Two Simple (and Not So Simple) Probability Activities

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The NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) has recommended that statistics and probability be included in the Grades 5 to 8 curriculum. Specifically, they recommend that students should

- model situations by devising and carrying out experiments or simulations to determine probabilities;
- model situations by constructing a sample space to determine probabilities;
- appreciate the power of using a probability model by comparing experimental results with mathematical expectations;
- make predictions that are based on experimental or theoretical probabilities;
- develop an appreciation for the pervasive use of probability in the real world (p. 109).

The introduction of probability concepts in the elementary grades (Grades 5 and 6) represents a significant addition to our present curriculum. Such a change necessitates further teacher education and the development of new teaching resources.

In this article we present two simple teaching activities that address two specific objectives taken from Alberta Education's new *Program of Studies* (1991). Both activities have adopted the familiar context of rolling dice and have been designed to stress simple probability principles, including the analysis of sample space as a means to predict and understand the outcomes of an experiment. In the first activity, The Biased Dice, four six-sided dice are used to introduce some important terms associated with probability experiments. In this activity each die has two unequal possible outcomes that control the movement of a colored marker making it easy to compare the various effects of the biased dice. In the second activity, The Mystery Dice, a problem-solving situation is presented through a computer simulation to stress the importance of sample space and the critical relationship between the conditions of the experiment and the range of possible outcomes. Each activity is described along with a discussion of the expected results, teaching hints, and a list of possible challenges or extensions. Answers to the challenges are not provided here but are left for the reader to enjoy figuring out.

Activity One: The Biased Dice

Objective

Develops probability concepts through experiments—identifies impossible events, certain events, uncertain events, equally likely events and unequally likely events (two or more events) (Alberta Education 1991).

Materials

Four blank six-sided dice, colored labels (dots) for file folders (blue, green, red and yellow), game board (see Figure 1) and four beans or other markers.

Activity

In this activity, the student begins by making four biased dice. Leave the first blank die blank. On two sides of the second die place a red dot. On four sides of the third die place a yellow dot, and on all six sides of the fourth die place a green dot.

To begin the activity, place the beans (or other markers) on the start squares of the game board. Roll all four dice simultaneously. Every time a dot of a given color is rolled, move the marker for that color one square further along its track. Continue rolling all four dice until one marker reaches the end of its track.

Questions

- *Predict which marker will make it to the end of its track first. Explain your prediction. Predict which color will finish second, third and fourth. Experiment to test your predictions.*
- *Using your experiment, give an example of each of the following: an impossible event, a likely event, a certain event, an uncertain event and two unequally likely events.*
- *Is it possible for the red marker to make it to the end of its track before the yellow marker does? Why or why not?*
- *Is it possible for the yellow marker to make it to the end of its track before the green marker? Why or why not?*
- *If you conduct the experiment again, will you get the same results? Why or why not?*

Discussion

In this activity, students are gaining experience with fundamental principles of probability. After a very few rolls, it will become obvious that a green dot will appear on every roll, while a blue dot will never appear. The yellow dot and red dot do not always appear, but the yellow dot appears more frequently than the red.

Understanding the results of this experiment requires considering the probability of rolling any one color in a given turn. This discussion may best proceed through an analysis of the *sample space* of each die. The sample space is defined as all the possible outcomes of an experiment. For each die, there are six possible sides that may surface when the die is rolled; therefore, there are six possible outcomes for each color and thus a sample space of six elements for each die.

Consider first the die with the green dots. We placed six dots on it and thus have a probability of 6 out of 6 for rolling a green dot. This is considered a *certain event*; that is, we know that whenever we roll this die a green dot will surface. We say that this die expresses a probability of 1 (or 6/6) of showing a green dot.

The die without any dots controls the movement of the blue marker. In our experiment the blue marker did not leave the start space. Because this die does not have any blue dots on it, it expresses a probability of 0 out of 6 (or simply 0) for rolling a blue dot (or any other color for that matter). This is considered an *impossible event*. That is, we know that no matter how many times we roll this die, it is impossible that a blue dot will ever surface.

The remaining two dice (the die with the red dots and the die with the yellow dots) provide us with examples of uncertain events. An *uncertain event* is one which cannot be guaranteed or predicted with any assurance. In the examples above, we knew that we would roll a green dot on each roll and that we would never roll a blue dot, no matter how many times we tried. We were certain about these events. With the remaining two dice, it was possible to roll a colored dot, but it was also possible to roll nothing, and thus we could not be certain about the actual outcomes. We call these uncertain events.

The die with the red dots provides us with an example of an *unlikely event*. On this die, we covered two sides with red dots, and thus we knew that the chance of rolling a red dot is 2 out of 6. It is more likely when rolling this die that we will have a blank side than a side with a red dot. This means that the rolling of a red dot is an unlikely event.

Now consider the remaining die, of which we covered four of the six sides with yellow dots. This die expresses the probability of 4 out of 6 for rolling a yellow dot. The chances of rolling a yellow dot are greater than the chances of having a blank side up with this die, and thus rolling a yellow dot is a *likely event*.

We have now described two likely events. The event of rolling a yellow dot is likely (4 chances out of 6), while that of rolling a green dot is very likely (in fact, certain) at 6 out of 6. These events are considered *unequally likely*. That is, they are both likely, but one is more likely than the other.

Some interesting questions arise in our analysis of the experiment. For example, is it possible for the red marker to make it to the end of its track before the yellow marker does? The occurrence of both the red dot and the yellow are uncertain events, and we cannot therefore answer this question with any guarantee; however, we also know that the event of rolling a yellow dot is more likely than the event of rolling a red dot (in fact, two times more likely). Therefore, it is *possible* for the red marker to reach the end point first, but also quite *unlikely*.

Is it possible for the yellow marker to reach the end of its track before the green marker does? We know that we will roll a green dot on each turn (a certain event), but the event of rolling a yellow dot is uncertain (although likely). If we were really lucky and rolled a yellow dot on every turn, then the green and yellow markers would reach the end point at the same time. This means that the yellow marker would never beat the green marker—the best it can do is finish at the same time (a tie). Because rolling a yellow dot is an uncertain event, while rolling a green dot is certain, it is probable that the green marker will almost always reach the finish line before the yellow marker (that is, a tie would be rare).

Would we get the same results if we ran the experiment again? This question is a fun question to ask in almost any probability experiment. In this case, we know that the green marker will always make it to the finish line, and we know that the blue marker will never make it to the finish line (in fact, it will never leave the start space!). It is more difficult to predict the behaviors of the red and yellow markers, however. The events of rolling either a red or yellow dot are both uncertain; therefore, if we were to run the experiment again, we would predict that both markers would make it somewhere along the track by the time the green marker finished, but not necessarily to the same space they reached in the previous experiment. Because rolling a red is less likely than rolling a yellow, it is *probable* (no guarantees however!) that the yellow marker will move further along its track than the red marker every time we conduct the experiment.

Hints for the Classroom

Hint One: Have students create the dice themselves; this is simple to do and helpful in classroom discussions. In creating the dice, the students will pay attention to what is actually found on the faces of the dice, an analysis that is critical to understanding the activity.

Hint Two: Ask many "why" questions while the students are conducting the experiment. You may wish to ask questions such as these:

- *Why does the green marker seem to move faster than the others?*
- *Why does the red marker seem to move faster than the yellow?*
- *Why does the blue marker never seem to move at all?*

"Why" questions help the students synthesize the information demonstrated through the experiment and force them to draw generalizations from the activity. Remember that it is not of particular interest whether

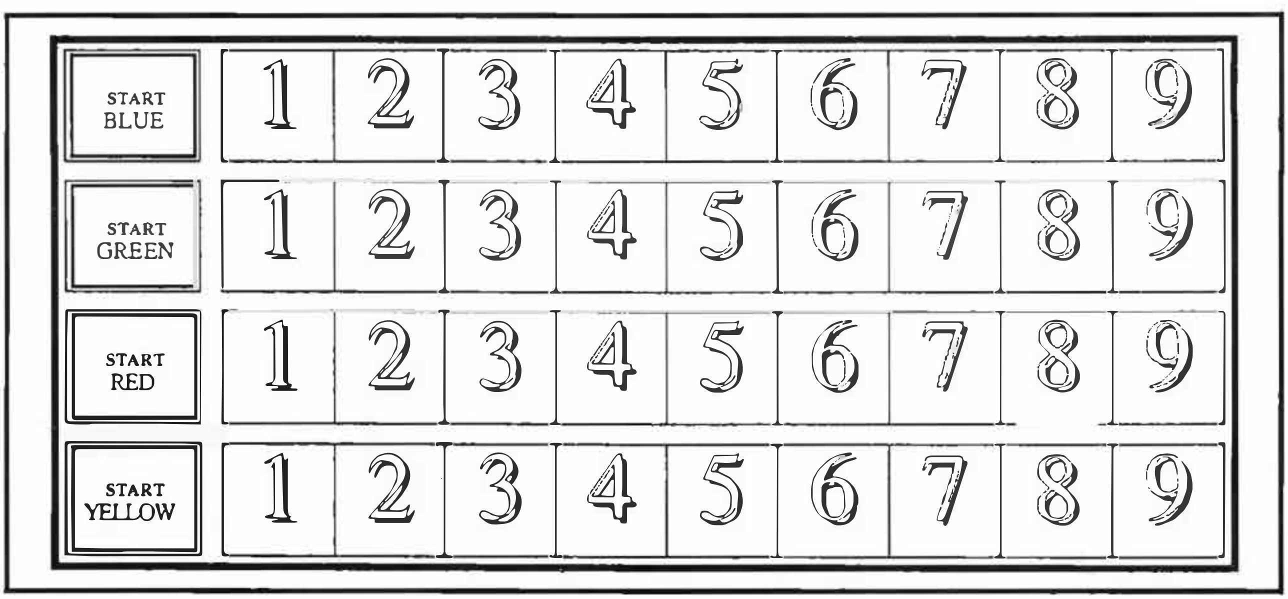

Figure 1, Game Board for the Biased Dice

or not students can correctly predict or describe the outcomes of the experiment; we are more interested in whether students can explain the outcomes. Ability to explain the outcomes of the experiment attests well to the students' mastery over and understanding of the experience. "Why" questions are helpful because they demand explanations.

Hint Three: Try to build the activity up from a simple to a more complex level. In rolling all four dice at the same time, students are forced to draw comparisons between four simultaneous events, and this large number of comparisons may be overwhelming for some of them. As an alternative, start with only two dice (say the die with the green dots and the die with the red dots) and conduct the experiment. When students become comfortable with their ability to explain the results, then you may introduce other dice (either adding a die or using different dice).

You may find it interesting to have the students complete the activity using only the dice with the red and yellow dots. It may happen that one of the groups has the red marker finish first, which could lead to an interesting discussion of the notion of chance.

Hint Four: A good test of how well students understand this activity may be found in asking them to construct some of their own dice. For example, ask the students to construct a die that (a) makes rolling a yellow dot an uncertain and unlikely event, or (b) makes rolling a green dot an uncertain but likely event. These tasks may be good tests of learning in that they require students to apply their new ideas in familiar contexts.

Challenges

Here are some ways to extend the activity as it has been described above. Find the answers for your own amusement.

- *Can you create a single six-sided die that makes rolling a yellow dot an uncertain but unlikely event and rolling a green dot an uncertain but likely event? How many such dice can you make?*
- *Can you create a single six-sided die that makes rolling a yellow dot a certain event and rolling a green dot an uncertain but likely event?*
- *Can a single six-sided die be created to show both a certain and an impossible event?*
- *Can a single six-sided die be created to show two equally likely events?*
- *Create new dice for the original experiment that would theoretically enable yellow to finish first, green second, blue third and red not at all. How many such combinations of dice can you make? Find several different possibilities.*

Activity Two: The Mystery Dice

Objective

Develops probability concepts through experiments—identifies all the possible outcomes of a probability experiment (Alberta Education 1991).

Materials

Hidden Dice program (see Figure 2), Apple II*e* computer, pencil, paper, one four-sided die, one six-sided die and one eight-sided die.

Activity

This activity is a computer simulation of rolling an unknown number of hidden dice. In this activity we are rolling four-, six- and eight-sided dice only. The computer secretly selects either one or two dice from a collection of two four-sided, two six-sided and two eight-sided dice. If the computer selects two dice, it never selects two of the same kind of die (that is, it will never pick two four-sided dice or two six-sided dice or two eight-sided dice). The computer rolls the dice for you and reports the value of the roll. If it is rolling only one die, then it simply tells you the value rolled on that die. If it is rolling two dice, then it adds the values of the two faces and reports this sum. By rolling the dice many times and observing the various outcomes of the rolls, you can figure out how many dice the computer is rolling, and whether the dice have four, six or eight sides.

Every time you ask the computer to roll the dice for you, it costs you one point. You may roll as often as you like, and the computer will keep track of your score. When you think you know how many dice the computer selected, and the number of faces on the die or the dice, you may opt to input your guess. If you guess incorrectly, 50 points will be added to your score. The computer will also draw a frequency histogram showing how often each value has been rolled, but this clue costs you 10 points. Of course, your objective is to discover the identity of the hidden dice with as low a score as possible.

Questions

- *What clue or clues tell you that the computer has selected only a single die?*

- *What clue or clues let you know that the computer has selected two dice?*
- *What clue or clues indicate that the computer has selected a six- and an eight-sided die?*
- *When you run the program the first five values the computer reports are 3, 7, 4, 1, 6. Do you know how many dice have been selected? Do you know which one or more dice have been selected? Explain.*
- *When you run the program the first five values the computer reports are 2, 2, 3, 4, 3. Do you have enough information to solve the problem (identify the number and nature of the one or more dice)? Explain.*
- *Could the computer generate these numbers in its report: 2, 1, 9, 5, 2? Explain.*
- *How does the sample space for the simultaneous rolling of a six- and a four-sided die differ from the sample space of an eight-sided die?*

Getting Started

You will need to begin this activity by entering the program into your Apple II*e* computer. This can be accomplished by first booting the machine with a Disk Operating System (DOS) disk, and then typing NEW and pressing return.

You are now ready to type the program as listed in Figure 2. After typing it in, try running the program to ensure that you have not made any typing errors. Any line that has an error has to be retyped. Once the program runs well from beginning to end, save the program on a formatted disk so that you do not have to retype and debug it each time. When you are ready to begin your experiment, you merely have to type the word RUN and press return.

Operation of the program is very simple. You will see a statement of the first value rolled at the top of the screen along with your score so far. You may now opt to roll the die or dice again (simply press the R key, but make sure the caps lock key is down), guess at the number and type of dice (press the G key), or have the computer draw a histogram of the values rolled to date (press the H key). If you opt to guess the number and type of dice, you will be asked the following questions:

NUMBER OF DICE: (to which you respond either 1 or 2)

If you indicate that there is only one die, the computer will ask:

NUMBER OF SIDES ON DIE: (you respond with 4 or 6 or 8)

If you indicate that there are two dice, the computer will ask how many sides for each die:

NUMBER OF SIDES ON FIRST DIE: (either 4 or 6 or 8)

NUMBER OF SIDES ON OTHER DIE: (either 4 or 6 or 8)

If you are correct you may opt to play again or quit. If you are incorrect you will be assessed 50 points and asked to choose between rolling, guessing again and drawing the histogram.

Figure 2: BASIC Program for the Mystery Dice Activity

```
10 REM WRITTEN BY A. CRAIG LOEWEN
20 REM MYSTERY DICE PROGRAM
30 REM JANUARY 10, 1992
40 DICE = INT(RND(1) * 2 + 1)
50 D1 = INT(RND(1) * 3 + 2) * 2
60 D2 = INT(RND(1) * 3 + 2) * 2
65 IF D1 = D2 THEN 60
70 IF DICE = 1 THEN D2 = 0
80 HOME: SC = 0: DIM R(16)
90 PRINT "DIE OR DICE SELECTED!":PRINT
100 X = INT(RND(1) * D1 + 1)
110 Y = INT(RND(1) * D2 + 1)
120 IF DICE = 1 THEN Y = 0
130 ROLL = X + Y: R(ROLL) = R(ROLL) + 1
140 SC = SC + 1
150 PRINT "VALUE ROLLED: ";: INVERSE: PRINT "*  ";ROLL;"  *": NORMAL
160 PRINT "SCORE:  ";SC
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Figure 2 continued overleaf

```
170 PRINT "---------------------------------------"
180 PRINT "ROLL (R), GUESS (G), OR HISTOGRAM (H)";
190 GET A$
200 IF A$ = "R" THEN PRINT: GOTO 100
210 IF A$ < > "G" THEN 560
220 HOME
230 PRINT "NUMBER OF DICE:  ";
240 GET A$
250 IF A$ < "1" OR A$ > "2" THEN PRINT CHR$(7);: GOTO 240
260 PRINT A$: A = VAL (A$)
270 IF A = 2 THEN 340
280 PRINT "NUMBER OF SIDES ON DIE:  ";
290 GET A$
300 IF A$ < > "4" AND A$ < > "6" AND A$ < > "8" THEN PRINT CHR$(7);: GOTO 290
310 PRINT A$: B = VAL (A$)
320 IF A = DICE AND B = D1 AND DICE = 1 THEN 480
330 GOTO 440
340 PRINT "NUMBER OF SIDES ON FIRST DIE:  ";
350 GET A$
360 IF A$ < > "4" AND A$ < > "6" AND A$ < > "8" THEN PRINT CHR$(7);: GOTO 350
370 PRINT A$: B = VAL (A$)
380 PRINT "NUMBER OF SIDES ON OTHER DIE:  ";
390 GET A$
400 IF A$ < > "4" AND A$ < > "6" AND A$ < > "8" THEN PRINT CHR$(7);: GOTO 390
410 PRINT A$: C = VAL (A$)
420 IF A = DICE AND B = D1 AND C = D2 AND DICE = 2 THEN 480
430 IF A = DICE AND B = D2 AND C = D1 AND DICE = 2 THEN 480
440 PRINT
450 PRINT "OOOOOPS!!": SC = SC + 50
460 PRINT "SCORE:  ";SC
470 GOTO 170
480 PRINT "CORRECT!!"
490 PRINT "SCORE:  ";SC
500 PRINT "---------------------------------------"
510 PRINT "PLAY AGAIN? (Y/N):  ";
520 GET A$
530 IF A$ = "Y" THEN RUN
540 IF A$ = "N" THEN HOME: END
550 PRINT CHR$(7);: GOTO 520
560 IF A$ < > "H" THEN PRINT CHR$(7);: GOTO 190
570 HOME
580 PRINT "HISTOGRAM:"
590 FOR X = 1 TO 16
600 IF X < 10 THEN PRINT "  ";
610 PRINT X;"|";
620 N = R(X): IF N > 35 THEN N = 35
625 IF N = 0 THEN PRINT: GOTO 660
630 FOR Y = 1 TO N
640 PRINT "*";
650 NEXT Y: PRINT
660 NEXT X
670 PRINT
680 SC = SC + 10: PRINT "SCORE:  ";SC
690 GOTO 170
```

Example

Assume that the program has been correctly placed in the computer's memory and has been started. The screen shows:

```
VALUE ROLLED: *4*
SCORE: 1
-------------
ROLL(R), GUESS(G), OR HISTOGRAM(H)
VALUE ROLLED: *1*
SCORE: 2
-------------
ROLL(R), GUESS(G), OR HISTOGRAM(H)
VALUE ROLLED: *7*
SCORE: 3
-------------
ROLL(R), GUESS(G), OR HISTOGRAM(H)
```

The first roll told you very little, but the second roll was important. It told you that the computer opted to roll only one die. (Why does a 1 tell you that?) The third roll was also important as it told you that the computer was rolling an eight-sided die. (Why does a 7 give this away?) You now know all the information you need to make your guess so you may select "G" and input your guess.

Discussion

As in the previous experiment, understanding this activity requires an analysis of sample space or the list of all the possible outcomes of an experiment. To determine the sample space in each case, you need merely ask what the lowest and highest values are that can be rolled. For example, with a single six-sided die, the lowest value is a 1, and the highest value is a 6. The sample space for a six-sided die includes six possible numbers: the values from 1 to 6. The sample space for a four-sided die will include all the values from 1 to 4. The sample space for an eight-sided die will include all the values from 1 to 8.

The sample space produced from rolling two dice is slightly more complicated, but the task can be simplified by asking what the smallest and largest possible sums are. The smallest sum can be determined by adding the two smallest values on the two dice. Likewise, the largest sum can be determined by adding the two largest values on the two dice. If you are rolling a four-sided and a six-sided die, then the sample space includes values ranging from 2 to 10. A six-sided and an eight-sided die together have a sample space with values ranging from 2 to 14. Notice that the value 1 can be rolled only with a single die. Therefore, if the computer reports that a 1 has been rolled, you know that the computer has selected only one die. We also know that the only combination of dice that can give us values of 13 and 14 are a six- and an eight-sided die together.

The problem that the computer presents can lead to some difficult questions. For example, the computer has reported several values ranging from 2 to 8; do we know what die or dice are being rolled? In this case the reported range alone does not identify the die or dice. It is possible that the computer is rolling an eight-sided die but, just by coincidence, the number 1 has yet to appear. However, it is also possible that the computer is rolling a four-sided and a six-sided die, but, just by coincidence, the numbers 9 and 10 have yet to appear. Because we do not yet know what dice are being rolled, we will have to roll again.

Consider another hypothetical situation. The computer has reported six values all ranging from 1 to 4; do we know what die or dice are being rolled? We know that the computer is rolling a single die given that the value 1 has appeared. We do not know whether the computer is rolling a four-, six- or eight-sided die. We must ask ourselves this question: Is it possible that the computer could roll a six-sided die six times, and not roll a 5 or 6? The answer is yes. Given that the roll of a die is a chance event, it is quite possible that after so few rolls we may not have seen either a 5 or 6. We must therefore ask ourselves another question: How many more rolls would convince us that the computer is rolling a four-sided die? Obviously, the larger the number of rolls made, the more likely it is that we would get a 5 or a 6 if the computer is rolling a six-sided die. The exact number of rolls necessary to convince each individual will vary: there is no single "correct" number of rolls. (Incidentally, statisticians would argue that approximately eight more rolls without obtaining a 5 or 6 would constitute a very good argument for rejecting the possibility of a six-sided die, but an explanation of their argument is beyond the scope of this paper.)

Hints for the Classroom

Hint One: When introducing this activity to students, start with a single six-sided die and ask a few important questions to establish the sample space (or range of possible outcomes) of rolling the die. For example: What is the largest possible number you could roll? What is the smallest possible number you could roll? How many different possible numbers could we roll? You may wish to familiarize students with the four- and eight-sided dice as well. Stress the fact that different outcomes result from rolling different dice.

Hint Two: After students are familiar with the range of possible outcomes from rolling a single die, introduce two dice. Again, we will need to determine our range of possible outcomes for rolling two dice. Be sure students understand how we determine the smallest possible outcome and the largest possible outcome.

Hint Three: Try playing the game a few times with the students as a whole while you (or a student) hide(s) behind a partition while rolling the dice. This activity familiarizes the students with the activity as well as allowing them to form initial hypotheses.

Hint Four: Create a low score list on the blackboard to emphasize that the task is to find out about the dice in as few rolls as possible. You will want to treat each combination of dice separately, as it is easier to identify a single die than to identify any combination of two dice.

Hint Five: You may wish to create a chart of the sample space for rolling two dice, such as for a four- and an eight-sided die (see Figure 3). This chart helps to clarify what the largest and smallest possible values are within the range of possible outcomes, and will help explain why some values are generated more often than others. For example, we notice in Figure 3 that there is only one way to roll a 2 (by rolling a 1 on each die), while there are four ways to roll a 7:

- roll a 1 on the four-sided die and a 6 on the eight-sided die
- roll a 2 on the four-sided die and a 5 on the eight-sided die
- roll a 3 on the four-sided die and a 4 on the eight-sided die
- roll a 4 on the four-sided die and a 3 on the eight-sided die

Therefore, if we conduct this experiment many times, we would expect that the value 7 would appear about four times more often than the value 2. The ability to create and interpret this sample space chart is important for the challenge activity described below.

Possible rolls of the eight-sided die

Possible rolls of the four-sided die		1	2	3	4	5	6	7	8
	1	2	3	4	5	6	7	8	9
	2	3	4	5	6	7	8	9	10
	3	4	5	6	7	8	9	10	11
	4	5	6	7	8	9	10	11	12

Figure 3: Sample Space for the Simultaneous Rolling of a Four- and an Eight-Sided Die

Challenges

When selecting two dice, the computer never selects two dice with the same number of sides, always two different dice. This criterion has been included so that one can determine the dice rolled through simple inspection of the range of possible outcomes without considering the frequency of each element within the sample space. If we remove this criterion, it becomes possible for the computer to pick two six-sided dice. Notice that the range of possible outcomes (2 to 12) for two six-sided dice is exactly the same as the range of possible outcomes for a four- and an eight-sided die. How then could we tell the difference between rolling a four- and an eight-sided die and rolling two six-sided dice?

To free the computer to select two dice with the same number of sides, simply remove line 65 from the program by typing the number 65 after the "}" prompt and pressing return. *Hint:* You will probably want to draw out the sample space for rolling two six-sided dice, and you will probably need to exercise the option to have the computer draw a histogram for you. Good luck!

References

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