Unsolvable Problems: Trisection of an Angle

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We have all learned in school how easy it is to bisect any angle with a straightedge and compass. However, trisection using a straightedge and compass is impossible for some angles, as P. L. Wantzel proved in 1847.

Certain specific angles can be easily trisected, but there is no general procedure that permits the construction, with Euclidean tools, of an angle that is exactly one-third of a given arbitrary angle. For example, because a 30° angle is easily constructed using only a straightedge and a pair of compasses, a 90° angle can be trisected.

We know that an obtuse angle can always be divided into one or more right angles and an acute angle. Hence, the problem of trisecting an arbitrary angle can be reduced, without loss of validity of the generality of the claim, to the task of trisecting an acute angle.

Let us assume we have an angle $3\theta = 60°$, which is to be trisected. From our knowledge of trigonometry, we know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ is a trigonometric identity. As we insert $\cos 3\theta = 60° = 1/2$, and substitute $\cos\theta = y/2$, we get $3\cos\theta = 3y/2$ and $4\cos^3\theta = y^3/2$, from which the irreducible cubic equation follows: $y^3 - 3y - 1 = 0$. The roots of this cubic equation cannot be constructed using Euclidean methods. We must therefore conclude that cos 20° is not constructible. Because the angle 60° cannot be trisected, we have thus produced one example that allows us to conclude that it is impossible to trisect an arbitrary angle using a straightedge and compass only.

The first attempts to trisect a general angle arose so long ago that historians are unable to find a record. Comfort with bisecting angles led naturally to attempts at trisecting angles. Only after many attempts at trisecting a general angle, restricted of course to the classical rules and tools, was it apparent that some other mathematical principles blocked the success of this endeavor.

Hippias of Elis, who lived in the fifth century B.C., was one of the first to attempt to solve the trisection problem. Frustrated, he devised a curve called the quadratrix, which allowed him to give an exact solution to the problem. However, this was not achieved with the use of a straightedge and a compass alone; it involved what is often referred to as a nonclassical solution. The history of the trisection problem reveals other nonclassical solutions.

Many people have been drawn to this powerful and fascinating problem, all attempting to solve it only to discover that trisecting a general angle using only a straightedge and compass is impossible. The hundreds of attempts in the past and present have shown that it is impossible to find, by a straightedge and compass construction, a root x of the trisection equation. This and the other unsolvable problems, when stripped of all implications, are hardly worth more than passing attention, but they have yielded fruitful discoveries in other mathematical fields.

Archimedes, Niomedes, Pappus, Leonardo da Vinci, Dürer, Descartes, Ceva, Pascal, Huygens, Leibniz, Newton, Maclaurin, Mascheroni, Gauss, Steiner, Charles, Sylvester, Kempe, Klein, Dickson—all of these, and hundreds more, have attacked the trisection problem directly or created mathematics by which substantial advances could be made toward a full understanding of the situation. Those who are still determined to show that the trisection of any angle is possible using only a compass and straightedge are well advised to first examine the existing proofs to see if mathematical mistakes have been made.

Bibliography

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