Unsolvable Problems: Quadrature of the Circle

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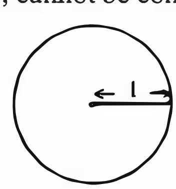
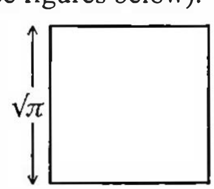
The Quadrature of the Circle, often referred to as squaring the circle, involves the construction of a square equal in area to a given circle, using only straight-edge and compass. This is one of three famous problems of antiquity that cannot be solved. (The other two will be dealt with in future issues of *delta-K*.)

Although the Greeks found themselves confronted by these problems which they could not solve, at least by the use of the unmarked ruler and the compass alone, it was not until the late 19th century that it was proved to be impossible. Its impossibility arises from the fact that π is a *transcendental* number, and therefore a length equal to $\sqrt{\pi}$, which is also transcendental, cannot be constructed (see figures below).





$A = \pi \times 1^2 = \pi$ $\qquad$ $A = \sqrt{\pi} \times \sqrt{\pi} = \pi$

In a circle with radius 1, the edge of the square would be $\sqrt{\pi}$.

Essentially there are three methods of attacking the problem: first, by the use of the ruler and compass only; second, by the use of higher plane curves; third, by such devices as infinite series, leading to close approximations.

Greek mathematicians seem to have found the insolubility of the first method, but they did not prove it. They were successful with the second method, but less skillful with the third.

Many noted mathematicians have attempted to square the circle. Antiphon (c. 430 BCE) tried the quadrature by inscribing a polygon and then doubling the number of sides successively until he approximately exhausted the area between the polygon and the circle.

Hippocrates of Chios (c. 460 BCE) also attempted the solution and was the first to actually square a curvilinear figure. Other noteworthy attempts at squaring the circle were made by Deinostratus (c. 350 BCE) and Archimedes, and they all focused on finding the best approximation of π.

Approximations of π were also attempted by many European mathematicians, including Fibonacci, Tycho Brahe and Vieta. Vieta (c. 1593) gave an interesting approximation of π using continued products for the purpose:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots$$

John Wallis (1655) gave the form:

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \times 9 \times 11 \times 11 \cdots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \times 10 \times 10 \times 12 \cdots}$$

Leibniz (1673) used the form:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

While William Jones (1706), an English writer, was the first to use π to stand for the ratio of circumference to diameter, it was F. Lindemann (1882) who proved the transcendence of π, thus showing the impossibility of squaring the circle by the use of ruler and compass alone.

Bibliography

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