Activities on the Natural Number Triangle: Sums and Quotients

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Consider the natural number triangle as shown in Figure I. Each row of the triangle, after the first, consists of one more natural number than the row above it. In general, the n^{th} row consists of the natural numbers 1, 2, 3, . . ., n. If the triangle were extended indefinitely, all columns consist of the natural numbers.

Figure I

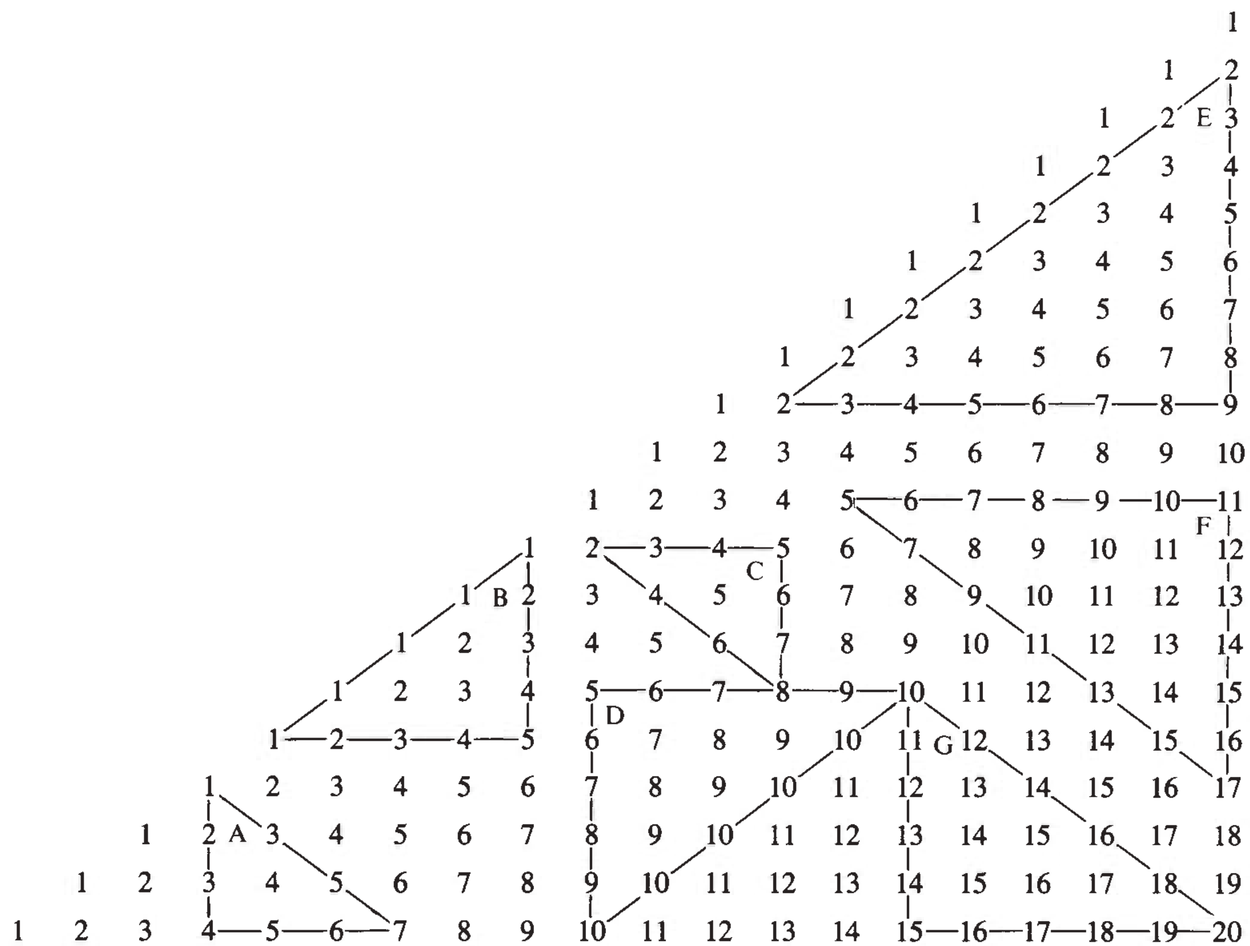
																			1
																		1	2
																	1	2	3
																1	2	3	4
															1	2	3	4	5
														1	2	3	4	5	6
													1	2	3	4	5	6	7
												1	2	3	4	5	6	7	8
											1	2	3	4	5	6	7	8	9
										1	2	3	4	5	6	7	8	9	10
									1	2	3	4	5	6	7	8	9	10	11
								1	2	3	4	5	6	7	8	9	10	11	12
							1	2	3	4	5	6	7	8	9	10	11	12	13
						1	2	3	4	5	6	7	8	9	10	11	12	13	14
					1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
				1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Note that the row sums generate the set of triangular numbers: 1, 3, 6, 10, 15, 21, 18. . . .
We shall draw selected geometric figures on the natural number triangle and compute sums and ratios.

Activity 1

Isosceles right triangles have been drawn on Figure II. The perpendicular sides lie on the rows and columns of the triangle.

Figure II



For each triangle, compute the following:

1. The sum of the numbers on the three vertices of the isosceles triangle. Call this sum V.

2. The sum of the numbers which lie in the interior of the triangle. Call this sum I. Also, count the number of interior numbers of each triangle and call this number N.

3. Compute:

   A $\frac{V}{I}$ B $\frac{3}{N}$

   (Recall that a triangle has three vertices.)

Table 1 reports the results of our computation for triangles A through G.

Table 1

Triangle	V	I	N	$\frac{V}{I}$	$\frac{3}{N}$
A	12	4	1	$\frac{12}{4}$	$\frac{3}{1}$
B	7	7	3	$\frac{7}{7}$	$\frac{3}{3}$
C	15	5	1	$\frac{15}{5}$	$\frac{3}{1}$
D	25	50	6	$\frac{25}{50}$	$\frac{3}{6}$
E	13	65	15	$\frac{13}{65}$	$\frac{3}{15}$
F	33	110	10	$\frac{33}{110}$	$\frac{3}{10}$
G	45	90	6	$\frac{45}{90}$	$\frac{3}{6}$

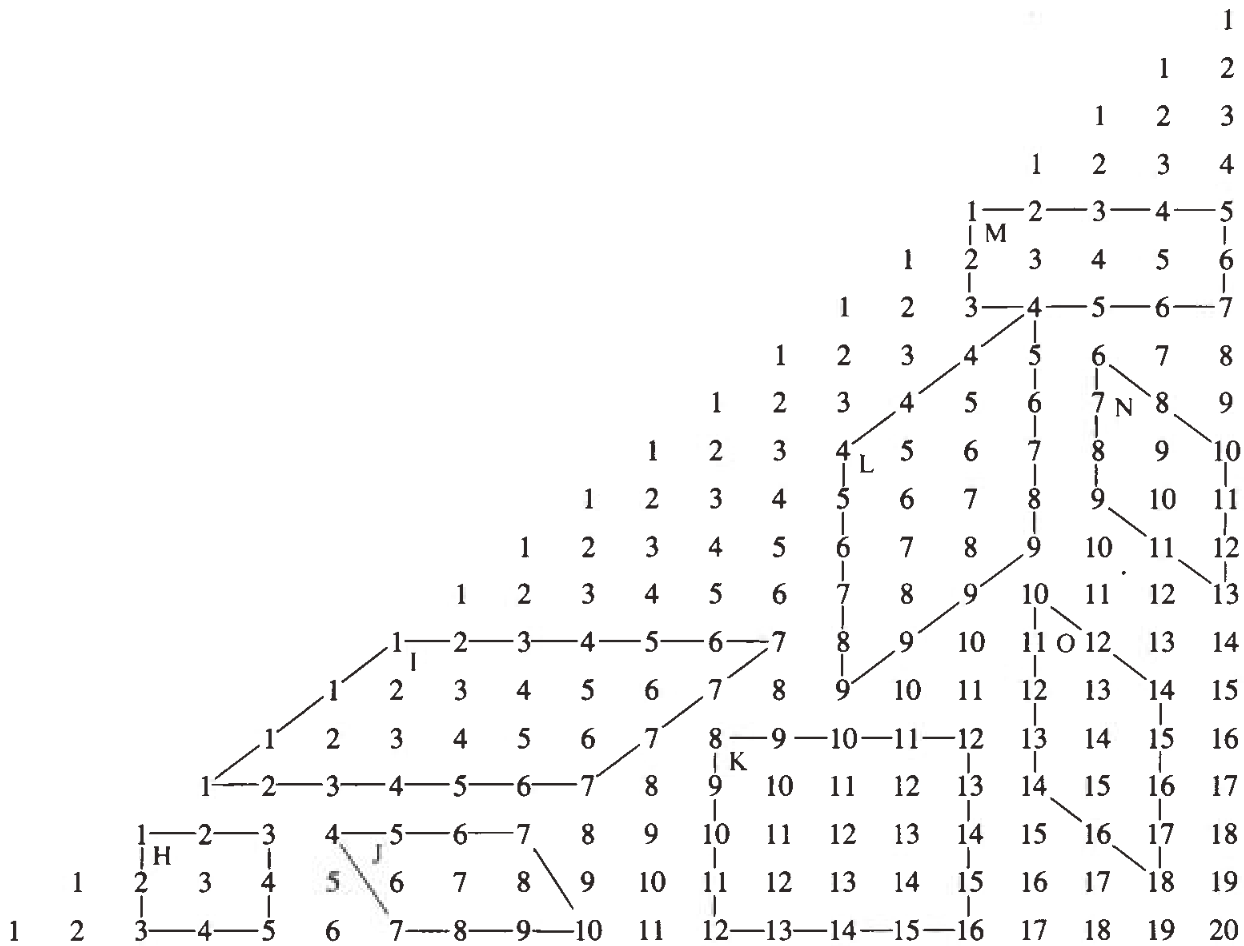
In each case, observe that $\frac{V}{I} = \frac{3}{N}$.

In other words, the ratio of the sum of the vertex numbers to the sum of the interior numbers equals the ratio of the number of vertex numbers (3) to the number of interior numbers. Have your students draw other isosceles right triangles to verify this.

Activity 2

Parallelograms have been drawn on Figure III. One pair of sides must lie on either the horizontal rows or the vertical columns of the number triangle.

Figure III



Follow the same steps as in Activity 1, except recall that a parallelogram has four vertices. Table 2 reports the results of our computations for parallelograms H through O.

Table 2

Parallelogram	V	I	N	$\frac{V}{I}$	$\frac{4}{N}$
H	12	3	1	$\frac{12}{3}$	$\frac{4}{1}$
I	16	40	10	$\frac{16}{40}$	$\frac{4}{10}$
J	28	21	3	$\frac{28}{21}$	$\frac{4}{3}$
K	48	108	9	$\frac{48}{108}$	$\frac{4}{9}$
L	26	52	8	$\frac{26}{52}$	$\frac{4}{8}$
M	16	12	3	$\frac{16}{12}$	$\frac{4}{3}$
N	38	19	2	$\frac{38}{19}$	$\frac{4}{2}$
O	56	42	3	$\frac{56}{42}$	$\frac{4}{3}$

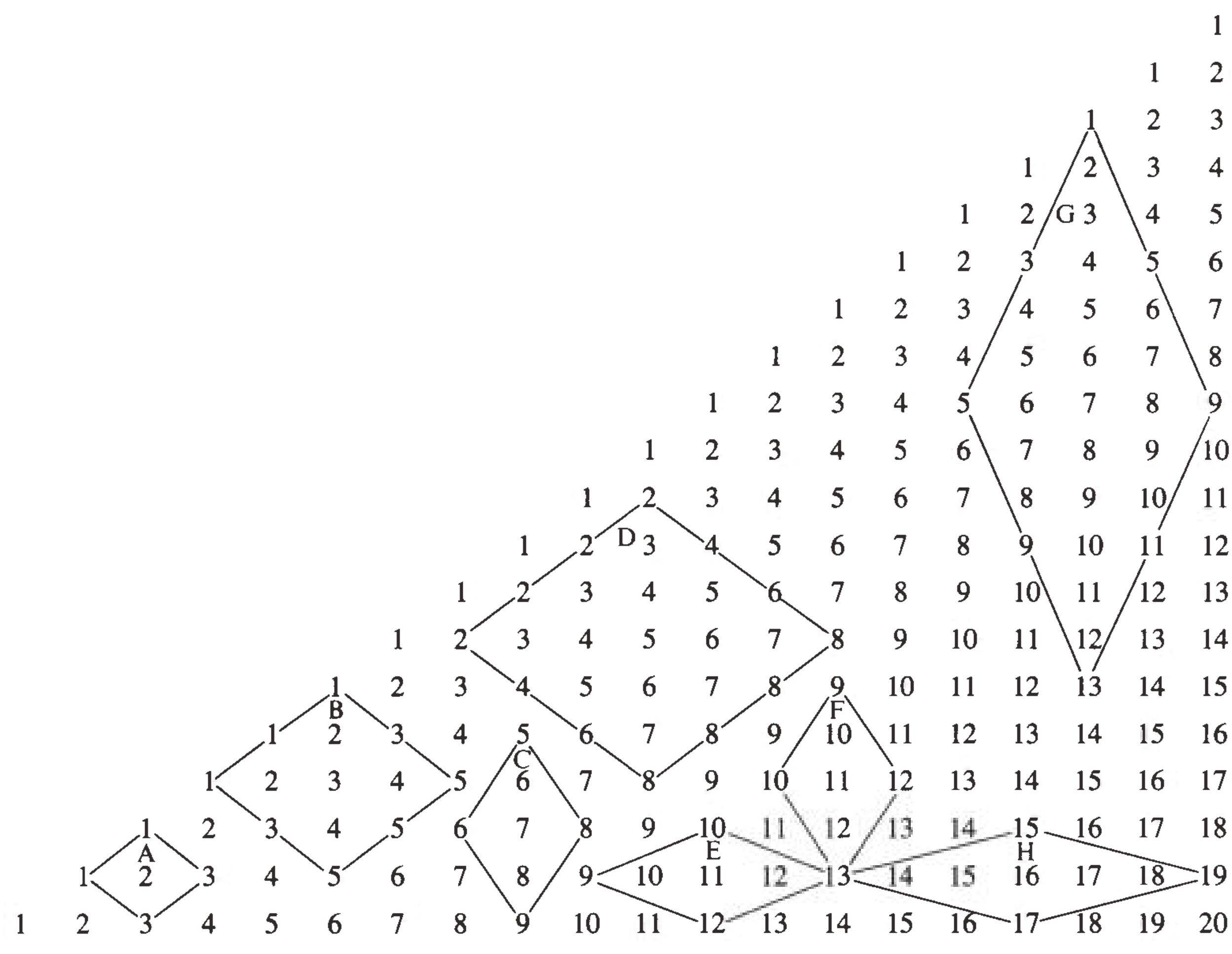
Observe that $\frac{V}{I} = \frac{4}{N}$,

that is, the ratio of the sum of the vertex numbers to the sum of the interior numbers equals the ratio of the number of vertex numbers (4) to the number of interior numbers. Have your students draw other parallelograms to verify this.

Activity 3

Diamonds have been drawn on Figure IV. The diagonals of the diamonds lie on the rows and columns of the number triangle.

Figure IV



Follow the steps of Activity 2. Table 3 reports the results of our computation for diamonds A through H.

Table 3

Diamond	V	I	N	$\frac{V}{I}$	$\frac{4}{N}$
A	8	2	1	$\frac{8}{2}$	$\frac{4}{1}$
B	12	15	5	$\frac{12}{15}$	$\frac{4}{5}$
C	28	21	3	$\frac{28}{21}$	$\frac{4}{3}$
D	20	65	13	$\frac{20}{65}$	$\frac{4}{13}$
E	44	33	3	$\frac{44}{33}$	$\frac{4}{3}$
F	44	33	3	$\frac{44}{33}$	$\frac{4}{3}$
G	28	147	21	$\frac{28}{147}$	$\frac{4}{21}$
H	64	80	5	$\frac{64}{80}$	$\frac{4}{5}$

Observe the pattern that Activity 2 holds. Challenges for the reader:

1. In Activity 1, the perpendicular sides of the isosceles right triangle were in rows and columns. If any of these conditions were changed, would the same patterns hold?

2. In Activity 2, would the same pattern hold if the sides of the parallelograms were not in rows or columns?