Random Variables: Simulations and Surprising Connections

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Many traditional ideas about the content of the secondary mathematics curriculum are being challenged. The assumption that mathematics can be taught as a series of unrelated skills and algorithms is giving way to an approach that emphasizes the connections among the many branches of study. Such topics as probability and statistics must be given greater prominence as important and practical areas of mathematical knowledge. Students need opportunities to collect and interpret data, simulate probabilistic situations, consider the relationship between theoretical and experimental probabilities, and gain an understanding of random variables (NCTM 1989). Integrating probability and statistics within the curriculum provides a number of interesting and elegant connections that help students develop an appreciation for the inherent beauty of mathematics. The following lesson on random variables incorporates class discussion and experimental activities in the practical and theoretical exploration of one such connection.

This lesson is designed for, and has been used with, advanced second-year-algebra students in Grades 11 and 12. Coins, regular dice, decahedral dice and calculators are used. The lesson involves introducing three random variables followed by considering an empirical and theoretical probability for each. Approximately one and one-half hours are required for these activities. The activities can be scheduled in a single extended time block or can be split over two 50-minute class periods.

Introducing Three Random Variables

The lesson begins with a class discussion, during which the students develop a working definition of a random variable. The exact wording of this definition may vary, but it should include the sense that the value of a random variable is determined by the result of a chance experiment. For example, the number occurring on the toss of a regular die is a random variable with possible values of 1, 2, 3, 4, 5 or 6, each having a probability of 1/6.

Next, the three random variables are introduced. The first, R_x, is the number of tosses needed to get a head in a series of coin flips. R_x has a value of 1 if a head occurs on the first flip, 2 if the first occurrence of a head is on the second flip and so on. The second random variable, R_y, is defined as the number of times a die is rolled to get a 1. For example, if a 1 comes up on the first roll, the value of R_y is 1, whereas if the first three rolls of the die are 2, 4 and 1, the value of R_y is 3. Finally, the third random variable, R_z, is defined as the number of times a decahedral, or ten-faced, die is rolled to get a 1.

Before the empirical phase of the lesson begins, the students are asked to guess the most likely value of each of these random variables. Many assume that the most likely value for each can be found by dividing the number of possible outcomes of a single trial of the coin or die by 2. Thus, they believe that the most likely values of R_x, R_y and R_z will be 1, 3 and 5, respectively. Other students begin with a similar idea but decide that the initial numbers should not be divided by 2, leading them to suggest the most likely values of 2 for R_x, 6 for R_y and 10 for R_z. These guesses are recorded so that they can be compared with the empirical and theoretical results obtained later in the lesson.

The Empirical and Theoretical Investigation of R_x

The students form pairs to investigate these random variables empirically. They begin by conducting the coin-flipping experiment and recording the values of R_x for 10 trials. Each pair's data are written on the chalkboard and compiled into a single table. This larger sample space provides a better basis for deriving the estimate of theoretical probabilities. Figure 1 shows an example of a completed worksheet on which students tallied their data and copied class totals for R_x and R_y.

Figure 1

Sample Results of Student Worksheet for Coin (R_x) and Die (R_y)

Outcome	Coin		Die	
	Tally	Class Total	Tally	Class Total
1	~~IIII~~	46	III	29
2	III	25	I	26
3	I	14	II	21
4		9	II	13
5	I	4		10
6		1	I	7
7		1	I	8
8		0	II	6
9		0		6
10		0	II	3
11		0		1
12		0		2
13				2
14				3
15				1
16				2
17				4
18				0
19			I	3
20				0
21				0
22				0
23				1
>23				2

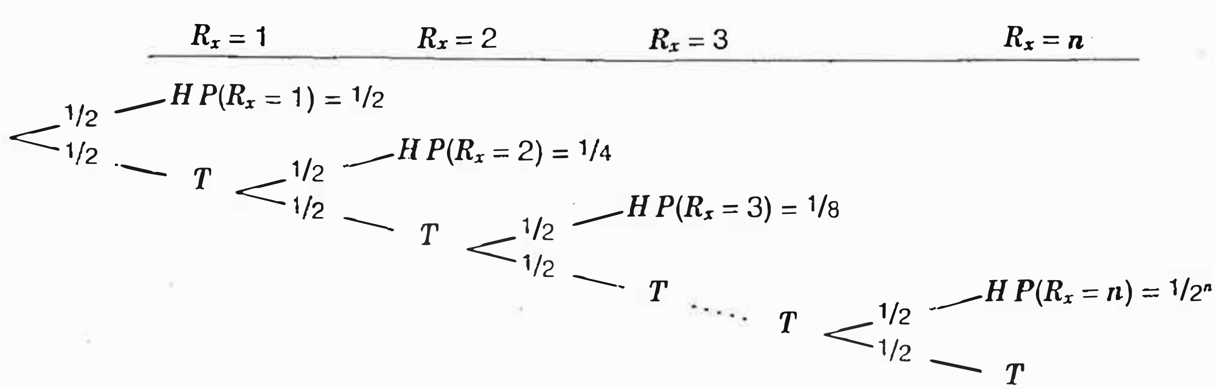
The class is encouraged to consider theoretical probabilities by predicting the likelihood of event H: getting a head on one toss of a fair coin. Most students realize that this probability is ½, that is, $P(\text{H}) = ½$. I explain that since the probability of heads on the first toss is ½, the probability that the random variable, R_x, equals 1 is also ½. The probability that the random variable takes on a value of 2 can then be found by considering the sequence of coin flips necessary for this outcome.

A tree diagram, as shown in Figure 2, demonstrates that the only way the random variable can equal 2 is if the first toss is tails and the second is heads. Since each toss is an independent event, the probability is ½ times ½, or 1/4. Similarly, the probability that $R_x = 3$ is equal to the probability of the event (tails, tails, heads) (TTH), or (½) (½) • (½) = 1/8. At this point, a pattern becomes clear to the students: the probability of any particular value of this random variable is ½ raised to the power of that value, $P(R_x = n) = (½)^n$. Figure 3 shows a worksheet on which students computed the empirical and theoretical probabilities of R_x.

After gathering the empirical probabilities and calculating the theoretical probabilities, students see that the most likely value of this random variable is 1. They also notice that the higher the value of the random variable, the less likely it is for that value to occur. Returning to Figure 2, it is interesting to observe that the probabilities of consecutive values of R_x form a geometric sequence: $P(R_x = 1) = ½$, $P(R_x = 2) = 1/4$, $P(R_x = 3) = 1/8$ and

Figure 2

Tree Diagram for R_x (Coin)



so on. Consequently, the corresponding infinite geometric series represents the probability that the value of the random variable will be some element of the set of positive integers. Since a head must occur eventually, the laws of probability provide an unusual justification showing why the sum of this infinite series must be 1:

$P(R_x = 1 \text{ or } R_x = 2 \text{ or } R_x = 3 \text{ or} \ldots)$
$= P(R_x = 1) + P(R_x = 2) + P(R_x = 3) + \cdots = 1.$

At this point, the students reconsider their original guesses concerning the most likely outcomes for R_y and R_z. Most students who had guessed that 1 was the most likely outcome for R_x do not change their original guesses for the other random variables. Surprisingly, even those who had guessed 2 for R_x tend not to change their guesses of 6 and 10 for R_y and R_z. They are either too stubborn to change or think that those who guessed 1 just "got lucky" in the coin-flipping experiment. Students are encouraged to think about the geometric sequence that corresponds to the probabilities of successive values of R_x. Occasionally, an insightful student will suggest that 1 will be the most likely outcome for R_y and R_z and that the probabilities of successive values of these random variables will decrease.

Exploring the Roll of the Die

Next, the students consider the second random variable, R_y, the number of rolls needed to obtain 1 when a die is tossed repeatedly. The students carry out this experiment, but more time is allotted for gathering data. This extra time is necessary because more time is needed for the "average" trial and because the decreased probabilities for the values of R_y imply that more trials will be needed to get a reasonably accurate picture of the distribution.

After each pair of students has carried out the experiment 15 times, the results are written on the chalkboard and compiled into a single table. To examine the theoretical probabilities of the different values of R_y, students are asked the probability of rolling a 1 on the first roll of the die. Most students realize that this probability is 1/6; therefore, $P(R_y = 1) = 1/6$. The probability that this random variable takes on a value of 2 can be found by determining the probability that the first roll of the die is "not 1" and the second roll is 1. Since $P(\text{not } 1) = 5/6$ and $P(1) = 1/6$, $P(R_y = 2) = P(\text{not } 1) * P(1) = (5/6)(1/6) = 5/36$. Similarly, $P(R_y = 3) = P(\text{not } 1) * P(\text{not } 1) * P(1) = (5/6)(5/6)(1/6) = 25/216$. Figure 4 shows a sample of student work in determining the empirical and theoretical probabilities of R_y.

Figure 3

Sample Results of Student Worksheet for Probability Calculations (Coin)

Outcome	Pair		Class			
	Freq.	Empirical Probability	Freq.	Empirical Probability	Theoretical Probability	Nearest .001
1	5	$\frac{5}{10} = .5$	46	$\frac{46}{100} = .46$	$\frac{1}{2}$	.500
2	3	$\frac{3}{10} = .3$	25	$\frac{25}{100} = .25$	$(\frac{1}{2})^2$	.250
3	1	$\frac{1}{10} = .1$	14	$\frac{14}{100} = .14$	$(\frac{1}{2})^3$	.125
4	0	0	9	$\frac{9}{100} = .09$	$(\frac{1}{2})^4$	.063
5	1	$\frac{1}{10} = .1$	4	$\frac{4}{100} = .04$	$(\frac{1}{2})^5$	.031
6	0	0	1	$\frac{1}{100} = .01$	$(\frac{1}{2})^6$	.016
7	0	0	1	$\frac{1}{100} = .01$	$(\frac{1}{2})^7$	.008
8	0	0	0	0	$(\frac{1}{2})^8$	.004
9	0	0	0	0	$(\frac{1}{2})^9$	.002
10	0	0	0	0	$(\frac{1}{2})^{10}$	.001
11	0	0	0	0	$(\frac{1}{2})^{11}$	.000
>11	0	0	0	0	$1-(1-(\frac{1}{2})^{11})$	.000

Figure 4

Sample Results of Student Worksheet for Probability Calculations (Die)

Outcome	Pair		Class			
	Freq.	Empirical Probability	Freq.	Empirical Probability	Theoretical Probability	Nearest .001
1	3	$\frac{3}{15} = .2$	29	$\frac{29}{150} = .193$	$\frac{1}{6}$	.167
2	1	$\frac{1}{15} = .067$	26	$\frac{26}{150} = .173$	$(\frac{1}{6})(\frac{5}{6})$	.139
3	2	$\frac{2}{15} = .133$	21	$\frac{21}{150} = .140$	$(\frac{1}{6})(\frac{5}{6})^2$	.116
4	2	$\frac{2}{15} = .133$	13	$\frac{13}{150} = .087$	$(\frac{1}{6})(\frac{5}{6})^3$	.096
5	0	0	10	$\frac{10}{150} = .067$	$(\frac{1}{6})(\frac{5}{6})^4$	.080
6	1	$\frac{1}{15} = .067$	7	$\frac{7}{150} = .047$	$(\frac{1}{6})(\frac{5}{6})^5$	.067
7	1	$\frac{1}{15} = .067$	8	$\frac{8}{150} = .053$	$(\frac{1}{6})(\frac{5}{6})^6$	.056
8	2	$\frac{2}{15} = .133$	6	$\frac{6}{150} = .040$	$(\frac{1}{6})(\frac{5}{6})^7$	.047
9	0	0	6	$\frac{6}{150} = .040$	$(\frac{1}{6})(\frac{5}{6})^8$	.039
10	2	$\frac{2}{15} = .133$	3	$\frac{3}{150} = .020$	$(\frac{1}{6})(\frac{5}{6})^9$	.032
11	0	0	1	$\frac{1}{150} = .007$	$(\frac{1}{6})(\frac{5}{6})^{10}$	.027
12	0	0	2	$\frac{2}{150} = .013$	$(\frac{1}{6})(\frac{5}{6})^{11}$	.022
13	0	0	2	$\frac{2}{150} = .013$	$(\frac{1}{6})(\frac{5}{6})^{12}$	.019
14	0	0	3	$\frac{3}{150} = .020$	$(\frac{1}{6})(\frac{5}{6})^{13}$	.016
15	0	0	1	$\frac{1}{150} = .007$	$(\frac{1}{6})(\frac{5}{6})^{14}$	.013
16	0	0	2	$\frac{2}{150} = .013$	$(\frac{1}{6})(\frac{5}{6})^{15}$	.011
17	0	0	4	$\frac{4}{150} = .027$	$(\frac{1}{6})(\frac{5}{6})^{16}$	.009
18	0	•	0	0	$(\frac{1}{6})(\frac{5}{6})^{17}$	.008
19	1	$\frac{1}{15} = .067$	3	$\frac{3}{150} = .020$	$(\frac{1}{6})(\frac{5}{6})^{18}$	.006
20	0	0	0	0	$(\frac{1}{6})(\frac{5}{6})^{19}$	.005
21	0	0	0	0	$(\frac{1}{6})(\frac{5}{6})^{20}$	.004
22	0	0	0	0	$(\frac{1}{6})(\frac{5}{6})^{21}$	.004
23	0	0	1	$\frac{1}{150} = .007$	$(\frac{1}{6})(\frac{5}{6})^{22}$	.003
>23	0	0	2	$\frac{2}{150} = .013$	[illegible]	.015

Continuing this argument produces the following:

$P(R_y = 1) = (1/6)$
$= 1/6 \approx 0.167$

$P(R_y = 2) = (5/6)(1/6)$
$= 5/36 \approx 0.139$

$P(R_y = 3) = (5/6)(5/6)(1/6)$
$= 25/216 \approx 0.116$

$P(R_y = 4) = (5/6)(5/6)(5/6)(1/6)$
$= 125/1296 \approx 0.096$

$P(R_y = 5) = (5/6)(5/6)(5/6)(5/6)(1/6)$
$= 625/7776 \approx 0.080$

Generalizing from this pattern, students determine that the nth term of this sequence includes $n - 1$ factors of 5/6 and one factor of 1/6, as shown by the tree diagram in Figure 5.

Consequently, it becomes clear that the most likely value of this random variable is also 1. Further, the formula for successive terms is the same as that for a geometric sequence with a first term of 1/6 and a common ratio of 5/6, that is, $a_n = a_1 r^{n-1}$. Applying the traditional formula for the sum of an infinite series,

$$S = \frac{a_1}{1 - r},$$

yields

$$S = \frac{\frac{1}{6}}{1 - \frac{5}{6}} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1.$$

This result is not surprising because each potential value of R_y is an element of the set of positive integers and

Figure 5
Tree Diagram for R_y (Die)

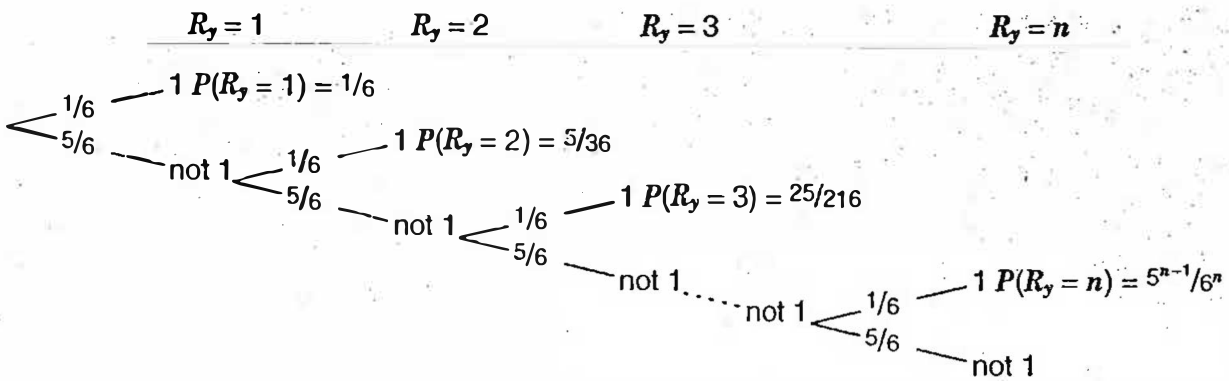
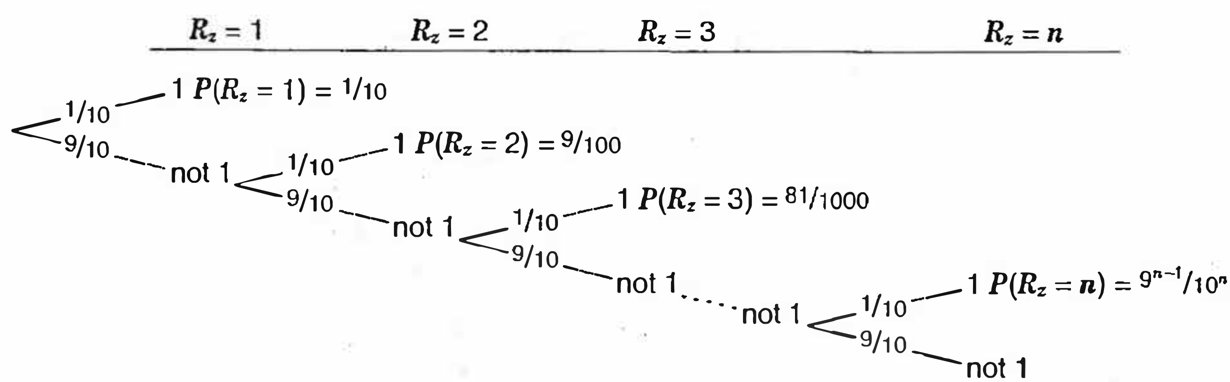


Figure 6
Tree Diagram for R_z (Die with 10 Faces)



$P(R_y = 1 \text{ or } R_y = 2 \text{ or } R_y = 3 \text{ or} \ldots) = P(R_y = 1) + P(R_y = 2) + P(R_y = 3) + \ldots = 1/6 + 5/36 + 25/216 + \ldots = 1.$

At this point many students are ready to change their guess concerning the most likely value of R_z. Their experiences with R_x and R_y enable them to generalize that 1 will be the most likely value of any random variable defined similarly.

R_z, The Decahedral Die

A consideration of R_z, the number of rolls needed to roll a 1 in a series of tosses of a decahedral die, provides students with practice using decimals. Students can collect and compile data in the same manner as in the previous two cases. If, however, most students have already concluded that 1 is the most likely outcome for R_z, the empirical aspects of this example can be omitted. Students complete this phase of the lesson by developing the subsequent probability table and the tree diagram shown in Figure 6.

$P(R_z = 1) = (.1)$
$P(R_z = 2) = (.9)(.1) = .09$
$P(R_z = 3) = (.9)(.9)(.1) = .081$
$P(R_z = 4) = (.9)(.9)(.9)(.1) = .0729$
$P(R_z = 5) = (.9)(.9)(.9)(.9)(.1) = .06561$

The probabilities of successive values of the random variable in this sequence are clearly decreasing, but students may wonder whether the sum of these terms is 1. "Maybe the values don't get small fast enough," one student hypothesized. If the Texas Instruments Explorer calculator is available, it can be used to help students investigate this result. By following the keystroke sequence .1, ☒ .9, ⊟, ⊟, ⊟, ⊟, . . . , students notice that each time = is pressed, the previous product is multiplied by .9, yielding the consecutive values of this geometric sequence. When ⊟ has been entered many times, the value of the term will be extremely small. This process should help convince students that, although the terms decrease more slowly than the other two random variables considered, the terms do become small enough fast enough to allow the sum to remain finite. More powerful calculators should not be used. Although they will eventually produce a result of 0, the time required to achieve this answer would be extremely frustrating.

Returning to a theoretical consideration of the sequence of partial sums, S_z, the students see that the addition of each successive term brings the sum 1/10 of the way from its current value to 1. Thus, as in the previous two cases, the formula for the sum of an infinite number of these terms yields

$$S = \frac{.1}{1 - .9} = \frac{.1}{.1} = 1.$$

Figure 7

Expected Values Using Partial Sums

$$\text{Coin: } \sum_{n=1}^{t} n\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1}$$

$S(1) = 0.5$
$S(2) = 1.0$
$S(3) = 1.375$
$S(4) = 1.625$
$S(5) = 1.78125$
$S(10) = 1.98828125$
$S(50) = 1.999999999999954$
$S(100) = 2$
$S(500) = 2$
⋮
$S(t)$

$$\text{Six-sided die: } \sum_{n=1}^{t} n\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}$$

$S(1) = 0.1666666666666667$
$S(2) = 0.4444444444444444$
$S(3) = 0.7916666666666666$
$S(4) = 1.177469135802469$
$S(5) = 1.57034670781893$
$S(10) = 3.415910673762469$
$S(50) = 5.993846450129437$
$S(100) = 5.999998720084609$
$S(500) = 5.999999999999996$
⋮
$S(t)$

$$\text{Ten-sided die: } \sum_{n=1}^{t} n\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{n-1}$$

$S(1) = 0.1$
$S(2) = 0.28$
$S(3) = 0.523$
$S(4) = 0.8146$
$S(5) = 1.14265$
$S(10) = 3.026431198$
$S(50) = 9.690773487560795$
$S(100) = 9.997078246122364$
$S(500) = 9.999999999999984$
⋮
$S(t)$

Extensions

These experiences can be enriched by having students study the expected value of each of the three random variables. I ask the students, "On average, how many times must a coin be tossed to get a head?" Similar questions can be phrased for the two types of dice. The expected value, E, of a random variable can be defined as the summation of the product of each successive value of the random variable with its probability, that is,

$$E = \sum_{n=1}^{\infty} nP(R = n).$$

When a random variable can assume an infinite number of values, as is the case with R_x, R_y and R_z, this sum can be approximated by adding these products for a relatively large number of terms. The partial sums shown in Figure 7 approach the values of E_x, E_y and E_z.

In each case, the expected value seems to converge on the number of possible outcomes for a single toss of the coin, roll of the ordinary die or roll of the decahedral die. That is, $E_x = 2$, $E_y = 6$ and $E_z = 10$. For any random variable defined similarly to R_x, R_y and R_z with q equally likely outcomes, it appears that

$$\sum_{n=1}^{\infty} n\left(\frac{1}{q}\right)\left(\frac{q-1}{q}\right)^{n-1} = q.$$

Figure 8

Proof that $\sum_{n=1}^{\infty}(n)\left(\frac{1}{q}\right)\left(\frac{q-1}{q}\right)^{n-1} = q$

$$E_q = \sum_{n=1}^{\infty} n\left(\frac{1}{q}\right)\left(\frac{q-1}{q}\right)^{n-1}$$

$$E_q = \left(\frac{1}{q}\right)\sum_{n=1}^{\infty} n\left(\frac{q-1}{q}\right)^{n-1}$$

Let

$$\frac{q-1}{q} = K.$$

Then

$$\left(\frac{1}{q}\right)\sum_{n=1}^{\infty} n(K)^{n-1}$$

converges by the root test, since

$$\lim_{n\to\infty}\left|\frac{A_n+1}{A_n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)\frac{K^n}{K^{n-1}}\right|$$

$$= K\lim_{n\to\infty}\left|\frac{n+1}{n}\right|$$

$$= K \text{ and } K < 1.$$

Thus, it will be sufficient to complete the proof by showing that

$$\sum_{n=1}^{\infty} nK^{n-1} = q^2.$$

Let

$$A_n = \sum_{n=1}^{\infty} nK^{n-1}.$$

Then

$$A_n = 1 + 2K + 3K^2 + 4K^3 + \cdots$$

$$= 1 + K + K^2 + K^3 + K^4 + \cdots$$

$$+ K + 2K^2 + 3K^3 + \cdots$$

$$= \frac{1}{1-K} + K(1 + 2K + 3K^2 + \cdots)$$

$$= \frac{1}{1-K} + K(A_n);$$

$$A_n(1-K) = \frac{1}{1-K};$$

$$A_n = \frac{1}{(1-K)^2}$$

$$= \frac{1}{\left(1-\frac{q-1}{q}\right)^2}$$

$$= \frac{1}{\left(\frac{q-q+1}{q}\right)^2}$$

$$= \frac{1}{\left(\frac{1}{q}\right)^2}$$

$$= q^2.$$

Hence

$$\frac{1}{q}\sum_{n=1}^{\infty} n\left(\frac{q-1}{q}\right)^{n-1} = q.$$

This empirical evidence led us to consider proving the assertion formally. Many advanced high school students can understand this proof (see Figure 8). Further, the progression of inductive to deductive reasoning illustrated is indicative of the way that mathematicians often work. It is important for students to attempt to emulate this path in their own mathematical reasoning.

Conclusion

The students enjoyed participating in this lesson. The opportunity to guess the most likely value of each of the three random variables hooked them, piquing their competitive spirit. During the collection of the empirical data, the students vocally rooted for outcomes that would show their guess to be correct. This motivation continued during the discussion phases of the lesson as students tried to convince one another of the accuracy of their conjectures. Finally, many students commented that it was pretty cool to find a geometric sequence in a situation in which they never would have expected it.

Reference

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

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Hose A can fill a basin in 40 minutes. Hose B can fill the basin in 30 minutes and hose C in 20 minutes. How long would it take to fill the basin if all three hoses were used at the same time?