# The Pythagorean Numbers

*Klaus Puhlmann*

The history of mathematics produced many brilliant and interesting figures, but Pythagoras probably ranks near the top. This is partly due to the mystery surrounding his life, partly from his own mysticism, partly from the brotherhood which he established and partly from his own unquestioned ability. He seems to have been born between the 50th and 52nd Olympics, to use the Greek system of chronology, or between 580 and 568 B.C.E. of our calendar. Although called a Samian, it is uncertain that he was born on the island of Samos. He spent several years of his life in Egypt, but it is not known for certain that Pythagoras came into contact with the Far East. However, evidence derived from the philosophy of Pythagoras seems to point to contact with the Far East. Following the years of travel, he sought out Crotona (south-eastern coast of Italy) as a favorable place for his school. Here he gathered 300 wealthy young men, established a brotherhood and transmitted his theories by word of mouth. Pythagoras based his philosophy upon the postulate that number is the cause of the various qualities of matter. His philosophy is best reflected in what Philolaus of Croton once wrote:

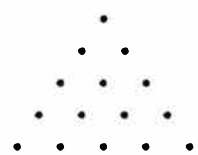
> In fact, everything that can be known has number. For it is not possible to conceive of or to know anything that has not.

This led Pythagoras to exalt arithmetic, as distinguished from logistic, out of all proportion to its real importance. It also led him to dwell upon the mystic properties of numbers and to consider arithmetic as one of the four degrees of wisdom—arithmetic, music, geometry and spherics (astronomy). While there are many philosophical theories attributed to the Pythagoreans, we shall now turn our attention to their mathematical accomplishments. It is important to note that none of the mathematical accomplishments can be attributed with certainty to Pythagoras himself, as nothing came down to us in writing. The following discovery is one of the mathematical accomplishments of the Pythagoreans:

## The Pythagorean Numbers

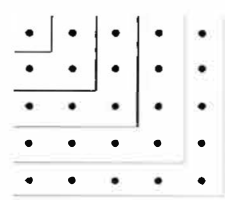
The Pythagoreans pictured the integers as groups of points like constellations. From such configurations one can read some remarkable number-theoretic laws. An example is the Triangular Numbers in Figure 1.

**Figure 1**



The rows in Figure 1 contain 1, 2, 3, 4, 5, ... points, and the number of points in an **n**-rowed triangle is the sum of the first **n** positive integers. For example, $1 + 2 = 3$, $1 + 2 + 3 = 6$, $1 + 2 + 3 + 4 = 10$, $1 + 2 + 3 + 4 + 5 = 15$, show that clearly. In this way, the Pythagoreans obtained the well-known sequence of Triangular Numbers: 1, 3, 6, 10, 15, 21, .... Even more remarkable are the laws which can be read off from a square array (see Figure 2).

**Figure 2**



One can see that to a square array with $n^2$ points one must add $n + 1 + n = 2n + 1$ points to get a square with $(n + 1)^2$ points. Thus it follows that:

$$n^2 + (2n + 1) = (n + 1)^2 \qquad (1)$$

From (1) we see that the difference between successive squares generates the sequence of odd numbers. With the help of (1) we can also find sets of numbers x, y and z which satisfy the Pythagorean equation.

$$x^2 + y^2 = z^2 \qquad (2)$$

To do this we need only make 2n + 1 in (1) a square. If we let $2n + 1 = m^2$, then

$$n = \frac{m^2 - 1}{2}, \quad n+1 = \frac{m^2 + 1}{2} \qquad (3)$$

(Please note that $m^2$ is odd, so the fractions in (3) are integers.)

Substituting (3) in (1) we get:

$$m^2 + \left(\frac{m^2 - 1}{2}\right)^2 = \left(\frac{m^2 + 1}{2}\right)^2 \qquad (4)$$

[Pythagoras, c. 540 B.C.E.]

Thus for m = 3, 5, 7, 9, ... equation (4) generates Pythagorean numbers **x, y** and **z**, that is, integers which satisfy equation (2):

| m | x | y | z |
|---|---|---|---|
| 3 | 3 | 4 | 5 |
| 5 | 5 | 12 | 13 |
| 7 | 7 | 24 | 25 |
| 9 | 9 | 40 | 41 |

There is a distinct possibility that the Pythagoreans discovered the celebrated Pythagorean theorem by such number-theoretic arguments rather than by comparison of areas. (Please note that the theorem on right triangles was known to the Indians, however, they had no proof of it.)

As we have seen, adding the successive positive integers leads to the Triangular Numbers. In contrast to adding the successive odd numbers (see Figure 2) leads to the squares:

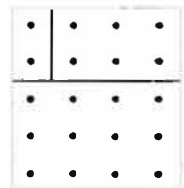
$1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

What happens when we add the successive even numbers?

$2+4+6+8+\ldots+2n = 2(1+2+3+4+\ldots n) = n(n+1)$,

as can be seen by considering a rectangular arrangement of n(n + 1) numbers. Figure 3 shows the first four rectangular numbers (2, 6, 8, 12) in such an array.

**Figure 3**



Adding odd numbers leads to a square schema, while adding even ones leads to a rectangle in which the ratio (n + 1)/n of sides depends on n.

The odd numbers generate a limited number of forms (see Figure 2), while the even ones generate a multiplicity of rectangles which are not similar (see Figure 3). The Pythagoreans deduced the following correspondence:

odd $\leftrightarrow$ limited

even $\leftrightarrow$ unlimited

Today, we would no longer subscribe to the Pythagorean's conclusion.

## Bibliography

Diels, H. *Die Fragmente der Vorsokratiker*. Hamburg, 1957.

Smith, D. E. *History of Mathematics, Volumes 1 and 2*. Dover: New York, 1958.

### Filling the Tank

Two pipes can be used to fill a tank with water. When both pipes are turned on, the tank fills in 9 minutes. When only pipe A is used, the tank is filled in 15 minutes. How long will it take to fill the tank if only pipe B is used?