate how the perception of angle as a turn can be used to develop the concept of angle. Teachers can adapt the activities for use at different grade levels.

Exploring the Concept of Angle

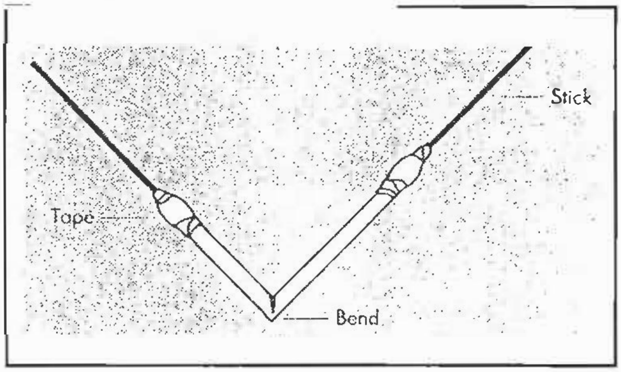
Prior to the introduction of the concept of angle, students need experiences that will help them understand the concept. Ask the students to face the front of the room. Then ask them to try to stay on the same spot on the floor while they turn until they face the front of the room again. Identify this movement as a full turn. Discuss things that turn, such as doorknobs, wheels on a car and hands on a clock. Ask the students to make a turn that is more than a full turn, then discuss how they knew that the turn was more than a full turn. Have the students face the front of the room and then turn until they face the back of the room. Ask them if they turned less than or more than a full turn. Allow them to describe the size of turn (for example, "I went halfway around"). Then introduce the term *half-turn*. Using different walls or objects to mark the beginning of each turn, have students make other half-turns and full turns. Ask, "Do you turn more if you make a full turn or a half-turn? How many half-turns make a full turn?" Also introduce quarter-turns through similar activities and questions and compare them with half- and full turns.

After students understand the language and relationships involved in making quarter-, half- and full turns, introduce the idea of angle as a sweeping motion. In one Grade 2 classroom, for example, a student pretended to be a robot following the directions given by the other students as it searched for an object hidden somewhere in the room. The robot held an arm straight out in front to emphasize the turns. Give a quarter-turn the special name *right angle*, being alert to helping students become aware that a right angle does not necessarily turn to the right. Students often generalize that there must be a "left angle." It is important that they realize that *right* describes the amount of turn, not the direction of turn.

Include activities that involve students in forming angles with their arms. They might, for example, use their arms to represent the hands on a clock. Have them find different ways to make right angles with their arms. They could, for example, start with both arms held straight out in front and sweep one arm to the left. One way to identify a right angle is to have a student stand at the corner of a table that has a square corner, extend both arms along one edge and then sweep out the corner with one arm.

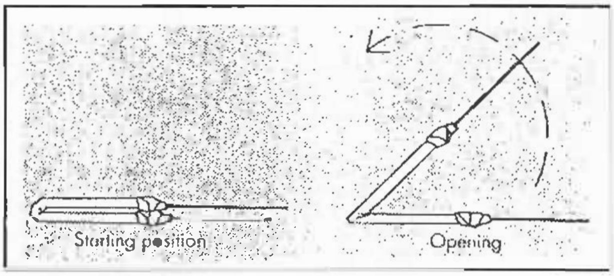
Turns using a flexible hand-held angle can be used to make the connection between turns made with the body and turns represented on paper. A flexible hand-held angle (Figure 2) can be made by inserting sticks into a bent drinking straw and taping them securely in place. By using sticks of different lengths or by bending the straw at various points, the teacher helps students realize that the lengths of the sticks are not relevant.

Figure 2
A Flexible Angle Made from a Drinking Straw



Have the students use the angle to demonstrate turns by starting with both sides touching and then rotating one side (Figure 3). Holding the flexible angle in the air, Grade 4 students easily showed right angles, angles greater than and less than a right angle, and straight angles. As students use the flexible angle, they will be constantly changing its orientation. Be sure that they notice that they have created a new angle only when they open or close the straws.

Figure 3
A Turn Using the Flexible Angle

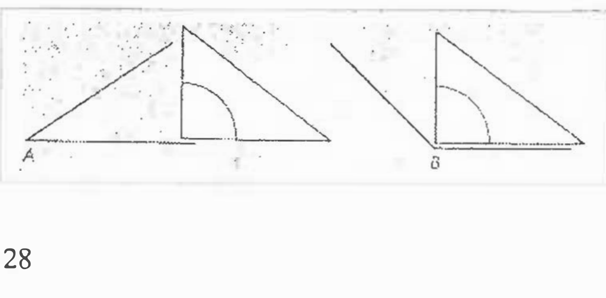


Comparing Angles

Comparisons of sizes of angles can initially focus on comparisons with right angles. Have students make an angle using their arms or the flexible angle and then describe it using language appropriate to their understanding. For example, a student might say, "My angle is more than a half-turn but less than a three-quarter turn" or "My angle is less than a right angle."

Comparing angles that are close in size necessitates creating static angles so that one angle can be placed on top of the other. The flexible straw is not very useful for making such comparisons because of the difficulty of keeping it from opening or closing as it is moved. It is important to note that, perceptually, dynamic and static angles are different. Students need to integrate the two types of angles as they build their concept of angle. Begin by looking at static angles that are right angles and creating a tool (for example, the corner of an index card) for comparing angles to a right angle. The tool can be used to identify corners that have right angles, such as the corner of the room or the corner of a book. After the students have a tool for measuring a right angle, ask them to use the tool to find right angles and angles that are greater than a right angle or less than a right angle. Existing language needs to be developed to describe the situation. For example, Kindergarten students said that a wedge placed in angles like those in Figure 4 did not fit in $\angle A$ but did fit in $\angle B$. Using examples, let students develop appropriate language to describe what it means to fit. Teachers may want to talk about fitting exactly.

Figure 4
Comparing a Right-Angled Wedge to Other Angles

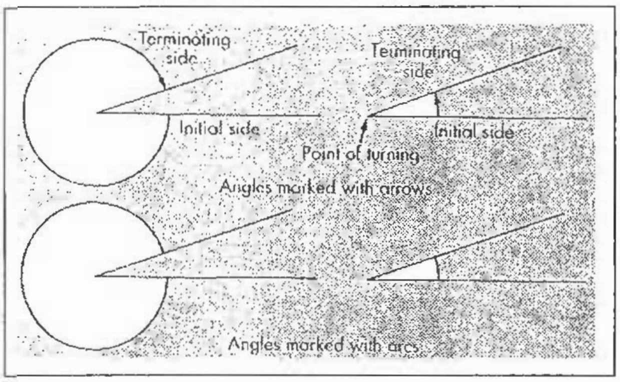


Static angles are created by drawing on paper or cutting wedges of paper. A transition from the dynamic way of thinking about angles to the static way of thinking about angles can be made by asking students to represent the turns on paper. Recording turns requires that students gain a deeper understanding of the nature of angles than they have been using. Teachers should help students focus on all four components of turning that are inherent in the dynamic situation:

- The point of turning
- The initial side of the angle
- The direction of the turn
- The terminating side of the angle

As students complete activities involving turns, these components need to be made explicit. Figure 5 shows an example of how these features of dynamic angles can be represented in static angles. The point of turning is called the vertex of the angle. Teachers should decide whether they want to introduce that terminology. An arrow is used for the purpose of identifying the initial side, terminating side and direction of turn. If the initial and terminating sides are not of interest, the angle can be indicated using arcs.

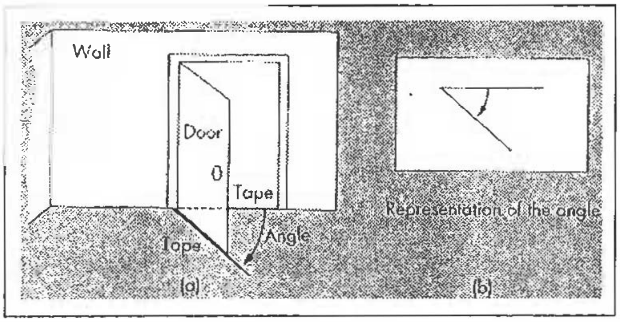
Figure 5
Representing Angles



The first step in an activity that makes the transition to representing angles on paper is to put a line of tape in the doorway to show the closed position of the bottom of the door. Move the door to demonstrate the swing arm of an angle. Stop the door at some point of its motion, bringing attention to the line along the bottom of the door. Use tape to mark this line on the floor (Figure 6a). Close the door and repeat the swing from the start line to the end line. Identify the point of turning. Next, open the door well past the end line of the first angle so that the students can see the angle marked on the floor. Copy the angle onto a large sheet of paper at the doorway and tape it to the chalkboard (Figure 6b). Have students visually estimate the size of the angle by drawing it on smaller-sized paper at their desks. Have them compare their angle to the angle marked on the floor and the representation on the chalkboard. To do this task, they can also cut out their angle and fit it in the angle made by the door.

Figure 6

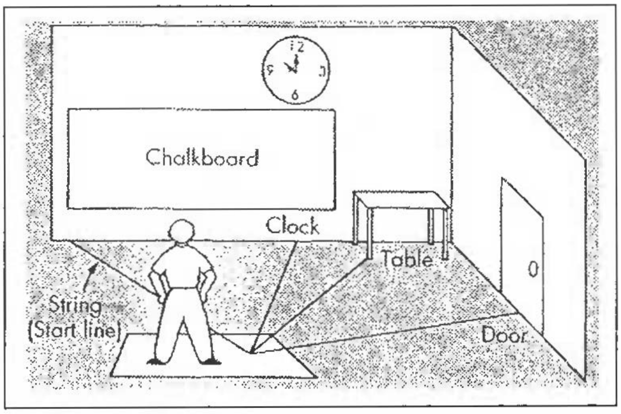
An Angle Made by a Swinging Door



Have students sweep out angles and represent them on paper. Begin by taping a large piece of paper on the floor. Then tape a string from a point marked on the paper to a point on the floor below the corner of the chalkboard. Tape strings to other objects as shown in Figure 7. Each student should stand on the paper, point along a string and sweep out an angle from the start line to another object in the room. Have students make a drawing of the angle by copying the angle marked on the floor. Label it with a description, such as "Angle from the clock to the table." After they have made several drawings, instruct the students to order the angles from smallest to largest. Ask students to identify angles that have the same amount of turn; for example, the angle from the corner of the chalkboard to the clock has the same amount of turn as the angle from the clock to the corner of the chalkboard. Have students compare their drawings to the angles represented on the floor by the strings.

Figure 7

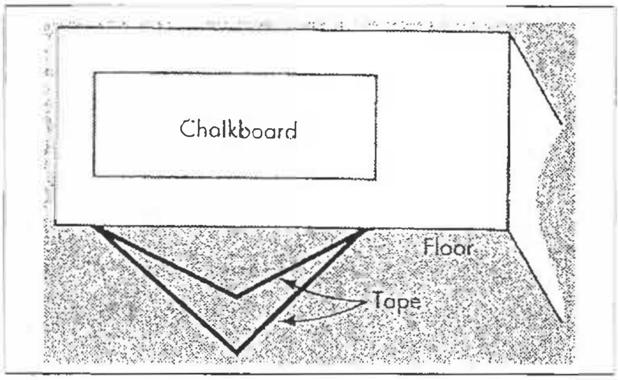
Angles Found in the Classroom



To emphasize the importance of the point of turning, mark two angles whose sides point at the same objects in the room, as shown in Figure 8. Using the flexible angle, have a student stand at the vertex of the angle that is farther from the objects. Have the student position the flexible angle so that the sides of the flexible angle point to the objects, matching the angle marked on the floor. Keeping the end points of the flexible angle aimed at the objects, have the student walk toward the second angle's vertex. Ask students to think about the two angles. Did the angle increase in size or decrease? If the student were to walk even closer to the objects, would the size of the angle increase or decrease?

Figure 8

Two Angles That Point to the Same Objects



Students' experiences in exploring the properties of polygons furnish opportunities to deepen their understandings of angle. As students work with the static angles of geometric figures, useful comparisons can be made by tracing the angles. The language and perceptions developed in thinking about angle as a turn, however, can still be applied. In a Grade 4 classroom, for example, finding the sum of the angle measures in a triangle was posed in terms of turning. The lesson began with the teacher's first reviewing the turning concept of angle, using the flexible angle described earlier and an overhead projector, demonstrating how turning relates to the three angles in a triangle. The students were asked, "How much turning is involved altogether?" After students made several conjectures and checked their conjectures by representing the angles as wedges and putting them together, they identified the sum as a half-turn. This experience lays the foundation for middle school lessons in which students develop the idea that the sum of the angle measures in a triangle is 180°.

Developing a Unit for Measuring Angles

Measuring an angle involves a comparison between the angle and an iteration of a unit angle. The angle in Figure 9, for example, has a measure of 3 units.

Conceptualizing angle as a rotation, or turn, allows the use of the circle as a basis for forming unit angles. The circle, representing a full turn, can be partitioned into wedges in the following way. Start by giving students a cutout circle marked in sixths and have them fold it in half along a diameter and then cut along the crease with scissors. Then using one of the halves (Figure 10), have them cut along the lines, dividing the half-circle into three equal parts. Have them fold each of the pieces in half and cut on the crease. They should have six congruent wedges that can be used to measure and draw angles.

Figure 9
Nonstandard Units

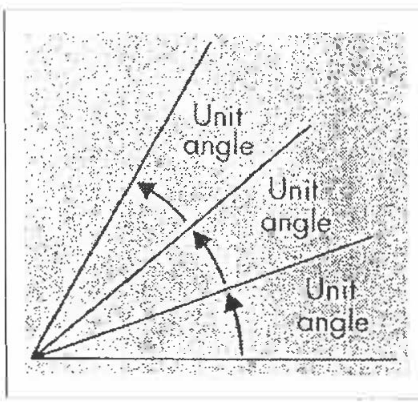
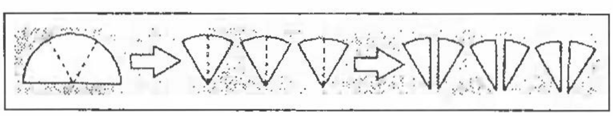


Figure 10
Creating Wedges



After students have experience using the wedges to measure angles, a protractor can be created from the other half-circle (Figure 11). Have the students fold the half-circle into three equal parts and then fold again in half. When they open the half-circle, the creases will mark the wedges. Have students use the protractor to measure angles by placing the edge of the protractor along one side of an angle with the centre on the vertex of the angle (Figure 12). Later, students may want to add scales to the protractor to make it easier to count the number of wedges.

Have students estimate the measures of angles that are not a whole number of wedges (for example, 1½ wedges). Discuss the difficulty of determining the measures of angles that are fractional parts of a wedge. Encourage students to offer suggestions for more precise measures.

Figure 11
Creating a Protractor

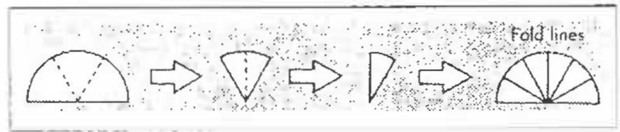
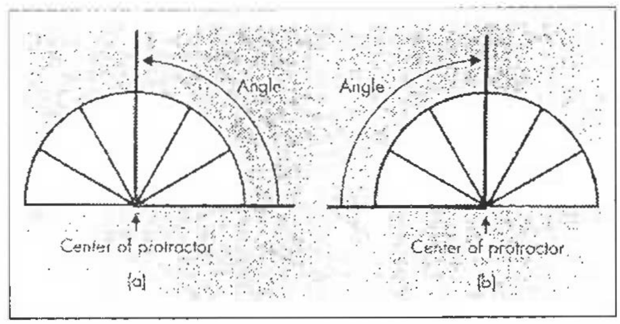


Figure 12
Using a Protractor



At this point, students have a foundation for moving to the study of degrees and the standard protractor. Although the Babylonians are credited with developing the standard unit of degrees by dividing the circle into 360 parts, more than one theory has been advanced about why this number of parts was used (Eves 1969). Encourage students to offer reasons why 360 parts might be a good choice. Relate the degree to relationships that the students have discovered. Ask, "If a whole circle has 360° (1 turn), how many degrees does a semicircle (½ turn) measure? How many degrees does a right angle (¼ turn) have?" Have students determine the number of degrees in each wedge of their wedge protractor. Teachers may want to have them label the wedges in degrees before giving them a standard protractor.

After transition activities, Grades 5 and 6 students are better able to learn to use a protractor. Development of the dynamic concept of angle helps students make judgments about the use of the protractor. For example, because the angle measures of 30° and 150° both appear on the protractor at the same mark, students need to make a judgment about whether the angle is greater than or less than a right angle. For more ideas on using a standard protractor, consult your textbook; Edwards, Bitter and Hatfield (1990); or Wilson (1990).

Summary

Students' lack of understanding of and attention to the concept of angle may be a result of neglecting the topics of angle and angle measurement in the early elementary school curriculum. Often the study of angle is limited to the study of static angles and the use of a protractor. The alternative of studying angles as both dynamic and static allows students to begin developing the concept of angle in Kindergarten and refine it at each grade level.

We have identified a four-step teaching strategy for planning instruction about angle and angle measure and have described activities for the first three steps: exploring the concept of angle, comparing angles and developing a unit to measure angles. The extended time needed for this development requires that students actively explore angles and their properties and applications throughout elementary school. The suggested sequence lays the foundation for introducing standard angle measure, which is not established if a traditional protractor is introduced too early.

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Ryan can drill the holes he needs in five minutes with a power drill or in 20 minutes with a hand drill. He starts with the power drill, but after two minutes it stops working and he finishes with the hand drill. How long does he work with the hand drill?

Let's Do It: Measurement for the Times

Mary Montgomery Lindquist and Marcia E. Dana

This is an article about measurement and multiplication. We chose this topic for three reasons. First, children need a variety of practice in multiplication. We hope that when children are working with problems in interesting, meaningful situations, they will be motivated to do the related computations thoughtfully and accurately. The individual activities do not contain a multitude of multiplication exercises—we are more interested in the quality than in the quantity of work done. Second, measurement is one of the main ways we use mathematics in everyday life. Thus, at the same time the children are practising multiplication, they are applying it. Third, mathematical skills often are taught separately and children do not see the relationships between them. Here we have provided ways to tie measurement and multiplication together.

The activities in this article are appropriate for children who need practice in multiplying numbers of any size. Although some are written for specific sizes of numbers, they can easily be modified for use with other numbers. Many of them can even be adapted to multiplication with decimals or common fractions. In these activities, we assume that the children know how to use rulers and scales, or how to compute the measure of the selected property—length, weight, area, volume or capacity. The activities are independent of each other and, hence, can be used in any way that suits you and your children.

The Latest Length

Lengths can be used to generate multiplication problems. We have suggested two activities, both of which ask the children to first measure and then multiply. In the first activity, the children measure something—the length of the classroom in decimetres, for example—and then find out how long the room would be if it grew to be 28 times as long as it is now. Tell the children that they are going to have a "megamare" and, in this dream, everything is 28 times as long, as high or as wide as it really is. Have the children measure different things and then find out how big those same things would be in their megamare. Provide a recording sheet for each child like the one in Table 1. Choose some appropriate objects and units of measure to get the children started, but leave a few choices of objects for the children to dream up.

Table 1

A Megamare "It" grew 28 times as long or as tall.		
"It"	Size "It" Is	Size in Dream
Your Height (cm)		
Room Length (dm)		
Pencil Length (cm)		
Width of Desk (dm)		
Length of Waist (cm)		
Length of Little Finger (mm)		
Thickness of Book (mm)		
Width of Doorway (dm)		

In the second activity, the children measure things in one unit (a chain) and then use multiplication to find the length of the object in a smaller unit (a paper clip). Give each child a chain of paper clips—say 13 paper clips—and a list of things to be measured. Have the children work in pairs to find the various lengths in paper clips. Since each pair of children has only a chain of 13 paper clips, they will need to find the lengths in chains and then multiply by 13 to find the lengths in paper clips. One list of distances that you might use is the following:

How high can each of you reach when you are

- lying on the floor,
- sitting on the floor,
- squatting on the floor,
- kneeling on the floor,
- sitting on a chair,
- standing with feet flat on the floor and
- standing on tiptoes?

If you wish to vary the multiplier, make chains of different lengths and label them *A*, *B*, *C* and so on. Then, place each chain with a direction card near the object to be measured. A direction card could read as follows: Use chain *C*. How many paper clips long is the cabinet door? The examples given here have involved multiplying by a two-digit number. They could easily be changed to require multiplying by only a one-digit number. In fact, if you choose the objects carefully, the same activities could provide practice for the multiplication facts.

Worldly Weight

Grams are grand for measurement/multiplication practice since small objects can weigh a large number of grams. You can pick the objects to be weighed to reflect the size of the numbers you want the children to use in multiplying: 1 digit × 1 digit (fact), 1 digit × 2 digits, 2 digits × 2 digits and so on. You will need a balance and gram weights or a scale that weighs in grams. Two suggestions for weighing activities that give multiplication practice follow.

The first activity is a game for two players called Gram Slam. Pick six objects to weigh and prepare a duplicating master of 12 cards similar to those shown in Table 2.

Table 2

36 rocks	25 jar lids	51 marbles
40 marbles	16 bolts	27 bolts
64 blocks	75 pencil erasers	50 jar lids
82 pencil erasers	23 blocks	18 rocks

Rules for Gram Slam

1. The two players cut out the cards, mix them up and deal them into two facedown piles.
2. Each player draws a card, weighs one of the objects listed on his or her drawn card and then multiplies to find the total weight.
3. The player whose card shows the larger total weight wins both cards.
4. Play continues until one player has won all the cards and has a "gram slam."

Encourage the children to weigh and multiply only when they need to. For example, if the two cards "36 rocks" and "16 bolts" are drawn, and it is clear that the rock weighs more than the bolt, 36 rocks will definitely weigh more than 16 bolts.

The other weight activity involves a story situation in connection with some objects to weigh. The Dress in Style Co. sends out shipments of different kinds of clothes. The clothes must be weighed and then the total weight for a shipment of a certain type of clothes is determined by multiplying. The information can be given in a table like Table 3.

Table 3

Dress in Style Co.			
Clothing	Weight (g)	Number Ordered	Total Weight (g)
Classy Coat		75	
Faddish Belt		29	
Trendy T-Shirts		80	
Posh Hat		66	
Super Socks		47	
Swanky Scarf		53	

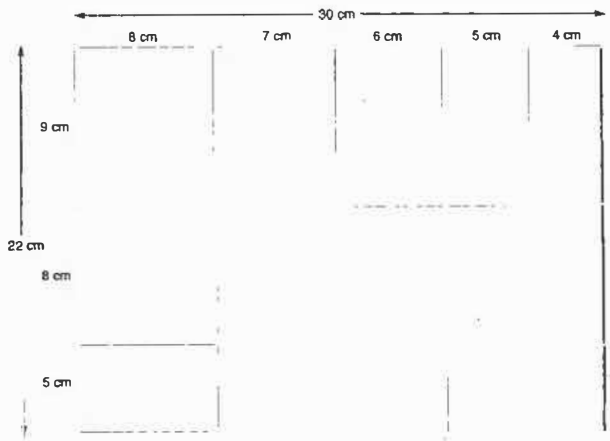
As before, you can choose the items to weigh and the numbers to multiply by to coincide with the type of multiplication problems the children need to practise. We have suggested a specific situation, but it would be easy to modify our suggestion to situations that use other objects. You can introduce the story situation to the class as a group or you can write it on a card; or you can put it on a duplicating master with a table on which the children can record their results.

Avant-Garde Area

Areas provide a natural way to practise multiplication. The first activity suggested here can be used to help develop the concept of the area of a rectangle while providing practice in multiplication. The next two activities depend on the children's knowing that the area of a rectangle can be found by multiplying.

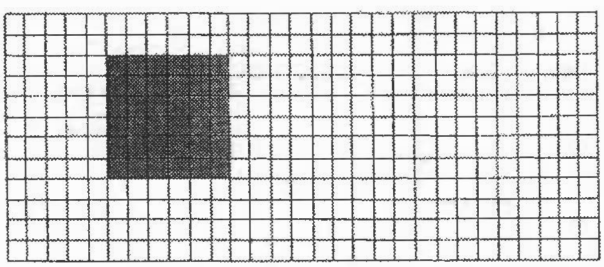
The first activity is a multiplication-fact game, Spaceship 79, designed for pairs of children. Each pair of children needs a transparent centimetre grid and a set of rectangular shapes. The centimetre grid can be made from centimetre graph paper and an overhead transparency. The set of rectangular shapes can be cut from a regular sheet of construction paper. (The pattern in Figure 1 is easy to cut and provides a variety of sizes.) Rectangles of other dimensions can also be used, if you want practice in other multiplication facts.

Figure 1



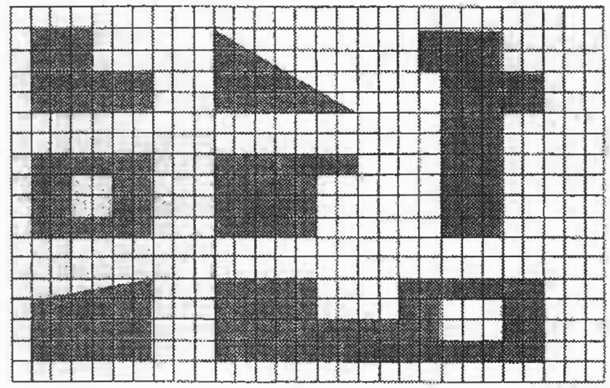
To play the game, each child picks a rectangle from out of the Space Treasury (a box, large envelope or the like), and then uses the transparent grid to find the area of his or her rectangle (Figure 2). The child with the larger area receives one point for each square centimetre of the difference between the two areas. The children continue the game, choosing two more rectangles, until one child has 79 points.

Figure 2



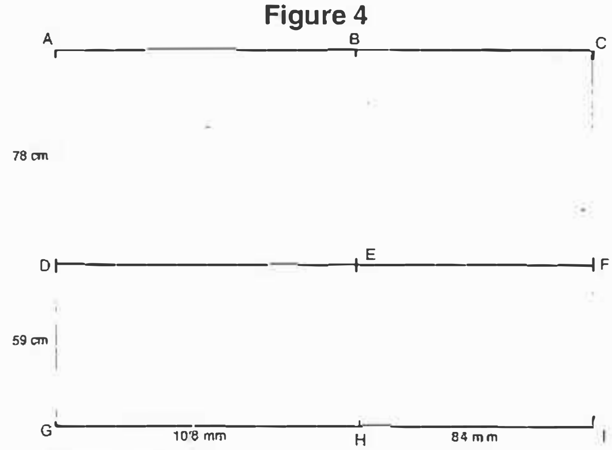
A few things can be noted about this game. First, the set of materials can be used in a variety of ways, such as having children put the rectangles in order by area, from the largest to the smallest. Second, children just beginning to use the grid may count all the squares instead of multiplying the length by the width. This is fine, but if your purpose is to have them practise multiplying, they will need to see how to use multiplication. Encourage them to look at the area of a rectangle, such as the one in Figure 2, as five rows of six squares; thus, $5 \times 6 = 30$. Third, the same game can be used for older children either by omitting the grid and using rulers, or by using the grid with shapes other than rectangles. The shapes in Figure 3, for example, require the children to break the figure into rectangular or triangular parts. (In making and cutting out shapes like those in Figure 3, make the vertical and horizontal sides a whole number of centimetres.)

Figure 3



The second activity has a puzzle to be solved before the task can be completed. It also requires measuring in millimetres, multiplying by three-digit numbers and knowing how to determine the area of a rectangle by multiplying. (You can alter the size of the numbers to be multiplied by making the rectangles whole numbers of centimetres and having the children measure them in centimetres.) This activity is designed for children to do individually. You could make a card, for only one child to do at a time, or a duplicating master, so many could be doing the activity. Figure 4 shows the instructions and the drawing. If you are making a duplicating master, leave space for answers. The dimensions are included for your reference only; do not include them on the children's copies. If children enjoy this activity, you might increase the number of rectangles (warning: adding two rectangles can increase the total number to 18), or you could use right triangles as in Figure 5.

Figure 4

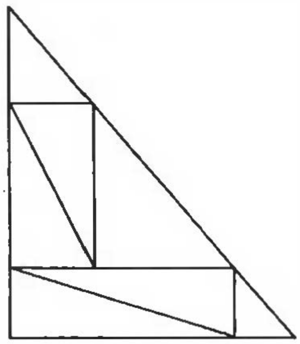


1. Find nine rectangles.
2. Just by looking at the rectangles, list them in order by area, from the smallest to the largest.

   smallest ______ ______

   ______ ______ ______ ______ ______ largest
3. Measure the rectangle in millimeters and find the areas in square millimeters. Check the areas against your list. Are you a good estimator of area?

Figure 5



For a third activity, you can have children use their ability to find areas of rectangles with real objects in the room—areas of bulletin boards, tabletops and desktops, and so on. Often we (teachers) limit finding areas to situations similar to the first two area activities; rectangles are drawn on paper, and only rectangles with sides a whole number of units are used. When children are finding areas of real objects, this is often not true. You can help children handle this by suggesting that they find an approximate area by rounding the lengths of the sides to the nearest unit, that they use a smaller unit or that they use decimals. The decision as to how to handle this will depend on the mathematics level of your children. Remember that measuring real objects is important. Children see how we really use measurement, and it helps develop the idea that all measures are approximate—we often mislead children about measurements when everything turns out "nice and even."

Another way to involve children in measuring areas of real things is to have a contest to see who can be the first to find four mystery objects. Choose and measure the areas of four rectangular things in your room—use things in a variety of sizes, such as a light-switch cover, a book, a desktop and a door. List the four areas without identifying the things themselves. Have the children, working in pairs, list 12 rectangular things that they think might be the mystery objects. When their lists are completed, they should measure the areas of the things on their lists to see if they have identified any of the mystery objects. The search also could be turned into a class project in which all cooperate to find the mystery objects.

Volume in Vogue

Once children are familiar with how to find the volume of a rectangular solid (box) by using its measurements, such problems can also be used for review or practice with multiplication. The first activity that we have suggested is an estimating game, The Cubic Question, which is played by one child. Collect and label—*A, B, C, . . ., G*—seven small boxes. (Each child playing the game will need a set of boxes.) Boxes for such things as pencils, pills, spices or other small things can be used. Make a worksheet like the one in Figure 6. This game gives children practice in estimating, measuring and multiplying, but not in competition with other children.

Figure 6

The Cubic Question

Look at boxes *A-G*. Without doing any measuring, just by looking at them, put them in order, from largest to smallest. Write the letters in order here:

___ ___ ___ ___ ___ ___ ___

Now measure and find the volume of each box in cubic centimeters. Do your multiplying in the spaces below:

A	B
C	D
E	F
G	Were your guesses about the order correct?

The size of the metric unit that you use can be varied to give practice with the numbers—one digit, two digits and so on—that are appropriate for your children. You can also vary the size of the boxes being measured. As with finding the areas of actual objects, the children will be confronted by boxes whose sides are not whole numbers of units. Possibly the easiest way to handle this in finding volume is to round off the dimensions to the nearest whole unit.

A second activity involves using larger boxes or even closets, cupboards or whole rooms. You will need to gear the metric units used to the size of the multiplication problem you want the children to practise. Very large boxes could be measured in cubic decimetres, for example, or even cubic metres. One way to get children to measure the volumes of large boxes is to begin with a guessing contest. Show the children a cardboard box and have them guess its volume in cubic centimetres. They can each record their own guess and need not share it with anyone. Ask for a volunteer to measure the length, width and height of the box. Then each child uses those measurements to find the volume. Check the answer together.

You could ask questions like these about the children's guesses:

- Whose guess was closer than 1,000 cm^3?
- Whose guess was closer than 500 cm^3?
- Who guessed too high?
- Who guessed too low?

Repeat the activity with another box. The children's guesses should be considerably more accurate this time since they can compare the second box to the one whose volume they already know. The activity can be done several times if the children enjoy it.

Another way to have children measure large solids is to separate the children into pairs. Then assign to each pair, or let them choose, large boxes or rectangular solids in the room, or even outside the room. The two children measure the lengths, widths and heights together, but each does the multiplication to serve as a check on the other.

Children can be very interested in how many cubic centimetres there are in a cupboard, a locker, a desk or even the classroom itself. A really ambitious project might consist of finding the volume of the whole school in cubic metres (or cubic decimetres if you want to get to *really* large numbers).

Contemporary Capacity

Activities in which children find the capacities of objects can be easily devised and provide much interesting multiplication practice. Again, you will want to choose units and containers to correspond with the sort of multiplication problem you want the children to practise.

The first activity involves some multiplication of very large numbers and will need to be modified if your children are not ready to handle them. Collect a cupful of as many of any of the following as you wish: lima beans, macaroni, popped popcorn, marshmallows, wrapped candy, marbles, crayons, pennies and so on. Place them at a math centre, with a large sign:

How Many in a Swimming Pool?

Pool your efforts with a friend. Dive right in and find out! Choose any kind of object at the math centre and follow the directions on the card.

Write a direction card with the following directions included:

1. Fill a cup with the objects you want to use.
2. Count how many of the objects are in the cup.
3. One pint holds two cups. How many objects would be in a pint?
4. One quart holds two pints. How many objects would be in a quart?
5. One gallon holds four quarts. How many objects would be in a gallon?
6. One bathtub holds about 70 gallons. How many objects would be in a bathtub?
7. A swimming pool might hold about 386 bathtubs. How many objects would be in a swimming pool?

The children can work in pairs (as the sign indicates), thus acting as a check for each other. If your children cannot handle such large numbers, you can stop the sequence at any step. Your final question could be, How many objects in a gallon? The children might be interested in comparing results for different objects: What is the difference between the number of lima beans in a swimming pool and the number of marshmallows in a swimming pool?

The second activity can be done at a centre by an individual child or by pairs of children, or with the whole class at one time, depending on how many materials are available. It requires a cylinder marked in millilitres and some small jars or bottles of a variety of sizes—spice bottles, mustard jars and pill bottles will all work. The children measure the capacities of the jars or bottles in millilitres, and then multiply each result by a given number. Label each bottle—*A*, *B*, *C*, . . .—and make a worksheet or card like that in Figure 7. (If you wish to see the children's computations, have them write them on a separate sheet.) Encourage children to use logic to avoid multiplication whenever reasonable.

Figure 7

Which Is Less?

Measure the capacity of each bottle in milliliters. Multiply to find out which of each of the following pairs holds the smaller number of milliliters, then circle it.

54 of bottle *A* 49 of bottle *B*	76 of bottle *B* 40 of bottle *C*
92 of bottle *A* 76 of bottle *C*	37 of bottle *D* 29 of bottle *A*
50 of bottle *B* 74 of bottle *D*	81 of bottle *C* 72 of bottle *C*
60 of bottle *E* 74 of bottle *D*	75 of bottle *B* 90 of bottle *E*

The activities in this article were built around the particular property in question. Many of them, however, could just as well be done with another property. For example, the game in the weight section could be played with length or capacity. You may even want to combine properties. Think what you could do with the ordering of boxes described in the volume section—they could be ordered by length, weight or surface area. Take the ideas and change them so they will *measure* up to what you need at that time.

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A pumpkin weighed 100 kg and was 99 per cent water. While the pumpkin was exposed to the sun, some of the water evaporated, so it was only 98 per cent water. How much did the pumpkin then weigh?

Calendar Math

Art Jorgensen

Below are the math problems for the month of September—one problem for each day of the month.

1. Tom bought nine apples and eight oranges. He spent \$2.29 altogether. If the oranges cost \$1.20, how much did he spend on apples?
2. How much did each orange cost?
3. How much did each apple cost?
4. If Tom gave the clerk a \$5 bill, how much change did he get back?
5. If Tom had bought 13 oranges, what would they have cost?
6. What is the better buy: six apples for 42¢ or nine apples for 54¢?
7. Jill bought some grapes. She gave the clerk a \$10 bill and got \$6.01 in change. How much did the grapes cost?
8. If grapes sell for \$1.33 per pound, how many pounds did Jill buy?
9. Watermelons sell for \$5.38 each. What would half a watermelon sell for?
10. A watermelon weighs 4.5 kg. If each person eats 0.5 kg, how many people will be able to eat from the watermelon?
11. Steak costs \$10.36 per kilogram and pork chops cost \$8.78 per kilogram. If Susan buys 1.5 kg of steak and 1 kg of pork chops, how much must she pay?
12. Bill can eat $^1/_3$ of a banana in three minutes. How long will it take him to eat the whole banana?
13. If Susan eats two plums on the first day and doubles the number she eats each day after that, how many plums will she eat on the third day?
14. How many plums will Susan have eaten altogether?
15. If the plums cost 9¢ each, how much money did Susan spend on plums?
16. Joe can eat a hamburger in 15 minutes. How long will it take him to eat six hamburgers?
17. Tanya ate ½ her candy on Monday, ¼ on Tuesday and $^1/_8$ on Wednesday. How much did she have left?
18. Orville ate ¼ of his chocolate bar. Ann ate $^5/_8$ of hers. How much more of her bar did Ann eat than Orville did of his?
19. Jan's garden is 10 m × 12 m. Ronald's garden is 8 m × 11 m. How much bigger is Jan's garden?
20. If Jan can dig 12 m^2 per hour, how long will it take her to dig her garden?
21. Rachel's apple tree is 5 m tall. Willie says his apple tree is half again as tall. How tall is Willie's apple tree?
22. When Robert was digging potatoes, he found that the first hill had two potatoes, the next hill had four potatoes, the next hill had eight potatoes and so on. If this pattern continued, how many potatoes were under the fifth hill?
23. If Robert dug four hills, how many potatoes did he dig altogether?
24. Peter's cantaloupe is 16 cm in diameter. What is its volume?
25. Raspberries cost \$6 for a 4 L basket. At that price, what would a 6 L basket cost?
26. Corncobs normally sell for \$4 a dozen. However, the grocery store has a 40 per cent reduction sale. What is the selling price of a dozen cobs?
27. A sausage 20 cm long will feed four people. A sausage 35 cm long will feed how many people?
28. Lou says he can buy seven kiwi fruits for 84¢. Leslie says she can buy nine kiwi fruits for 99¢. Who gets the better buy and by how much per kiwi fruit?
29. Three lemons can make enough juice for 2 L of lemonade. How many lemons would be necessary to make 16 L of lemonade?
30. Cherries were first in the store on June 3. Peaches did not become available for another 35 days. What day were peaches available?

Answers

1. $1.09
2. 15¢
3. 12¢
4. $2.71
5. $1.95
6. Nine apples for 54¢
7. $3.99
8. 3 lbs.
9. $2.69
10. Nine people
11. $24.32
12. Nine minutes
13. Eight plums
14. 14 plums
15. $1.26
16. 90 minutes, or 1½ hours
17. $\frac{1}{8}$
18. $\frac{3}{8}$
19. 32 m^2
20. 10 hours
21. 7.5 m
22. 32 potatoes
23. 30 potatoes
24. 2,143.57 cm^3
25. $9
26. $2.40
27. Seven people
28. Leslie, by 1¢ per kiwi fruit
29. 24 lemons
30. July 8

A farmer cultivated twice as much land this year as last year. Last year the weather was perfect, but this year unfavourable weather conditions caused him to lose 1/3 of his crop. What was the ratio of this year's crop production to last year's?

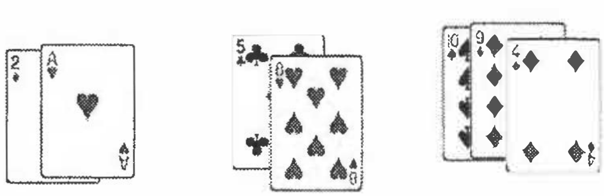
Activities for the Middle School Math Classroom: Card Games

A. Craig Loewen

Like dice games, card games are typically learned early in life. Because these games are learned so early and are so commonly played, they present an opportunity to build on prior knowledge in the math classroom. Unlike many manipulative-based games, card games are highly familiar to students and, thus, easy to introduce and implement. Further, the general availability of cards makes possible at-home practice activities with parents and siblings, thus building the home–school connection.

Card games (even those favoured by very young children) typically have several mathematical applications. Many childhood card games, such as Go Fish! and Rummy, have at their core early mathematical concepts such as sorting and sequencing. Topics in probability also often surface because card games usually revolve around trying to collect certain combinations from randomly arranged decks. It is easy to see how card games could lead to discussions containing words such as *likely, unlikely* and *impossible*. When I was playing Go Fish! with one of my own children, he announced that he knew what the last card in the pile had to be and that he knew what cards I held in my hand because those were the only cards left in the game. I lost.

Card games can often be easily adapted to higher grade levels. Cards are particularly well suited (no pun intended) to games involving integers. We need only define black cards as positive values and red cards as negative values to create an effective model for integer expressions. For example, here are collections of cards each with a value of -3:



Simple card arrangements can be used to show algebraic expressions and equations. For example, here is a model of the expression $2x - 4$:

In this example, two cards with the same value and colour are placed facedown beside a red 4. If these cards together have a value of 8, what is the value on the facedown cards? With the added information, the cards represent the equation $2x - 4 = 8$.

The drawback of using a typical card deck is that it contains aces and face cards. If these cards are included, they are usually assigned number values (that is, ace = 1, jack = 11, queen = 12 and king = 13), but these values can be difficult for students to remember. It may be easier to remove these cards from the deck or to use a special deck that includes the values 1–13 (with no aces or face cards).

In general, the simplicity, familiarity and flexibility of card games make them an excellent tool for instruction and reinforcement in the mathematics classroom.

Suggestions and considerations for integrating games in the math classroom include the following:

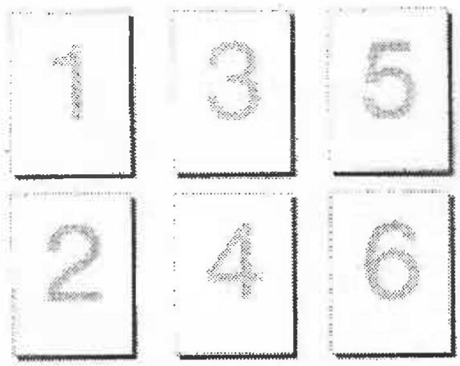
- Use a game at the end of a unit for review or at the beginning of a lesson to motivate students and help them recall prior knowledge.
- Be prepared for a more active and noisier classroom environment. Motivating activities (and games are highly motivating!) are often noisier than more traditional activities.
- Try to collect games that can be easily adapted to various levels. Highly flexible games also let you apply a familiar game structure to various math concepts. Using flexible games reduces the amount of time necessary for developing and introducing games.

Fraction Switch

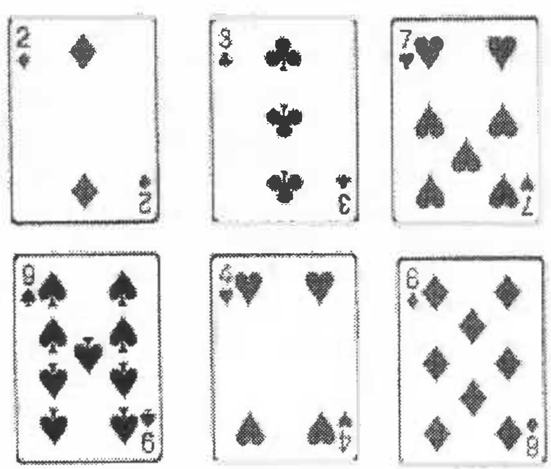
Objective: Compare and/or order proper fractions and decimals to hundredths (Alberta mathematics program of studies, Number [Number Concepts], Grade 5, Outcome 9)
Materials: Deck of cards (no jack, queen or king; ace = 1)
Players: Two

Rules

1. Shuffle the cards well and deal six cards facedown to each player. Place the remaining cards facedown to form a draw pile.
2. Without looking at the cards, each player arranges his or her six cards in a pattern as shown below:



   Cards 1 and 2 form a fraction with 1 in the numerator and 2 in the denominator; likewise, cards 3 and 4 form a fraction, and cards 5 and 6 form a fraction. The player then turns the cards over.
3. On a turn, a player draws the top card from the draw pile and, if so desired, replaces one of his or her six cards with it. The player then puts the replaced card (or the drawn card if it is not wanted) in a discard pile.
4. The first player to build three proper fractions (that is, fractions in which the numerator is less than the denominator), ordered from least to greatest, wins. Below is an example of a winning hand:



5. If the draw pile is depleted before any player has finished building the three fractions, shuffle the discard pile, place it facedown to form a new draw pile and continue play.

Adaptations

- For a more significant challenge, play the game with all the cards facedown. A player may not turn the cards over until he or she believes that they are correctly sequenced.
- Allow improper fractions (that is, fractions in which the numerator is greater than the denominator).
- Have each player build three equivalent fractions rather than sequenced fractions.

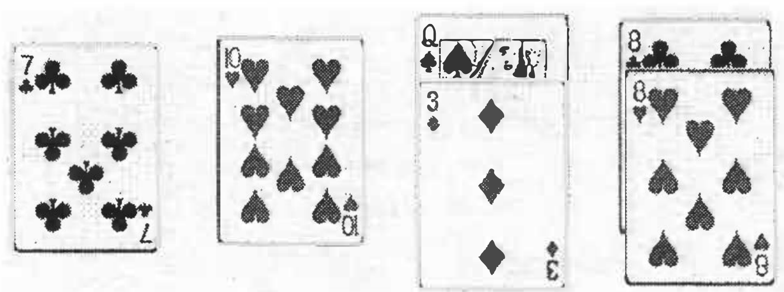
Construction

Objective: Generate and extend number patterns from a problem-solving context (Alberta mathematics program of studies, Patterns and Relations [Patterns], Grade 5, Outcome 4)
Materials: Deck of cards (ace = 1), four-sided die
Players: Two

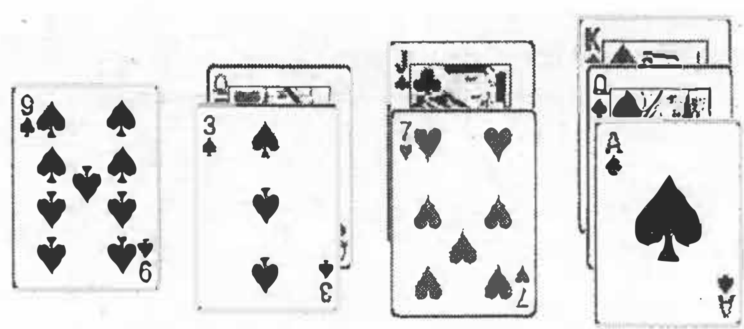
Rules

1. Shuffle the cards well and deal three cards to each player. Place the remaining cards facedown to form a draw pile.
2. Turn over the top card of the draw pile. This is the value at which the pattern must start. If the top card is a face card, move it to the bottom of the deck and turn over the next card. Roll the four-sided die. This is the increment between elements in the pattern. For example, if the top card is a 3 and the die shows 2, the pattern will be 3, 5, 7, 9, 11, 13, ...
3. On a turn, a player plays as many cards as he or she can, as long as the cards follow the pattern. If a player cannot play a card, he or she draws a card from the draw pile but cannot play the card on that turn.
4. A face card has a value of 10 when placed with another card. For example, a jack with a 4 (place the 4 on top with the jack showing underneath) represents 14. Examples are shown below.

   These cards show the pattern 7, 10, 13, 16, . . .:



   These cards show the pattern 9, 13, 17, 21, . . .:



5. The first player to get rid of all his or her cards wins. If the draw pile is depleted before any player has won, all players count the number of cards in their hands. The player with the fewest cards wins. Note: It is very difficult to get rid of all the cards, and in some patterns it may be impossible!

Adaptations

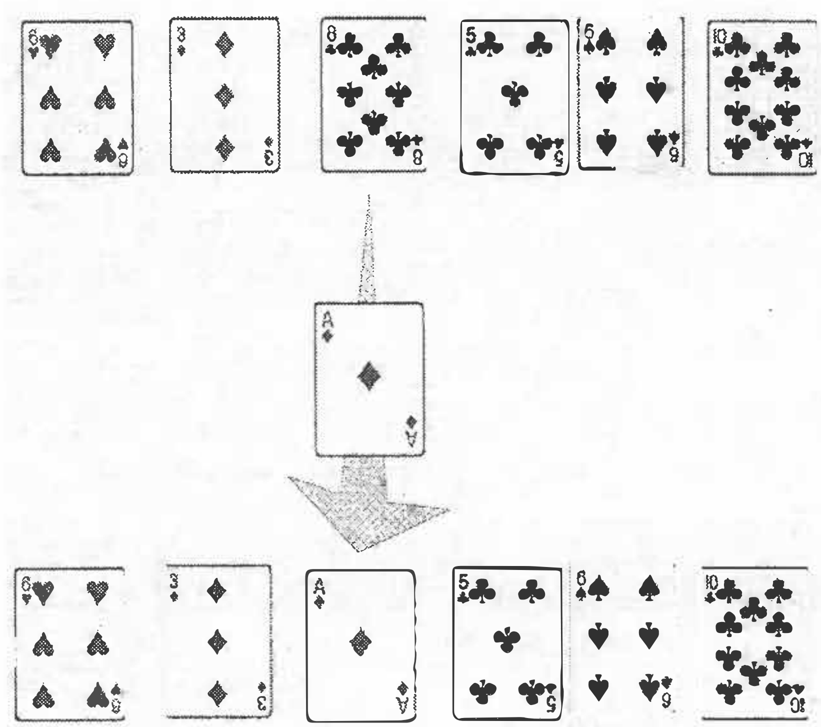
- Have more than one pattern on the go. A player who draws a king may start a new pattern.
- For a simpler game, play such that the cards must be lined up in suits (that is, a row of ♠, ♣, ♥ and ♦ laid in ascending order starting with the ace).

Switch

Objective: Compare and order integers (Alberta mathematics program of studies, Number [Number Concepts], Grade 7, Outcome 12)
Materials: Deck of cards (jack, queen and king removed; ace = 1)
Players: Two to four

Rules

1. Shuffle the cards and deal six cards faceup to each player. The cards must form a row and be placed from left to right in the order dealt. Place the remaining cards facedown to form a draw pile.
2. The first player draws the top card from the draw pile and, if so desired, replaces any one of his or her six cards with it. If the player does not want the card, he or she places it in a discard pile. Subsequently, players may draw from either the draw pile or the discard pile. Play passes to the left.
3. In this game, red cards have a negative value and black cards have a positive value. The first player to correctly order (from left to right) six cards from least to greatest wins. Below is an example of a winning sequence:



4. If the draw pile is depleted before anyone has won, shuffle the discard pile and place it facedown to form a new draw pile. Play continues as before.

Adaptations

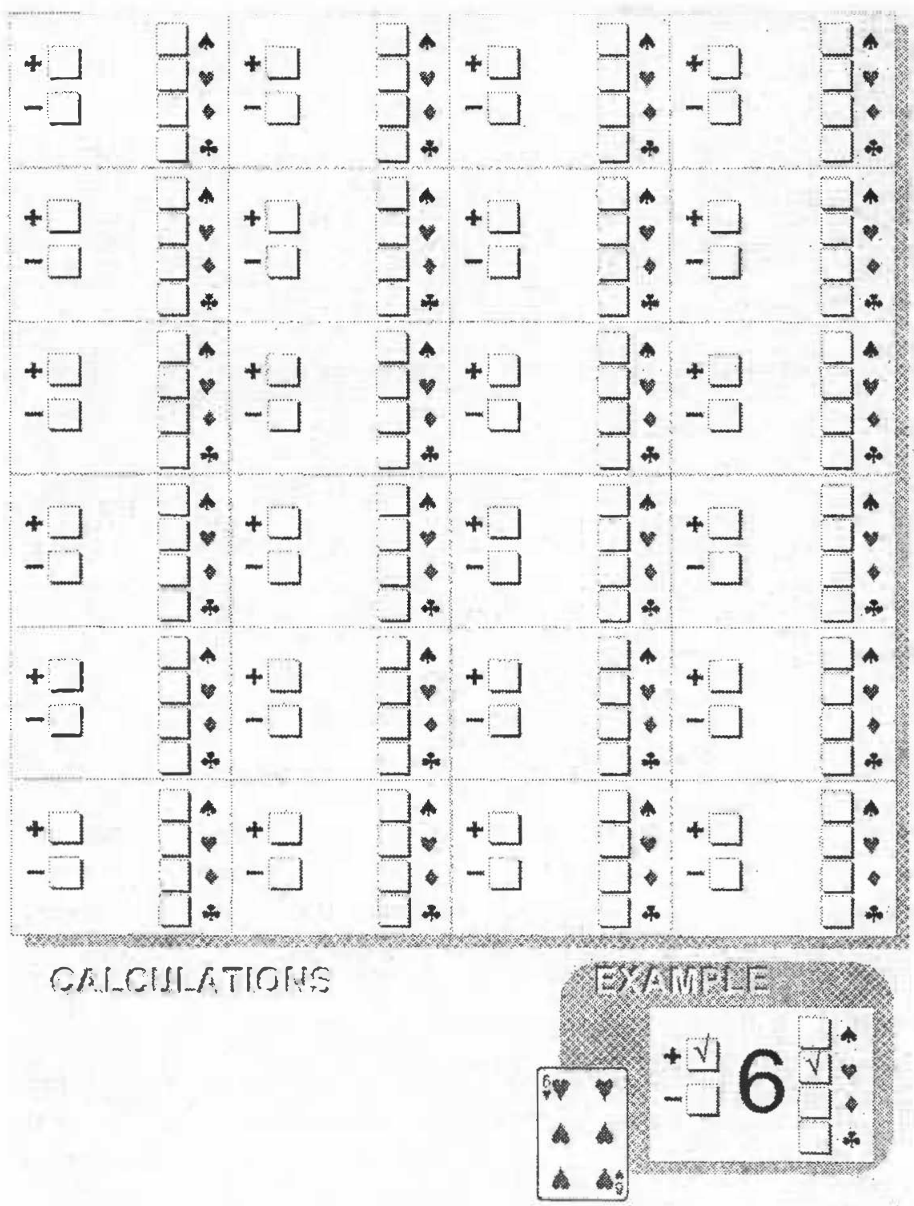
- Specify that, in the sequence, three cards must be red and three must be black.
- Allow players to draw only from the draw pile.

Square Deal

Objective: Distinguish between a square root and its decimal approximation as it appears on a calculator (Alberta mathematics program of studies, Number [Number Concepts], Grade 8, Outcome 8)
Materials: Deck of cards (jack, queen and king removed; ace = 1; one joker included), calculator, Square Deal game board
Players: Two or more

Rules

1. Shuffle the cards and place them facedown to form a draw pile. Turn over the top card in the pile, show it to all players and set it aside as the target value.
2. On a turn, a player takes a card from the top of the pile and either keeps it (adding the value to or subtracting the value from the value the player has accumulated thus far) or gives it to an opponent (who must then add the value to or subtract the value from the value he or she has accumulated thus far). Play passes to the left.
3. If a player's sum drops below 0, the player must drop out of the game.
4. Players continue drawing cards and adding or subtracting values until a player draws the joker. At this point, all players must use the calculator to find the square roots of the values they have accumulated.
5. The players now compare their results with the target value on the card set aside at the beginning of the game. The player with the value closest to the target value wins.



Adaptations

- To simplify the game, eliminate the target value. See who can be the first to reach a value with a square root greater than 10.
- Have the player who draws the joker reset his or her calculator to 0. Play continues until a player exactly meets the target value. Cards are reshuffled if necessary.

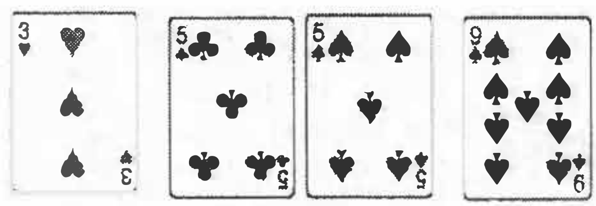
The Mean Game

Objective: Construct sets of data given measures of central tendency and variability (Alberta mathematics program of studies, Statistics and Probability [Data Analysis], Grade 8, Outcome 6)
Materials: Deck of cards (jack, queen and king removed; ace = 1)
Players: Two to four

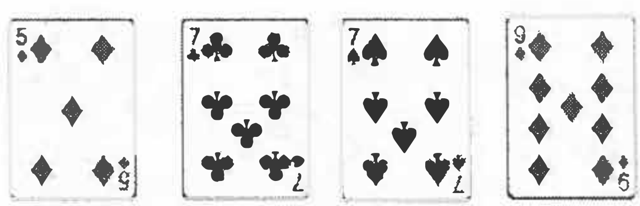
Rules

1. Shuffle the cards well and deal one card facedown to each player. Deal four more cards faceup to each player. Each player may then turn over the facedown card, which represents the target mode. Place the remaining cards facedown to form a draw pile.
2. The first player draws the top card from the draw pile. The player may replace any one of his or her four faceup cards with that card. If the player does not want the card, he or she places it faceup in a discard pile. Subsequently, players may draw from either the draw pile or the discard pile.
3. The first player to create a set of cards with a mode equal to the target mode wins.
4. Play the game two more times, looking for the median and then the mean. Note that it is impossible to create a mean of 1 or 10 with this deck of cards. If a player is dealt a 1 or a 10 as a target card, he or she should switch it with any one of the four faceup cards or randomly draw a new target card from the deck. It is possible to build a set of cards with a median of 1 or 10 or a mode of 1 or 10.

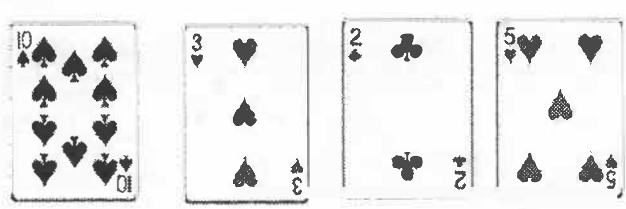
   These cards represent a mode of 5:



   These cards represent a median of 7:



   These cards represent a mean of 5:



Adaptations

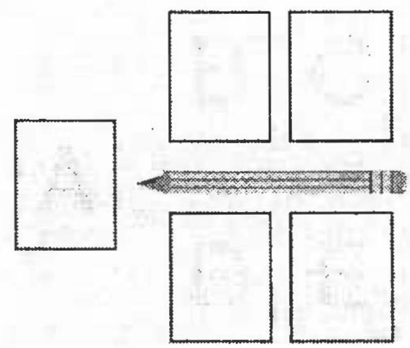
- Include the jack, queen and king, but assign them very high or very low values (for example, king = 1,000 or queen = –1,000).
- Eliminate the target card and allow players to collect cards until a player believes that he or she has a mean, median or mode greater than that of any other player. This player then knocks the table instead of taking a turn, thus signalling the final round of play.
- Allow a player to substitute his or her target card for any of the four faceup cards at any time during the game. The substituted card now becomes the target card.

Half 'n' Half

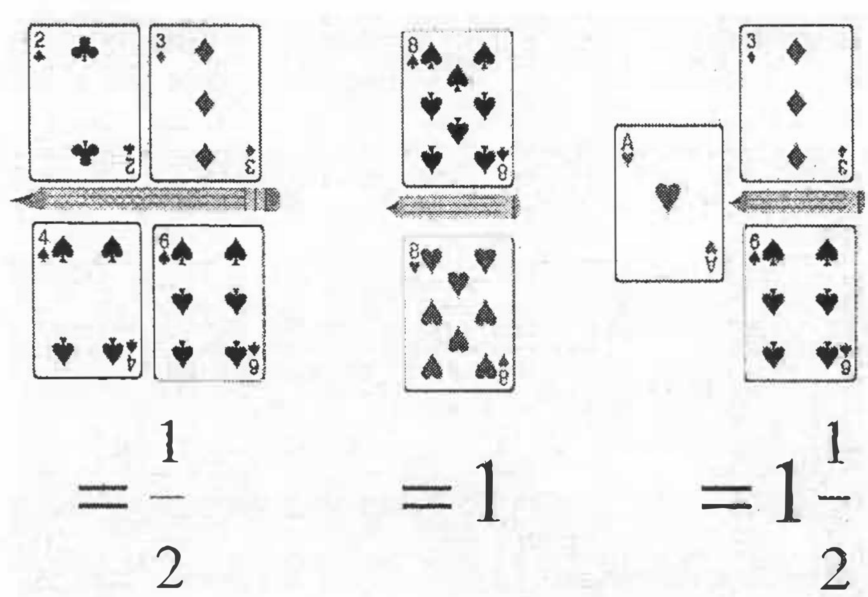
Objective: Demonstrate and explain the meaning of improper fractions and mixed numbers (Alberta mathematics program of studies, Number [Number Concepts], Grade 6, Outcome 9)
Materials: Deck of cards (10, jack, queen and king removed; ace = 1), pencil
Players: Two or more

Rules

1. Shuffle the cards well and place them facedown to form a draw pile.
2. The youngest player goes first. On a turn, a player draws a card from the draw pile and places that card in any of the five locations indicated below:



   A player does not need to have a card in each of the five locations to complete a fraction. A card may be used to start a new place value, or it may be placed on top of an existing card (replacing the covered card) or underneath an existing card (essentially discarding the drawn card). Use a pencil for the fraction line (that is, the line that separates the numerator from the denominator). Play passes to the left.
3. The first player to collect and arrange cards to construct a value equal to ½, then 1, then 1½ wins. At right are some models that, built in turn, would win the game.



4. A card covered by another card remains out of play until the end of the game. If the draw pile is depleted before any player has built all the fractions, the player closest to correctly building the final fraction wins.

Adaptations

- Create a discard pile and have players return unused cards to the discard pile. Players may draw from the draw pile or the discard pile.
- Have players build much larger fractions.
- Play the game as a form of solitaire. What is the smallest number of cards you need to turn over, one at a time, to build the four fractions?

Not-So-Common Multiples

Objective: Recognize, model, identify, find and describe common multiples, common factors, least common multiple, greatest common factor (Alberta mathematics program of studies, Number [Number Concepts], Grade 6, Outcome 4)
Materials: Deck of cards (ace, jack, queen and king removed), calculator
Players: Two or more

Rules

1. Shuffle the cards well and turn them facedown to form a draw pile. Turn over the top three cards and calculate the sum to find the target number. For example, the target number shown by the cards below is 18:

   The three cards can then be put aside or returned randomly to the deck.
2. The first player draws a card from the top of the draw pile. The player may keep the card or put it in a discard pile. Subsequently, players may draw from either the draw pile or the discard pile. Play passes to the left.
3. Each player builds a set of cards, adding to the set or substituting one card for another until the sum is a multiple of the target number. A winning multiple must be at least twice the target number. The first player to build this multiple (and prove it with the help of a calculator) wins.

Adaptation

- Turn over only two cards at the start of the game. Players then attempt to build a card set with a sum that is a common multiple of the sum of the two numbers.

Divisibility Rules

Objective: Use divisibility rules to determine if a number is divisible by 2, 3, 4, 5, 6, 9 or 10 (Alberta mathematics program of studies, Number [Number Concepts], Grade 7, Outcome 3)
Materials: Deck of cards (7, 8, jack, queen and king removed; ace = 1), 10-sided die
Players: Two or more

Rules

1. Roll the 10-sided die twice to create a two-digit number. The first roll specifies the value in the 10s place; the second roll specifies the value in the units place. For example, rolling a 9 and then a 2 creates the value 92.
2. Shuffle the cards and place them facedown to form a draw pile.
3. On a turn, a player turns over one card. If the value of the card divides evenly into the two-digit number already determined, the player keeps the card. Otherwise, the player discards the card.
4. When the draw pile is depleted, the player who has collected the most cards wins.
5. If a player keeps or discards a card erroneously, the player who catches the error gets to steal the card.

Adaptations

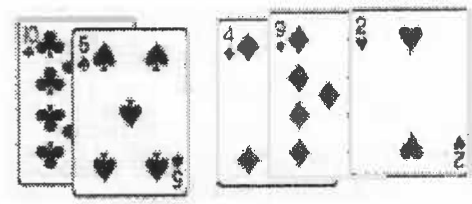
- To increase the challenge, determine (using the die) another two-digit number at the start of the game. A player then keeps a card if its value divides into either two-digit number. If the card's value divides into both numbers, the player steals a card from an opponent.
- Instead of counting cards at the end of the game, find the value of the cards. The highest sum wins.

Integer Addition

Objective: Represent integers in a variety of concrete, pictorial and symbolic ways (Alberta mathematics program of studies, Number [Number Concepts], Grade 7, Outcome 11)
Materials: Deck of cards (ace, jack, queen and king removed)
Players: Two or more

Rules

1. Shuffle the cards and place the deck facedown to form a draw pile.
2. The first player draws a card from the top of the draw pile and decides whether to keep or discard it. Subsequently, players may draw from either the draw pile or the discard pile. If a player draws from the draw pile, he or she may keep the card or discard it. If the player draws from the discard pile, he or she must keep the card until the end of the game. Kept cards are placed faceup in front of the player. Play passes to the left.
3. Red cards represent negative values and black cards represent positive values. Each player continues adding cards to his or her set until a player has collected a set of cards with a sum of 0. This player is declared the winner. The set of cards at the right would constitute a winning hand.



Adaptations

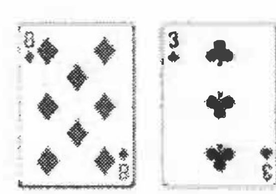
- Select a target number other than 0.
- Allow players to have a maximum of four cards in their hands. Once four cards have been collected, players may only exchange, not add, cards.
- Play with closed hands (that is, players hold their cards so they cannot be seen by other players).

Integer Challenge

Objective: Add, subtract, multiply and divide integers concretely, pictorially and symbolically (Alberta mathematics program of studies, Number [Number Operations], Grade 7, Outcome 16)
Materials: Deck of cards (ace, jack, queen and king removed)
Players: Two

Rules

1. Shuffle the cards thoroughly and give each player approximately half the deck.
2. The players simultaneously turn over their top cards and place them such that both players can see both cards.
3. The red cards represent negative values and the black cards represent positive values. The first player to state the sum of the two cards takes both cards and adds them to the bottom of his or her deck. For example, the first player to say, "Negative five" would claim both of the cards at right.
4. Play continues until a player has captured all the cards (or until time runs out). The player with the most cards wins.



Adaptations

- Play with three or more players, who each flip over a card. The first player to state the correct sum of all the cards takes the cards.
- Have players multiply, rather than add, the values on the cards.
- Have players identify the distance between the two values on a number line rather than adding or multiplying.

Edmonton Junior High Mathematics Contest 2000

Andy Liu

The Edmonton Junior High Mathematics Contest is designed to challenge the top 5 per cent of Grade 9 math students in Edmonton. The annual contest is run by a group of mathematics teachers and is written by Andy Liu.The main sponsors are the Association of Professional Engineers, Geologists and Geophysicists of Alberta (APEGGA), IBM, MCATA, Edmonton Public Schools and Edmonton Catholic Schools. After the contest, the top 50 students are recognized at a dinner banquet also attended by parents and teachers.

Part 1: Multiple Choice

1. What is the value of $\sqrt{2} - \sqrt{8} + \sqrt{18}$?

   (a) 0 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) $3\sqrt{2}$

   Solution: We have $\sqrt{2} - \sqrt{8} + \sqrt{18} = \sqrt{2} - 2\sqrt{2} + 3\sqrt{2} = 2\sqrt{2}$. **Answer: (c)**

2. Donald mixed x kg of peanuts and y kg of raisins to make a mixture. The peanuts cost \$5/kg and the raisins cost \$4/kg. When the cost of the peanuts rose by 10 per cent and the cost of the raisins dropped by 15 per cent, Donald's total cost for making the mixture remained the same. What was the ratio $x{:}y$?

   (a) 2:3 (b) 5:6 (c) 6:5 (d) 3:2

   Solution: The original cost (in dollars) for Donald was $50x + 40y$, and the new cost was $55x + 34y$. Equating the two, we have $6y = 5x$, so that $x{:}y = 6{:}5$. **Answer: (c)**

3. Alice is twice as old as Betty, and Betty is seven years younger than Cecilia. The total of their ages is 67. How old is Betty?

   (a) 10 (b) 12 (c) 15 (d) 20

   Solution: Let Betty's age be x. Then Alice's age is $2x$, and Cecilia's age is $x + 7$. From $2x + x + x + 7 = 67$, we have $4x = 60$, or $x = 15$. **Answer: (c)**

4. What is the solution to the equation $\sqrt{3x+10} = -x$?

   (a) –2 (b) 5 (c) –2 and 5 (d) no solution

   Solution: Squaring the equation, we have $3x + 10 = x^2$ or $(x - 5)(x + 2) = 0$. Hence, the solutions are –2 and 5. **Answer: (c)**

5. Let x be any number; $a = 2{,}000x + 1{,}999$; $b = 2{,}000x + 2{,}000$; and $c = 2{,}000x + 2{,}001$. What is the value of $a^2 + b^2 + c^2 - ab - bc - ca$?

   (a) 3 (b) 4 (c) 6 (d) dependent on x

   Solution: Note that $a = b - 1$ and $c = b + 1$. Hence $a^2 + b^2 + c^2 = a(a - b) + b(b - c) + c(c - a) = -a - b + 2c = 3$. **Answer: (a)**

6. The numbers a and b satisfy $1 > -b > a > 0 > b$. Which of the following inequalities must be correct?

   (a) $1 - b > 1 + a > -b > a$ (b) $1 + a > 1 - b > -b > a$ (c) $1 + a > a > 1 - b > -b$ (d) $1 - b > -b > 1 + a > a$

   Solution: In (b), $1 + a > 1 - b$ is wrong. In (c), $a > 1 - b$ is wrong. In (d), $-b > 1 + a$ is wrong. On the other hand, since $-b > a$, we have $1 - b > 1 + a$. Since $1 > -b$ and $a > 0$, we have $1 + a > -b$. **Answer: (a)**

7. No two of the numbers a, b and c are equal, and none of the numbers x, y and z is equal to 0. Which of the following statements must be correct if $\frac{x}{b-c} = \frac{y}{c-a} = \frac{z}{a-b}$?

   (a) $ax + by + cz = 0$ (b) $ax = by = cz$ (c) $x + y + z = 0$ (d) $x = y = z$

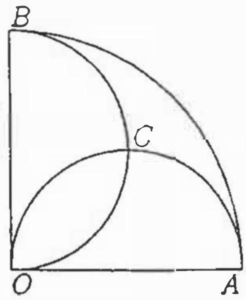
   Solution: Let the common value of the three given fractions be r. Then $x = br - cr$, $y = cr - ar$ and $z = ar - br$. Hence $x + y + z = 0$. **Answer: (c)**

8. In ΔABC, $3\angle A > 5\angle B$ and $2\angle B \geq 3\angle C$. What kind of triangle is ABC?

   (a) acute triangle (b) right triangle (c) obtuse triangle (d) inconclusive

   Solution: Let $\angle A = 25x°$. Then $\angle B < 15x°$ and $\angle C \leq 25x°$. Hence $180° = \angle A + \angle B + \angle C < 50x°$ so that $\angle A = 25x° > 90°$. **Answer: (c)**

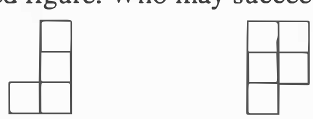
9. The diagram below shows a circular arc AB with centre O, such that OA is perpendicular to OB. Two semicircles with respective diameters OA and OB meet at C. Which has the larger area—the football-shaped region OC or the scythe-shaped region ABC?



(a) OC (c) equal
(b) ABC (d) inconclusive

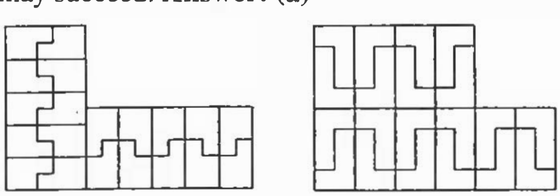
Solution: Let the area of OC be x and the area of ABC be y. The other two regions obviously have the same area, and let their common value be z. Let $OA = 2$. Then $x + y + 2z = \frac{1}{4}\pi(2)^2 = \pi$ while $x + z = \frac{1}{2}\pi(1)^2$. Hence $x + y + 2z = 2(x + z)$ or $x = y$. **Answer: (c)**

10. The diagrams below show an L-shaped figure and a P-shaped figure. Leon wants to build a similar larger P-shaped figure using several identical smaller copies of the L-shaped figure, while Paul wants to build a similar larger L-shaped figure using several identical smaller copies of the P-shaped figure. Who may succeed?



(a) both (b) only Leon (c) only Paul (d) neither

Solution: The diagram below shows that both may succeed. **Answer: (a)**



Part 2: Numeric Response

1. If $2^8 + 4^4 + 16^2 + 256 = 2^x$, what is the value of x?

Solution: We have $2^8 + 4^4 + 16^2 + 256 = 2^8 + 2^8 + 2^8 + 2^8 = 2^2 \times 2^8 = 2^{10}$. **Answer: 10**

2. In a contest, there are 80 per cent more boys than girls, and the average of the girls is 20 per cent higher than that of the boys. If the overall average is 75, what is the average of the girls?

Solution: Let the girls' average be $6x$ so that the boys' average is $5x$. Now the ratio of girls to boys is 5:9. Hence, $5(6x) + 9(5x) = (5 + 9)75$ or $x = 14$. Then, $6x = 6(14) = 84$. **Answer: 84**

3. The number of red balls is more than the number of green balls but less than twice the number of green balls. Each green ball costs \$2, and each red ball costs \$3. If the total cost of the balls is \$60, what is the total number of balls?

Solution: Let there be x red balls and y green balls. Then $y < x < 2y$ and $3x + 2y = 60$. Note that x must be divisible by 2 and y by 3. Let $x = 2u$ and $y = 3v$. Then $3v < 2u < 6v$ and $u + v = 10$. If $u \leq 6$, then $v \geq 4$ and $3v \geq 2u$. If $u \geq 8$, then $v \leq 2$ and $2u \geq 6v$. Hence, $u = 7$, $v = 3$, $x = 14$ and $y = 9$. Then, $x + y = 14 + 9 = 23$. **Answer: 23**

4. If both p and $p^5 + 5$ are prime numbers, what is the smallest value of $n > 5$ such that $p^n + n$ is also a prime number?

Solution: Since $p^5 + 5$ is a prime number, p must be even. Since p is also a prime number, we must have $p = 2$ so that $p^5 + 5 = 37$. In order for $p^n + n$ to be a prime number also, n must be odd. When $n = 7$, we get 135, which is divisible by 5. When $n = 9$, we get 521, which is less than the square of 23. Since 521 is not divisible by any of the prime numbers 2, 3, 5, 7, 11, 13, 17 and 19, it is a prime number itself. **Answer: 9**

5. If $x + y = 5$, $y + z = 8$ and $z + x = 7$, what is the value of $2x + 3y + 5z$?

Solution: Adding the given equations, we have $2(x + y + z) = 20$ or $x + y + z = 10$. Subtracting from this the given equations one at a time, we have $z = 5$, $x = 2$ and $y = 3$. Hence, $2x + 3y + 5z = 2(2) + 3(3) + 5(5) = 38$. **Answer: 38**

6. If $x + \frac{1}{x} = 3$, what is the value of $\frac{x^2}{x^4 + x^2 + 1}$?

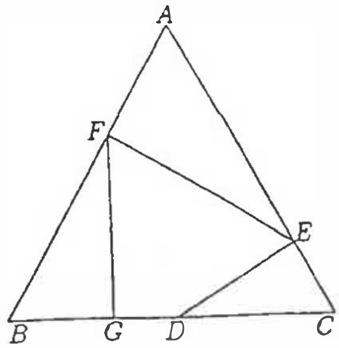
Solution: Squaring the given equation, we have $x^2 + 2 + \frac{1}{x^2} = 9$ so that $x^2 + 1 + \frac{1}{x^2} = 8$. Dividing the denominator of $\frac{x^2}{x^4 + x^2 + 1}$ by the numerator, we obtain $x^2 + 1 + \frac{1}{x^2}$. **Answer: 1/8**

7. How many integers x satisfy the inequality $\frac{4}{\sqrt{3} + \sqrt{2}} < x < \frac{60}{\sqrt{170} - \sqrt{2}}$?

Solution: We have $1 = \frac{4}{2 + 2} < \frac{4}{\sqrt{3} + \sqrt{2}} < \frac{4}{1 + 1} = 2 < 5 = \frac{60}{13 - 1} < \frac{60}{\sqrt{170} - \sqrt{2}} < \frac{60}{12 - 2} = 6$. Thus the acceptable values are 2, 3, 4 and 5. **Answer: 4 integers**

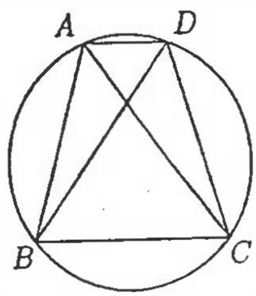
8. The diagram below shows an equilateral triangle ABC of area 128. D is the midpoint of BC. E is the point on CA such that DE is perpendicular to CA. F is the point on AB such that EF is perpendicular to AB. G is the point on BC such that FG is perpendicular to BC. What is the area of the quadrilateral $DEFG$?

Solution: Note that triangles CDE, AEF and BFG are similar to one another. We also have $CD = 2CE$ and $AE = 2AF$. Let $BC = 16x$. Then $CD = 8x$, $CE = 4x$, $AE = 12x$, $AF = 6x$ and $BF = 10x$.



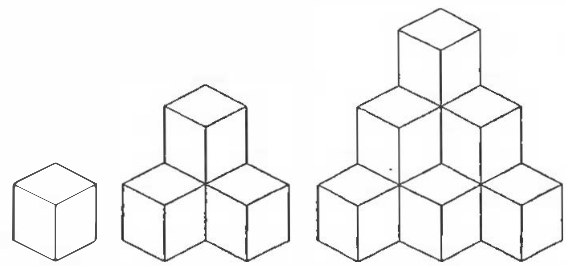
Since $BC = CD$, the area of ACD is 64. Since $AE = 3CE$, the area of CDE is 16. Since $CD{:}AE = 8{:}12$, the area of AEF is 36. Since $AE{:}BF = 12{:}10$, the area of BFG is 25. Hence, the area of $DEFG$ is $128 - 16 - 36 - 25 = 51$. **Answer: 51**

9. The diagram below is a rough sketch of a quadrilateral $ABCD$ inscribed in a circle. The arcs AB, BC and CD have the same length. The diagonals AC and BD meet at E, and $\angle AED = 70°$. What is the measure of $\angle ABD$?



Solution: From the equal arcs, we have $\angle ACB = \angle CAB = \angle DBC$. Since $\angle CEB = \angle AED = 70°$ and $\angle CEB + \angle ACB + \angle DBC = 180°$, each of those three equal angles has measure 55°. Now $\angle ABD = \angle AED - \angle CAB$. **Answer: 15°**

10. The diagram below shows the first three of a sequence of towers. After the first, the next tower is obtained from the preceding one by adding a unit cube on top of each stack of cubes plus a row of unit cubes at the base in front. There are 1, 4 and 10 unit cubes in the first three towers, respectively. How many unit cubes are needed to build the sixth tower?



Solution: To go from the $(n - 1)$st tower to the nth, we add $1 + 2 + \ldots + n = \frac{1}{2}n(n + 1)$ unit cubes. Since the third tower requires 10 unit cubes, the number of unit cubes needed to build the sixth tower is $10 + \frac{1}{2}[(4 \times 5) + (5 \times 6) + (6 \times 7)]$. **Answer: 56**

Each week Susan's class takes a 30-item spelling test. If Susan scored 20 on the first test, what is the lowest she can score on the second test in order for the average of her first three tests to be 26?

The Pacific Institute for the Mathematical Sciences/ University of Alberta Math Fair

Ted Lewis

Close to 700 elementary and junior high students from 25 schools visited the Pacific Institute for the Mathematical Sciences (PIMS)/University of Alberta Math Fair on November 5, 2002. This all-day event, held in Dinwoodie Lounge in the U of A's Students' Union Building, was sponsored by PIMS and presented by Venera Hrimiuc and Ted Lewis's Math 160 students. In the other half of the lounge, Andy Liu ran a problem-solving session. The students spent about two hours at the event, splitting their time between the math fair and the problem-solving session.

The math fair is part of the curriculum of Math 160, a course for elementary education students. There were 26 hands-on puzzles for the visiting students to try. The puzzles were all mathematical and were designed so that the students did not need to solve them with a pencil and paper—just their wits. Although the puzzles were aimed at K–6 students, many of the junior high students also found the puzzles entertaining. Part of the challenge for the university students was adapting their puzzles on the fly to a variety of grade levels. They met the challenge with great success.

In his problem-solving session, Andy taught the students some mathematical games. Of course, many were willing and eager to play against Andy. He never lost, and the problem-solving session was appropriately dubbed the "Math Unfair."

A booklet about math fairs is available through the U of A branch of PIMS (contact Shirley Mitchell). If you are interested in holding your own math fair or would like to attend a workshop on math fairs, contact Andy or Ted (in Edmonton) or Indy Lagu or Sharon Friesen (in Calgary):

Ted Lewis: tlewis@math.ualberta.ca
Andy Liu: al3@gpu.srv.ualberta.ca
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Sharon Friesen: sharon.friesen@shaw.ca

Fun with Mathematics—Challenging the Reader

Andy Liu

Each issue of delta-K *will contain problem sets, which will also be posted on the MCATA website (www.mathteachers.ab.ca).*

The spring issue will contain a set of problems for January, March and May, and the fall issue will contain a set of problems for September and November. Teachers and students are invited to participate by submitting the full solution to each problem by the deadline stated on the website.

Note that the solutions to the problem sets will be published in delta-K *only, with the fall issue containing the solutions for the January, March and May problems and the spring issue containing the solutions for the September and November problems.*

Submit your full solutions to Andy Liu, Department of Mathematics, University of Alberta, Edmonton T6G 2G1; fax (780) 492-6826, e-mail aliumath@telus.net.

January 2003 Problems

1. The numbers 1, 2, . . ., 16 are placed in the cells of a 4×4 table as shown below:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

One may add 1 to all numbers of any row or subtract 1 from all numbers of any column. How can one obtain, using these operations, the table shown below?

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

2. There are four kinds of bills: \$1, \$10, \$100 and \$1,000. Can one have exactly half a million bills worth exactly \$1 million?
3. The king intends to build six fortresses in his realm and to connect each pair with a road. Draw a diagram of the fortresses and roads so that there are exactly three intersections and exactly two roads cross at each intersection.
4. If each boy purchases a pencil and each girl purchases a pen, they will spend a total of 1¢ more than if each boy purchases a pen and each girl purchases a pencil. There are more boys than girls. What is the difference between the number of boys and the number of girls?
5. A six-digit number from 000000 to 999999 is called lucky if the sum of its first three digits is equal to the sum of its last three digits. How many consecutive numbers must we have to be sure of including a lucky number if the first number is chosen at random?
6. Two players play the following game on a 9×9 chessboard. They write in succession one of two signs in any empty cell of the board: the player making the first move writes a plus sign (+) and the other player writes a minus sign (–). When all the squares of the board are filled, the scores of the players are tabulated. The number of rows and columns containing more plus signs than minus signs is the score of the first player, and the number of all other rows and columns is the score of the second player. What is the highest number of points the first player can gain in a perfectly played game?

March 2003 Problems

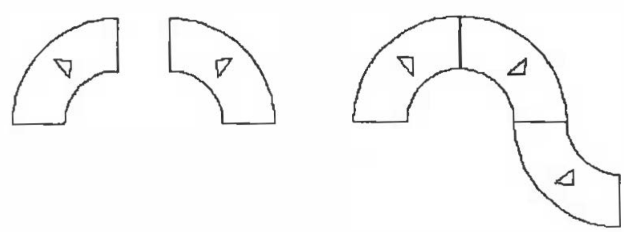
1. Initially, there is a 0 in each cell of a 3×3 table. One may choose any 2×2 subtable and add 1 to all numbers in it. Can one obtain, using this operation a number of times, the table shown below?

4	9	5
10	18	12
6	13	7

2. A teacher plays a game with 30 students. Each writes the numbers 1, 2, . . ., 30 in any order. Then the teacher compares the sequences. A student earns a point each time the same number appears in the same place in the sequences of that student and of the teacher. It turns out that each student earns a different number of points. Prove that at least one student's sequence is the same as the teacher's.
3. Is it possible to write the numbers 1, 2, . . ., 100 in a row so that the difference between any two adjacent numbers is not less than 50?
4. Do there exist two nonzero integers such that one is divisible by their sum and the other is divisible by their difference?
5. A game starts with a pile of 1,001 stones. In each move, choose any pile containing at least two stones and remove one of them, and then split any pile containing at least two stones into two nonempty piles, which need not be of equal size. Is it possible for all remaining piles to have exactly three stones after a sequence of moves?
6. A square castle is divided into 64 rooms in an 8×8 configuration. Each room has a door on each wall and a white floor. Each day, a painter walks through the castle repainting the floors of all the rooms he passes, so that white is changed to black and vice versa. Can he do this so that, after several days, the floors in the castle will be coloured like a chessboard?

May 2003 Problems

1. A jury makes up problems for an Olympiad, with a paper for each of Grades 7–12. The jury decides that each paper should consist of seven problems, with exactly four of them not appearing on any other paper of the Olympiad. What is the greatest number of distinct problems that could be included in the Olympiad?
2. A six-digit number (from 000000 to 999999) is called lucky if the sum of its first three digits is equal to the sum of its last three digits. Prove that the number of lucky numbers is equal to the number of six-digit numbers with a digit sum of 27.
3. Given 32 stones of distinct weights, prove that 35 weightings on a balance are sufficient to determine which are the heaviest and the second heaviest.
4. Find two six-digit numbers such that the number obtained by writing them one after another is divisible by their product.
5. Two players play a game of wild tic-tac-toe on a 10×10 board. They take turns putting either an X or an O in any empty cell on the board. Both players can use X or O, and not necessarily consistently. A player wins the game by making three identical symbols appear in consecutive cells horizontally, vertically or diagonally. Can either player have a winning strategy? If so, is it the player who moves first or the one who moves second?
6. Each section of tracks in a model railway is a quadrant of a circle directed either clockwise or counterclockwise, as shown below in the diagram on the left. One may assemble the track only in such a way that the directions of the sections are consistent along the whole track, as illustrated below in the diagram on the right. If such a closed track can be assembled using given sections, prove that this is no longer the case if one clockwise section is replaced by a counterclockwise section.



"Aha!" Problem Solving: Solution Begging for Meaning

Jerry Ameis

A key recommendation of the National Council of Teachers of Mathematics (NCTM 2000) *Principles and Standards for School Mathematics* concerns problem solving. In relation to classroom practice, a useful way to conceptualize problem solving is (1) teachers creating a classroom climate that facilitates learning in a problem-solving way and (2) students participating in two types of mathematical problem solving—"Aha!" problem solving and "Eureka!" problem solving.

A classroom climate of problem solving can be associated with the notion of teacher as facilitator. Facilitating is generally taken to mean setting up a classroom environment where children can ask and answer their own questions and building openness, reflective thinking, risk taking and investigation into lessons and activities. In such a climate, learning itself is thought of as a problem to be solved.

In the second aspect of problem solving, students solve mathematical problems using "Aha!" problem solving or "Eureka!" problem solving. Other labels for these two types of problem solving are used. For example, some curriculum documents refer to them, respectively, as routine problem solving and nonroutine problem solving.

"Aha!" problem solving involves an "Aha! I know how to do this!" reaction to a problem. This occurs when someone has seen a similar problem before and recognizes that its solution can be readily applied to the current problem. In other words, the problem solver almost automatically knows how to solve a problem because he or she has solved something like it before.

"Eureka!" problem solving involves a "Eureka! I finally figured it out!" reaction to a problem. This occurs when someone finally solves a problem after searching for and trying strategies, such as guess-and-check or think-backward strategies, that might be useful in solving the problem. In this case, a search for strategies is necessary because no convenient model or solution path is readily available to apply to the problem. This kind of problem solving involves much sweat and tears.

Both types of mathematical problem solving are useful and equally valuable. An analogy of going to see a doctor about an illness may help to illustrate the point. When you go to the doctor, you do not want the doctor to say, "Oh dear, that's an interesting illness. I wonder what it is and how to cure it." Rather, you would prefer that the doctor say, "Aha! I saw that illness just yesterday. Here is what is needed to cure it." However, there are times when an illness cannot be readily diagnosed and, therefore, investigation is required. You want the doctor to be able to deal with that circumstance as well.

"Aha!" Problem Solving

In the curricular sense, "Aha!" problem solving involves knowing whether to add, subtract, multiply, divide or use ratio to solve a problem. In the curricular view, problem solving must serve a socially useful function and have immediate and future payoff. For this reason, "Aha!" problem solving is not old-fashioned. Rather, it is a vital part of our lives. Children do "Aha!" problem solving as early as age five or six. They combine and separate toys or money in their normal activities. Adults are regularly called upon to do simple and complex "Aha!" problem solving. For example, a sales promotion in a store advertises a jacket selling for 20 per cent off the regular price of $125.98. If you are interested in buying the jacket, you might think, *Aha! I know that .20 × $125.98 is about $25. This means that the jacket will cost me about $100. Yup, I can afford it!*

This is a good opportunity to clarify an important matter: the ability to estimate .20 × $125.98 and subtract $25 from $125 is not the critical characteristic of the curricular sense of "Aha!" problem solving. Rather, the critical characteristic is realizing that you need to do those calculations to solve the problem. Actually doing the calculations is only the necessary final step in solving the problem. In other words, knowing what arithmetic to do, not actually doing it, is the critical characteristic of the curricular sense of "Aha!" problem solving.

Knowing what arithmetic to do is directly related to understanding the meanings of the four arithmetic operations and the concept of ratio. Although there are only four arithmetic operations—adding, subtracting, multiplying and dividing—these four operations have more than four distinct meanings (for more detail, refer to www.uwinnipeg.ca/~jameis/New%20Pages/EYR23.html). For example, *subtract* has at least three meanings: "take away," "compare two sets" and "find the change from one measurement to another." Consider the following situations. Each can be mathematically modelled by the number sentence 7 – 3 = 4, yet each involves a quite different meaning of *subtraction:*

- I had seven cookies in my lunch bag. I gave three away. Now I have four cookies.
- I have $7. Sam has $3. I have $4 more than Sam.
- The temperature at 9 a.m. was 3°C. At noon, the temperature was 7°C. The change in temperature was 4°C.

If students do not understand the meanings of the arithmetic operations and of ratio, "Aha!" problem solving can easily become a quagmire of confusion for them. Consider the following problem presented to a group of Grades 4 and 5 students:

> Harry went on a trip to the Nile River. His task was to completely fill his crocodile-egg incubator. On the first day of his journey, Harry came upon a crocodile nest. He dug up 23 unhatched crocodile eggs and placed them in the incubator. Harry walked five paces and dug up some more unhatched crocodile eggs, which he quickly placed in the incubator as well. Harry's task was now done: he had completely filled his egg incubator with 40 unhatched crocodile eggs. How many eggs did Harry find at the second crocodile nest?

The students had been taught two problem-solving strategies: (1) guess and check and (2) look for a pattern. The responses of three students typify the thinking of the other students in the class.

Student 1 read the problem carefully, underlined all the numbers and then said, "Easy numbers!" The student's answer of 44 was incorrect (although he strongly insisted that it was correct). His calculations were as follows: 40 + 5 = 45; 45 – 23 = 22; 22 × 2 = 44. To explain the work, the student stated that, because there were 40 eggs, it was necessary to add "the little number" (the 5) and then subtract 23. That answer then had to be multiplied by 2 because the problem had stated the word *second* somewhere.

Student 2 did not do the problem, explaining that the problem did not make any sense because it did not fit either of the problem-solving strategies learned in class.

Student 3 made a guess-and-check chart, but did not do any guessing. She obtained the correct answer of 17 by subtracting 23 from 40 but could not explain the subtraction strategy. When asked if she had done any guessing, she replied no. When asked why she had made a guess-and-check chart, she replied, "All word problems are done by guess and check."

The three student responses illustrate several things, including the inadequacy of the guess-and-check and look-for-a-pattern strategies in solving problems such as the one about the crocodile eggs. These strategies have little value in "Aha!" problem solving. They are, however, quite useful for "Eureka!" problem solving. The responses also illustrate the problem-solving difficulties that arise when students do not understand the meanings of the arithmetic operations.

The strategies that children often use for "Aha!" problem solving take a variety of forms. The following is a representative list (Sowder 1988):

1. Find the numbers and add (or multiply or . . .). The choice tends to be dictated by recent activities or by what the child feels comfortable with.
2. Guess the operation to be used and see what you get. This is a form of guess and check.
3. Look at the numbers; they will "tell" you which operation to use (for example, the numbers 63 and 59 suggest add or subtract, while 25 and 5 suggest divide).
4. Try all the operations and select the most reasonable answer.
5. Look for key words that tell you what to do.
6. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication, and select the more reasonable answer. If smaller, try both subtraction and division and select the more reasonable answer.
7. Choose the operation whose meaning fits what is happening in the problem.

The strategy of looking for key words warrants special comment because of its prevalence. Using this strategy, if the word *minus* appears in a problem, you subtract to get the answer; if the word *more* appears, you add to get the answer. Looking for key words is inappropriate for at least three reasons.

First, the strategy has little to do with understanding what the problem is actually about. Therefore, transfer to more complex problems—those that contain more than one step—tends to be difficult. For complex problems, merely picking out key words as indicators of what to do often leads to incorrect solutions.

Second, in contrast to the problems typically found in mathematics textbooks and resource books, many real-world problems do not contain key words. In life, there are rarely any flashing neon signs that signal what to do to solve a problem. What does the problem solver do, then, if all he or she knows is to look for key words?

Finally, key words can be misleading. For example, *more* is not necessarily a signal to add. Consider the following:

> Johnny had 139 candies. He ate 12 of them. Then he ate 15 more. How many candies does Johnny have now?

In this problem, adding would lead to an interesting (and incorrect) answer.

All key words can be misleading, even the word *total*. Is *total* always a signal to add? Consider the following two problems:

> Bucky the squirrel was storing acorns for the winter. Bucky had a large family. He hid 235 acorns in each of 985 holes. What was the total number of acorns Bucky hid?

> Sparks kept all his money in a piggy bank. When he counted it one day, he found that he had $345. He decided to put 2/3 of his money in a real bank. What was the total amount of money he put in the real bank?

To further appreciate the trap that key words can be for students, consider this example:

> Four out of five coaches recommend heavy-weight lifting as an important part of becoming an athlete. What percentage of coaches do not recommend it?

A student quickly answered the problem correctly by saying that the answer was 20 per cent. The teacher asked the student to explain how he had figured out the answer. The student replied, "I multiplied 4 by 5 because *of* means 'multiply.'"

Of the strategies in Sowder's (1988) list, only the seventh—"Choose the operation whose meaning fits what is happening in the problem"—will consistently lead to success with "Aha!" problem solving. That strategy involves identifying what is going on in the problem and then representing it mathematically. Mathematical forms such as number sentences can be used to represent what is going on in a problem. In the problem about the crocodile nest, eggs were combined in a "put together" sense, which can be modelled mathematically by the number sentence $23 + ? = 40$.

Research shows that good problem solvers identify what is going on in a problem rather than searching for the algorithm (the arithmetic process) to apply. This research strongly suggests that, to develop student proficiency in "Aha!" problem solving, teachers should use a teaching approach that involves modelling/representing the events or circumstances of problems.

A study found that modelling is a useful strategy for "Aha!" problem solving as early as Kindergarten (Carpenter et al. 1993). The Kindergarten students in the study had spent a year solving a variety of basic word problems related to addition, subtraction, multiplication and division. They used concrete materials to model the events of the stories. Overall, the children demonstrated remarkable ability in solving the word problems. Most of them solved most of the problems. They did as well as or better than children in Grades 1 and 3 who solved similar problems but did not consistently use modelling as the problem-solving strategy. The Kindergarten students' success can be explained by their consistent use of modelling to represent the actions of the problems in a way that reflected what was going on in the story.

The approach recommended here for developing proficiency in "Aha!" problem solving requires that students deeply learn the necessary conceptual tools—the meanings of the arithmetic operations and of ratio. The students use these tools to understand what is going on in a problem and, based on that understanding, to represent what is going on through a number sentence. Then, they decide which algorithm to use and perform that algorithm, using a calculator, paper and pencil, or mental arithmetic.

What is happening in the problem and the algorithm do not have to match. Consider the following problem:

> Sleepy the dwarf had some jewels. On the way home, he gave 123 away. Now he has 767 jewels. How many did he have to begin with?

The problem involves giving away jewels. The number sentence that best represents that action is $? - 123 = 767$, not $123 + 767 = ?$. Nothing has been put together or combined; rather, something (123 jewels) has been removed, and that involves the "take away" meaning of *subtraction*. However, the convenient algorithm to use for obtaining the answer involves addition.

Transforming the number sentence ? – 123 = 767 to ? = 123 + 767 turns the arithmetic operation from subtraction into addition. The transformation is justified by the inverse relationship between addition and subtraction.

At this point, the reader may be thinking, *I do the problem about Sleepy by simply adding the numbers. Why can't children?* That is a valid question that needs to be addressed. Once children deeply understand the meaning of an arithmetic operation, they should be encouraged and expected to take shortcuts with it. The meanings of the operations are best seen as models for viewing the world. They are not truths; rather, they are useful or not-useful tools. Once a tool is understood, it can be applied in a variety of ways. The problem about Sleepy could be conceptualized as "What I have left combined with what I gave away gives me what I started with." The number sentence for that would be 767 + 123 = ?. That way of thinking is a shortcut to solving the problem. Taking shortcuts should be encouraged once children deeply understand the meanings of the arithmetic operations. But shortcuts are not likely to be used consistently or reliably if children do not understand what is going on in the problem in the first place.

To conclude, here is a problem for the reader:

> George is opening a pizza restaurant. He must think small at first because his resources are limited. He decides to sell only two-topping pizzas, with one meat topping and one vegetable topping. He offers four meat toppings: pepperoni, sausage, ham and salami. Because George believes more is better, he offers five vegetable toppings: onions, mushrooms, green peppers, olives and tomatoes. George wants to make a menu listing all the combinations of two-topping pizzas he sells. What is the total number of pizzas he should list on the menu?

Write the number sentence that best represents what is going on in the problem. Explain why you wrote that number sentence, based on one of the meanings of the arithmetic operations (see www.uwinnipeg.ca/~jameis/New%20Pages/EYR23.html).

References

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Sowder, L. *Concept-Driven Strategies for Solving Story Problems in Mathematics*. Washington, D.C.: National Science Foundation, 1988.

The average of five numbers is 20. If x and y are two of the numbers, what is the average of the other three numbers in terms of x and y?

The Ancient Problem of Trisecting an Angle

Sandra M. Pulver

The problem of trisecting an angle dates back to the ancient Greeks, and as early as the fifth century B.C., Greek and Muslim geometers devoted much time to this puzzle. This problem is one of the three famous geometric problems of antiquity, which also include doubling the cube and squaring the circle. These three great construction problems of geometry could not be solved using an unmarked straightedge and compass stone, the only implements sanctioned by the ancient Greeks. But it was not until the 19th century that advances in the algebra of the real-number system allowed us to make instruments that made possible these constructions that were impossible with the straightedge and compass alone.

The problem of trisecting an angle is the simplest of the three famous problems to comprehend, and because the bisection of an angle presented no difficulty to the geometers of antiquity, there was no reason to suspect that its trisection might prove impossible.

The multisection of a line segment with Euclidean tools is a simple matter, and it may be that the ancient Greeks were led to the trisection problem in an effort to solve the analogous problem of multisection of an angle. Or, perhaps more likely, the problem arose in efforts to construct a regular nine-sided polygon, where the trisection of a 60° angle is required.

The angle trisection problem is not entirely unsolvable using the classical method of compass and straightedge. The Greeks knew this, but they were searching for a generalized construction (as in angle bisection) that could be used to trisect any angle.

Actually, an infinite number of angles can be trisected. Among this group are angles whose degree measure equals $360/n$, where n is an integer not evenly divisible by 3. For example, a 90° angle can be trisected because n in $360/n$ is 4, which is not evenly divisible by 3. Figure 1 shows a trisected 90° angle. The trisection of the 90° angle can be done quite simply using the following method:

> Construct a 90° angle, $\angle AOB$. Then, draw arc AB. Without changing the size of the compass opening, place the compass at point B and draw an arc intersecting arc AB at point C. Line OC is the line trisecting right angle $\angle AOB$. Line OD also trisects right angle $\angle AOB$ using the same method outlined above and placing the compass at point A. Line OD also bisects 60° angle $\angle COB$.

However, an infinite number of angles cannot be trisected by means of compass and straightedge. These are angles whose degree measures are equal to $360/n$, where n is an integer divisible by 3. For example, a 60° angle cannot be trisected because n, in $360/n$, would be 6, which is divisible by 3. To prove that general angle trisections are impossible with just an unmarked straightedge and compass, we use the special case of a 60° angle (see Figure 2). Suppose $\angle COA$ = 60° and $\angle BOA$ = 20°. For the proof, we use the following trigonometric identity:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Let $3\theta = 60°$ and let $x = 2\cos\theta = 2\cos 20°$. Then,

$$\cos 60° = \frac{x^3}{2} - \frac{3x}{2}$$

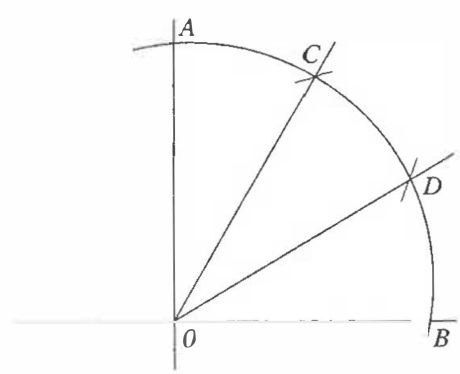
$$2\left(\frac{1}{2}\right) = \left(\frac{x^3}{2} - \frac{3x}{2}\right)2$$

$$1 = x^3 - 3x$$

$$x^3 - 3x - 1 = 0.$$

This cubic equation is irreducible. Thus, its roots cannot be constructed with a straightedge and compass. From this, we can conclude that the construction of trisecting the general angle cannot be performed with straightedge and compass alone.

Figure 1
Trisection of a 90° Angle, $\angle AOB$



An ingenious method for trisecting angles was presented by Archimedes, who used the marking of two points on a straightedge to mark off a line segment (not using the classical rules of compass and straightedge). Figure 3 shows a trisected 60° angle.

Archimedes' method is as follows. Draw $\angle AED$. Through the angle, draw a semicircle with a radius the same length as DE. Extend DE to the right. With the compass open to the length of radius DE, hold the legs of the compass against the straightedge and hold the straightedge so it passes through point A. Adjust the straightedge until the points marked by the compass intersect points B and C (where BC is equal to DE). Arc BF is 1/3 of arc AD. Because central angles are congruent in degree measure to their intercepted arcs, $\angle BEF$ is 1/3 of the degree measure of $\angle AED$.

Another method of trisecting an angle is using the conchoid of Nicomedes. Figure 4 helps to define *conchoid*.

Nicomedes took a fixed point O, which is d distance from a fixed line AB, and drew OX parallel to AB and OY perpendicular to OX. He then took any line OA through O and on OA made $AP = AP' = k$, a constant. Then the locus of points P and P' is a conchoid. The equation of the curve is

$$(x^2 + y^2)(x - d)^2 - k^2x^2 = 0.$$

To trisect a given angle, let $\angle YOA$ be the angle to be trisected. From point A, construct AB perpendicular to OY. From point O as pole, with AB as a fixed

Figure 2
Attempted Trisection of a 60° Angle, $\angle COA$

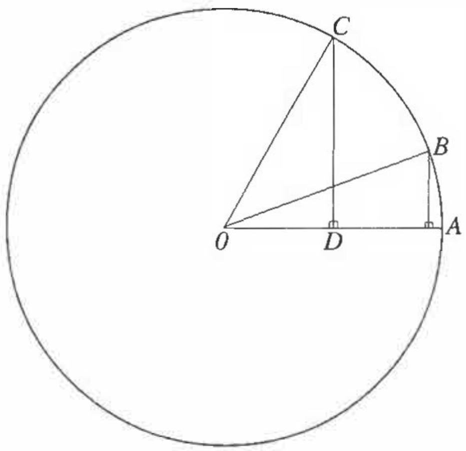


Figure 3
A Trisected 60° Angle, $\angle AED$

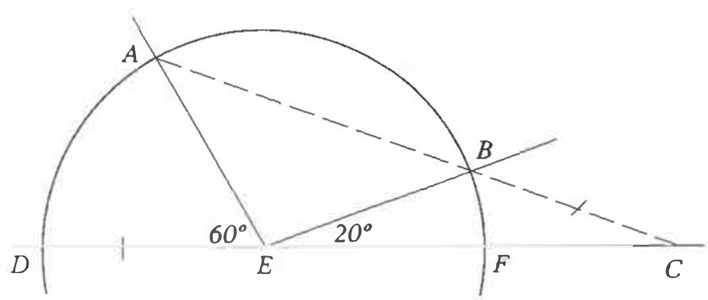
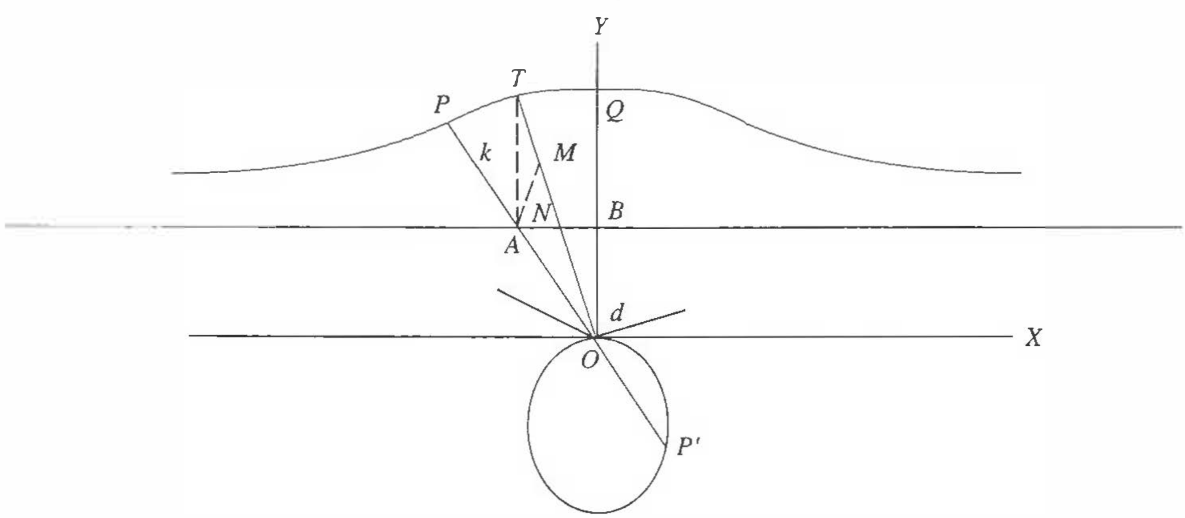


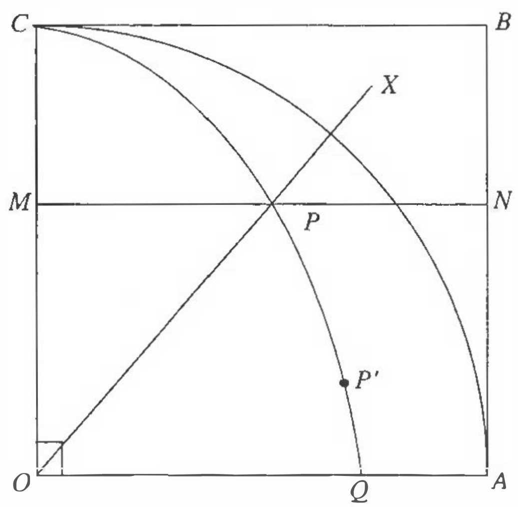
Figure 4
Conchoid of Nicomedes



straight line, $2(AO)$ as a constant distance, construct a conchoid to meet OA produced at P and to cut OY at Q. At A, construct a perpendicular to AB meeting the curve at T. Draw OT and let it cut AB at N. Let M be the midpoint of NT. Then $MT = MN = MA$. But $NT = 2(OA)$ by construction of the conchoid. Hence, $MA = OA$. Hence, $\angle AOM = \angle AMO = 2\ \angle ATM = 2\ \angle TOQ$. That is, $\angle AOM = \frac{2}{3}\ \angle YOA$, and $\angle TOQ = \frac{1}{3}\ \angle YOA$.

Hippias of Elis wrestled with this problem and, realizing the inadequacy of the ruler-and-compass method, resorted to other devices. These involved the use of curves other than the circle. The one employed by Hippias was the quadratrix, so called because it serves as well for the problem of quadrature (squaring the circle) as for dividing an angle into three or more equal parts. The quadratrix of Hippias may be defined as follows (see Figure 5). Let the radius OX of a circle rotate uniformly about the centre O from OC to OA, with $\angle COA$ forming a right angle. At the same time, let a line MN parallel to OA move uniformly parallel to itself from CB to OA. The locus of the intersection P of OX and MN is the quadratrix.

Figure 5
Quadratrix of Hippias



In the trisection of an angle, X is any point in the quadrant AC. As the radius OX revolves at a uniform rate from OC to OA, MN always remains parallel to OA. Then if MN is one nth of the way from CB to OA, the locus of point P (the intersection of OX and MN) is one nth of the way from OC around to OA. If, therefore, we make $CM = 1/3(CO)$, MN will cut CQ at a point P such that OP will trisect the right angle. In the same way, by trisecting OM, we can find a point P' on CQ such that OP' will trisect $\angle AOX$, and so for any other angle. This method evidently applies to the multisection as well as the trisection of an angle.

I have cited only three of the most ancient methods. There are many other techniques for trisecting an angle using tools such as a tomahawk or a mira.

Ingenious mathematicians of recent times have developed original methods to trisect angles. Leo Moser of the University of Alberta trisected angles with the use of an ordinary watch. He discovered that if the minute hand passed over an arc equal to four times the measure of the angle to be trisected, the hour hand would move through an arc exactly 1/3 the measure of the given angle to be trisected. Alfred Kempe, a London lawyer, developed a linkage method of folding parallelograms so that the two opposite sides cross.

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On Circumscribed Circles

David E. Dobbs

My recent article (Dobbs 2003) proposed some enrichment material for the typical precalculus course by developing several methods to study angle bisectors and inscribed circles of triangles. It seems natural to ask if the subject of circumscribed circles of triangles can also provide enrichment material for precalculus. This article shows how that may be done. The insight that such coverage is possible at the precalculus level is not new. For instance, Smith's (1956, 101) classic treatise includes an exercise asking for an equation of the circumscribed circle of a given triangle. Because Smith's text does not suggest a method to work that exercise, it is of some interest to find several such methods, and I do so here.

One way to proceed is to develop methods for finding the perpendicular bisector of a line segment, because the centre K of the circumscribed circle of a triangle Δ is the intersection of the perpendicular bisectors of the sides of Δ. For this reason, I devote the next section to developing two methods for constructing perpendicular bisectors of segments. Of course, once we have found K, we can use the distance formula to find the radius r of the circumscribed circle, because r is the distance from K to any vertex of Δ. Then, given K and r, we can move directly to the standard form equation of the circumscribed circle. The associated discussion in this section reinforces several precalculus topics: standard form equations of circles, midpoint formula, equations of lines, solution of systems of two linear equations in two unknowns, slope and the "negative reciprocals" criterion for perpendicularity. Note that the last of these topics is so fundamental that it can be proved in at least four ways in a precalculus course (Dobbs and Peterson 1993, 39, 41, 338, 427). To make matters concrete and more user-friendly, I give a numerical illustration of the methods here and in the following section by applying them to a particular triangle Δ.

In the final section of the article, I turn matters around by giving two methods for directly finding an equation for the circumscribed circle of a given triangle. Of course, with such an equation in hand, we can recover the coordinates of the centre and the radius of this circle by completion of squares. Both methods in this section are necessarily more algebraic than the methods developed in the earlier section. The first of these algebraic methods involves solving a system of three linear equations for three unknowns and thus reinforces an important topic from precalculus/algebra. The second method can be approached through Cramer's rule, thus reinforcing the study of determinants (and, possibly, matrices). In a closing comment, I draw an analogy between the second algebraic method and an old, but often overlooked, method for finding an equation of the line through two given points.

Two Methods for Finding Perpendicular Bisectors

Suppose we are given two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the Euclidean plane. There is an obvious, direct way to find the perpendicular bisector of the line segment P_1P_2. For this method, first recall from the midpoint formula (Dobbs and Peterson 1993, 34–35), a nice application of the theory of proportion and similar triangles, that the midpoint of the segment is $Q[(x_1 + x_2)/2, (y_1 + y_2)/2)]$. Next, we need only write an equation for the line that passes through Q and is perpendicular to P_1P_2.

Let us illustrate the above method by finding an equation for the circumscribed circle of ΔABC, given the vertices $A(-9, 11)$, $B(2, -4)$ and $C(6, 8)$. (Readers of Dobbs [2003] will surely recognize this "random" triangle.) The midpoint of the segment AB is $(-7/2, 7/2)$, and the slope of AB is $(-4 - 11)/[2 - (-9)] = -15/11$. By the "negative reciprocals" result, the perpendicular bisector of AB has slope 11/15 and, thus, by the point-slope form of the equation of a line, has equation

$$y = \frac{11}{15}\left[x - \left(-\frac{7}{2}\right)\right] + \frac{7}{2}$$

or, equivalently,

$$11x - 15y + 91 = 0.$$

Similarly, one verifies that the midpoint of the segment CA is (–3/2, 19/2) and that the perpendicular bisector of this segment is $5x - y + 17 = 0$.

By solving the system of linear equations

$$\begin{cases} 11x - 15y + 91 = 0 \\ 5x - y + 17 = 0 \end{cases}$$

for the unknowns x and y, one finds the coordinates (h, k) of the centre K of the circumscribed circle of ΔABC to be (–164/64, 67/16) = (–41/16, 67/16). The radius r of this circle is the distance between K and any vertex, say C, and so, by the distance formula,

$$r^2 = \left[6 - \left(-\frac{41}{16}\right)\right]^2 + \left(8 - \frac{67}{16}\right)^2 = \frac{22{,}490}{256} = \frac{11{,}245}{128}.$$

Then the standard form equation of the circumscribed circle is

$$(x - h)^2 + (y - k)^2 = r^2,$$

that is,

$$\left[x - \left(-\frac{41}{16}\right)\right]^2 + \left(y - \frac{67}{16}\right)^2 = \frac{11{,}245}{128}$$

or, equivalently, $x^2 + y^2 + (41/8)x - (67/8)y - 255/4 = 0$.

I turn next to the second method promised in the title of this section. This method depends on the following fact from Euclidean plane geometry (a prerequisite for the typical precalculus course): given distinct points P_1 and P_2 in the plane, a point Q in that plane is on the perpendicular bisector of the segment P_1P_2 if and only if Q is equidistant from P_1 and P_2. (The proof of this fact is a familiar application of congruence criteria: use side–angle–side and either side–side–side or hypotenuse–side.) This fact justifies my earlier comment that the centre of the circumscribed circle of a triangle is the intersection of the perpendicular bisectors of the sides of that triangle. Next, we can use this fact to find the perpendicular bisector of P_1P_2, as follows. By the distance formula, a point $Q(x, y)$ is on this perpendicular bisector if and only if

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2.$$

It is clear that algebraic simplification of the preceding equation leads to the desired linear equation (because the terms in x^2 and y^2 cancel). Rather than write the general form of this linear equation, let us illustrate it by returning to the data examined above.

Consider the segment AB, given $A(-9, 11)$, $B(2, -4)$. The above method yields that the perpendicular bisector of AB is given by

$$[x - (-9)]^2 + (y - 11)^2 = (x - 2)^2 + [y - (-4)]^2$$

or, equivalently (after cancellation of the terms in x^2 and y^2), $22x - 30y + 182 = 0$. This equation is equivalent to $11x - 15y + 91 = 0$, thus agreeing with the equation found by the first method. (It is interesting that our foray into quadratic equations has led to an arguably faster way to find this linear equation!) I encourage the reader to practise the second method to recover the earlier equation for the perpendicular bisector of the segment CA. Of course, with these two equations in hand, one can proceed as above to find the centre, radius and standard form equation for the circumscribed circle of ΔABC.

Two Algebraic Methods

Suppose that we are given three noncollinear points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ in the Euclidean plane (that is, the vertices of some triangle Δ). One way to find an equation for the circle passing through these three points (that is, the circumscribed circle of Δ) is to solve for the coefficients A, B and C in an equation, $x^2 + y^2 + Ax + By + C = 0$, for this circle. (Recall that circles are characterized as the graphs of equations of this form such that $A^2 + B^2 > 4C$.) For this, we solve the system of linear equations

$$\begin{cases} x_1^2 + y_1^2 + Ax_1 + By_1 + C = 0 \\ x_2^2 + y_2^2 + Ax_2 + By_2 + C = 0 \\ x_3^2 + y_3^2 + Ax_3 + By_3 + C = 0 \end{cases}$$

for the unknowns A, B and C. By completion of squares, we then find the centre $K(h, k)$ and radius r of this circle. We can then easily find equations for the perpendicular bisectors of the sides of Δ.

I next illustrate the procedure for the usual data, the triangle with vertices (–9, 11), (2, –4) and (6, 8). For this example, the above system of equations is

$$\begin{cases} (-9)^2 + 11^2 - 9A + 11B + C = 0 \\ 2^2 + (-4)^2 + 2A - 4B + C = 0 \\ 6^2 + 8^2 + 6A + 8B + C = 0 \end{cases}.$$

Any of the standard methods for solving such a linear system leads to the unique solution $A = 41/8$, $B = -67/8$ and $C = -255/4$, thus agreeing with the result of the methods in the preceding section. By completing squares, we can rewrite this equation as

$$\left[x - \left(-\frac{41}{16}\right)\right]^2 + \left(y - \frac{67}{16}\right)^2 = \frac{11{,}245}{128}$$

from which we recover the facts that K is (–41/16, 67/16) and r is determined as the principal square root of $r^2 = 11{,}245/128$. Finally, to identify the perpendicular bisector of one of the sides of the given triangle, say of AB, one of many possible ways to proceed would be to write the two-point form of the equation of the line through K and the midpoint of AB. We then obtain

$$y = \frac{\frac{67}{16} - \frac{7}{2}}{\frac{-41}{16} - \frac{-7}{2}}\left(x - \frac{-41}{16}\right) + \frac{67}{16}.$$

This equation simplifies to $352x - 480y + 2{,}912 = 0$, or equivalently, $11x - 15y + 91 = 0$, thus agreeing with a calculation in the preceding section.

In presenting the above algorithm, I glossed over one theoretical point—namely, how we can be sure, once we have solved for the unknowns A, B and C, that the equation $x^2 + y^2 + Ax + By + C = 0$ actually represents a circle. One answer depends on the following two observations: (1) by completion of squares, we see that the graph of any equation of this form is either a circle, a singleton set (that is, a set consisting of just one point) or the empty set and (2) the graph of this equation does pass through the three distinct points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$, because the above system of linear equations is satisfied and, hence, must be a circle.

I turn next to the second "algebraic" method promised in the title of this section. Quite simply, this method presents the following equation for the circle passing through three given noncollinear points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ in the plane:

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

For our usual example, the above equation is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ (-9)^2 + 11^2 & -9 & 11 & 1 \\ 2^2 + (-4)^2 & 2 & -4 & 1 \\ 6^2 + 8^2 & 6 & 8 & 1 \end{vmatrix} = 0.$$

By expanding the determinant along its first row, we can rewrite this equation as

$$192(x^2 + y^2) + 984x - 1{,}608y - 12{,}240 = 0$$

or, equivalently,

$$x^2 + y^2 + (41/8)x - (67/8)y - 255/4 = 0,$$

thus agreeing with the result already obtained twice above by other methods.

Why is the above determinental method valid in general? To answer this question, first notice that the proposed equation is satisfied by each of the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ because the determinant vanishes for any square matrix having two equal rows. Moreover, by the above comments, if the proposed equation is truly quadratic, then its graph must be a circle (because we now know that it is neither a singleton set nor empty). Finally, the proposed equation is truly quadratic. Indeed, expanding along the first row of the determinant appearing in this equation, we see that the coefficient of $x^2 + y^2$ is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

This determinant is nonzero (thus, the proposed equation is truly quadratic) for a fundamental geometric reason. In fact, the absolute value of this determinant can be shown to be twice the area of ΔABC (see Dobbs and Peterson [1993, 537]). Verification of this assertion makes for an accessible computational exercise early in a precalculus course (and a much easier, more conceptual exercise later for a student who knows about the crossproduct of vectors).

I next give another justification for the above determinental method. Still working in the Euclidean plane, let $Q(x, y)$ be another point on the circle passing through the three noncollinear points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$. Consider the system of linear equations

$$(x_2 + y_2)S + xA + yB + 1 \bullet C = 0$$
$$(x_1^2 + y_1^2)S + x_1A + y_1B + 1 \bullet C = 0$$
$$(x_2^2 + y_2^2)S + x_2A + y_2B + 1 \bullet C = 0$$
$$x_3^2 + y_3^2 + x_3A + y_3B + 1 \bullet C = 0$$

for the unknowns S, A, B and C. Because the four points Q, P_1, P_2 and P_3 all lie on some circle $x^2 + y^2 + Ax + By + C = 0$, there is a nontrivial solution for these unknowns (in which $S = 1$). Consequently, by Cramer's rule, the coefficient matrix of the above linear system has its determinant equal to 0. The statement that this determinant equals 0 is precisely the proposed determinental method; and we can see that the proposed equation is truly quadratic (that is, after we expand along the top row, the coefficient of $x^2 + y^2$ is nonzero) as explained above.

In closing, I note that the second "algebraic" method can be modified to give an equation for the line passing through any two distinct points, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. (This observation was made by Nathan Mendelsohn in a lecture I attended in 1962, but I have not seen it elsewhere. Nor have I seen the analogous description of an equation for the sphere passing through four given noncoplanar points, but that would not be as useful, because it involves a 5×5 determinant.) More specifically,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

is an equation for the line passing through the two given points P_1 and P_2. As above, the verification can proceed in either of two ways: (1) invoke Cramer's rule or (2) note that the expansion of the determinant gives a nontrivial linear equation satisfied by both the given points. Because the first postulate of Euclid's *Elements* states that exactly one line passes through any given pair of distinct points, we are done. In addition, by appealing to the very foundations of Euclidean geometry, this algebraic activity has served to illustrate the unity of mathematics.

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A is 25 per cent of *B*, and *B* is 30 per cent of *C*. What percentage of *C* is *A*? What percentage of *A* is *C*?

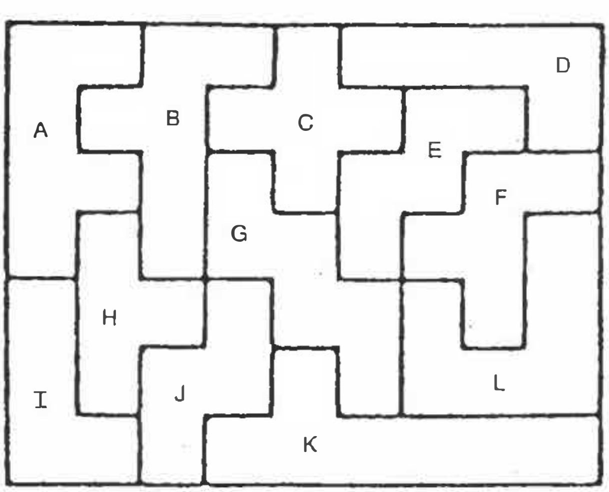
Probabilities: An Example of Linking Mathematical Ideas

Bonnie H. Litwiller and David R. Duncan

Teachers are always seeking situations in which mathematical ideas can be connected. In this article, we discuss ways in which a rectangular puzzle can be used to connect geometry and probability. These activities assume a background in geometry and a familiarity with the basic language of probability. They can be done by the whole class, in groups or for individual recreation at the teacher's discretion.

The puzzle to be discussed consists of 12 pieces that may be arranged to form a rectangle (see Figure 1).

Figure 1



Let us assign numerical values to the rectangle. Assume it is a 9×7 rectangle; then the perimeters and areas can be found (see Figure 2).

Figure 2

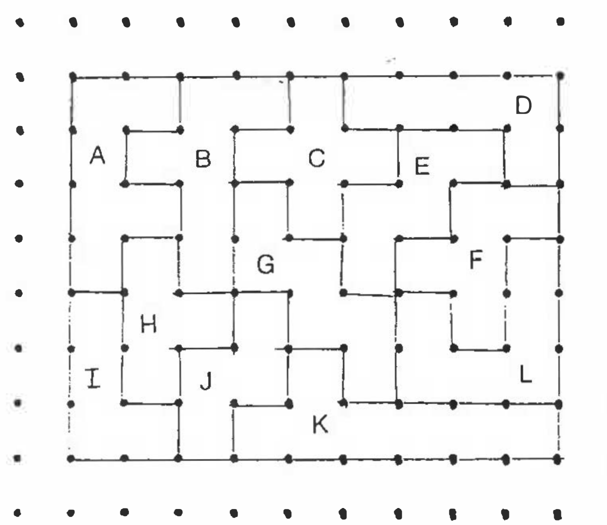


Table 1 shows the perimeter and area of each of the 12 pieces.

Table 1

Figure	Area (in square units)	Perimeter (in units)
A	6	14
B	6	14
C	5	12
D	5	12
E	5	12
F	5	12
G	6	14
H	4	10
I	4	10
J	4	10
K	7	16
L	6	14

We can now discuss links to probability. Suppose we cut the rectangle into its 12 pieces.

Problem 1

If we select two pieces at random, with replacement (that is, the first piece is returned before the second piece is selected), what is the probability that their areas are equal? What is the probability that their perimeters are equal? There are 144 possible pairings, which are listed in Table 2.

Area

Of the 144 equally likely pairs in Table 2, the following 42 pairs contain figures of equal area: AA, AB, AG, AL, BA, BB, BG, BL, CC, CD, CE, CF, DC, DD, DE, DF, EC, ED, EE, EF, FC, FD, FE, FF, GA, GB, GG, GL, HH, HI, HJ, IH, II, IJ, JH, JI, JJ, KK, LA, LB, LG and LL.

The probability of drawing two pieces with the same area is, thus, $\frac{42}{144}$, or 0.29.

Perimeter

Of the 144 equally likely pairs in Table 2, 42 contain figures of the same perimeter (the same 42 pairs as in the area problem). Thus, the probability of drawing two pieces with the same perimeter is $\frac{42}{144}$, or 0.29.

Problem 2

Let us redo the first problem, this time without replacement (that is, the first piece is not replaced before the second is drawn). Table 2 must be adjusted to eliminate all pairs in which the same piece is selected twice (see Table 3).

There are 132 pairs remaining.

Table 2

AA	AB	AC	AD	AE	AF	AG	AH	AI	AJ	AK	AL
BA	BB	BC	BD	BE	BF	BG	BH	BI	BJ	BK	BL
CA	CB	CC	CD	CE	CF	CG	CH	CI	CJ	CK	CL
DA	DB	DC	DD	DE	DF	DG	DH	DI	DJ	DK	DL
EA	EB	EC	ED	EE	EF	EG	EH	EI	EJ	EK	EL
FA	FB	FC	FD	FE	FF	FG	FH	FI	FJ	FK	FL
GA	GB	GC	GD	GE	GF	GG	GH	GI	GJ	GK	GL
HA	HB	HC	HD	HE	HF	HG	HH	HI	HJ	HK	HL
IA	IB	IC	ID	IE	IF	IG	IH	II	IJ	IK	IL
JA	JB	JC	JD	JE	JF	JG	JH	JI	JJ	JK	JL
KA	KB	KC	KD	KE	KF	KG	KH	KI	KJ	KK	KL
LA	LB	LC	LD	LE	LF	LG	LH	LI	LJ	LK	LL

Table 3

	AB	AC	AD	AE	AF	AG	AH	AI	AJ	AK	AL
BA		BC	BD	BE	BF	BG	BH	BI	BJ	BK	BL
CA	CB		CD	CE	CF	CG	CH	CI	CJ	CK	CL
DA	DB	DC		DE	DF	DG	DH	DI	DJ	DK	DL
EA	EB	EC	ED		EF	EG	EH	EI	EJ	EK	EL
FA	FB	FC	FD	FE		FG	FH	FI	FJ	FK	FL
GA	GB	GC	GD	GE	GF		GH	GI	GJ	GK	GL
HA	HB	HC	HD	HE	HF	HG		HI	HJ	HK	HL
IA	IB	IC	ID	IE	IF	IG	IH		IJ	IK	IL
JA	JB	JC	JD	JE	JF	JG	JH	JI		JK	JL
KA	KB	KC	KD	KE	KF	KG	KH	KI	KJ		KL
LA	LB	LC	LD	LE	LF	LG	LH	LI	LJ	LK	

Area

The 42 pairs of the same area in the first problem must now be reduced by eliminating the 12 pairs in which a letter is repeated, leaving 30 pairs. The probability of drawing two pieces with the same area is then $\frac{30}{132}$, or 0.23.

Perimeter

The same reasoning holds for finding the probability of drawing two pieces with the same perimeter. Again, the probability of matching perimeters is equal to that of matching areas: $\frac{30}{132}$, or 0.23.

Interesting Exercises for the Teacher and Students

1. If you throw a dart at a rectangular puzzle board, what is the probability of hitting any one of the seven pieces? For example, $P(G) = \frac{6}{63}$.
2. Use the fundamental principle of counting from probability to determine the number of figure pairs with the same area or the same perimeter.
3. Find and investigate other situations in which geometry and probability can be linked.

How much water should be added to 800 cm^3 of a 70 per cent solution of boric acid to make it a 40 per cent solution of boric acid?

Conversion from Fahrenheit to Celsius Simplified?

Klaus Puhlmann

My friend had just returned from the United States after a four-week study tour. His family organized a welcome-home party and invited family members and friends. Everyone was eager to hear about the study tour.

I asked my friend to tell us in what ways he found the United States different from Canada. My friend, a mathematics teacher, focused on the system of measurement in his comparison, and he responded, "The Americans do not use the metric system. For example, distances are measured not in kilometres but in miles, and weights are measured not in kilograms but in pounds. They use gallons, not litres, for capacity measurements, and those gallons are different from the imperial gallons once used here in Canada. The most problematic issue is the measurement of temperatures. In the U.S., temperatures are measured in degrees Fahrenheit, and in Canada we measure temperatures in degrees Celsius. To convert the temperature from Celsius to Fahrenheit, one must multiply the degrees measured in Celsius by $\frac{9}{5}$ and then add 32 to the product. For example, a temperature of 0°C is equivalent to 32°F, and 100°C is equivalent to 212°F."

My friend's wife smiled as she listened, and she said, "But the conversion is pretty simple. If one is given a temperature in degrees Fahrenheit"—she wrote a three-digit number on a piece of paper—"one simply crosses out the first digit and places it at the end of the number. The new number is the converted value in degrees Celsius."

I simply could not accept this simplistic method of converting temperatures from degrees Fahrenheit to degrees Celsius and, therefore, checked it for myself. I could not believe it when the result proved to be correct. I was perplexed. Still, I remained unconvinced that this method would hold true for all temperature values selected, partly because my friend's wife had a big grin on her face. I kept trying this method for various temperature values, but no matter what other value I chose, I could not make it work. My friend's wife had selected the only three-digit temperature value for which this simplistic conversion method works.

Do you know which three-digit temperature value my friend's wife wrote on her paper?

$$\boxed{32°\text{F} = 0°\text{C}}$$

Abraham de Moivre (1667–1754)

Natali Hritonenko

Mathematics brings beauty and diversity to our lives. It changes our world of thinking and feeling. Therefore, it is not surprising that mathematicians are unique and have incredible lives. Some of them were famous in life; others got recognition only after death. Often, the latter predicted mathematics discoveries before they were observed by the scientists after whom they are named. Sometimes history takes a turn and the truth comes out.

For a long time, the normal distribution was called the Gaussian distribution, after German mathematician Carl Friedrich Gauss (1777–1855). It is true that Gauss discussed this distribution in 1809, but French mathematician Abraham de Moivre first announced it in 1733. To avoid "an international question of priority," Sir Francis Galton suggested the adjective *normal* for the distribution in 1877, and Karl Pearson recommended the routine use of that term, although it has the effect of leading people to believe that all other distributions of frequency are in some way abnormal. The normal distribution was not the only distribution described by De Moivre. In 1718, he exposed a distribution later named the Poisson distribution after French mathematician and physicist Siméon Poisson (1781–1840), who described it only in 1830. Thinking of and talking about these distributions, we usually forget to mention their inventor.

Abraham de Moivre was born in Vitry, near Paris, France, on May 26, 1667. His life was full of adventure, innovation and mystery. After spending five years at a Protestant academy in Sedan, De Moivre studied logic at Saumur and at the Collège de Harcourt in Paris. Then, as a French Protestant, he was compelled to leave France on the revocation of the Edict of Nantes in 1685. At the age of 18, he immigrated to England with great hopes and expectations. As a foreigner, he could not find employment other than being a private tutor of mathematics. However, in 1710 he was appointed to the commission set up by the Royal Society to review the rival claims of Isaac Newton (1642–1727) and Gottfried Wilhelm von Leibniz (1646–1716) to be the discoverer of calculus. Given that De Moivre was friends with Newton, it is clear where the Royal Society's support lay!

De Moivre ranked high as a mathematician. In the later years of his life, Newton responded to mathematical inquiries by saying, "Go to Mr. De Moivre; he knows these things better than I do."

De Moivre's power as a mathematician lies in his analytic rather than geometric investigation. He pioneered the development of analytic geometry and the theory of probability. In 1718, he published *The Doctrine of Chance*. The definition of *statistical independence* appears in this book together with many problems with dice and other games. He also investigated mortality statistics and the foundation of the theory of annuities.

In *Miscellanea Analytica* (1730), De Moivre presented the first version of a famous formula later called Stirling's formula:

$$N! \sim [2\pi/(N+1)]^{1/2} E^{-(N+1)} (N+1)^{(N+1)}$$

that produces better approximations for $N!$ as N gets bigger. This formula was wrongly attributed to James Stirling (1692–1770), although De Moivre used it in 1733 to derive the normal distribution as an approximation of the binomial one. Stirling only pointed out some errors in the primary formula suggested by De Moivre. Even Stirling related the episode in a letter to Gabriel Cramer (1704–1752) in September 1730. In the second edition of *Miscellanea Analytica* in 1738, De Moivre gave credit to Stirling for improving the formula. But we still name this famous formula after Stirling.

De Moivre is remembered for the formula named after him,

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx,$$

which changed the mathematical world, took trigonometry into analysis and is still used in different applications.

Despite De Moivre's scientific eminence, his main income came from tutoring mathematics. He spent much of his free time answering questions about games of chance for rich patrons at a tavern on St. Martin's Lane in London. He died in poverty.

Such an extraordinary person could not simply pass away. Shortly before his death, De Moivre made his last, most prominent prediction. He declared that it would become necessary for him to sleep 15 minutes longer each subsequent night and that he would die on the day, calculated from the arithmetic progression, that he slept for 24 hours. He was right! The day after he had slept almost 24 hours, he slept exactly 24 hours and then passed away in his sleep.

Each side of a square is 60 cm long. If a rectangle has a width of 30 cm and the same area as the square, what is its perimeter?

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