Unsolvable Problems: Doubling the Cube

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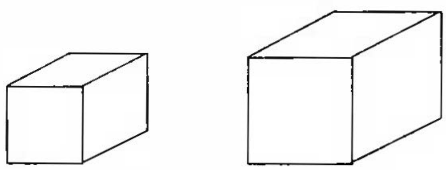
Doubling the cube, often referred to as duplicating the cube, involves finding (using only straightedge and compass) the edge of a cube whose volume is twice that of a given cube.



The problem requires solving the equation $x^3 = 2a^3$ for x. This is impossible because the cube root of 2 cannot be expressed in terms of radicals of index 2, and only such numbers can be evaluated by means of straightedge and compass alone.

Hippocrates of Chios (c. 460 B.C.) showed that the problem of duplicating the cube resolves itself into finding two mean proportionals between two given lines. If $a:x = x:y = y:b$, then $x^2 = ay$ and $y^2 = bx$; hence, $x^4 = a^2y^2 = a^2bx$, or $x^3 = a^2b$. If $b = 2a$, then $x^3 = 2a^3$. That is, the cube of edge x will then have double the volume of a given cube with edge a. Because we have the three equations $x^2 = ay$ (parabola), $y^2 = bx$ (parabola) and $ab = xy$ (hyperbola), we can evidently solve the problem by finding the intersection of two parabolas or of a parabola and a hyperbola (Smith 1958, 313).

Throughout the centuries, mathematicians such as Archytas (ca. 400 B.C.), Plato (ca. 380 B.C.), Eudoxus (ca. 370 B.C.), Apollonius (ca. 225 B.C.) and Diocles (ca. 180 B.C.) have solved the problem using various methods. Several modern writers and mathematicians such as Vieta, Descartes, Fermat and Newton have suggested a number of solution methods for duplicating the cube. Finding two mean proportionals, as suggested by Descartes, is required in solving the problem. Newton (1707) suggested several methods but preferred one that made use of the limaçon of Pascal.

Despite all the attempts, no one has ever duplicated a given cube using only straightedge and compass. Those who dare to try it, in light of the evidence that it is impossible, should study the mathematical proofs to see if they can detect an error.

Reference

Smith, D. E. *History of Mathematics*. Vol. II. New York: Dover, 1958. 313–16.