Applying a Fundamental Property of Integration

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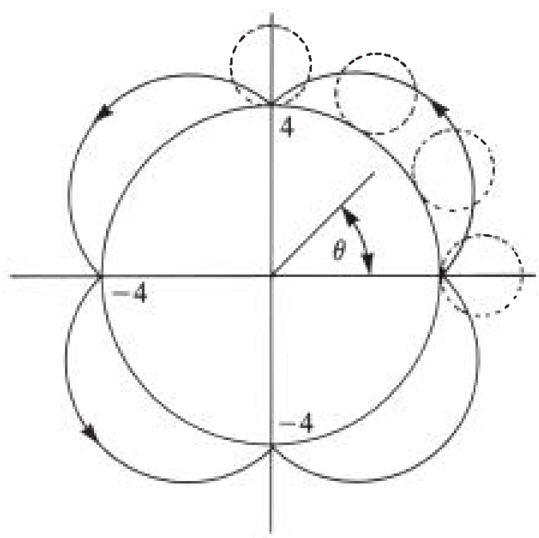
An interesting problem, in which students must ensure that the integrand is always positive between the limits of integration, is the following:



A circle of radius 1 is rolling around the circumference of a circle of radius 4. The path of a point on the circumference of the smaller circle is given by

$x = 5 \cos \theta - \cos 5\theta$

$y = 5 \sin \theta - \sin 5\theta$, tracing out an epicycloid.

Find the distance traveled by the point in one complete trip about the larger circle (that is, the arc length or perimeter of the epicycloid) (Larson and Hostetler 1982, 579).

Using the formula for arc length in parametric form:

If a curve is given by $x = f(\theta)$ and $y = g(\theta)$ (where f' and g' are continuous or the interval $[a,b]$), then the arc length of the curve over this interval is given by:

$$S = \int_{\theta = a}^{\theta = b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2} \; d\theta.$$

For example, we have

$$S = \int_{\theta = o}^{\theta = 2\pi} \sqrt{\left(\frac{d\,x}{d\,\theta}\right)^2 + \left(\frac{d\,x}{d\,\theta}\right)^2} \quad d\,\theta$$

$$S = \int \sqrt{(-5 \sin \theta + 5 \cos 5\,\theta)^2 + (5 \cos \theta - 5 \sin 5\,\theta)^2 \; d\,\theta}$$

$$= 5 \cdot \int \sqrt{2 - 2 \sin \theta \sin 5\,\theta - 2 \cos \theta \cos 5\,\theta} \; d\,\theta$$

$$S = 5 \cdot \int_{0}^{2\pi} \sqrt{2 - 2 \cos 4\,\theta} \quad d\,\theta$$

$$= 5 \cdot \int \sqrt{4 \sin^2 2\,\theta} \quad d\,\theta$$

$$= 10 \cdot \int \sqrt{\sin^2 2\,\theta} \quad d\,\theta$$

$$= 10 \cdot \int \left| \sin 2\,\theta \right| \quad d\,\theta\,.$$

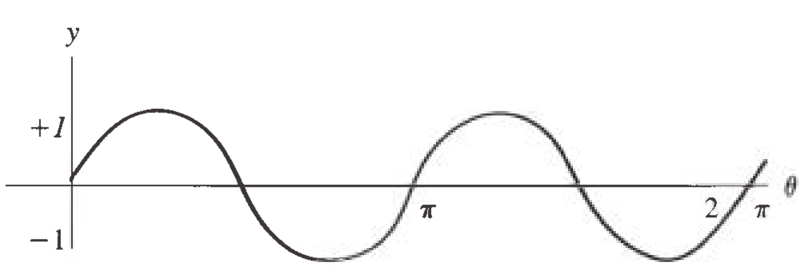
Without thinking, students now integrate between the limits of 0 and 2π, that is

$$S = \frac{10}{2} \int \sin 2\,\theta \cdot 2\,d\,\theta$$

$$= -5 \cdot \quad \cos 2\,\theta \Big]_{0}^{2\pi}$$

$$= -5 \cdot \quad (1-1) = 0, \text{ which is incorrect.}$$

Here, $\sin 2\,\theta$ must be positive on the interval of integration, that is, it must be broken up into parts. $\sin 2\,\theta$ has period $\frac{2\pi}{2} = \pi$, that is,



$\sin 2\,\theta$ is negative from $\frac{\pi}{2}$ to π, and again from $\frac{3\pi}{2}$ to $2\,\pi$.

Because of symmetry of this integral, we can change limits to $4\int_0^{\pi/2} \sin 2\theta \, d\theta$ to ensure that integral will always be positive on interval.

(Alternately, we could have used line $\pi \leq \theta \leq \frac{3\pi}{2}$ and multiplied by 4.)

$$\text{Therefore, } S = (4)\,(10)\int_0^{\frac{\pi}{2}} \sqrt{\sin^2 2\theta}\; d\theta$$

$$= 40\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{2}} \sin 2\theta \, 2 \, d\theta$$

$$= -20 \cos 2\theta \Big]_0^{\frac{\pi}{2}}$$

$$= -20\,[\cos \pi - \cos 0]$$

$$= -20\,[(-1) - \emptyset]$$

$$= 40$$

Reference

Larson, Roland, and Robert Hostetler. *Calculus with Analytic Geometry.* D.H. Heath and Company, Lexington, Massachusetts, 1982, 579.