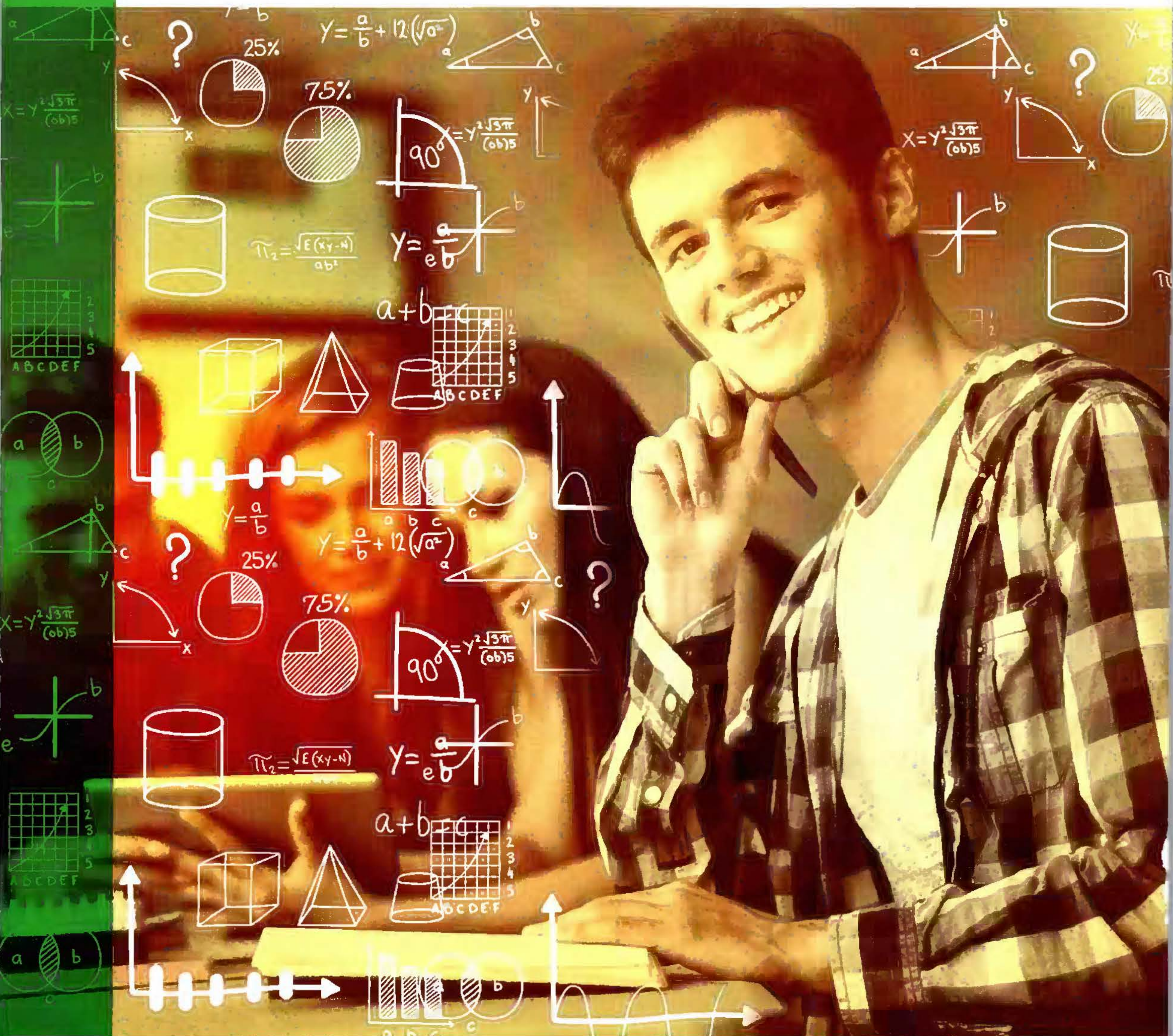
delta-k

Journal of the Mathematics Council of the Alberta Teachers' Association

Volume 53, Number 1 January 2016



Mathematical ideas = Springboard to learning

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. Peer-reviewed articles are normally 8–10 pages in length.
4. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
5. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
6. All manuscripts should be submitted electronically, using Microsoft Word format.
7. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
8. References and citations should be formatted consistently using *The Chicago Manual of Style*'s author-date system or the American Psychological Association (APA) style manual.
9. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal. The editor will provide this letter on request.
10. Letters to the editor, description of teaching practices or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Lorelei Boschman, c/o Medicine Hat College, Division of Arts and Education, 299 College Drive SE, Medicine Hat, AB T1A 3Y6; e-mail lboschman@mhc.ab.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 53, Number 1 January 2016

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 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Lorelei Boschman, c/o Medicine Hat College, Division of Arts and Education, 299 College Drive SE, Medicine Hat, AB T1A 3Y6; e-mail lboschman@mhc.ab.ca. EDITORIAL AND PRODUCTION SERVICES: Document Production staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in CBCA Education. ISSN 0319-8367

Individual copies of this journal can be ordered at the following prices: 1 to 4 copies, $7.50 each; 5 to 10 copies, $5.00 each; more than 10 copies, $3.50 each. Please add 5 per cent shipping and handling and 5 per cent GST. Please contact Distribution at Barnett House to place your order. In Edmonton, dial 780-447-9432; toll free in Alberta, dial 1-800-232-7208, ext 432.

From the Editor's Desk

Lorelei Boschman

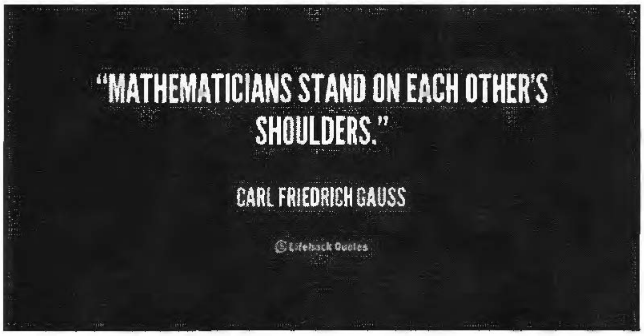
As this is my first issue as editor of *delta-K,* I would like to introduce myself. I am excited to work on *delta-K* and look forward to reading, discussing and sharing current, notable and engaging mathematics ideas with teachers in Alberta and beyond.

Here's a quick bit about my background: I received my bachelor of education and master of education degrees in 1988 and 1997, respectively, from the University of Lethbridge. I began teaching in 1988 at a K–8 school in Barons, Alberta. In 1989 I became a Grades 9–12 math teacher at Medicine Hat High School, where I taught all levels of high school math for the next 21 years. In 2008 I began teaching sessionally at Medicine Hat College; this turned into a full-time assignment in 2010 teaching education, and math and science classes. I am currently teaching the Math Curriculum and Instruction course, and supervising field experiences in the Mount Royal University Bachelor of Education program at Medicine Hat College. I also have had many professional experiences in terms of program development, leadership roles and professional growth.

I would like to sincerely thank Gladys Sterenberg for her excellent work as past editor of *delta-K*. Her expertise, involvement, time commitment and dedicated thought are truly valued. I'm sure we will continue to hear from Gladys as she moves into other roles and interests within the educational field.

This issue incorporates interesting discussion about education reform, the history of mathematics and formative assessment to encourage thought. The feature article, "Addressing the Misconceptions of the Equal Sign," by Brad Weible, discusses an in-depth debate, research presentation and implications of various student understanding levels of our familiar symbol. Also included in this issue are practical teaching ideas about blueprinting assessments, by Greg Wondga, and two cross-curricular units of study—the first on the topic of "homesteading" in Grade 4, by Dalyce Harrison, and the second on the topic of a business-marketing project in Grade 9, by Katherine Weber. The past three math contests, interesting problem-solving moments, a website highlight and two book reviews round out this issue. It is so encouraging to have the opportunity for these discussions and growth.

Continue to send in submissions of all types; sharing with other math teachers around the province is essential! So many of you have so much to give and share. Above all else, learn, grow and enjoy being the mathematicians that you all are. Your students are the benefactors!



Should (or Does) Mathematics Education Research Inform Our Mathematics Teaching Practices?

Est-ce que la recherche en didactique des mathématiques informe (ou devrait informer) notre enseignement des maths?

Chantal Buteau, Nadia Hardy and Joyce Mgombelo

Does (should) mathematics education research inform our mathematics teaching practices (Biesta 2010)? This remains a widespread debate among the mathematics education community; it stems from a perceived gap between research in mathematics education and its potential to change teaching practices. It is in this context that the theme of the one-day Canadian Mathematics Education Study Group (CMESG) preconference on "Mathematics Education Research and Mathematics Teaching: Illusions, Reality, and Opportunities," hosted at Brock University in May 2013, was framed. At the event, some 90 math educators, 47 CMESG delegates and 43 local community members came together to reflect on the gap between mathematics education research and the practices in mathematics classrooms.

La réflexion a eu lieu par le biais d'un exposé plénier par John Mason (Lerman 2010) et des groupes de travail pour chaque niveau d'enseignement. Elle a été guidée par les questions suivantes: Quelle recherche informe l'enseignement des mathematiques? Comment cette recherche vientelle à être mise en pratique? Quels sont les problèmes et les lacunes? Dans cet article. nous résumons les principales idées discutées à propos du niveau universitaire et l'exposé plénier. Nous concluons avec quelques questions et préoccupations au sujet des questions soulevées par le conférencier plénier et les participants.

A Summary of John Mason's Keynote: Responsive and Responsible Teaching (Lerman 2010)

Responsive teaching involves thoughtful response to learner behaviour, informed by principles and assumptions about learning. The teacher, however, may not have a discourse to justify actions or choices.

Responsible teaching is when the choices to act and the expected results, both during preparation and the minute-by-minute flow of the classroom, can be clearly articulated and justified through the use of technical terms, assumptions and values.

Mathematics education, as a research domain, has the responsibility to promote responsible teaching, stimulating teachers to bring assumptions, actions and other practices to articulation so that they can be compared and used to justify or challenge established practices. This claim is founded on the growing evidence that teaching practices are effective only when teachers understand the principles underpinning them. This understanding and the articulation and justification of choices require a certain degree of acquaintance with mathematics learning theories and their values and assumptions as they concern the lived experiences of mathematics learners.

En ce sens, de la recherche en didactique qui peut apporter quelque chose d'utile au développement curriculaires et aux pratiques en classe devrait mettre en évidence des actions et sensibilités a l'égard de ces actions, tout en montrant clairement comment elles s'alignent avec les hypothèses et les valeurs sur l'apprentissage des mathématiques (i.e., avec les théories de l'apprentissage). et la façon dont elles se déroulent dans des situations particulières, sans essayer de prétendre que ceci ou cela "fonctionne": voici pourquoi.

There are two major gaps in the research-practice domain of mathematics education: between researchers and policy-makers, and between researchers and teachers. The pragmatics of education means that policy-makers, leaders and teachers would like to know what works and what does not. But all attempts to turn theories into recipes for action will flounder because human beings are agentive organisms not machines, and trained behaviour may only work in local conditions. Certainly, training mechanical aspects of human behaviour can be successful, but only temporary; as soon as conditions change, trained behaviour becomes useless without educated awareness to guide it, which is why drill and practice can get learners through tests and even a few examinations, but leaves them feeling they don't understand, and vulnerable to changed conditions. This applies both to teachers and to learners. Cause and effect is not a dominant mechanism when human beings are involved. There are no practices that work independently of the context and conditions, and the variables involved cannot be specified sufficiently precisely, or perhaps even enumerated, so as to guarantee results. If research in mathematics education is taken as a tool for deciding between different teaching actions, then great care is needed to discern the relevant conditions and context that make those actions work.

From this perspective, valuable results in mathematics education research are, for example, those that reveal different

- ways in which learners reconstruct procedures from fragments of other procedures,
- ways that learner attention can be induced to shift toward what is mathematically significant and
- ways of inducing learners to encounter challenges they would not otherwise have considered.

But how does such research come to be put into practice? Effective mediation between research and practice involves the design of curricula and tasks clearly aligned with assumptions and values about learning, so teachers (and students) may articulate and justify their practices. This may lead to challenging assumptions, predispositions and perspectives about teaching and learning, but the goal is that this questioning takes place within a supportive and sustained environment so that changes in the discourse go hand in hand with changes in practices and changes in perspective. What matters are the learner experiences and the teacher in relation to both learners and mathematics. Mathematics education has a long history of looking for simple cause and effect, for silver bullets that will have an immediate and significant impact on learner performance. But teaching is a caring profession, and teaching mathematics is about caring passionately for learners and for mathematics, and for relationships between people and mathematics.

The underlying metaphor of education as a factory based on the mechanism of simple cause and effect needs to be challenged at every level. Maintaining complexity, respecting human beings as agentive, desirous and value directed, and respecting mathematics as a mode of enquiry and world perspective requires ongoing elaboration and support.

For more details, see Mason's various writings and publications at his website: http://mcs.open.ac.uk/jhm3/.

Mathematics Teaching at University Level: How Big of a Gap Is There Between Mathematics Education Research and Teaching Practices?

The focus of the working group (WG) was on the teaching of introductory mathematics courses (mostly calculus and linear algebra) as it was claimed that the issues around teaching these courses are significantly different than those around the teaching of advanced mathematics courses (such as abstract algebra or measure theory), and time was insufficient to address all. The WG participants (a group of 12 professors, researchers and graduate students) stressed several aspects of a perceived gap between research and the realities of the everyday teaching of these courses. In particular, they pointed out that related research often constrains its own capability of being put into practice since it considers ideal situations that are far from the realities of classrooms and students, and of teachers and their constrained practices. For example, research seldom takes into account students' diverse backgrounds, class size and time, and other institutional constraints such as how powerless instructors of introductory courses are to alter or influence curriculum design and everything it involves. (They don't choose content, often not even the order in which content is presented, and they have little or no control at all over assessment, textbook choice and so on.)

À la question de quelle recherche pourrait être "utile" pour l'enseignement. un groupe de participants a plaidé pour la recherche normative, i.e .. qui prescrit une approche pédagogique qui "fonctionne": "si on enseigne de telle ou tells façon les élèves apprendront/comprendront." Une discussion s'en est suivie, à partir des critiques formulae, lors de la présentation plénière. á l'endroit du paradigme de recherche "cause et effet." Les didacticiens en mathématiques qui participaient au groupe de travail (qui étaient pour la plupart également professeurs d'université de mathématiques) ont reconnu la frustration des professeurs d'université qui sentent leur liberté académique—qui devrait inclure la liberté d'explorer et de remettre en question les hypothèses institutionnelles et personnelles, les prédispositions et les perspectives sur l'enseignement et l'apprentissage—plus souvent qu'autrement bafouée par les pratiques institutionnelles actuelles.

Further Comment by the Authors

The view that research in mathematics education should be of prescriptive nature, providing scientific evidence that such and such teaching approach works or doesn't work seems to have gained popularity among the many educational stakeholders. The keynote speaker argued against a cause-and-effect approach to research in (mathematics) education, emphasizing its humanistic and social nature. Other researchers have expressed further worries with this approach (for example, Biesta 2010; Lerman 2010): the question of what works and what doesn't assumes that the ends of education are given, and that the only relevant questions to be asked are about the most effective and efficient ways of achieving those ends. Focusing on what works makes it difficult, if not impossible, to ask questions of what it should work for and who should have a say in determining the latter.

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Lerman, S. 2010. "Theories of Mathematics Education: Is Plurality a Problem?" In *Theories of Mathematics Education*, ed B Sriraman and L English, 99–109. Springer-Verlag: Berlin, Heidelberg.

Note: The preconference was supported by the Fields Institute, Brock University and Pearson Publishing Company. The working group was led by Nadia Hardy (Concordia University). The summary presented here is based on notes taken during interactions in the working group.

Chantal Buteau and Joyce Mgombelo are associate professors from Brock University, Saint Catharines, Ont; and Nadia Hardy is an associate professor from Concordia University, Montréal, QC.

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Comments from Your Executive

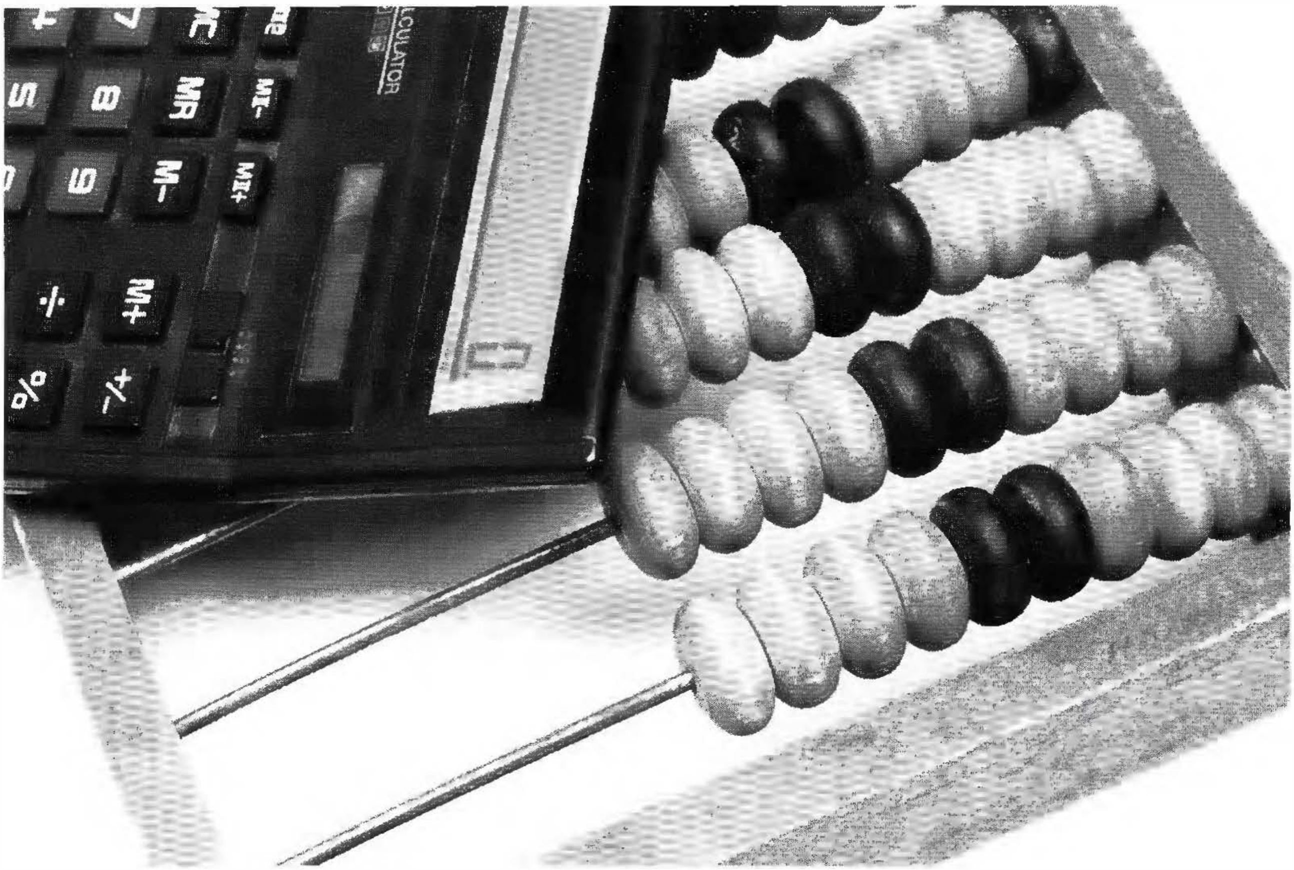
I found this to be an interesting article as I have been one to sit on the fence regarding best teaching practices based on research and/or on education personnel experiences to use in a high school mathematics classroom. For me, whether from presentations, discussions or reading, I evaluate how one or maybe more ideas can or will improve my teaching practice for my students in a whole class or for one student. From the phrase "what works for one does not always work for another" is what I think all teachers, parents, tutors, administrators, curriculum developers, government education departments and so on need to keep in mind.

To be the most effective in teaching mathematics to others is being knowledgeable in the subject of mathematics first. Next, one must become very familiar with the student's abilities in mathematical concepts as well in how a student learns. As one is exposed to research and other mathematical resources, one will include and/or adapt the ideas in one's teaching practice. To improve one's teaching practice, the best resource is the student's written work, whether presented in written form or orally.

Donna Chanasyk is the secretary of MCATA and department head of Special Programs at Paul Kane High School in St Albert.

Why Use History in a Mathematics Classroom?

Glen Van Brummelen



We've all experienced the problem. Given the mass of material we are required to cover in our math classes, it seems all but impossible to find avenues for creativity in our lectures. When small time windows open up, we tend to show extra problems, or new applications, or some favourite theoretical wrinkle that we had been saving for such an occasion. Why bring in history? It takes time and effort, and displaces other subjects. What's the advantage?

Simply put, history provides a path for the entire mathematical experience. Typically, our students are asked to solve problems and prove theorems, a limited part of what mathematicians do. The full story involves motivation: what is the context within which the subject arose, and why is it so appealing that it deserves our attention? Next is research: once the problem is identified, how do we articulate lines of attack that have already been made that might be adapted to the new situation? Third is critical thinking: how do we transition from received knowledge to new situations? Finally, we have implications: how does the solution affect us, the academic community or society? Good history of mathematics synthesizes all these aspects. Bringing it into the classroom can provide for our students a much broader and deeper mathematical experience. Most crucially, history is a natural means to attain these goals: we follow real people, who struggled as our students do, and eventually (usually) triumphed. We learn best through stories, and true stories are often the best ones.

Some Examples

Motivation

All mathematical subjects arose due to some need, either from within mathematics or from outside of it. Trigonometry was invented in ancient Greece to convert geometric models of the motions of the planets into quantitative predictions. Today, it is still a significant tool for moving back and forth between geometry and numerical measurement. Now, the need for some subjects may not have been the same in the past as it is today. Logarithms, for example, were invented in the early 17th century as a calculation device for astronomers to reduce the work involved in finding products and roots of numbers. Today our computing power renders this use obsolete. Nevertheless, this historical route can provide a meaningful context for students' first exposure to the subject; the benefits of the theory are obvious, even if its original motivation is no longer active.

Research

Coming to terms with methods that have been devised to attack difficult problems is, by definition, a study in history. Examples begin as early as ancient Babylon, where the geometric practice of "completing the square" seems to have led to solutions of problems related to quadratic equations. In differential equations, the discovery of a function that is its own derivative ($f(x) = e^x$) was exploited repeatedly to solve many problems from the early 18th century onward, and it even helped lead to a unification of exponential and trigonometric functions through complex analysis.

Critical Thinking

Every mathematical community makes shared decisions about the validity and power of various competing approaches. For instance, medieval Indian mathematics valued solutions that we might describe as approximate or iterative, while ancient Greek and medieval Islamic mathematicians preferred direct arguments and calculations. One of the most difficult concepts for modern students to understand is that such commitments are also present today. In order to think creatively, one needs to make informed judgments about alternate avenues of attack; one must know what the community's rules are before one decides to bend or break them. An example of such a struggle is the work of the 12th-century Iranian astronomer al-Samaw'al, who rejected an instance of ancient Greek use of approximation to calculate trigonometric tables. Nevertheless, he still needed to produce the tables. His creative solution was to redefine the base circle to contain not 360, but 480 degrees, which bypassed the need for approximation!

Implications

It is often said that the most powerful mathematical results are those that lead to new and interesting questions or that open mathematics to new applications. Witnessing the enlargement of the social role of certain types of mathematics can be a meaningful lesson in measuring its cultural significance. For instance, in early modern Europe the unification of trigonometry with logarithms brought mathematics into the hands of surveyors, architects and navigators. This transformed mathematics from a primarily theoretical discipline into an engine that eventually helped to reshape modern culture through science and technology. Students aware of these cosmic shifts are better able to place themselves in the intellectual landscape and to act in their profession with more reflectiveness.

There is one additional aspect of mathematical work that history can support: communication. Since history encompasses entire narratives from initial conception to final product and societal impact, there is a unique opportunity here to improve students' ability to write and otherwise present ideas. Students can write essays; they can make presentations on the background and significance of subdisciplines; they can write short-answer responses to questions about the significance of and interconnections between theories. It is usually difficult to find opportunities to improve mathematics students' rhetorical skills; history provides a powerful solution.

Challenges

Although the potential benefits of history are diverse, several dangers must be avoided.

- Misunderstanding history as mere biography: Textbooks often give snapshots of mathematicians' lives and works in the page margins, mistakenly believing they have done a service to history. They have not. Many of these biographies are unrelated in any direct way to the narrative in the text, so they unintentionally reinforce the tacit misconception that the mathematics itself is ahistorical. Genuine history in the classroom should be part of the presentation of the mathematics; its benefits can only be realized with deeper integration.
- Entering history without sufficient depth: The history of mathematics is a deeply challenging

endeavour, requiring sophistication in two disciplines with very different aims and modes of thought. Unfortunately, not everything one finds in the library or online is reliable, either historically or mathematically. The mathematics teacher should consult reliable sources; looking up reviews in professional journals is an effective way to screen out low-quality content.

- Assuming that history is a universal panacea: Although history is helpful in learning many mathematical concepts, assuming that it always leads to positive results is dangerous. Choose moments where the historical context genuinely interacts with the subject and is appropriate to students' concerns and maturity levels.

Places to Start

For topics in the undergraduate curriculum, there is no better place to begin than Victor Katz's history of mathematics textbooks. For accuracy, mathematical rigour and thorough coverage, they are unsurpassed; and they provide many connections to the rest of the literature. At an elementary level, consider William Berlinghoff and Fernando Gouvea's *Math Through the Ages* (2nd edition). Finally, the MAA Notes series has published a number of volumes of historical episodes ready for classroom use, edited by Victor Katz, Amy Shell-Gellasch, Dick Jardine and others.

Many modern theories of education attempt to address the plague of passivity in our students by promoting active educational experiences, such as the Moore method and inquiry-based learning. History provides the kind of engagement these innovations attempt to foster. However, history can also enhance the traditional mathematics classroom. By considering the entire cycle of mathematical development, and by asking students not merely to perform calculations but also to reflect upon them, history makes students more powerful, more thoughtful and more significant. In short, it makes them better mathematicians.

Glen Van Brummelen is coordinator of mathematics at Quest University in Squamish, BC. He is (twice) past president of the Canadian Society for History and Philosophy of Mathematics, and is currently the MAA governor-at-large for Canadian members. He is author of The Mathematics of the Heavens and the Earth: The Early History of Trigonometry *(Princeton 2009) and* Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry *(Princeton 2013), the latter based on his course at Quest.*

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Comments from Your Executive

Like most things that are more than 2,000 years old, mathematics has a rich history that is replete with interesting characters and unlikely events. And yet, after 12 years of school mathematics very few people know any of that history. Indeed, even after four more years of mathematics at university, it is possible to know very little of its history. In the article, Professor Glen Van Brummelen gives some good reasons why the history of mathematics should be incorporated into our classes. There is also good advice on how to do it. Equally important are his words on how not to do it.

Indy Lagu is the postsecondary mathematics representative of MCATA and a full-time professor at Mount Royal University in Calgary.

Improving Student Achievement in Mathematics Through Formative Assessment in Instruction

A joint position paper of the Association of Mathematics Teacher Educators (AMTE) and the National Council of Supervisors of Mathematics (NCSM).

It's really not surprising that formative assessment works so well. What is surprising is how few teachers use the process.
—James Popham, "Formative Assessments 'Advocatable' Moment"

Our Position

The National Council of Supervisors of Mathematics (NCSM) and the Association of Mathematics Teacher Educators (AMTE) affirm the centrality of research-based, mathematically focused, formative assessment—a key element in the national effort to improve mathematics proficiency. Formative assessment needs to be intentionally and systematically integrated into classroom instruction at every grade level. This requires adequate attention in the preparation of new teachers of mathematics and in the continuing education and professional development of current teachers.

What Is Formative Assessment?

Formative assessment is a process of gathering evidence within the stream of instruction in order to inform teaching and learning (Black et al 2004).

To be considered formative, the evidence must be "elicited, interpreted, and used by both teachers and learners" (Wiliam 2011, 43). In contrast, summative assessment is used to evaluate progress and achievement, assign grades and appraise programs. "Formative assessment involves getting the best possible evidence about what students have learned and then using this information to decide what to do next" (p 50). "In a classroom that uses assessment to support learning, the divide between instruction and assessment blurs. Everything students do—such as conversing in groups, completing seatwork, answering and asking questions, working on projects, handing in homework assignments, even sitting silently and looking confused—is a potential source of information about how much they understand" (Leahy et al 2005). "When classroom practice is based on formative assessment, teachers and students together develop a framework for what can be expected in students' learning, for what it means to move toward intended mathematics learning goals, and for a common goal of continuous and progressive learning. Formative assessment is a crucial tool for simultaneously improving classroom practice and students' performance" (Petit and Zawojewski 2011).

Evidence from Research and Practice That Supports Our Position

There is a growing body of research emphasizing the use of formative assessment in classroom instruction as a means to improve student achievement. In their synthesis of studies, Black and Wiliam (1998) note evidence of greater student achievement in classrooms where teachers use such techniques. Similar findings are replicated in a meta-analysis by Ehrenberg et al (2001). In particular, they report the impact of formative assessment on student achievement being four to five times greater than the effect of reducing class size.

Additionally, in an analysis and synthesis of studies, Leahy et al (2005) identify strategies supporting the use of formative assessment:

- Clarifying and sharing learning intentions and criteria for success.
- Engineering effective classroom discussions, questions and learning tasks.
- Providing feedback that moves learners forward.
- Activating students as the owners of their learning.
- Activating students as resources for one another.

Clarifying and Sharing Learning Intentions and Criteria for Success

One technique to clarify and share learning intentions and criteria shown to positively impact student achievement is when students analyze their work as they proceed through a task using explicitly stated criteria for performance (White and Fredrickson 1998). Another strategy is to give students anonymous samples of student work, such as from another class or different year, on a task that requires students to do such work. Students review and analyze the samples and communicate what is good in the better samples and what is lacking in the weaker ones.

Implicit to this strategy is explicitly stating and engaging students in the mathematics goal of a lesson, task or activity. Understanding and being able to articulate the mathematics goal provide students with a clear idea of where they are going and enables them to reflect on progress toward the goal rather than aimlessly working through a lesson.

Engineering Effective Classroom Discussions, Questions and Learning Tasks

This strategy involves three interrelated activities: (1) engaging students in tasks and activities that provide insights into their thinking; (2) teachers and students listening and analyzing student discussions and artifacts interpretatively, not just from an evaluative perspective; and (3) implementing instructional strategies designed to engage all students in tasks, activities and discussions (Wiliam 2011).

Wiliam (2011) suggests only two reasons to ask questions in a classroom: "one, to cause thinking and two, to provide information for the teacher about what to do next" (p 70). To do this, the task should be selected based on its ability to reveal student thinking and understanding around important mathematics concepts and practices. In addition, teachers should consider the potential of a task to reveal student progress along a developmental progression and its potential to elicit misconceptions and common errors. Engineering effective classroom discussions, questions and learning is also dependent on both teacher's and student's ability to listen interpretatively; that is, not just listening for the right answers but listening for evidence about student thinking to inform the next instructional steps.

"High engagement classroom environments appear to have a significant impact on student achievement" (Wiliam 2011, 81). When students are highly engaged, they are absorbed in activities, tasks and discussions using techniques, such as think-pair-share, wait time, cold calling, sharing student-generated solutions and all student response systems such as mini whiteboards and exit cards. These and other instructional strategies provide teachers many opportunities to check for understanding during or right after a lesson, rather than waiting for homework, quizzes and tests for evidence of what sense students are making of the mathematics.

Providing Feedback That Moves Learners Forward

It has been known for some time that just checking answers as right or wrong and giving scores, negatively impacts student learning as compared to the practice of asking students to revisit their work (Bangert-Drowns et al 1991).

When done correctly, feedback can result in students reflecting and rethinking their mathematics, while increasing their effort and motivation.

Providing feedback linked to learning criteria and mathematical goals provides information that is actionable by the student and has been shown to have positive effects on student learning. Conversely, feedback that results in less effort or lowering goals has shown decreases in performance (Kluger and DeNisi 1996). Wiliam (2011) summarizes this idea stating: "feedback functions formatively only if information fed back to the learner is used by the learner to improve performance" (p 120). Effective feedback strategies will cause students to think, rather than react emotionally.

All feedback, whether given as students are working on a task, activity, during classroom discussions or after an assignment is completed, should be focused, causing the student to take action. Comments such as think or try again or good work do not result in increased motivation, therefore, do not often result in increased student achievement.

Activating Students as the Owners of Their Learning

Students must have opportunities to be involved and be responsible for all aspects of their learning. Using such techniques as self-assessment with a provided rubric or student and teacher codeveloped rubric is one way to make learning a shared experience.

Activating Students as Resources for One Another

Many teachers have found that asking students to review, analyze and provide feedback (not grade) another student's work is sometimes easier than analyzing one's own work. Working on this strategy provides a stepping stone to analyzing one's own work more efficiently and effectively.

There are important areas of consideration for implementing the five aforementioned strategies. First, thoughtful and intentional planning for formative assessment begins with a goal for each lesson and determining criteria for success that is clear as to what should be accomplished. Then rich tasks and activities are selected that will engage all students in discussions, while providing opportunities for constructive feedback, and establishing ways for students to monitor their own progress toward the learning goal. Additionally, planning requires revisiting and reworking lesson plans in an effort to implement formative assessment on a more regular basis. Moreover, thoughtful, intentional planning involves developing a lesson that will elicit student thinking in relationship to the mathematical goal.

Also important to implementing the five formative assessment strategies is teacher knowledge of mathematics' learning trajectories also referred to as learning progressions. Studies by Clements et al (2011), Carpenter et al (1989), Clarke (2004) and Clarke et al (2001) have found that professional development focused on the instructional use of learning progressions results in improved student achievement. The findings also suggest that knowledge of learning progressions in the use of formative assessment has the potential to strengthen the interpretation of evidence of student work to inform instruction and learning. (For a brief explanation of learning progressions/trajectories by researcher Douglas Clements, go to www.youtube.com/watch?v=GNBi4xhXevo.)

All strategies and techniques are bound together by the fact that they impact instruction and learning. Although there is not a prescription for where to start and what strategy to use, it is important that formative assessment is part of one's practice. Implementing formative assessment in classrooms works best if teachers start with where they are and move to where they want to be (Leahy et al 2005).

How NCSM and AMTE Members Can Implement Our Position

As leaders, NCSM and AMTE members must work to ensure that preservice and inservice teachers, administrators and other stakeholders in districts and states have knowledge of the research-based practices involved in formative assessment. In order for formative assessment to be intentionally and systematically integrated into classroom instruction, major effort is needed.

Members of NCSM and AMTE are strongly encouraged to provide professional development in the skilful use of formative assessment so that preservice and inservice teachers

- understand how to implement the previously discussed formative assessment strategies;
- use knowledge of the mathematics education research including learning progressions to inform instructional decision making;
- use activities and tasks that elicit student understanding;
- expand and improve questioning and classroom discourse;
- provide opportunities for analysis of student work and instructional decision making;
- implement strategies to engage all students in rich activities, tasks and discussions;
- provide productive oral and written feedback that moves learning forward; and
- incorporate peer and self-assessment opportunities in the classroom.

Additionally, NCSM leaders are strongly encouraged to

- use NCSM's *PRIME Principles and Indicators for Mathematics Education* to guide the work of formative and summative assessment;
- provide ongoing support for teachers as they plan formative assessment within professional learning communities (PLCs);
- assure that facilitators of professional development model the use of formative assessment instructional strategies;
- provide teachers with tools and resources, such as learning progressions and item and lesson banks;
- provide professional development for school administrators in order to

a) create opportunity and time for teachers to meet and collaborate;
b) provide opportunities for teachers to report progress, for example, at staff meetings;
c) incorporate formative assessment into the school improvement plans; and

- ensure that local policies support the implementation of formative assessment and that those policies such as pacing guides and interim assessments do not detract from the effective use of formative assessment.

Additionally, members of AMTE are strongly encouraged to

- assure that preservice teachers have experienced the use of formative assessment by their instructors in preservice classes;
- focus on research that deepens understanding of effective formative assessment practices in mathematics classrooms;
- require preservice teachers to intentionally and systematically incorporate formative assessment in the writing of lesson plans;
- provide preservice and inservice teachers opportunities during clinical experiences to reflect on formative assessment's impact on student learning;
- work with K–12 partners to provide professional development on formative assessment to inservice educators;
- provide support and guidance to school administrators on how best to support teachers as they implement formative assessment in their classrooms;
- provide support and guidance to school administrators on local policies that support the effective use of formative assessment in schools;
- provide tools and resources such as learning progressions and item and lesson banks that support the implementation of research-based formative assessment to preservice and inservice teachers; and
- publish for the purpose of supporting a greater understanding of the effective use of formative assessment.

One of a series of position papers of the National Council of Supervisors of Mathematics mathedleadership.org.

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Comments from Your Executive

This formative assessment position paper correctly identifies the challenge of bridging research and practice. While most teachers understand that formative assessment is an essential element of classroom practice, many struggle to implement more formative assessment into their math classes. Formative assessment can be embedded directly into lessons so that it is virtually indistinguishable from instruction. When schools ask me to work with them on assessment, the most common request I get is to help them come up with ways to embed formative assessment more seamlessly into their lessons. In Alberta, we are fortunate to have the Alberta Assessment Consortium, whose purpose is to promote good classroom assessment practice. Their website (www.aac.ab.ca) is a great place to start the hunt for formative assessment materials for math classrooms. A study they recently completed looked at formative assessment specifically in high school math classrooms. All the resources developed in that project are on the AAC website.

John Scammell is the president of MCATA and assistant principal at S Bruce Smith Junior High School in Edmonton.

Quick! Draw!

Adele Hanlon

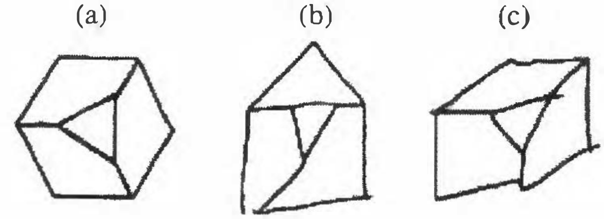
Spatial reasoning plays a significant role in mathematics learning; we use mental images to see patterns and relationships between concepts. Quick Draw (Wheatley 2007) sharpens spatial and geometric reasoning as well as develops flexible thinking. The process is simple: Show a geometric figure for three seconds, then remove it from sight; students draw the figure from memory.

Initiate student discourse by asking, What did you see? and How did you draw it? Students' drawings and interpretations of the figures will vary greatly. Descriptions from a fifth-grade class for Figure 1a ranged from "a triangle within a hexagon" to "a square prism with a corner removed." Sharing diverse descriptions and drawing strategies will allow students to see other perspectives. Repeated exposure to the activity will help students develop and talk about spatial reasoning, as one student's initial representation (see Figure 1b) of the Quick Draw figure shows weeks later (see Figure 1c).

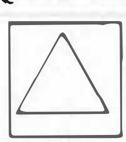
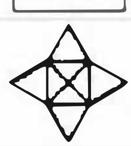
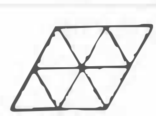
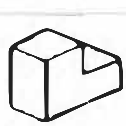
Adapt to the learning needs of students in K–6 by choosing figures of appropriate complexity (for example, individual shapes, overlapping shapes, two-dimensional shapes, three-dimensional shapes) as well as by increasing or decreasing the amount of time for the activity and the number of times the figure is shown (see the table of tasks). Make the task more challenging by asking students to transform the image mentally (for example, to mentally turn the original image 90 degrees counter-clockwise) before drawing it. Include concepts of number area, perimeter, volume and coordinate geometry by drawing figures on graph paper.

Figure 1

(a) (b) (c)



Quick! Draw! Tasks

Grade level	Quick Draw	Geometric Reasoning Focus	Extensions
K		Identify and describe shapes.	Turn the figure upside down. Is the shape inside still a triangle?
1–3		Analyze and compare shapes and their attributes.	How many triangles do you see? What types of triangles do you see?
4–5		Classify shapes by properties. Infer relationships among shapes.	Can rhombuses be classified as squares? Can squares be classified as rhombuses?
6		Represent 3-D figures using nets.	Create a net of this figure and use the net to calculate the surface area.

Reference

Wheatley, G H. 2007. *Quick Draw: Developing Spatial Sense in Mathematics*. Bethany Beach, Del: Mathematics Learning.

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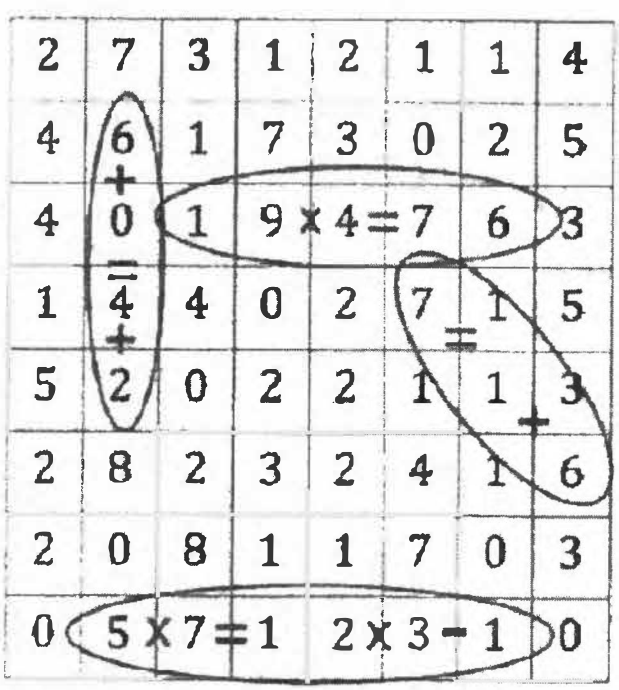
Number Sleuth

Cathery Yeh

Children love scavenger hunts and have rich experiences finding hidden objects in pictures as well as words in word searches. The Number Sleuth activity builds on these experiences and encourages problem solving and computation as students hunt for hidden equations in a number puzzle. The object of the game is simple: uncover all the hidden equations, which may be placed horizontally, vertically or diagonally. For example, a second-grade student circled the digits 6, 0, 4 and 2 and wrote the equation $6 + 0 = 4 + 2$. Equations can be written directly on the puzzle or on a separate sheet. Adapt the activity for different grades by modifying the array size and placing digits in the array on the basis of the operations of focus. For emerging mathematicians, keep the array small and have students circle number pairs or trios that equal five or ten. Construct arrays using digits 0–5 to build fluency to and from five. Encourage middle- and upper-grade learners to include equations with two or more operations. For example, a fourth-grade student created the equation $5 \times 7 = 12 \times 3 - 1$ using the digits 5, 7, 1, 2, 3 and 1. Challenge learners to use positive and negative integers, integer exponents and fractions.



Game Variations

- Choose a target number and have students find expressions equalling that value.
- Students can create their own number sleuth activity to share with classmates.

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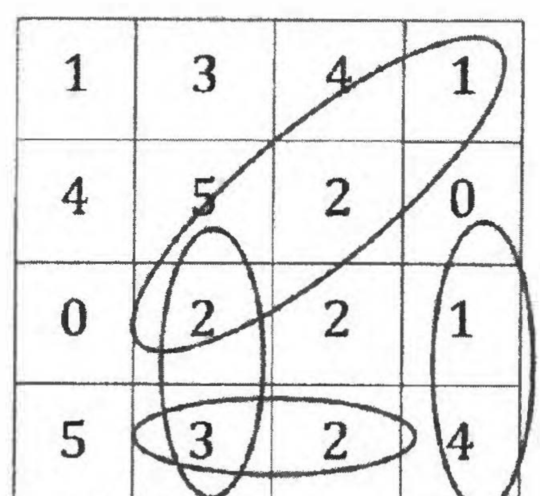
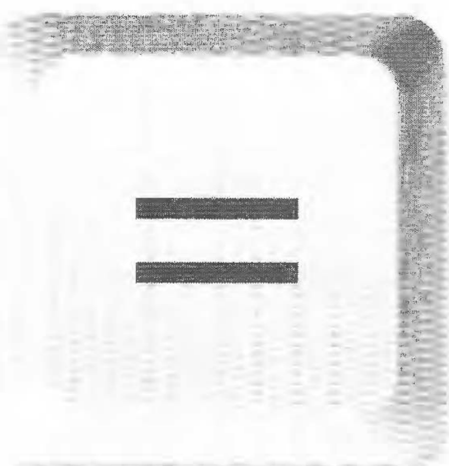


Addressing the Misconceptions of the Equal Sign

Brad Weible



Throughout my eight years of teaching mathematics in the Grades 4 and 5 classroom, I have been consistently surprised at areas of the curriculum where my students have been challenged. Every year I would give lessons on solving equations using larger numbers in the traditional vertical or horizontal format that can be found in most classrooms. Typically after reviewing the steps for solving these equations, my students would become quite competent solving a variety of equations with 3- or 4-digit numbers. I always felt I was a very successful mathematics teacher because of the success my students achieved until we started the Patterns and Relations unit and more specifically the lessons on variables and equations. Year after year, I was caught off guard by students who could easily solve the equation 42×23, but these same students could not immediately identify the missing number in $56 = \square \times 8$. Naturally, I was concerned about the confusion my students encountered regarding basic fact equations. My curiosity prompted me to investigate the available research on what misconceptions exist when students are introduced to pre-algebraic thinking. I was surprised to find a deep body of literature that examines how students understand the equal sign.

Background

The understanding of equivalency is a key concept that is essential for all levels of mathematics from early elementary grades to university programs. Ma (1999) refers to one of her teachers who considered the equal sign "the soul of mathematical operations" (p 111). The Alberta Education program of studies also identifies the importance of understanding equality. In Grade 1, students are expected to "describe equality as a balance and inequality as an imbalance, concretely and pictorially (0 to 20)" and "record equalities, using the equal symbol" (Alberta Education 2007, 31). In spite of the early expectations for students to develop an understanding of equality, students in elementary grades continue to show a misunderstanding of how the equal sign is used. In the subject of mathematics, the equal sign as a symbol can be used in many different situations such as "a computational result, as in $2 + 2 = 4$; an identity, as in $(x + y)(x - y) = x^2 - y^2$; assignment of a rule to a function, as in ; a substitution, as in $x = ½$; and so on" (Jones and Pratt 2012, 2). All of these examples require the equal sign to show a relationship, yet students spend most of their first years studying mathematics by focusing on the computational strategy. It is generally understood that this emphasis on using the equal sign to complete equations has promoted an operational understanding of the equal sign. In the operational understanding, the equal sign is seen as a signal to find the answer to the operation that precedes it. The equal sign is used as found on a calculator where an operation is entered and the equal sign is pressed to get an answer.

Researchers emphasize that students need to develop a relational understanding of the equal sign. A relational understanding of the equal sign takes on a broader meaning that shows the relationship between the operations or numbers on the left and right side of the symbol. When the symbol is misconceived as an operator, the equal sign typically demonstrates to "get the answer" as opposed to a relational meaning of "is the same as." Small (2013) suggests that students should understand that "the equality symbol sign should be viewed as a way to say that the same number has two different names, one on either side of the equals sign" (p 625). Faulkner (2009) is even more assertive in her stance that using the phrase "the same as" is not strong enough when describing equality. She believes that students require language that is more specific because "two trucks may be equal in weight to an elephant, but they certainly aren't the same as" (p 26). When looking at $2 + 5 = 4 + 3$, students should be able to see that they are not the same, but they are "equal in value" (p 26). Mann (2004) defines the equal sign as "a symbol that indicates that a state of equality exists and that the two values on either side of the equal sign are the same. It does not mean that the answer is coming or that the answer is on the other side of the sign" (p 65). From the variety of research available on this topic, it seems clear that students and often teachers are not aware of the meaning of this symbol which is "vital to successful algebraic thinking and is one of the big ideas of algebra about which students should reason" (p 26). Mann and other researchers suggest that the proper understanding of equality and the equal sign should be taught in elementary grades in order to prevent students from developing misconceptions.

Elementary Implications

In a situation presented by Falkner, Levi and Carpenter (1999), Grade 6 teachers were asked to present their students with the problem: $8 + 4 = \square + 5$ (p 232). The results were surprising to many of the teachers because out of 145 Grade 6 students, 84 per cent of the students thought the missing number was 12, 14 per cent of the students thought the number was 17 and none of the students chose the correct value of 7. Saenz-Ludlow and Walgamuth (1998) identified similar results with Grade 3 students. The students were asked to solve the equation $246 + 14 = \_\_\_\_ + 246$. Although the teacher in this situation used this equation as a warm-up and expected it to be fairly simple, it led to many discussions about what the answer should be. The students did not see a relationship between the numbers on the left and right side of the equal sign. They instead focused on adding the numbers on the left to get the answer (p 167). McNeil and Alibali (2005) examined why students continue to demonstrate an operational understanding. They infer that students are exposed to the same operational patterns, and they are not able to create new strategies when they encounter different, nonstandard equations.

Another common issue that can occur with equations is using the equal sign to keep a running total (Kieran 1981). In the false equation, $16 + 4 = 20 - 5 = 15$, a person will keep a running total when solving different steps in a word problem. This procedure does not demonstrate equivalence though and can cause confusion when a student must show a relationship between two sides of the equal sign. Saenz-Ludlow and Walgamuth (1998) observed this with the Grade 3 students who were working on the equation $246 + 14 = \_\_\_\_ + 246$. Some students saw the solution as $246 + 14 = 260 + 246 = 506$. These children were not using the equal sign to show "quantitative sameness on both sides of it" and instead used the symbol as a "separator of their sequence of operations" (p 177). To demonstrate equivalency properly, a teacher should make sure the student writes each equation separately such as $16 + 4 = 20$ and $20 - 5 = 15$. Ma (1999) discusses how teachers she observed in the United States differed in their opinion of the running total as opposed to teachers in China. She gives the example of a teacher in the United States who often accepts a running total because "they (students) did the calculational order correctly and got the correct answer" (p 111). On the other hand, the Chinese perspective is very rigorous in regards to how mathematical symbols are used and the Chinese teacher would not accept two different values on the left and right side of the symbol (p 177). People will often use the false equation as a shortcut during calculations, but it is important for teachers to refrain from modelling this technique while teaching the meaning of equivalency.

Researchers (Hiebert 1989; McNeil and Alibali 2005) suggest that students may have too much exposure completing operations with symbols before developing a proper understanding of the symbols. In the Alberta Education program of studies (2007), understanding equality is found in the Patterns and Relations strand, but Grade 1 students are also expected to develop an understanding of addition and subtraction of numbers concretely, pictorially and symbolically. Kieran (1981) states that "many children learn fairly quickly to read and write the elementary written symbolism of simple arithmetic, but do not necessarily understand it the same way we do" (p 318). It is this early lack of attention to the foundational skills that leads to the misconceptions and issues with more complex equations later in higher grades. Falkner, Levi and Carpenter (1999) discuss an experiment where a kindergarten teacher presents her class with the problem 4 + 5 = □ + 6. Her students unanimously thought that the number 9 should go in the box. When the teacher modelled the same equation with unifix cubes, the students were able to recognize that a grouping with a stack of four blocks and a stack of five blocks was not the same as a grouping with a stack of nine blocks and a stack of six blocks (pp 232–34). This experiment demonstrates how primary age children have an understanding of equality when using concrete objects, but they are not able to relate this to the symbolic representation. As children continue to work with equations, it appears as though they also continue to develop habits that prevent them from having a relational understanding of the equal sign. Kieran (1981) discusses how elementary school children will argue that an equation written as □ = 4 + 3 is actually written the wrong way (p 318). Children become so accustomed to reading equations in this order that they are conditioned to do the calculation from left to right.

The students in all grades who had a relational understanding had more success at solving the algebraic equations often because these students were more likely to use an algebraic strategy.

Hattikudur and Alibali (2010) propose that through comparison of different relational symbols, students will "recognize more abstract commonalities" (p 17) and therefore develop a better understanding of the equal sign. In their study with Grades 3 and 4 students, groups that received lessons comparing symbols such as <, > and = demonstrated more of a relational understanding of the equal sign than groups that received lessons on just the equal sign as an operational symbol. This suggests that it is useful for students to group the equal sign with relational symbols such as < or > rather than with operational symbols. Also, the researchers state, "Students in the comparing symbols group were also more likely than students in the equal sign group to correctly recognize nonstandard equations as making sense" (p 28).

Symbols are essential for mathematics, and they "offer a convenient and powerful way to represent mathematical situations and to manipulate mathematical ideas" (Hiebert 1989, 39). He goes on to suggest that students need to create "sound meanings" of symbols before they can be used and manipulated in problem situations (p 39). Similar to the experiment of the kindergarten teacher in the article by Falkner, Levi and Carpenter (1999) with the unifix cubes, Hiebert (1989) states, "We need to design instruction so that we can help students connect the new knowledge they are acquiring about written symbols with the understandings they possess from experiences outside and inside of school" (p 40). McNeil and Alibali (2005) concluded that the elementary age children in their study had become so dependent on the operational patterns they had learned that even after receiving a lesson on the meaning of the equal sign and equations, the children continued to rely on their previous operational patterns and did not solve nonstandard equations correctly. Children need opportunities to demonstrate an understanding of how to balance concrete objects on a scale, and they should be taught how to apply this understanding to the equal sign before learning to do operations.

Middle School Implications

Middle school is typically a time when a student's misconception of the equal sign can cause confusion in mathematics as more complicated algebraic thinking is introduced. Alibali et al (2007, 222) assert, "Developing an understanding of the equal sign has typically been considered mathematically straightforward—after its initial introduction during students' early elementary school, little, if any, instructional time is explicitly spent on the equal sign in later grades." Alibali et al continues by stating that research actually seems to show that students at all grade levels do not demonstrate a proficient understanding of the equal sign. Students in middle school who have had many years of experience with solving equations and using the equal sign continue to use this symbol with an operational understanding. Knuth et al (2006) studied middle school students' success with algebraic equations and if this had a connection to their understanding of the equal sign. The students in all grades who had a relational understanding had more success at solving the algebraic equations often because these students were more likely to use an algebraic strategy. The researchers conclude, "We argue that there is a clear need for continued attention to be given to the notion of equality in the middle school grades" (p 310).

Alibali et al (2007) provide a concise explanation of the importance of developing a relational understanding. "A relational view of the equal sign allows students to interpret equations appropriately, and appropriate interpretations can guide judgments about the equivalence of equations" (Alibali et al 2007, 235). Through a longitudinal study they discovered that even middle school students continue to show an operational understanding. The students' development of a relational understanding is only gradual from Grades 6 to 8 with some students still retaining the operational understanding. Alibali et al (2007, 241) identified that "students' performance varied as a function of when they had acquired a relational understanding of the equal sign."

In a study of middle school teachers, Asquith et al (2007) examined what teachers think their students know about the equal sign. The six teachers from Grade 7 predicted that 73 per cent of the students would have a relational understanding when the actual number was only 37 per cent. It was also observed in this study that many teachers did not recognize that when students have an operational view of the equal sign that it would prevent them from being successful in math. There appears to be a disconnection between how students understand the equal sign and what teachers assume that they know. Alibali et al (2007) emphasize that middle school mathematics must give attention to equivalence and the equal sign, and this attention "should include varied and regular opportunities for students to develop a relational understanding" (p 245). According to McNeil et al (2006) students were more likely to show a relational understanding when they were presented with nonstandard equations, specifically equations with operations on both sides of the equal sign.

The Equal Sign in Textbooks

Before presenting suggestions for how teachers can promote the relational understanding in the classroom, there is evidence that the typical classroom resources do not provide sufficient material for assisting teachers. The operations-equals-answer equations is traditionally the most commonly used equation and "is thought to promote an operational interpretation of the equal sign," and the operations on both sides equation is thought to be the most effective at showing a relational interpretation (McNeil et al 2006, 371). Matthews et al (2012) examined different types of equations and concluded that "not all non-standard equation formats are equally challenging" and "the more an equation varies from the standard $a + b = c$ format, the more difficult it is likely to be" (p 338). They also discovered that "equations with operations on both sides were more difficult for children to solve than those that involved operators on a single side only" (p 339). In their examination of textbooks, McNeil et al (2006) discovered that operations on both sides equations made up only 5 per cent of equations used. Powell (2012) analyzed kindergarten through Grade 5 textbooks and found that "the majority of equations across kindergarten to fifth grade fall into the standard category (operation on the right side)" (p 642). When nonstandard equations were used, they were mainly used as operation-right-side equations.

Powell (2012) took this study further by examining the teacher's manuals that are provided with each textbook series and analyzing how these manuals suggest that the equal sign should be explained to students. Throughout the teacher's manuals the definitions and explanations given for the equal sign are consistent and show a relational meaning. "No curriculum, however, provides the same definition at all grade levels, and some curricula provide different definitions across grade levels or within the same grade level" (p 642). This means that teachers in a school may not be using consistent vocabulary when instructing students, which can cause confusion as the students move to new grades.

Li et al (2008) compared US and Chinese textbooks. In general, the Chinese textbooks first provided instruction for numbers and values up to 10, then moved on to introduce the equal sign as a way to compare numbers. "After introducing the equal sign, addition and subtraction were introduced where students were provided with both standard and non-standard forms to understand both operations and the equal sign" (p 206). Sixth grade Chinese students were almost three times more likely to provide correct answers to equations such as $6 + 9 = \_\_\_\_ + 4$ and $\_\_\_\_ + 3 = 5 + 7 = \_\_\_\_$. The information gathered from the analysis of textbooks shows that teachers will need to facilitate opportunities in the classroom to promote the understanding of the equal sign.

What Teachers Can Do

There are advantages for providing students with specific equal sign instruction. According to Mann (2004) the opportunity to introduce the concept of equivalence should not wait until middle school. She suggests that "teachers should help students in elementary school come to recognize the equals sign as a symbol that represents equivalence and balance" (p 65). Falkner, Levi and Carpenter (1999) discuss how to incorporate lessons about the equal sign in first and second grade classes. In the first grade classroom, the students will decide if various number sentences are true or false such as $8 = 8$ or $8 + 2 = 10 + 4$. These statements will lead to class discussions about what a number sentence can look like and what the equal sign means. Powell and Fuchs (2010) identified how providing word-problem tutoring combined with equal-sign instruction and practice with open equations (for example, $8 = 4 + x$) increased a Grade 3 student's ability to use a relational understanding and to solve nonstandard equations. Students will often require specific instruction about the equal sign regardless of the grade they are in, and there are a number of ways this can be achieved.

The students in all grades who had a relational understanding had more success at solving the algebraic equations often because these students were more likely to use an algebraic strategy.

Children's picture books are used as a way to introduce algebraic relationships in early elementary grades (Leavy, Hourigan and McMahon 2013; Lubinski and Otto 2002). *How Many Snails?*, by Paul Giganti (1988), presents various groups of pictures that can be categorized in different ways to show relationships between the quantities. Lubinski and Otto (2002) describe a page in the book that has yellow, pink and white flowers with black, yellow or orange centres. During the class discussion, the students are able to represent the different flowers with the equation 9 + 4 + 2. They also create the equation 9 + 3 + 3 to represent the different centre colours. The teacher can then discuss the relationship between these two equations. Leavy, Hourigan and McMahon (2013) use the picture book *Equal Shmequel*, by Virginia Kroll (2005), to also show relationships. This book uses wilderness animals that need to balance the two sides of a tug-of-war game in order to make the game equal. The pictorial representations can be very useful as reinforcing mathematics concepts for students.

A common tool for providing concrete examples of equality is the pan balance (Barlow and Harmon 2012; Mann 2004; Ellis and Yeh 2009). Students can easily manipulate unifix cubes or other objects on either side of a pan balance as demonstrated by Falkner, Levi and Carpenter (1999). Barlow and Harmon (2012) emphasize the importance of using pan balances as a way to provide students with the "opportunity to think about equivalence and balance without rushing toward representing the relationships with symbols" (p 98). Mann (2004) uses balance representation with see-saws to have the students create "Seesaw Rules" (p 66). These statements highlighted the essential understandings that are necessary for balancing objects on a seesaw. After creating the rules, the class then transfers to make connections between the see-saw and the equal sign.

Ellis and Yeh (2009) take the pan balance a step further with their mobile-balance puzzle. Individual numbers are further separated to keep the whole equation balanced. When shown as a balance, numbers are split to represent the different arms of the balance (that is, 12 = 12 would be split into arms showing 8 + 4 = 6 + 6 and this would be split into further arms showing 8 + 2 + 3 = 6 + 3 + 3).

After students become familiar with using the equal sign symbol, they need to practise writing different equations that represent the same values as suggested by MacGregor and Stacey (1999). They give examples of a number of problems that can lead to students to create these different equations such as 24 = 2 + 10 + 2 + 10, 24 = 4 × 6 = 6 + 6 + 6 + 6 = 12 + 12 and 3 + 9 + 3 + 9 = 4 + 8 + 4 + 8. "Seeing the reasons behind such relationships requires a generalization about properties of numbers that is deeply algebraic" (p 80).

Creating opportunities for students to understand the meaning of the equal sign will be the responsibility of the mathematics teachers. Blanton and Kaput (2005) performed a case study on a Grade 3 teacher who displayed "robustness" in her ability to adapt her teaching into opportunities to explore algebraic thinking. The researchers observed that "algebraic reasoning tasks were not mathematical 'field trips' but were woven in the daily fabric of instruction" (p 440). It is the responsibility of the teacher to develop their own algebraic thinking skills in order to have this ability to incorporate it into daily mathematics practice. Blanton and Kaput conclude, "Elementary teachers must develop algebra 'eyes and ears' as a new way of both looking at the mathematics they are teaching and listening to students' thinking about it" (p 440).

Conclusion

From my experience, teachers and students generally feel as though the equal sign is a straightforward concept that does not require much attention. Research has proven overwhelmingly that this is not the case, and many students are progressing through school with only the operational understanding. This concept requires attention from mathematics teachers across all grade levels in order to ensure the success of students in developing algebraic thinking. It will be essential for schools and teachers to understand how to incorporate these concepts into the classroom mathematics practice.

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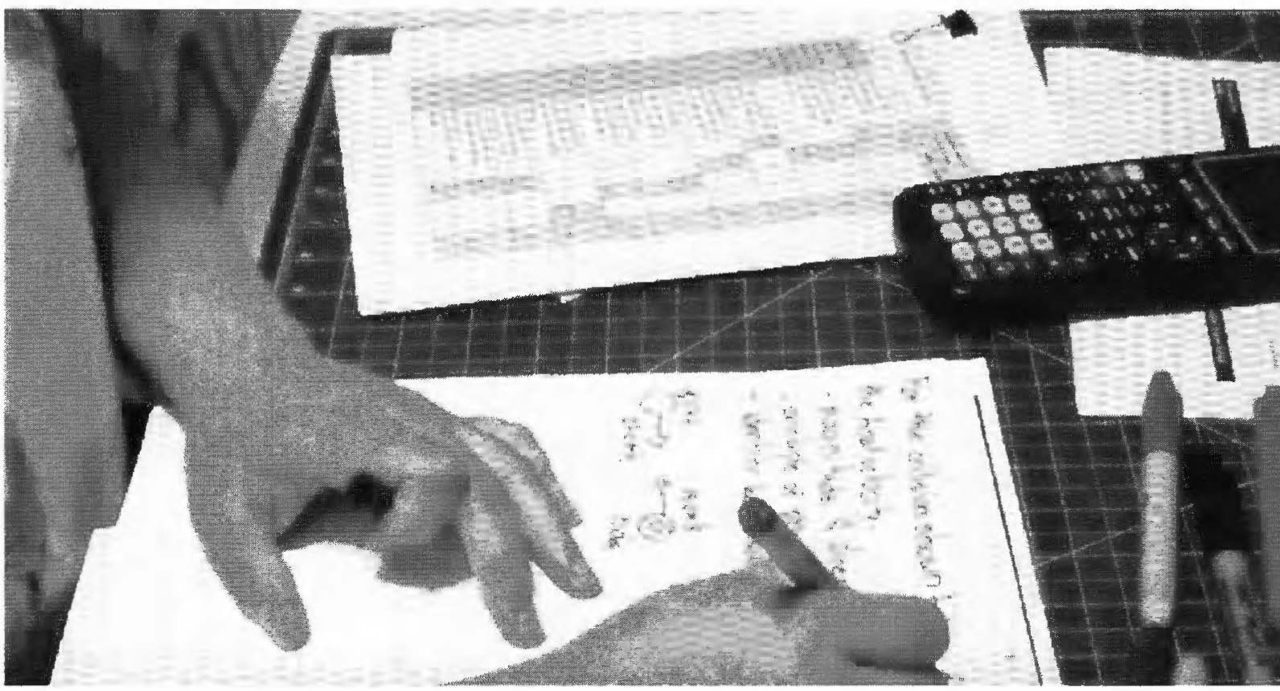
Brad Weible teaches Grades 7 and 8 mathematics at Foothills Academy in Calgary. He previously taught Grades 4 and 5 where he discovered his passion for helping his students feel excited about mathematics. After completing this final paper for his master's program, he now takes any opportunity to educate students and teachers about how the equal sign can be so misunderstood.

If You Are Not Blueprinting Your Assessments, Read This Article

Greg Wondga



The task of blueprinting is a matter of analyzing the learner outcomes in a program of studies (POS) and constructing an assessment tool, or series of assessment tools, to measure student proficiency based on those outcomes. Good blueprinting practice considers the verbs in the outcomes, the cognitive level implied by the outcomes, and strives to use appropriate weightings to assess all outcomes with equity in mind. My experience with blueprinting has caused a profound mental shift in assessment practice for me, as well as many of my colleagues who have shared this experience with me.

If one had to identify a golden rule when it comes to assessment, it would be that teachers must report student achievement based solely on the outcomes in the Alberta POS. For example, a teacher cannot evaluate a student's proficiency at memorizing the genetic code in Science 9 because no such outcome exists in the POS. In theory this makes sense in that all students in Alberta should experience reliable instruction and assessment for a given program. This golden rule, however, brings up some interesting questions. For example, if students misbehave; for example, cheat on a test, miss class or refuse to complete an assessment, should they receive a reduced mark? If so, what outcomes are reflected in this reduction? Should some outcomes have greater weightings than others? How is a final mark generated based on all of the evidence of student performance? Is it valid to review concepts only minutes prior to an assessment? Different teachers will have conflicting opinions on the answers to these questions. It is important for educators to find clarity about what a mark is actually meant to describe.

If we were to agree to the golden rule, we would need to have the ability to identify the outcomes that students perform well in as well as those that students experience difficulty. Regardless of a student's mark, the teacher should be able to answer the question, "Which outcomes from the Alberta POS are met and not met?" In order to make sure that our assessments fit with the POS, teachers must analyze the outcomes and design assessments in a purposeful manner. I have done this task and found it to be both incredibly taxing yet extremely enlightening.

The first step of blueprinting is to analyze each learner outcome and interpret its meaning. It is likely that many outcomes from any given POS will be interpreted differently by different teachers. Group consensus is a great way to dive into the outcomes and reach a common understanding of what they mean and how they should be assessed. One way to encourage conversation about learner outcomes is for teachers to identify cognitive levels (CL) in both outcomes and assessment tools. I focused on three cognitive levels; knowledge (K), comprehension/application (C/A) and high mental activities (HMA).

- K-level outcomes ask students to identify, list, describe, classify and so on. These outcomes could be met by looking the information up in a book or accessing the Internet. These outcomes test student memory of basic facts.
- C/A-level outcomes require students to comprehend concepts and therefore apply their knowledge to new situations. In these outcomes, students must be able to interpret information in order to solve a new problem.
- HMA-level outcomes assess students' ability to articulate their own original thoughts about something. They may be asked to justify, create, analyze or evaluate.

Discourse about CL provides an opportunity for teachers to unpack the outcomes and gain a closer understanding of what it is students should be expected to do. It is this clarity that is of greatest benefit to teachers.

Summative assessment tools are created that focus specifically on the end, the outcomes. Whether these assessment tools are tests, discussions, debates or presentations, they must all perform the same task: to measure proficiency of the outcomes and nothing else. Teachers must justify that the assessment tool measures student achievement around the outcomes as described in the POS and the blueprint template is ideal for this. Assessment tools that test knowledge, skills or attitudes outside of those outcomes are never used as evidence for reporting student achievement. For example, in the Science 7 POS in the Structures and Forces unit, students must "interpret examples of variation in the design of structures that share a common function, and evaluate the effectiveness of the designs (e.g., compare and evaluate different forms of roofed structures, or different designs for communication towers)." The CL of this outcome is HMA because students must interpret and evaluate. Therefore, an assessment of students' ability to identify designs of structures is inadequate to meet the criteria of this outcome. However, an outcome stating that students must "identify points of failure and modes of failure in natural and built structures (for example, potential failure of a tree under snow load, potential failure of an overloaded bridge)" has all three possibilities for CL. It would be rich discussion for teachers to reach consensus on the CL of this outcome because justification for all three levels is possible. The discussion about outcomes reveals a variety of interpretations that a professional learning community can draw from to blueprint assessments.

Teachers must analyze the outcomes and design assessments in a purposeful manner.

The next advantage of blueprinting is that teachers gain a clear vision of the end, and therefore are better able to consider the means. For example, at one school we constructed a Grade 9 math test on the Patterns and Relations strand. The test was perfectly blueprinted in that every outcome was assessed at the appropriate CL and the outcomes were evenly weighted. No questions were easier or more difficult than what is indicated by the outcomes in the POS. We wrote the test afterward to discover that the questions seemed to be more difficult due to the increase in HMA-level assessment. The obvious next step was to ask questions about how to change teaching practice. The process of blueprinting has a profound effect on professional reflection, readiness to accept new ideas and initiating a change in practice.

Figure 1 shows a blueprinted assessment tool using questions picked from the released materials of Alberta provincial achievement tests. In this case, all questions are at the CL implied by the outcomes, and the weighting of questions is appropriate for each outcome.

Figure 1: Blueprint of a test made from a cohort of teachers that was designed to match the outcomes in the Patterns and Relations strand of the Alberta program of studies.

Math 9 Patterns and Relation Unit Exam Blueprint

Program of Studies Outcome		Cognitive Level of Outcome/Task			Weight
Strand	Specific	K	C/A	HMA	
PAR	1		2013:mc21 2013:nr1		2
PAR	2	2013. MC38	2010 MC35,36	key PR pq 4	4
PAR	3		2010 MC 2 2013 mc 17		2
PAR	4		2010 MC 6, NR 8	2010 MC 8,29	4
PAR	5		2010mc23		1
PAR	6	(WR) taken from 2010 mc21	(WR) taken from 2010 mc21 2013 MC29		3
PAR	7	2013nr9	2013mc3, 39		3

Many questions are raised when engaging in this process. Can blueprinting be used to better pinpoint where students have difficulty? Can self-assessment practices be incorporated to make students involved in the process of unpacking the outcomes, appraising their own work and setting goals to improve? Can adapted assessments be blueprinted to have lower CL sections that are formative and a summative section that follows the CL implied by the outcomes? Should we reconsider the implementation of mandatory final exams containing mostly K-level outcomes? How can we include the front matter of the POS in a blueprint to develop an assessment plan that incorporates skills and attitudes?

Below is a self-assessment tool that uses a blueprint of the test to help students analyze their performance and set goals to improve.

Many teachers struggle with finding a strategy to allow students opportunities to improve their grades as knowledge is developed after assessments are graded. Retesting has obvious trade-offs such as increased teacher workload, more time devoted to summative assessment and the possibility of further decreasing motivation to learn. One strategy that effectively utilized blueprinting to increase marks was to provide students with opportunities to change the

Figure 2: Self-assessment tool using a blueprint of the Number strand of Mathematics 6.

Grade 6 Mathematics
Number Strand Analysis

Task

As we go over the test, **circle** the questions that you answered correctly.

Specific Outcome	Cognitive Level			Weight
	Knowledge	Comprehension /Application	Higher Mental Activity	
1. Demonstrate an understanding of place value, including numbers that are: • greater than one million	MC 12			1
1. Demonstrate an understanding of place value, including numbers that are: • less than one thousandth.	MC 1			1
2. Solve problems involving whole numbers and decimal numbers.		MC 4		1
3.Demonstrate an understanding of factors and multiples by: • determining multiples and factors of numbers less than 100		MC 2, MC 11		2
3.Demonstrate an understanding of factors and multiples by: • identifying prime and composite numbers	MC 3			1
3.Demonstrate an understanding of factors and multiples by: • solving problems using multiples and factors.		MC 2, MC 4		2
4.Relate improper fractions to mixed numbers and mixed numbers to improper fractions		MC 7		1
5. Demonstrate an understanding of ratio, concretely, pictorially and symbolically.		MC 6		1
6. Demonstrate an understanding of percent (limited to whole numbers), concretely, pictorially and symbolically.		MC 9		1
7. Demonstrate an understanding of integers, concretely, pictorially and symbolically		MC 10		1
8.Demonstrate an understanding of **multiplication** of decimals (1-digit whole number multipliers).		NR 1		1
8.Demonstrate an understanding of **division** of decimals (1-digit natural number divisors).		NR 2		1
9. Explain and apply the order of operations, excluding exponents, with and without technology (limited to whole numbers)			MC 8	1

Analyze your Data

1. What specific outcome did you do well on? Explain why.

2. What specific outcome do you need to work on? What were some possible reasons for your difficulties?

marks for the outcomes. Any student can raise his or her grade by making a video showing proficiency with the outcome(s) that were unsatisfactory. Students uploaded their instructional videos on YouTube and shared the links with me. Three benefits of this approach became clear. Students took ownership of their own learning and achievement. Making the videos required a higher level of thinking, and students developed permanent knowledge through this practice. Finally, I had a valid collection of evidence to support raising marks.

Blueprinting provides a degree of clarity to grading

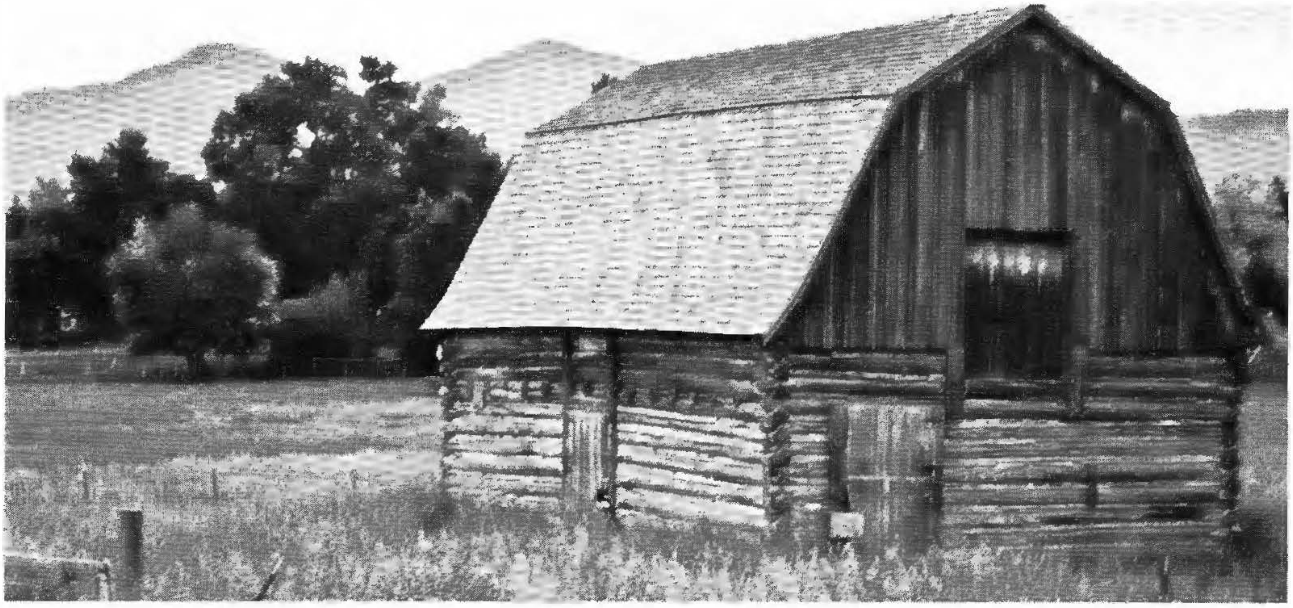
Reporting on student achievement through a single value seems insufficient in the same way that a letter grade fails to adequately describe a patient's health. Blueprinting provides a degree of clarity to grading in that each outcome is graded separately and a final grade is determined based on proficiency with the outcomes. When students are involved in this process, they become the drivers of their own learning and motivation is increased. Although the amount of work to assess this way is significant, I would not fall back on the ways I assessed previously.

Greg Wondga has been teaching secondary biology, chemistry and physics since 1997 in Edmonton, Alberta, with a brief one-year exchange to Queensland, Australia, in 2012. Greg is currently a teacher consultant for the department of curriculum and resource support at Edmonton Public Schools. In this role, Greg is responsible for meeting the professional learning needs of teachers in this district, researching best practices and consulting with educators and educational leaders in the implementation of district changes in practice. Prior to this, he has been a classroom teacher, field experiences associate with the University of Alberta Faculty of Education, human resources consultant, curriculum coordinator and assistant principal. In the many roles that Greg has filled, he has had the opportunity to learn from an incredible range of master educators. Greg has an infectious passion for sharing successful teaching and learning strategies. He was invited by the Rotary Club Belize Literacy Project to lead a group of teachers to plan and implement professional learning sessions for Belizean teachers. As well, Greg is a member of the ATA Science Council as the Division III director. Greg holds a master's degree of education in secondary science with a focus on constructing conceptual understanding. He also holds a bachelor of science and a bachelor of education degree.

Integration of Mathematics into Alberta's Homesteading History

Unit: Early Settlement of Alberta, Grade 4

Dalyce Harrison



Rational

As we explore Alberta's history during early settlement, students role play the work done by homesteaders. First the students find their land and roll a die to find out what is on the land, based on fractions. Then the work that the students do through the progression of numerous activities earns them agricultural units (AU) for the final auction. The results of the auction determine what fraction of the homesteaders survived. This is not a lesson but a unit.

Assessment

Mathematics

- Demonstrate an understanding of addition of numbers up to 10,000 and related subtraction of numbers.
- Demonstrate an understanding of multiplication (2- or 3-digit by 1-digit) to solve problems.
- Demonstrate an understanding of fractions less than or equal to one.
- Represent and describe decimals (tenths and hundredths).
- Relate decimals to fractions and fractions to decimals.
- Demonstrate an understanding of regular and irregular 2-D shapes by recognizing that area is measured in square units (square metre).
- Visualization—Visual reasoning is an important component of number, spatial and measurement sense.
- Spatial sense—Spatial sense enables students to communicate about shapes and objects and to create their own representations.

Social Studies

- Recognize how the diversity of immigrants from Europe and other continents has enriched Alberta's rural and urban communities.

- What movement or migration within Canada contributed to the populating of Alberta?
- How did European immigration contribute to the establishment of communities in Alberta in the late 19th and early 20th centuries?
- How did the arrival of diverse groups of people determine the establishment and continued growth of rural and urban communities?
- How are agriculture and the establishment of communities interconnected?
- What led to Alberta joining Confederation?

Science

- Wheels and Levers
- Building Devices and Vehicles that Move

Fine Arts

- Sketching of homesteader artifacts

Language Arts

- Reading comprehension (poetry, journal entries, letters, picture books) and writing to express ideas (poetry, letters and journal entries).

Essential Question

- Why did some homesteaders survive while others failed?

Introduction/Hook

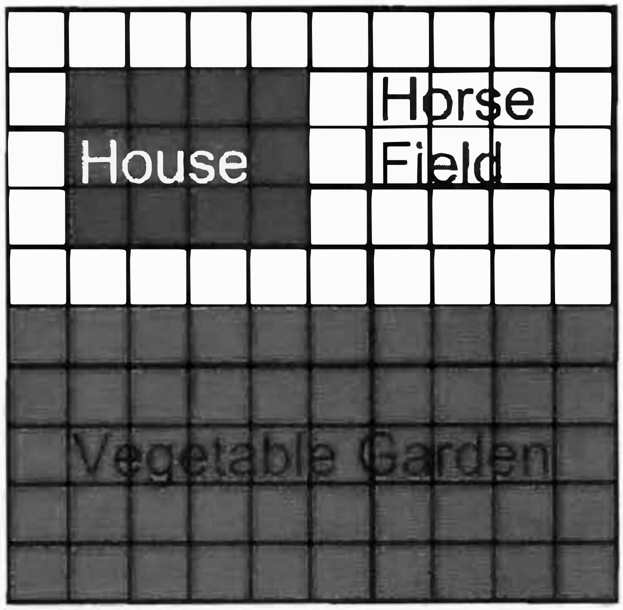
The students are going to be homesteaders. As such, they will each receive a square piece of land that is divided into tenths. The students will roll a die to determine what tenth of their land will be water, grassland, forest or rock. We will explore how rural land addresses work so that the students can find their address in our community. All of the quarter sections (square pieces of paper) will be hung in the hall to record the growth of our rural community.

Body

1. The students will earn agricultural units (AU) based on their work in this unit. AUs are recorded as pieces of yellow paper on their quarter section. AUs may be earned as one-quarter, one-half, three-quarters or one full AU, depending on how closely the work approaches the goal. One-tenth of land can hold five AUs and only grassland can support agricultural units. If a student runs out of grassland, then they can pay one AU to clear forest or rock to make more grassland.
2. The main source of agricultural units (AU) is from progressing through numerous activities in the unit. They consist of 20 activities that students complete individually. These 20 activities include mathematics, reading comprehension, writing, homesteader artifacts and 10 science experiments (from our unit on simple machines). Performance on the individual activities results in AUs for the land. The mathematical activities build on spatial awareness, fractions and decimals. Three such activities follow.

Activity 4—Your Yard

You are a homesteader with a yard measuring 10 metres by 10 metres. Your yard has a house, vegetable garden and horse field. Here is a map of your yard:



Complete the following chart:

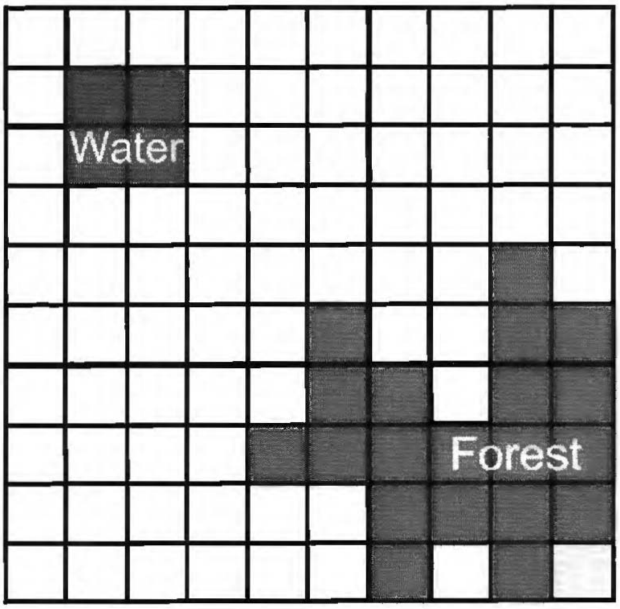
	Area (m^2)	Fraction of Yard	Decimal
House			
Vegetable Garden			
Horse Field			

Activity 8—Two Brothers

Two brothers, Brother Abe and Brother Bob, are dividing a piece of land in half.

- They each need half of the water.
- They each need half of the forested area.
- They each need half of the grassland (white).
- They want to build only one straight fence.
- Below is the map of the land of the two brothers.

How can they divide the land into two equal pieces? Draw the line.



Complete the chart. One square = 1 unit.

	Area of Water	Area of Forest	Area of Grass
Brother Abe			
Brother Bob			
Total Land			

How do you know the brothers each have half?

____

____

Activity 16—Garden Plan

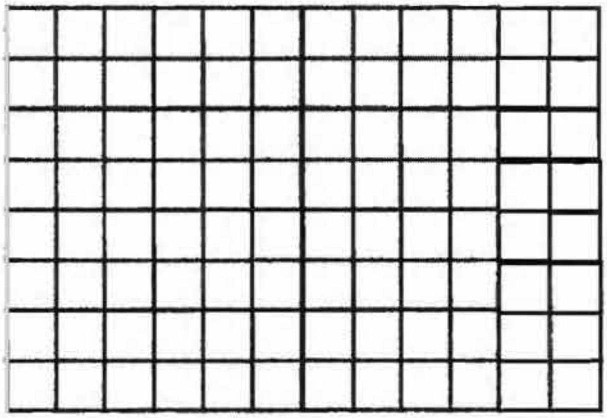
As a homesteader, you have a garden that is 12 metres by 8 metres. You want to plant potatoes, carrots, turnips, corn and peas.

You want to plant

- potatoes in a rectangle with an area of 24 m^2, so the potato hills are easier to maintain;
- corn in a rectangle with an area of 24 m^2, so the wind has a greater chance of pollinating all of the corn flowers;
- peas in the longest row possible, so you only have to run one length of mesh for the vines to grow up; and
- carrots and turnips in an equal area.

Below is a map of your garden plan.

Use coloured pencil crayons and a legend to show your plan.



3. As part of our science unit on wheels, levers and devices that move, students will plan, construct and test a catapult and a vehicle that rolls down a ramp. The students will each be given a credit at the general store of $50 or one-half AU. As they need to purchase materials for their constructions, they will do the subtraction from the money they have available. If they run out of money at the general store, they will need to sell the store one-half AU for an additional $50.
4. As homesteaders, students will need to build a house. Prior to building the house, students will explore the concept of the metre square. We will explore the actual size of a homesteader house. Students will calculate how much space is required to build a house for a single person, consisting of a bed, stove, table, chest and pantry. Often, this comes out to approximately 12 to 18 meters square. Students experiment with fractions of a square metre (such as one-half or one-quarter of a square metre). Students will then draw a map of their homestead house, using criteria for the house size

and furniture inside the house. A thumbnail sketch of their house will take up one-tenth of their land, so it must be built of either sod, logs or stone.

Conclusion and Final Project

In addition to the culminating projects for language arts and social studies, the culminating project for mathematics takes the form of an auction. First, students explore the concept of an auction through video clips and a picture book. Then students take the number of remaining agricultural units to sell at the auction. Students can choose what fraction of their AU is wheat or cattle. There are two different auctions, so students can decide what fraction of their cattle and wheat are sold at the first or the second auction. The price for one AU of wheat and one AU of cattle is determined by rolling a die. The students then calculate how much money they made by multiplying and adding. (Based on probability of the prices on the die, approximately half of the students should survive as a homesteader, which reflects what happened in Alberta historically.)

Teams, Individual and Class

As a team, students took their square metre on a hunt through the school for 2-D shapes that are less than, equal to or greater than the referent. Many of the cooperative learning strategies listed below used a partner. The students conducted their science experiments and math activities individually. Full-class discussions included picture books, the vote on confederation and explorations of life as a homesteader.

Differentiated Instruction and Cooperative Learning

PWIM (picture word induction model), line up (with fractions and decimals), Quiz-Quiz-Trade (naming decimals), Think-Pair-Share, Round Robin, Rally Coach, Placemat (What do you know about ½?), the Frayer Model, Graffiti Board, Math Court (pros and cons of various strategies).

Learning Styles/Multiple Intelligences

Visual (picture books, graphs, PWIMs, posters, videos, sketching artifacts), auditory (picture books read aloud, class discussions, videos), kinesthetic/physical (line up, hands-on science experiments), linguistic (poetry and journal entries), logical/mathematic (math activities), interpersonal (class discussions, Round Robin, Think-Pair-Share, picture books), intrapersonal (individual responses, story writing), existential (concepts of survival and diversity).

Number of articles Dalyce Harrison has submitted to delta-K *= 1*

Number of years Dalyce Harrison has been teaching Grade 4 in Medicine Hat = 10

Number of years Dalyce Harrison's family has been ranching in southern Alberta = 100

Number of times Dalyce Harrison tries exploring math in a new way with students = ∞

Project-Based Learning: The BUX Market Project Grade 9

Katherine Weber

This project's aim was to join Grade 9 social studies, language arts and math in an authentic and engaging all-encompassing project. Students created a business to market and sell the same product in an open market to the rest of the school. The rest of the school population—teachers, students and support staff—had BUX (the Share Index of the Budapest Stock Exchange) to spend in the market. (This is the only currency allowed in the market.) Groups only had one hour in the market to sell their product, but from here they gathered statistics and totalled their profit, and two weeks later they would have to present their business or company to a board of potential investors. Students did premarket surveys, analyzed data and investigated the behaviours of teenage consumers. The potential investors on the final panel were all community members from various areas: business experts and owners, marketers, professional entrepreneurs and so on. We utilized as many community experts as possible in this project, many who would come back and join us on the expert investment panel at the end of the project. The panel of experts would question the students on the viability of their company and how their investments would be used to improve or expand their current operations.

Rationale and Math Objectives

When looking to integrate a cross-curricular project or to transition a classroom into a project-based focus, teachers have reservations, believing it will be more work than traditional learning. However one needs to keep in mind based on what we know about the changing 21st-century learner that this traditional approach may only work for a small portion of the class. Authentic engagement in a meaningful and real-world task allows students to do math without really being aware that they are doing math. Also, bringing professional mentors from the community to show how they use math in their everyday lives can be less work for teachers and extremely beneficial for students. Students who succeed in math and know how math and the other subjects interrelate take this knowledge with them to their professional careers,

Social Studies 9.2
Students will demonstrate an understanding and appreciation of how economic decision making in Canada and the Unites States affects quality of life, citizenship and identity.

Scarcity Simulation and Reflection
Popcorn simulation—What is scarcity? What problems does it create? Is it necessary for our market to function? How can we diminish the cost of scarcity?

Experts and Mentors
- Small business owners (various)
- Marketing (Cathi Hobbins 780-518-3067)
- Community Futures Grande Prairie and Region (Leah Holler 780-814-5340)

Blog Response (Online)
Consumerism and Teens—Mindless Spenders or Voting Power?

Case Studies of Market Effects on Oil
How has the recent oil crash affected our society in Alberta?
Understand why oil crashed and what repercussions this has on our area.

BUX Market Project

Students will create a business to sell the same product as their peers in an open market to other students in the school. Students will have varying amounts of money to spend. Businesses will have to keep track of the product they sell, how much profit they make and analyze their demographics for analysis. This market will last one hour, but marketing will take place the weeks beforehand.

Math 9
- Describe the effect of bias, use of language, ethics, cost, time and timing, privacy, cultural sensitivity on the collection of data.
- Demonstrate an understanding of the role of probability in society.
- Select and defend the choice of using either a population or a sample of a population to answer a question.
- Develop and implement a project plan for the collection, display an analysis of data by (1) formulating a question for investigation, (2) choosing a data collection method that includes social considerations, (3) selecting a population or sample, (4) collecting the data, (5) displaying the collected data in an appropriate manner and (6) drawing conclusions to answer the questions.

Social Studies 9
- What is a market economy?
- How do the economies of the USA and Canada differ? How are they similar?
- What is scarcity?
- What is consumerism?
- Why do people buy certain products?
- What role should governments play in regulating business in our society?
- Taxation: How does this affect businesses? People? Why is it important and what does it provide?

Language Arts 9
Analyze creative uses of language and visuals in popular culture, such as advertisements, electronic magazines and the Internet; recognize how imagery and figurative language, such as metaphors, create a dominant impression, mood and tone.
Express the themes of oral, print or other media texts in different forms or genres.
Evaluate the effectiveness of different types of media texts for presenting ideas and information.
Assess adequacy, accuracy, detail and appropriateness of oral, print and other media texts to support or further develop arguments, opinions or points of view.
Develop coherence by relating all key ideas to the overall purpose of the oral, print or other media text.
Communicate ideas and information in a variety of oral, print and other media texts, such as media scripts, multimedia presentations, panel discussions and articles.
Communicate ideas and information in a variety of oral, print and other media texts, such as media scripts, multimedia presentations, panel discussions and articles.
Identify and experiment with some principles of design that enhance the presentation of texts.
Select, organize and present information to appeal to the interests and background knowledge of various readers or audiences.

Driving Questions
Math 9: How do we as small business owners create a business plan for a public audience to gain financial backing of our products?
Language Arts: How can we as small business owners develop marketing materials that are organized, planned out and purposefully written to sell products for a formal company?
Social Studies: How do market and mixed economies affect businesses who operate in them? How are we all part of the consumer cycle?

Formative Assessment
Business proposal feedback from experts
Discussion (online and in class)
Group choices—reflection
Self-assessment. Midpoint check-in. Self-assessment final.
Feedback from experts.

Cross-Curricular Competencies
- Working in groups
- Collaboration
- Critical thinking
- Creative thinking
- Entrepreneurial spirit
- Ethical citizenship

Summative Assessment
Presentation of business plans to "Dragons" to get backing and funding for company.
- Rubric for all classes based on content of materials—letter graded A, B, C

not just to get into college or university. During this project we covered a multitude of anticipated and unanticipated objectives. Next year, we would like to formally incorporate those unintentionally met objectives. This project covered the following objectives:

1. Graph a linear relation, analyze the graph and interpolate or extrapolate to solve problems.
2. Describe the effect of the following on the collection of data:
   - Bias
   - Use of language
   - Ethics
   - Cost
   - Time and timing
   - Privacy
   - Cultural sensitivity
3. Select and defend the choice of using either a population or a sample of a population to answer a question.
4. Develop and implement a project plan for the collection, display and analysis of data by
   - formulating a question for investigation,
   - choosing a data collection method that includes social considerations,
   - selecting a population or a sample,
   - collecting the data,
   - displaying the collected data in an appropriate manner and
   - drawing conclusions to answer the question.
5. Cover most or all Information and Communication Technology (ICT) outcomes for this division.

Timeline

This project began on March 11, 2015, and culminated with the expert panel on April 18, 2015. We utilized social studies, language arts and math blocks to become general project blocks, which gave us about 770 minutes a week. Each subject area had its own checklist of formative and summative items with dates attached, which allowed students and teachers to know the progress of each group.

Teacher Collaboration

Teachers met during lunch hours to discuss group progress, mini lessons, expert visits and so on. These meetings were essential to ensure success for students in the long run. We met for 30 minutes one to two times a week for seven weeks. We also utilized Google Drive to share a common folder and create rubrics and assignments together, which allowed us to meet and collaborate without actually getting together.

Results

The amount of student growth during this project was very tangible in our project groups. Not all students were as deeply engaged as others were, but all students found success. For example, two students who were on Knowledge and Employability math programming fully integrated into their project groups and provided meaningful work in their area of expertise, such as graphing or analyzing the statistics of a student survey. This was true success for us as a teacher cohort because all students invested in the project, not just those who enjoyed math or were normal high achievers. Also, quite a few students who were normally average academic achievers shone through and developed a deep-seeded interest in the mathematical and business-oriented side of this project. They seemed very motivated by the fact that math would be helpful to them as entrepreneurs or business professionals. We really understood how deeply the students grasped the mathematical concepts in this project when they began to describe the results of their business at the Dragons' Den panel review (our finale). For example, students provided graphs of projected profits based off of their earnings from the first market, used data collected to justify projected growth of their businesses, used and applied language like bias, ethics and cultural sensitivity in their presentations to the professional panel. It was truly amazing to see how deeply they grasped these concepts and not only understood but applied them in a meaningful way. In the end, this project did take some work on the part of the teachers. However, the traditional approach to teaching these topics may work to have some students understand, but rarely would you see the level of engagement we did during our project-based approach.

Katherine Weber grew up on a farm in northern Alberta and returned to her hometown after university to teach and coach. In her spare time she enjoys the outdoors, rock climbing, fishing and gardening with her family. Katherine also has a huge passion for social justice initiatives and doing volunteer work in her community. She is in her fifth year of teaching and her current teaching assignment is in junior high at St Mary's School in Sexsmith, Alberta. Katherine is an educational technology enthusiast and loves to learn about new educational initiatives. She has a blog where she writes on topics of technology and project-based learning: http://houseofweber.weebly.com/blog.

Alberta High School Mathematics Competition 2014/15

Part 1

Question 1.
When the repeating decimal $0.\overline{6}$ is divided by the repeating decimal $0.\overline{3}$, the quotient is

(a) $0.\overline{2}$ (b) 2 (c) $0.\overline{5}$ (d) 0.5 (e) none of these

Question 2.
The number of two-digit positive integers such that the sum of the two digits is 12 is

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

Question 3.
Let $a > b > 0$ be prime numbers. Of the following five numbers, the one which cannot be equal to $a - b$ is

(a) 41 (b) 42 (c) 43 (d) 44 (e) 45

Question 4.
Chau and Matt are picking berries. On Monday Matt picks twice as many kilograms of berries as Chau does, and on Tuesday Chau picks twice as many kilograms of berries as Matt does. Between them, they pick a total of 30 kilograms of berries over the two days. The number of kilograms of berries Chau picks over the two days is

(a) 14 (b) 15 (c) 16 (d) some number greater than 16

(e) not uniquely determined

Question 5.
For every set of five of the numbers 1, 2, ..., 2014, Lac writes down the smallest of the five numbers. The largest number she writes down is

(a) 5 (b) 2009 (c) 2010 (d) 2014 (e) none of these

Question 6.
The sum of four consecutive integers is n. The largest of these four numbers is

(a) n (b) $\frac{n}{4}$ (c) $\frac{n-2}{2}$ (d) $\frac{n+6}{4}$ (e) none of these

Question 7.
Let $f(n) = 2n^3$ for any positive integer n. For any odd number $m \geq 3$, the largest positive integer k such that 2^k divides $f(f(f(m)))$ is

(a) 4 (b) 8 (c) 12 (d) 13 (e) not uniquely determined

Question 8.
Let f be a function such that $f(x) + 3f(\frac{1}{x}) = x^2$ for any nonzero real number x. Then the value of $f(-3)$ is

(a) $-\frac{77}{72}$ (b) $-\frac{37}{36}$ (c) $-\frac{25}{24}$ (d) $-\frac{13}{12}$ (e) none of these

Question 9.
The number of integer pairs (m, n) such that $mn = m + n$ is

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

Question 10.
The number of five-digit positive integers such that the digits are alternately odd and even (either odd-even-odd-even-odd or even-odd-even-odd-even) is

(a) 10×5^5 (b) 2×5^5 (c) 5^5 (d) 9×5^4 (e) none of these

Question 11.
Let a, b and c be real numbers such that $a + b + c = 5$ and $\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} = 6$. The value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$ is

(a) 21 (b) 23 (c) 25 (d) 27 (e) not uniquely determined

Question 12.
P is a point inside an acute triangle ABC. D, E and F are the feet of the perpendiculars from P on BC, CA and AB respectively. If $BD = 1$, $DC = 10$, $CE = 6$, $EA = 9$ and $AF = 13$, the length of FB is

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Question 13.
Three candles which can burn for 30, 40 and 50 minutes respectively are lit at different times. All three candles are burning simultaneously for 10 minutes, and there is a total of 20 minutes in which exactly one of them is burning. The number of minutes in which exactly two of them are burning is

(a) 35 (b) 45 (c) 70 (d) 90 (e) none of these

Question 14.
P is a point inside a convex quadrilateral $ABCD$ of area 168 such that $PA = 9$, $PB = PD = 12$ and $PC = 5$. The perimeter of the quadrilateral is

(a) 38 (b) 56 (c) 58 (d) 60 (e) 62

Question 15.
For any real numbers x and y, the minimum value of $x^4 - 5x^2 + y^2 + 2x + 2y + 2xy + 6$ is

(a) -5 (b) -4 (c) -1 (d) 0 (e) none of these

Question 16.
Four cages are arranged in a 2×2 formation. Each cage contains some chickens and some rabbits (some cages may contain only chickens or only rabbits). The total number of heads in the two cages in the first row is 60. The total number of legs in the two cages in the second row is 240. The total number of heads in the two cages in the first column is 70. The total number of legs in the two cages in the second column is 230. The minimum number of animals in all four cages is

(a) 120 (b) 128 (c) 145 (d) 180 (e) none of these

Solutions (Part 1)

Question 1.
When the repeating decimal $0.\overline{6}$ is divided by the repeating decimal $0.\overline{3}$, the quotient is

(a) $0.\overline{2}$ (b) 2 (c) $0.\overline{5}$ (d) 0.5 (e) none of these

Solution:
We have $0.\overline{6} \div 0.\overline{3} = \frac{2}{3} \div \frac{1}{3} = 2$.The answer is **(b)**.

Question 2.
The number of two-digit positive integers such that the sum of the two digits is 12 is

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

Solution:
There are 7 such numbers, namely 39, 48, 57, 66, 75, 84 and 93. The answer is **(c)**.

Question 3.
Let $a > b > 0$ be prime numbers. Of the following five numbers, the one which cannot be equal to $a - b$ is

(a) 41 (b) 42 (c) 43 (d) 44 (e) 45

Solution:
We have $43 - 2 = 41$, $47 - 5 = 42$, $47 - 3 = 44$ and $47 - 2 = 45$. In order for $a - b = 43$, we must have $b = 2$ but then $a = 45$ is not a prime number. The answer is **(c)**.

Question 4.
Chau and Matt are picking berries. On Monday Matt picks twice as many kilograms of berries as Chau does, and on Tuesday Chau picks twice as many kilograms of berries as Matt does. Between them, they pick a total of 30 kilograms of berries over the two days. The number of kilograms of berries Chau picks over the two days is

(a) 14 (b) 15 (c) 16 (d) some number greater than 16

(e) not uniquely determined

Solution:
Let Chau pick x kilograms of berries on Monday and Matt pick y kilograms of berries on Tuesday. Then $3x + 3y = 30$ so that $x + y = 10$. The number of kilograms Chau picks over the two days is $x + 2y$. This expression can take on infinitely many values if all we know is that $x + y = 10$. For example we may have $x = 0$ so that $y = 10$ and $x + 2y = 20$; or we may have $x = 4$ so that $y = 6$ and $x + 2y = 16$; and so on. The answer is **(e)**.

Question 5.
For every set of five of the numbers 1, 2, ..., 2014, Lac writes down the smallest of the five numbers. The largest number she writes down is

(a) 5 (b) 2009 (c) 2010 (d) 2014 (e) none of these

Solution:
The largest minimum occurs when Lac chooses the numbers 2010, 2011, 2012, 2013 and 2014, and writes down 2010. The answer is **(c)**.

Question 6.
The sum of four consecutive integers is n. The largest of these four numbers is

(a) n (b) $\frac{n}{4}$ (c) $\frac{n-2}{2}$ (d) $\frac{n+6}{4}$ (e) none of these

Solution:
Let $n = (k-3) + (k-2) + (k-1) + k = 4k - 6$. Then $k = \frac{n+6}{4}$. The answer is **(d)**.

Question 7.
Let $f(n) = 2n^3$ for any positive integer n. For any odd number $m \geq 3$, the largest positive integer k such that 2^k divides $f(f(f(m)))$ is

(a) 4 (b) 8 (c) 12 (d) 13 (e) not uniquely determined

Solution:
We have $f(m) = 2m^3$, $f(2m^3) = 2^4m^9$ and $f(2^4m^9) = 2^{13}m^{27}$. The answer is **(d)**.

Question 8.
Let f be a function such that $f(x) + 3f(\frac{1}{x}) = x^2$ for any nonzero real number x. Then the value of $f(-3)$ is

(a) $-\frac{77}{72}$ (b) $-\frac{37}{36}$ (c) $-\frac{25}{24}$ (d) $-\frac{13}{12}$ (e) none of these

Solution:
Replacing x by -3 and then by $-\frac{1}{3}$ in the given equation, we obtain $f(-3) + 3f(-\frac{1}{3}) = 9$ and respectively $f(-\frac{1}{3}) + 3f(-3) = \frac{1}{9}$. Solving for $f(-3)$ we get $f(-3) = -\frac{13}{12}$. The answer is **(d)**.

Alternative Solution: Replacing x by $\frac{1}{x}$ in the given equation, we obtain $f(\frac{1}{x}) + 3f(x) = \frac{1}{x^2}$. Solving for $f(x)$ we get $f(x) = \frac{3-x^4}{8x^2}$. Hence $f(-3) = -\frac{39}{36} = -\frac{13}{12}$.

Question 9.
The number of integer pairs (m, n) such that $mn = m + n$ is

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

Solution:
Clearly $n \neq 1$, so that $m = \frac{n}{n-1} = 1 + \frac{1}{n-1}$. Since m is an integer, we can only have $n = 0$ or 2, with $m = 0$ and 2 respectively. The answer is **(c)**.

Question 10.
The number of five-digit positive integers such that the digits are alternately odd and even (either odd-even-odd-even-odd or even-odd-even-odd-even) is

(a) 10×5^5 (b) 2×5^5 (c) 5^5 (d) 9×5^4 (e) none of these

Solution:
For the odd-even-odd-even-odd pattern, we have 5^5 such numbers. For the even-odd-even-odd-even pattern, we have 4×5^4 such numbers. The total is 9×5^4. The answer is (d).

Question 11.
Let a, b and c be real numbers such that $a + b + c = 5$ and $\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} = 6$. The value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$ is

(a) 21 (b) 23 (c) 25 (d) 27 (e) not uniquely determined

Solution:
Note that

$$\begin{aligned}\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &= \frac{5-(b+c)}{b+c} + \frac{5-(c+a)}{c+a} + \frac{5-(a+b)}{a+b} \\ &= \frac{5}{b+c} + \frac{5}{c+a} + \frac{5}{a+b} - 3.\end{aligned}$$

Hence $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 5 \times 6 - 3 = 27$. The answer is (d).

Question 12.
P is a point inside an acute triangle ABC. D, E and F are the feet of the perpendiculars from P on BC, CA and AB respectively. If $BD = 1$, $DC = 10$, $CE = 6$, $EA = 9$ and $AF = 13$, the length of FB is

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Solution:
By Pythagoras' Theorem, $BD^2 - DC^2 = (BP^2 - PD^2) - (CP^2 - PD^2) = BP^2 - CP^2$. Similarly, $CE^2 - EA^2 = CP^2 - AP^2$ and $AF^2 - BF^2 = AP^2 - BP^2$. Summing these equations yields $(BD^2 + CE^2 + AF^2) - (DC^2 + EA^2 + FB^2) = 0$. Hence $FB^2 = (1^2 + 6^2 + 13^2) - (10^2 + 9^2) = 5^2$ so that $FB = 5$. The answer is (e).

Question 13.
Three candles which can burn for 30, 40 and 50 minutes respectively are lit at different times. All three candles are burning simultaneously for 10 minutes, and there is a total of 20 minutes in which exactly one of them is burning. The number of minutes in which exactly two of them are burning is

(a) 35 (b) 45 (c) 70 (d) 90 (e) none of these

Solution:

Count the number of minutes in which each candle is burning and add them together. The ten minutes of simultaneous burning contributes 30 minutes to this. and there are 20 minutes of individual burning. The total must be 30+40+50=120 minutes, so the time when exactly two candles are burning must contribute the remaining 120-30-20=70 minutes. Thus the answer must be 70/2=35.

Alternative Solution: Let the first two candles burn simultaneously for a minutes, the first and the third for b minutes and the last two for c minutes. Then the first candle burns alone for $20-a-b$ minutes, the second for $30-a-c$ minutes and the third for $40-b-c$ minutes. Hence $90+2(a+b+c)=20$ so that $a+b+c=35$. This may be realized if $a=0$, $b=20$ and $c=15$. The three candles are lit at minutes 30, 0 and 15 respectively. The answer is **(a)**.

Question 14.

P is a point inside a convex quadrilateral $ABCD$ of area 168 such that $PA=9$, $PB=PD=12$ and $PC=5$. The perimeter of the quadrilateral is

(a) 38 (b) 56 (c) 58 (d) 60 (e) 62

Solution:

Note that $AC \leq AP+PC=14$, $BD \leq BP+PD=24$ and the area of $ABCD$ is at most $\frac{1}{2}AC \cdot BD \leq 168$. Thus we must have equality, so that P is the point of intersection of the diagonals which must be perpendicular to each other. By Pythagoras' Theorem, $AB=AD=\sqrt{9^2+12^2}=15$ and $CB=CD=\sqrt{5^2+12^2}=13$. Hence the perimeter of $ABCD$ is $2(15+13)=56$. The answer is **(b)**.

Question 15.

For any real numbers x and y, the minimum value of $x^4-5x^2+y^2+2x+2y+2xy+6$ is

(a) -5 (b) -4 (c) -1 (d) 0 (e) none of these

Solution:

Since $x^4-5x^2+y^2+2x+2y+2xy+6=(x^2-3)^2+(x+y+1)^2-4$, minimum value is -4, attained at $x=\pm\sqrt{3}$ and $y=-1\mp\sqrt{3}$. The answer is **(b)**.

Question 16.

Four cages are arranged in a 2×2 formation. Each cage contains some chickens and some rabbits (some cages may contain only chickens or only rabbits). The total number of heads in the two cages in the first row is 60. The total number of legs in the two cages in the second row is 240. The total number of heads in the two cages in the first column is 70. The total number of legs in the two cages in the second column is 230. The minimum number of animals in all four cages is

(a) 120 (b) 128 (c) 145 (d) 180 (e) none of these

Solution:

There are 230 legs in the two cages in the second column. Since $230=4\times 57+2$, they may come from as few as 58 animals, namely, 57 rabbits and 1 chicken. Hence the minimum number of animals in all four cages is 70+58=128. If we have 55 animals in the first cage in the first row, 0 chickens and 5 rabbits in the second cage in the first row, 15 chickens and 0 rabbits in the first cage in the second row, and 1 chicken and 52 rabbits in the second cage in the second row, we have 128 animals overall. The answer is **(b)**.

Alberta High School Mathematics Competition 2014/15

Part 2

Problem 1.
Find the number of isosceles triangles of perimeter 2015 such that all three sides are **odd** integers.

Problem 2.
Find all pairs (m, n) of positive integers such that $m^3 - n^3 = 5mn + 43$.

Problem 3.
Let $f : [0, 4] \to [0, \infty)$ be such that $f(4) = 2$ and $f(x + y) \geq f(x) + f(y)$ for any real numbers x and y in the closed interval [0,4] such that $x + y \leq 4$.

(a) Suppose that $0 \leq x \leq y \leq 4$. Show that $f(y) \geq f(x)$.

(b) Show that $f(x) \leq x$ for any x in [0,4].

Problem 4.
E and F are points on the sides CA and AB, respectively, of an equilateral triangle ABC such that EF is parallel to BC. G is the intersection point of medians in triangle AEF and M a point on the segment BE. Prove that $\angle MGC = 60°$ if and only if M is the midpoint of BE.

Problem 5.
Karys is helping her father move basketballs from his car to the gymnasium. She carries either 3 or 4 basketballs each trip, while her father carries 6 or 7 basketballs each trip. Altogether Karys makes 15 more trips and carries 15 fewer basketballs than her father.

(a) Determine the minimum number of basketballs that Karys carries.

(b) Determine the maximum number of basketballs that Karys carries.

Solutions (Part 2)

Problem 1.
Find the number of isosceles triangles of perimeter 2015 such that all three sides are **odd** integers.

Solution:
Let m, m and n be the lengths of the sides of the triangle. Then $2m+n=2015$. We should have $2m>n$, hence $n\leq 1007$. Thus $(n, 2m) = (1, 2014), (3, 2012), (5, 2010), \cdots, (1005, 1010), (1007, 1008)$, or taking into account that m is odd, $(n, m) = (1, 1007), (5, 1005), \cdots, (1005, 505)$. The number of pairs is $(1007-505)/2+1=252$ and hence there are 252 triangles.

Problem 2.
Find all pairs (m, n) of positive integers such that $m^3-n^3=5mn+43$.

Solution:
Since $m>n$, $k=m-n$ is a positive integer. Then $(k+n)^3-n^3-5(k+n)n=43$ or

$$(3k-5)n^2+k(3k-5)n+k^3=43 \tag{1}$$

We cannot have $k=1$ because then (1) simplifies to $n^2+n+21=0$ which has no real solutions.Thus we assume $k\geq 2$, which means that $3k-5>0$. From (1), we get $k^3<43$, and so we have $k\leq 3$. If $k=2$, then (1) simplifies to $0=n^2+2n-35=(n-5)(n+7)$, with the positive integral solution $n=5$. If $k=3$, from (1) we get $0=n^2+3n-4=(n-1)(n+4)$, with the positive integral solution $n=1$. Hence there are two such pairs, namely, $(m, n) = (7, 5)$ and $(4, 1)$.

Problem 3.
Let $f:[0,4]\to[0,\infty)$ be such that $f(4)=2$ and $f(x+y)\geq f(x)+f(y)$ for any real numbers x and y in the closed interval $[0,4]$ such that $x+y\leq 4$.

(a) Suppose that $0\leq x\leq y\leq 4$. Show that $f(y)\geq f(x)$.

(b) Show that $f(x)\leq x$ for any x in $[0,4]$.

Solution:

(a) If $0\leq x\leq y\leq 4$ then $f(y)=f(x+(y-x))\geq f(x)+f(y-x)\geq f(x)$.

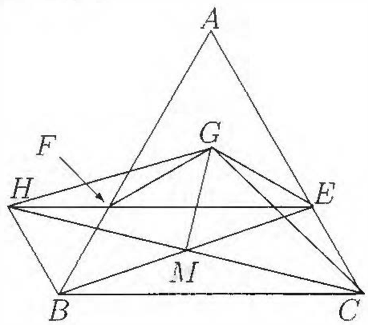
(b) If x is in $[2,4]$, then $f(x)\leq f(4)=2\leq x$. On the other hand, if x is in $(0,2)$, then $\frac{4}{x}-\frac{2}{x}=\frac{2}{x}>1$. Hence there is a positive integer n such that $\frac{2}{x}<n<\frac{4}{x}$ or $2<nx<4$, so that $f(nx)\leq nx$. Also, $f(nx)=f((n-1)x+x))\geq f((n-1)x)+f(x)\geq f((n-2)x)+2f(x)\geq\cdots\geq nf(x)$ and hence $f(nx)\geq nf(x)$. Consequently $nf(x)\leq f(nx)\leq nx$, that is, $f(x)\leq x$ for any x in $(0,2)$. Also, from $f(0)+f(0)\leq f(0)$ we obtain $f(0)\leq 0$, and since $f(0)\geq 0$, we get $f(0)=0$. Therefore $f(x)\leq x$ for any x in $[0,4]$.

Problem 4.

E and F are points on the sides CA and AB, respectively, of an equilateral triangle ABC such that EF is parallel to BC. G is the intersection point of medians in triangle AEF and M a point on the segment BE. Prove that $\angle MGC = 60°$ if and only if M is the midpoint of BE.

Solution:

Let H be on the extension of EF such that $BCEH$ is a parallelogram. Then $\angle BHE = \angle BCE = 60°$ and $\angle BFH = \angle AFE = 60°$, so FBH is also an equilateral triangle, and in particular $HF = FB = EC$. Also $\angle HFG = 180° - \angle GFE = 180° - 30° = 150°$. and similarly $\angle CEG = 150°$. thus $\angle HFG = \angle CEG$. Since $\angle GFE = 30° = \angle GEF$. we know $FG = GE$. Hence triangles HFG and CEG are congruent (by SAS) so that $GH = GC$ and $\angle HGF = \angle CGE$. It follows that $\angle HGC = \angle HGF + \angle FGC = \angle CGE + \angle FGC = \angle EGF = 180° - \angle GFE - \angle GEF = 180° - 30° - 30° = 120°$. Now M is the midpoint of the diagonal BE of the parallelogram if and only if it is the midpoint of the diagonal HC. Since triangle HGC is isosceles, this is equivalent to GM being the bisector of $\angle HGC$. In other words, $\angle MGC = 60°$.



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Karys is helping her father move basketballs from his car to the gymnasium. She carries either 3 or 4 basketballs each trip, while her father carries 6 or 7 basketballs each trip. Altogether Karys makes 15 more trips and carries 15 fewer basketballs than her father.

(a) Determine the minimum number of basketballs that Karys carries.

(b) Determine the maximum number of basketballs that Karys carries.

Solution:

Suppose that Karys carries n basketballs in total. Then she will make at least $\frac{n}{4}$ and at most $\frac{n}{3}$ trips. Her father carries $n + 15$ basketballs in total, so he makes at least $\frac{n+15}{7}$ and at most $\frac{n+15}{6}$ trips. Thus the difference between the number of trips Karys makes and the number of trips her father makes is at least $\frac{n}{4} - \frac{n+15}{6}$ and at most $\frac{n}{3} - \frac{n+15}{7}$. Thus we get

$$\frac{n}{4} - \frac{n+15}{6} \leq 15 \quad \text{and} \quad \frac{n}{3} - \frac{n+15}{7} \geq 15,$$

which simplify respectively to

$$2n - 60 \leq 360 \quad \text{and} \quad 4n - 45 \geq 315.$$

Thus $90 \leq n \leq 210$.

(a) It is possible for Karys to carry 90 basketballs in total and for her father to carry 90+15=105 basketballs, because Karys can carry 3 basketballs at a time for 30 trips, while her father carries 7 basketballs at a time for 15 trips, so Karys indeed makes $30 - 15 = 15$ more trips than her father.

(b) It is not possible for Karys to carry 210 basketballs in total. To do this she would need at least $\frac{210}{4} = 52.5$ trips, so at least 53 trips. Her father would take at most $\frac{225}{6} = 37.5$ trips, so at most 37 trips. So the difference in the number of trips is at least $53 - 37 = 16$, not 15. Similarly, 209 basketballs carried by Karys and 224 carried by her father is not possible either, because we get $\lceil\frac{209}{4}\rceil = 53$ and $\lfloor\frac{224}{6}\rfloor = 37$. However, 208 for Karys and 223 for her father is possible, because now $\lceil\frac{208}{4}\rceil = 52$ and $\lfloor\frac{223}{6}\rfloor = 37$, and $52 - 37 = 15$. Karys carries 4 basketballs at a time for 52 trips for a total of 208 basketballs, while her father carries 6 basketballs at a time for 36 trips and 7 basketballs at a time for 1 trip, for a total of $36 \cdot 6 + 7 = 223$ basketballs in 37 trips.

Remark:

Similarly we can check that Karys could carry 207 basketballs, but it turns out that 206 basketballs carried by Karys (and 221 by her father) is again not possible, because $\lceil\frac{206}{4}\rceil = 52$ and $\lfloor\frac{221}{6}\rfloor = 36$, and $52 - 36 = 16$.

Edmonton Junior High Mathematics Competition 2014/15

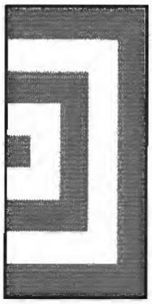
Part A: Multiple Choice

Each correct answer is worth four points. Each unanswered question is worth two points to a maximum of three unanswered questions.

1. Which of these numbers is greater than its reciprocal?
   (a) $-1.\overline{5}$
   (b) 0.995
   (c) -99.9%
   (d) $0.\overline{3}$
   (e) $\frac{2}{5}$

2. What number is doubled when $\frac{3}{4}$ of it is subtracted from 99?
   (a) 32
   (b) 36
   (c) 40
   (d) 44
   (e) 52

3. A target is made of dark and white strips of equal width as shown at the right. If a dart is thrown and lands randomly inside the target, what is the probability that it will land on white?



   (a) $\frac{2}{5}$
   (b) $\frac{3}{8}$
   (c) $\frac{4}{9}$
   (d) $\frac{1}{2}$
   (e) $\frac{1}{3}$

4. How many two-digit whole numbers less than 40 are divisible by the product of its digits?
   (a) 5
   (b) 4
   (c) 3
   (d) 2
   (e) More than 5

5. A florist has 72 roses, 90 tulips and 60 daffodils, and uses all of them to make as many identical bouquets as possible. How many flowers does the florist put in each bouquet?
   (a) 6
   (b) 18
   (c) 24
   (d) 29
   (e) 37

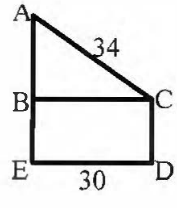
Part B: Short Answer

6. A rectangle has an area of 48 cm^2 and a perimeter of 28 cm. What is the length of the rectangle's diagonal, rounded to the nearest whole centimetre?

7. When a two-digit number is multiplied by the sum of its digits, the product is 952. What is the two-digit number?

8. Twenty-six people are seated in a circle and are lettered alphabetically from A to Z. Beginning with person A and proceeding in a clockwise direction, each alternate person leaves the circle. What is the letter of the last person to leave?

9. In the rectangle BCDE, BC = 30 cm. A is on the extension of EB, and AC=34 cm. The area of triangle ABC is 30 cm^2 less than half of the area of BCDE. What is the perimeter of the quadrilateral ACDE?



10. The age of a tortoise is 52 years more than the combined age of two elephants. In 10 years, the tortoise will be twice as old as the two elephants combined. How old is the tortoise now?

11. The angle bisectors of the two acute angles of obtuse triangle, ΔXYZ, intersect at point W. The measure of $\angle Z$ is 98°. What is the measure, in degrees, of $\angle XWY$?

12. Maria purchased a number of peaches and apples. The mean mass of the peaches is 170 g. The mean mass of the apples is 140 g. The mean mass of all the fruit is 152 g. What is the ratio of the number of peaches to apples purchased?

13. Two sides of a scalene acute triangle measure 12 cm and 13 cm. If the length of the third side is also an integer, then how many lengths are possible for the third side to be?

14. What is the largest n such that n^n is an n-digit number?

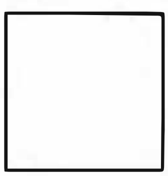
Part C: Short Answer

15. Consider the 2014 digit number consists of 2013 nines followed by 1 one.

$$\overbrace{99\ldots99}^{2013}1$$

The smallest factor is 1 and the largest factor is the number itself. Let M be the second smallest factor and N be the second largest factor. What is the sum of the digits of M and N?

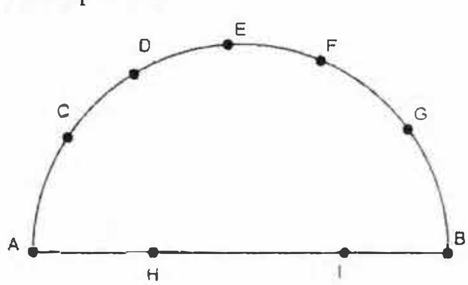
16. $ABCD$ is a square with $AC = 49.5$ cm. P is a point inside $ABCD$ such that $PB = PC$, and the area of triangle PCB is one-third of the area of $ABCD$. What is the length, in cm, of PA? Round your answer off to the nearest integer.



17 A three-digit number is equal to 17 times the product of its digits, and the hundreds digit is 1 more than the sum of the other two digits. Find all such three-digit numbers.

18. A magazine receives 32 articles, of length 1, 2, ..., 32 pages, respectively. The first article starts on page 1 and all other articles start on the page after the preceding article. The articles may be arranged in any order. What is the maximum number of articles that can start on an odd-numbered page?

19. The diagram shows nine points. How many triangles are there whose vertices are chosen from the nine points?



Solutions

1. (a)$= -\frac{14}{9} < \frac{-9}{14}$

(b)$= \frac{199}{200} < \frac{200}{199}$

(c)$= \frac{-999}{1000} > \frac{-1000}{999}$

(d)$= \frac{1}{3} < 3$

(e)$= \frac{2}{5} < \frac{5}{2}$

The correct answer is (c).

2. Let n be the number.

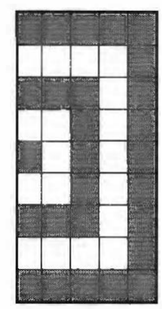
$$2n = 99 - \frac{3}{4}(n)$$

$$\frac{11n}{4} = 99$$

$$n = 36$$

The correct answer is (b).

3. The shape can be divided into 45 individual squares. The white squares are $\frac{18}{45} = \frac{2}{5}$ of the entire target.
The correct answer is (a).



4. There are exactly five of them: 11, 12, 15, 24 and 36.
The correct answer is (a).

5. The GCF (72, 90, 60) = 6. This gives 6 bouquets with 12 roses, 15 tulips and 10 daffodils; a total of 37 flowers in each bouquet.
The correct answer is (e).

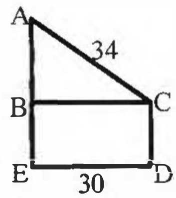
Part B: Short Answer

6. Knowing that L(W) = 48 and L+W = 14, we have L = 8 and W = 6. The diagonal is $\sqrt{8^2 + 6^2} = 10$ cm.

7. The prime factorization of 952 is $2 \times 2 \times 2 \times 7 \times 17$. Two-digit divisors are 14, 17, 28, 34, 56 and 68. Checking all cases, we have 952=68(6+8).

8. After the first round, BDFHJLNPRTVXZ are left, and the next to go is B. After the second round, DHLPTX, and the next to go is H. After the third round, DLT are left and the next to go is L. After the fourth round, DT are left and the next to go is D. After the fifth round, only T is left.

9. In the rectangle BCDE, BC=30 cm. A is on the extension of EB, and AC=34 cm. By Pythagoras' Theorem, AB=16 cm and the area of triangle ABC is 240 cm^2. Hence the area of BCDE is 2(240+30) or 540 cm^2, so that CD=18 cm. The perimeter of the trapezoid ACDE is 34+18+30+18+16=116 cm.



10. Suppose the tortoise is x years old and the two elephants together are y years old. Then $x - y = 52$. In 10 years' time; $x + 10 = 2(y + 20)$. Hence $y + 52 = 2y + 30$ so that $y = 22$ and $x = 74$.

11. In degrees, $\angle$X plus $\angle$Y is 82, $\angle$WXY + $\angle$WYX is 41 and $\angle$XWY is 139.

12. Let p = number of peaches and a = number of apples.

    This gives $\frac{170p+140a}{p+a} = 152$ or 18p = 12a.

    Thus the ratio of p:a = 2:3.

13. Let ABC be the triangle where AC = 13 cm and BC = 12 cm.

    When AB = 5 cm, we have a right angle triangle at $\angle$ABC.

    When $\angle$ACB = 90°, AB = 17.69.

    We now have 5 < third side < 17.69 so that it is an acute triangle. This gives 12 possible lengths for the third side from 6 to 17 cm. In order to be scalene, we need to eliminate both 12 and 13 from the list; hence, there are a total of 10 possible lengths for the third side.

14. A quick check would reveal that 10^{10} would give a total of 11 digits. In fact when n is greater than 10, the resulting power will always have more than n digits. Thus the greatest $n = 9$ giving 9^9 = 387420489 (9 digits).

Part C: Short Answer

15. First, we know the number is not divisible by 3 as it divides into all the 9s except the last digit of 1.

    As for 7, it will divide into six 9s evenly. The longest string of 9s would be 2010 digits. This leaves 9,991 which 7 do not divide evenly.

    Eleven divides into pairs of 99 but won't divide evenly into 91.

    The number 13 will go into six 9s evenly. Similar to 7, it does not divide evenly into 9,991.

    Seventeen divides evenly into a string of sixteen 9s. Leaving thirteen 9's followed by a 1, which 17 does not divide evenly.

    Nineteen divides evenly into a string of eighteen 9s. Leaving fifteen 9s followed by a 1, which 19 also does not divide evenly.

    Similarly 23 divides evenly into a string of twenty-two 9s. Leaving eleven 9s followed by a 1, which 23 does not divide evenly.

    The next prime number to try is 29. Like the previous quotients, 29 divides evenly into a number made up of a string of twenty-eight 9s. The quotient is 0344827586206896551724137931. The sum of its digits is 126. This also means there are 71 sets of this number giving a total of 1988 digits of 9s. We still have a number with 25 digits of 9s followed by a 1. A total of 26 digits left. Fortunately, 29 multiplied by the quotient less the last two digits of "31" results in exactly 25 digits of 9s followed by a 1. This concludes that 29 divides evenly into the original number.

    To recap, the second smallest factor is 29. The sum of the digits is 2 + 9 = 11.

    The second largest factor is a number of the form

    $$\overbrace{0344827586206896551724137931}^{71\ sets}$$
    $$0344827586206896551724137 9$$

    The sum of the digits is 126 × 72 – 4 = 9068.

    Therefore the total sum of the digits of M and N is 11 + 9,068 = 9,079.

Note from the Committee

Below are several related problems on the 2014-digit number n consisting of all 9s except for a 1 as its last digit.

Problem 1
Prove that n is a composite number.

Problem 2
Prove that n is not a square.
Problem 2 is needed to set up the next problem. A positive integer which is not a square has an even number of positive divisors because they form pairs whose product is n. If n is a square, which means that $\sqrt{n}$ is a positive integer, then it is paired with itself. It counts as only one divisor, making the total number of divisors odd. By problem 2, n has $2k$ positive divisors for some positive integer k, namely, $1 = d_1 < d_2 < \cdots < d_k < d_{k+1} < \cdots < d_{2k-1} < d_{2k} = n$.

Problem 3
Find the combined digit sum of d_k and d_{k+1}.

Problem 4
Find the combined digit sum of d_2 and d_{2k-1}.

Problem 1 was proposed for this year's contest. Since numerical answers were desired, it was intended to be modified as problem 3. However, the problem was worded so that problem 4 became what was actually asked.

Solution to Problem 1
Note that $n = 10^{2.014} - 9 = (10^{1.007})^2 - 3^2 = (10^{1.007} + 3)(10^{1.007} - 3)$. Since each factor is clearly greater than 1, n is a composite number.

Solution to Problem 2
Note that $n = 999... 991 = 999... 9 \times 100 + 91 \equiv 0 + 3 = 3 \pmod 4$ since $100 \equiv 0 \pmod 4$. Since all squares are congruent to 0 or 1 (mod 4), n is not a square.

Solution to Problem 3
Note that $d_k = 10^{1.007} - 3 = 999... 997$ so that its digit sum is $1,006 \times 9 + 7 = 9,061$. On the other hand, $d_{k+1} = 1,000... 003$ so that its digit sum is $1 + 3 = 4$. Hence the combined digit sum is $9,061 + 4 = 9,065$.

The rest of the article is devoted to the solution to problem 4.

Clearly, $d_2 \neq 2$ or 5. By the tests of divisibility, it is neither 3 nor 11. If $d_2 = 7$, then we must have $10^{2.014} \equiv 9 \equiv 2 \pmod 7$. Now $10 \equiv 3 \pmod 7$, $10^2 \equiv 3 \times 3 = 9 \pmod 7$, $10^3 \equiv 3 \times 9 = 27 \equiv 6 \pmod 7$, $10^4 \equiv 3 \times 6 = 18 \equiv 4 \pmod 7$, $10^5 \equiv 3 \times 4 = 12 \equiv 5 \pmod 7$ and $10^6 \equiv 3 \times 5 = 15 \equiv 1 \pmod 7$. It is not necessary to go on any further. This is because $2,014 = 335 \times 6 + 4$, so that $10^{2.014} = (10^6)^{335} \times 10^4 \equiv 1^{335} \times 4 = 4 \neq 2$. It follows that $d_2 \neq 7$.

A key step in the above argument is that $10^k \equiv 1 \pmod 7$ for some positive integer k, which happens to be 6. How do we know that such a k always exists, if we replace 7 by another prime number? Let us understand why $k = 6$ for the prime number 7. Suppose we wish to convert the fraction $\frac{1}{7}$ into a decimal. By long division, we find that $\frac{1}{7} = 0.\overline{142857}$, a decimal expansion consisting of repeating blocks of the six digits 142857.The reason that there are six digits is that when we divide by 7, the only possible remainders are 0, 1, 2, 3, 4, 5 and 6. Here 0 will not appear since no power of 10 is divisible by 7. By the time we have seen each of the non-zero remainders once, repetition must start. Thus the repeating block of decimal digits has length at most 6. In this case, it happens to be exactly 6. This means that $\frac{1}{7} = \frac{142857}{999999}$ so that 999999 is divisible by 7. It follows that $10^6 \equiv 1$.

In a similar manner, we can prove that $d_2 \neq 13$, 17, 19 or 23. We know that $10^{12} \equiv 1 \pmod{13}$, $10^{16} \equiv 1 \pmod{17}$, $10^{18} \equiv 1 \pmod{19}$ and $10^{22} \equiv 1 \pmod{23}$. As it turns out, $10^6 \equiv 1 \pmod{13}$, but the other powers, namely, 16, 18 and 22, cannot be reduced. Since $10^4 \not\equiv 9 \pmod{13}$, $d_2 \neq 13$.

Now $2,014 = 125 \times 16 + 14$ but $10^{14} \not\equiv 9 \pmod{17}$, $2,014 = 111 \times 18 + 16$ but $10^{16} \not\equiv 9 \pmod{19}$, and $2,014 = 91 \times 22 + 12$ but $10^{12} \not\equiv 9 \pmod{23}$. Hence $d_2 \neq 17$, 19 or 23.

The next candidate for d_2 is 29. We know that $10^{28} \equiv 1$, but perhaps one of 10^2, 10^4, 10^7 and 10^{14} may be too. In modulo 29, we have $10^2 = 100 \equiv 13$, $103 \equiv 10 \times 13 = 130 \equiv 14$, $10^4 \equiv 10 \times 14 = 140 \equiv 24$, $10^7 \equiv 14 \times 24 = 336 \equiv 17$ and $10^{14} \equiv 17^2 = 289 \equiv 28$. So this does not happen. Since $2,014 = 28 \times 71 + 26$, what we need is $10^{26} \equiv 9$ Now $10^5 \equiv 24 \times 10 = 240 \equiv 8$, $10^{25} \equiv 8^5 = 32,768 \equiv 27$ and $10^{26} \equiv 10 \times 27 = 270 \equiv 9$. This is exactly what we want. We are lucky that $n = 10^{2.014} - 9 = (10^{1.007} + 3)(10^{1.007} - 3)$ has a prime factor as small as 29. Each of $10^{1.007} + 3$ and $10^{1.007} - 3$ has more than 1,000 digits. Even if they were not prime numbers, they could have been products of prime numbers with over 500 digits. It would be very diffcult to find d_2 then. From $d_2 = 29$, we have $d_{2k-1} = \frac{n}{29}$. There remains only the trivial matter of determining their combined digit sums, via the following long division:

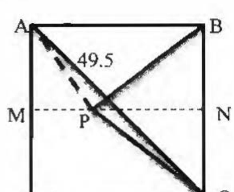
9999999 999999 9999999 9999999 / 29 =
344827586206896551724137931

The sum of the digits of the quotient is 126, and there are 71 such blocks. In the last incomplete block, the quotient is without the last 2 digits 3 and 1. It follows that the digit sum of d_{2k-1} is $126 \times 71 + 122 = 9,068$. Since the digit sum of d_2 is 11, the combined digit sum is 9,079. The solution of the following two problems are left to the readers.

Problem 5
Determine which of $10^{1.007} + 3$ and $10^{1.007} - 3$ is divisible by 29.

Problem 6
For what year $y > 2,014$ would the second smallest positive divisor of $10^y - 9$ be
(a) 7 (b) 13 (c) 17 (d) 19 (e) 23?

16. Let L be the length of one side of the square.



Using Pythagorean property, we have $2L^2 = 49.5^2$. This gives $L^2 = 1225.125$, $L = 35$cm

As well, knowing that 3 times the area of ΔPCB is equal to L^2, we have $3\left(\frac{\overline{PN} \times 35}{2}\right) = 35^2$, or $\overline{PN} = 23\frac{1}{3}$ and $\overline{PM} = 11\frac{2}{3}$

It follows that $\overline{PA} = \sqrt{17.5^2 + 11\frac{2}{3}^2} = 21\text{cm}$

17. Let a, b, c be the three digits not necessarily different. As well, we should only consider product that is less than $999 \div 17 = 59$. Since we have the hundreds digit 1 more than the sum of the other two digits, we could use the following table to sort out the three digits.

Therefore, only one such number exists and it is 816.

Alternate solution:

The number is divisible by any of its digits. Using its hundreds digit, the quotient is greater than 100 and less than 111. It is also a multiple of 17, so that it has to be 102. Now $102 = 17 \times 6$. So the last two digits are 1 and 6 or 2 and 3. It is easy to check that 861, 632 and 623 are not multiples of 17 but 816 is.

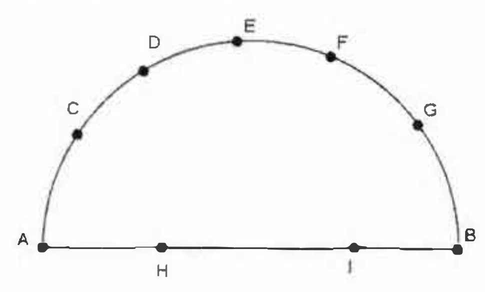
Original number	a	b	c	Product abc
	9	1	7	63
	9	2	6	108
	9	3	5	135
	9	4	4	144
17(48) = 816	8	1	6	48
	8	2	5	80
	8	3	4	96
17(35) = 595	7	1	5	35
17(56) = 952	7	2	4	56
	7	3	3	63
17(24) = 408	6	1	4	24
17(36) = 612	6	2	3	36
17(15) = 255	5	1	3	15
17(20) = 340	5	2	2	20
17(8) =136	4	1	2	8
	3	1	1	3

18. Put all 16 articles of even length first, so that they all start on odd-numbered pages. Of the other 16, half of them will start on odd-numbered pages, for a total of 24. This cannot be higher because an article of odd length changes the parity of the starting page number. This parity must change at least 15 times so that at least 8 articles must start on even-numbered pages.

19. There are six ways to choose two points from the straight line and each pair can form a triangle with each of the points on the curve; this gives $6 \times 5 = 30$ triangles.

There are 10 ways to choose two points from the curve and each pair can form a triangle with each of the points on the straight line; this gives $10 \times 4 = 40$ triangles.

Last, all three vertices can be chosen from the curve alone. There are 10 ways to do so. In total, there are $30 + 40 + 10 = 80$ triangles.



Calgary Junior High Mathematics Competition 2014/15

NAME: ______________________________ GENDER: ______
PLEASE PRINT (First name Last name) (optional)

SCHOOL: ______________________________ GRADE: ______
(9,8,7,...)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer: and PART B long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given. PART A has a total possible score of 45 points.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- Hint: Read all the problems and select those you have the best chance to solve first. You may not have time to solve all the problems.

MARKERS' USE ONLY	
PART A ____ ×5	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.

THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

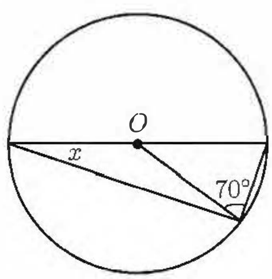
PART A: SHORT ANSWER QUESTIONS (Place answers in the boxes provided)

A1 At a bus station, a bus leaves at 8:00 am and a new bus leaves every 7 minutes after that. At what time does the first bus after 9:00 am leave? A1

A2 If we mix one litre of lemonade that contains 4% lemon with two litres of lemonade that contains 10% lemon, what is the percentage of lemon in the resulting three litre mixture? A2

A3 At the swimming pool last week, on each day there were ten fewer people than twice the number of people on the previous day. There were 130 people at the pool on Friday. How many people were at the pool on the previous Tuesday? A3

A4 Given the circle below with centre O, find the angle x in degrees. A4

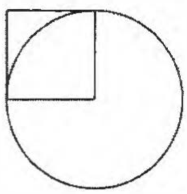


A5 Firmamint boxes of chocolates contain 17 with hard centres and 5 with soft centres. Sweetart boxes contain 7 with hard centres and 11 with soft centres. If I buy one Firmamint box, how many Sweetart boxes must I buy so that the total number of hard centres is equal to the total number of soft centres? A5

A6 Sagal and Xi leave home at the same time to walk to the park which is 6 km away. Sagal walks at 1 km/hr for 2 km, then at 2 km/hr for 4 km. Xi walks at 1 km/hr for 4 km, then at 2 km/hr for 2 km. Sagal arrives at the park at noon. At what time does Xi arrive?

A6

A7 In the following figure the square has one corner in the centre of the circle and two sides are tangent to the circle. How many times larger is the area of the circle than the area of the square?



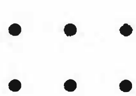
A7

A8 Below, the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are to be filled into the nine smaller squares so that every number is used exactly once. If the sum of each row and the sum of each column is at most 15, what must the value of x be?

	x	
		7
8		

A8

A9 How many triangles (with positive area) are there which have their three corners as points chosen from the 2×3 grid shown?



A9

PART B: LONG ANSWER QUESTIONS

B1 An Egyptian grid is a square of numbers so that all numbers in the outside ring are 1's, all numbers in the next inner ring are 2's, all numbers in the next inner ring are 3's, and so forth. The following are the Egyptian grids of sizes 1, 2, 3, 4, 5, 6, respectively. What is the sum of the entries of an Egyptian grid of size 9? The answer should be given as a whole number.

$$1 \qquad \begin{matrix}1&1\\1&1\end{matrix} \qquad \begin{matrix}1&1&1\\1&2&1\\1&1&1\end{matrix} \qquad \begin{matrix}1&1&1&1\\1&2&2&1\\1&2&2&1\\1&1&1&1\end{matrix} \qquad \begin{matrix}1&1&1&1&1\\1&2&2&2&1\\1&2&3&2&1\\1&2&2&2&1\\1&1&1&1&1\end{matrix} \qquad \begin{matrix}1&1&1&1&1&1\\1&2&2&2&2&1\\1&2&3&3&2&1\\1&2&3&3&2&1\\1&2&2&2&2&1\\1&1&1&1&1&1\end{matrix}$$

B2 Archibald runs round a 300 metre circular race track at 7 km/hr, while Beauregard runs at 8 km/hr. Suppose they start at the same time at the same place, but run in opposite directions.

(a) How long in minutes will it be before they first meet?

(b) If they keep running, will they ever meet at the point where they started, and if so, after how many minutes?

B3 There are 2015 balls in 1000 boxes.

(a) Each box contains 1, 2, or 3 balls.

(b) The number of boxes containing exactly one ball is greater than 308.

(c) The total number of balls in boxes containing more than one ball is greater than 1705.

How many boxes contain exactly 1, 2, and 3 balls, respectively?

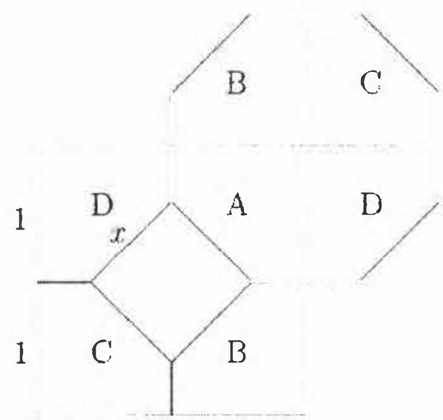
B4 A **preven number** is an integer that uses each digit in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ at most once, both starts and ends with a single digit that is prime or even, and each pair of consecutive digits forms a two-digit number which is prime or even.

For example, 8347 is preven since its first digit is even, its last digit is prime, and any two consecutive digits (83, 34, 47) are either even or prime. On the other hand, 8743 is not preven since 87 is neither even nor prime. The number 8343 is also not preven since it has a repeated digit.

(a) Find a four-digit preven number larger than 8347. The larger your four-digit preven number is, the more marks you may earn.

(b) Find a preven number which is as large as possible. The larger your number is, the more marks you may earn.

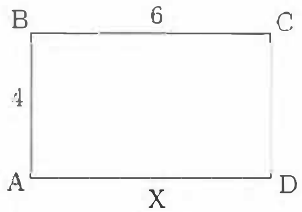
B5 A square with edge length 2 is cut into five pieces: a square of edge length x, and four congruent pieces, A, B, C, and D which are reassembled to form an octagon which is regular, that is, has all its eight edges equal in length.



(a) What is x?

(b) Which piece has larger area: the square with edge length x or the piece labelled by A?

B6 Ellie is on her side of the tennis court (which is a 4 metres by 6 metres rectangle ABCD), practising serving from the midpoint X of the baseline AD. When there are three balls lying in her court she walks in straight lines to pick them up, from X to one ball, then to a second ball, then to the third ball and back to X. For example, if there were two balls at B and one at C, she could travel XBBCX for a total distance of 5+0+6+5=16 metres, or she could go XBCBX for a distance of 5+6+6+5=22 metres.



(a) Suppose the three balls are at points A, B and C. What is the shortest distance Ellie could walk to pick up the three balls? What is the longest distance Ellie could walk to pick up the three balls?

(b) Find places on the court for the three balls to be located so that the ratio

$$\frac{\text{longest distance Ellie could walk}}{\text{shortest distance Ellie could walk}}$$

is at least 1.5. The larger a ratio you find, the better your mark will be. (For extra credit, prove that your ratio is as large as possible.)

Solutions (Part A)

A1 At a bus station, a bus leaves at 8:00 am and a new bus leaves every 7 minutes after that. At what time does the first bus after 9:00 am leave?

A1
9:03 AM (or 9:03)

Solution: The following buses leave at 8:07, 8:14, 8:21, 8:28, 8:35, 8:42, 8:49, 8:56, 9:03.

A2 If we mix one litre of lemonade that contains 4% lemon with two litres of lemonade that contains 10% lemon, what is the percentage of lemon in the resulting three litre mixture?

A2
8%

Solution: There is 0.24 litres of lemon in the three litre mixture. Thus, the mixture contains $0.24/3 = 0.08$ or 8% lemon.

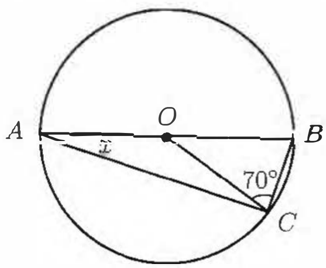
A3 At the swimming pool last week, on each day there were ten fewer people than twice the number of people on the previous day. There were 130 people at the pool on Friday. How many people were at the pool on the previous Tuesday?

A3
25

Solution: On Thursday there were $(130 + 10)/2 = 70$ people. On Wednesday there were $(70 + 10)/2 = 40$ people. On Tuesday there were $(40 + 10)/2 = 25$ people.

A4 Given the circle below with centre O, find the angle x in degrees.

A4
20°



Since $OA = OB = OC$, triangles OAC and OBC are isosceles. Hence, $\angle OCA = x$ and $\angle OBC = 70°$. As $\angle COB = 40°$, we have $\angle AOC = 140°$. Thus, $x + x = 40°$ implying $x = 20°$.

A5 Firmamint boxes of chocolates contain 17 with hard centres and 5 with soft centres. Sweetart boxes contain 7 with hard centres and 11 with soft centres. If I buy one Firmamint box, how many Sweetart boxes must I buy so that the total number of hard centres is equal to the total number of soft centres?

A5
3

Solution 1: The difference of 12 between the numbers of hard centres and soft centres in the Firmamint box must be made up by the Sweetart boxes, which can reduce the difference by 4 per box, so 3 boxes of Sweetarts are required.

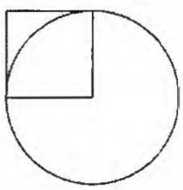
Solution 2: f Firmamint boxes contain $17f$ hard and $5f$ soft, while s Sweetart contain $7s$ hard and $11s$ soft. If $17f + 7s = 5f + 11s$, then $12f = 4s$ and you need exactly 3 times as many Sweetart as Firmamint.

A6 Sagal and Xi leave home at the same time to walk to the park which is 6 km away. Sagal walks at 1 km/hr for 2 km. then at 2 km/hr for 4 km. Xi walks at 1 km/hr for 4 km. then at 2 km/hr for 2 km. Sagal arrives at the park at noon. At what time does Xi arrive?

A6
1:00 PM
(or 1:00)
(or 13:00)

A7 In the following figure the square has one corner in the centre of the circle and two sides are tangent to the circle. How many times larger is the area of the circle than the area of the square?

A7
π
(or 3.14)



Solution: The answer is π and can be calculated assuming that the radius is r and then dividing the area of the circle by the area of the square, $\frac{\pi r^2}{r^2}$.

A8 Below, the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are to be filled into the nine smaller squares so that every number is used exactly once. If the sum of each row and the sum of each column is at most 15, what must the value of x be?

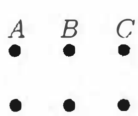
A8
9

	x	
		7
8		

Solution: Observe that if the number 9 is placed in any small square other than x then we have a row or column that sums to at least 16.

A9 How many triangles (with positive area) are there which have their three corners as points chosen from the 2×3 grid shown?

A9
18



Solution: There must be exactly two vertices from either the top row or the bottom row. If there are two vertices from the top row, we could have AB, AC or BC, any of which could be matched with one of the three vertices in the bottom row. This gives $3 \times 3 = 9$ distinct triangles. By symmetry, using two vertices in the bottom row and one in the top row gives 9 triangles. Thus, there are $9 + 9 = 18$ triangles in total.

Solutions (Part B)

B1 An Egyptian grid is a square of numbers so that all numbers in the outside ring are 1's, all numbers in the next inner ring are 2's, all numbers in the next inner ring are 3's, and so forth. The following are the Egyptian grids of sizes 1, 2, 3, 4, 5, 6, respectively. What is the sum of the entries of an Egyptian grid of size 9? The answer should be given as a whole number.

$$\begin{matrix}1\end{matrix} \quad \begin{matrix}1&1\\1&1\end{matrix} \quad \begin{matrix}1&1&1\\1&2&1\\1&1&1\end{matrix} \quad \begin{matrix}1&1&1&1\\1&2&2&1\\1&2&2&1\\1&1&1&1\end{matrix} \quad \begin{matrix}1&1&1&1&1\\1&2&2&2&1\\1&2&3&2&1\\1&2&2&2&1\\1&1&1&1&1\end{matrix} \quad \begin{matrix}1&1&1&1&1&1\\1&2&2&2&2&1\\1&2&3&3&2&1\\1&2&3&3&2&1\\1&2&2&2&2&1\\1&1&1&1&1&1\end{matrix}$$

Solution.
The answer is 165. One could brute force it or use $1^2 + 3^2 + 5^2 + 7^2 + 9^2$ or $4[8(1) + 6(2) + 4(3) + 2(4)] + 5 = 16(2 + 3 + 3 + 2) + 5 = 165$.

B2 Archibald runs round a 300 metre circular race track at 7 km/hr, while Beauregard runs at 8 km/hr. Suppose they start at the same time at the same place, but run in opposite directions.

(a) How long in minutes will it be before they first meet?

(b) If they keep running, will they ever meet at the point where they started, and if so, after how many minutes?

Solution.
Their relative speed is 15 kph, so they will cover 0.3 km in 0.3/15 hours, or 1.2 minutes, Archibald having covered 140 metres and Beauregard 160 metres. They will meet every 1.2 minutes, and $140n$ and $160n$ will be exact multiples of 300 just when $7n$ and $8n$ are multiples of 15. So they meet at their starting point for the first time, when $n = 15$, after $15 \times 1.2 = 18$ minutes.

B3 There are 2015 balls in 1000 boxes.

(a) Each box contains 1, 2, or 3 balls.

(b) The number of boxes containing exactly one ball is greater than 308.

(c) The total number of balls in boxes containing more than one ball is greater than 1705.

How many boxes contain exactly 1, 2, and 3 balls, respectively?

Solution.
There are at least 309 boxes with exactly 1 ball, and at least 1706 balls in boxes containing 2 or 3 balls. This accounts for at least $309 + 1706 = 2015$ balls, which is all of them. So in fact there must be exactly 309 boxes containing one ball, and exactly 1706 balls in the other $1000 - 309 = 691$ boxes. If each of these 691 boxes contained just 2 balls, that would account for $2 \times 691 = 1382$ balls, leaving $1706 - 1382 = 324$ balls to be the third ball in some of these 691 boxes. So 309 boxes have just one ball, 324 boxes have three balls, and the remaining $691 - 324 = 367 (= 1000 - 309 - 324)$ boxes have two balls.

B4 A **preven number** is an integer that uses each digit in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ at most once, both starts and ends with a single digit that is prime or even, and each pair of consecutive digits forms a two-digit number which is prime or even.

For example, 8347 is preven since its first digit is even, its last digit is prime, and any two consecutive digits (83, 34, 47) are either even or prime. On the other hand, 8743 is not preven since 87 is neither even nor prime. The number 8343 is also not preven since it has a repeated digit.

(a) Find a four-digit preven number larger than 8347. The larger your four-digit preven number is, the more marks you may earn.

(b) Find a preven number which is as large as possible. The larger your number is, the more marks you may earn.

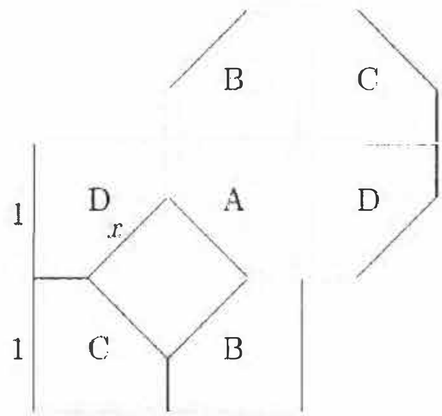
Solution.
For reference, the two-digit primes are

$$\{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}.$$

(a) The largest is 8976, followed by 8974, 8973, 8972, 8967, 8964 and 8962.

(b) Any nine-digit preven number must begin with 5 since no two-digit prime or even number ends in 5. The largest possible is 598674312, followed by 598674132, 598674123, 598673412, 598673142 and 598673124.

B5 A square with edge length 2 is cut into five pieces: a square of edge length x, and four congruent pieces, A, B, C, and D which are reassembled to form an octagon which is regular, that is, has all its eight edges equal in length.



(a) What is x?

(b) Which piece has larger area: the square with edge length x or the piece labelled by A?

Solution.

(a) Four edges of the octagon (the sloping ones) are of length x. The other four are of length $2 - x\sqrt{2}$. Since the octagon is regular, we have $x = 2 - x\sqrt{2}$. Therefore,

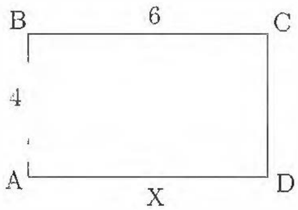
$$x = 2/(1+\sqrt{2}) = 2(\sqrt{2}-1) \approx 0.8284271.$$

(b) The area of one of the four congruent pieces is

$$\begin{aligned} \frac{1}{4}(4 - x^2) &= 1 - \left(\frac{x}{2}\right)^2 \\ &= 1 - (2 - 2\sqrt{2} + 1) \\ &= 1 - 2 + 2\sqrt{2} - 1 \\ &= 2(\sqrt{2} - 1) \\ &= x \end{aligned}$$

while the area of the square is x^2. Since $x < 1$, we have $x^2 < x$, so the square is smaller in area.

B6 Ellie is on her side of the tennis court (which is a 4 metres by 6 metres rectangle ABCD), practising serving from the midpoint X of the baseline AD. When there are three balls lying in her court she walks in straight lines to pick them up, from X to one ball, then to a second ball, then to the third ball and back to X. For example, if there were two balls at B and one at C, she could travel XBBCX for a total distance of 5+0+6+5=16 metres, or she could go XBCBX for a distance of 5+6+6+5=22 metres.



(a) Suppose the three balls are at points A, B and C. What is the shortest distance Ellie could walk to pick up the three balls? What is the longest distance Ellie could walk to pick up the three balls?

(b) Find places on the court for the three balls to be located so that the ratio

$$\frac{\text{longest distance Ellie could walk}}{\text{shortest distance Ellie could walk}}$$

is at least 1.5. The larger a ratio you find, the better your mark will be. (For extra credit, prove that your ratio is as large as possible.)

Solution.
(a) There are six paths to check: XABCX. XACBX. XBACX, XBCAX. XCABX. XCBAX. The shortest is length 18m. The longest is length $14 + 2\sqrt{13}$.

(b) We can get as close to 2 as we like by taking two balls close to A (or close to any other point) and one close to X.

To prove this, let XABCX be the minimum-length route, where X is Ellie's starting and ending point, and A,B,C are the positions of the three balls. Let XDEFX be the maximum-length route (or any route, actually), where A,B,C=D,E,F. Let e be one of the edges XD, DE, EF, FX. Then e is either (i) an edge of XABCX or (ii) connects X to B or A to C. If (i), then the length of e is at most half the length of XABCX. Suppose (ii), and that e connects X to B (wlog). Then the length of e is at most the smaller of the two paths XAB and XCB, and thus is at most half the length of XABCX. This holds for each of the four edges of XDEFX, thus the entire route XDEFX is at most twice as long as XABCX, as claimed.

The Odds Are You Will Relate to the Book, *Struck by Lightning*, by Jeffrey S Rosenthal

Reviewed by Cheyenne Murphy

The book *Struck by Lightning: The Curious World of Probabilities*, by Jeffrey S Rosenthal, was an entertaining and thoughtful book. The book has 17 relatively short chapters and includes a "final exam" at the end of the book that measures your probability perspective. The book's 17 chapters are short and easy to read, tying one chapter to the next in a cohesive manner. The key concept of this book is probability, the odds and oddities of chance and randomness in our everyday life. The author gives an entertaining yet sophisticated look at the world of probabilities. Jeffrey S Rosenthal explains the world of probabilities, the mechanics of randomness in everyday life and teaches us how to develop our own perspective on probability.

This book contains information to strengthen our decision-making skills by examining how statistics and statistical analysis are used in our daily lives. We see statistical analysis of some sort in the everyday decisions that we make and especially in the media. This book dug deeply into the statistical analysis that is behind medical studies and how it can be skewed to make us believe what they want us to believe in order to want their product. As the authors of "Statistical Errors in Medical Research—A Review of Common Pitfalls" (2007) stated "standards in the use of statistics in medical research are generally low." Also, that a "growing body of literature points to persistent statistical errors, flaws and deficiencies in most medical journals." Some of these flaws and specific examples are examined in this book, which made it easy to understand this issue.

I really enjoyed the chapter that discussed crime rates and trends. This chapter was very informative, and I found that I agreed with the author when he talked about how we need to know what we are reading when charts and other data analysis are used in a media story. This is something that many people seem to miss and can easily affect public opinions. He uses easy-to-read language that can be understood by most people, even the mathematically challenged. I did not find anything in this book that I completely disagreed with. Each chapter examined different probability aspects relating to the topic being discussed, and each chapter made sense to me. I especially like how the author uses humour in many cases to help explain the points he is making in a chapter. The author puts these humorous sections in as asides, which really adds to the experience of reading and relating to the topics being discussed.

This is a good, quick-read sort of math book that would be appropriate for anyone who wants to learn more about randomness, the world of probabilities and statistical analysis. I think that anyone in Grade 9 or higher who is interested in these topics would get the greatest value from this book. This book will allow those who read it to increase their knowledge of these topics through easy-to-read language and without all the math equations that could be involved. It is comprehensive yet clear, which should appeal to a vast audience.

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Strasak, A, Q Zaman, K Pfeiffer, G Gobel and H Ulmer. 2007. "Statistical Errors in Medical Research—A Review of Common Pitfalls." *Swiss Medical Weekly* 137, 44–49. http://people.stat.sfu.ca/~tloughin/Stat Errors in Medicine.pdf (accessed December 1, 2014).

Everyday Mathematics—Book Review of *Life by the Numbers*, by Keith Devlin

Reviewed by Katelyn Valkenburg

Throughout our universe, there is an unseen world of mathematics, an idea that Keith Devlin calls "the invisible universe," the main concept of Keith Devlin's *Life by the Numbers* (Devlin 1998, 1). Devlin shows us the invisible universe of mathematics and how crucial it is to our everyday lives. The curiosity, imagination and creativity of people around the world, throughout the centuries, helped to get us to where we are today. As this book is based on the TV miniseries, *Life by the Numbers* (Plonka 1998), Devlin used the same categorization of mathematical themes for his chapters but was also able to add and expand on the concepts in the show. The first chapter, one of the easiest to follow, gives a good overview of the book, but is very concise with mathematical concepts. The other themes in this book consist of the connections between mathematics, art and special effects, patterns in nature, medicine, sports and training, discovering our world through creating maps, exploring our universe, probability in casino games, insurance, economic predictions and technology. These themes that affect our everyday lives are explained with mathematics: "[humans] use mathematics to investigate things that the eye cannot see" (p 2). Therefore, we could say that mathematics gives us the means to help us understand our world.

I found *Life by the Numbers* to be a very approachable and easy read. This book is straightforward and without mathematical jargon. The mathematical concepts were explained with enough detail that most readers could follow along. Mathematical equations were not used to try to explain concepts, which is what makes this book so approachable. Devlin (1998) explicitly states that throughout the book "you won't find any formulas or problems" (p vii). As someone who enjoys mathematics, I found this book intriguing. However, since it does not have any symbolic mathematics, some of the concepts left me wanting mathematical proof. Based on the purpose of this book, I understand why Devlin did not include any mathematical equations or proofs. The fact that there are no mathematical symbols or equations could be the biggest strength of *Life by the Numbers* because someone who isn't mathematically inclined can still follow along without getting lost in the equations, which was Devlin's intent. *Life by the Numbers* may inspire those who enjoy mathematics to further explore the concepts presented. Since this book does not go into great depth about the concepts, you may need to seek outside sources. Devlin has included a section in the back to get you started with further readings.

Similar to Devlin, Brough and Calder (2014) also believe that "mathematics is everywhere" (p 9). In *Making Mathematics Meaningful*, Brough and Calder (2014) discuss the importance of using real-life examples to teach mathematics. Real-life examples make learning mathematics less abstract and more tangible because readers can make connections to their own lives (p 9). As a teacher candidate, my eyes were opened to the importance of introducing real-world applications when teaching mathematics. The book explains that most things have either a mathematical explanation or use mathematics to make them work, so after reading it, I learned about a wide range of examples of where mathematics applies in our everyday lives, which I can bring with me into the classroom. Mathematics is at work behind the composition of a work of art, the creation of a movie that you enjoyed recently and that amazing curveball that causes a batter to strike out in a baseball game. Knowing the mathematics behind our everyday lives helps us see patterns. We can also use mathematics to discover new patterns. Some of the concepts that Devlin discusses are much more advanced than those taught in primary and lower secondary school. Therefore, using some of these concepts with young children may not be useful or appropriate.

After reading this book, I agree with Devlin's opinion that mathematics is an invisible universe. Mathematics is in our everyday lives, but we don't see it explicitly. Devlin shows his readers that mathematics isn't just about the numbers, it is about the patterns that are all around us; numbers are just a means by which we can make sense of those patterns. I think anxiety about mathematics comes from the confusion of trying to decipher numbers and symbols used as representations for abstract concepts. Devlin (1998) makes a great comparison between mathematics and music in his first chapter. In the same way that mathematics uses symbols to represent abstract concepts, music has symbolic representation through musical notes. Our minds require some type of tangible way to represent things; that is why mathematics exists. If there wasn't some kind of standard way to communicate our thoughts or make our abstract ideas more tangible, it would be difficult to express ourselves (pp 16–19). Throughout this book, we see the vast number of patterns that now make sense to us through the symbolic representation of mathematics.

I agree with Devlin (1998) in that this book is for anyone who believes "that mathematics has little to do with life," "that mathematics is just about numbers," "that mathematics was all worked out centuries ago" or for anyone who is "curious about life" and is willing to get a taste of what is inside the invisible universe (p vii). I would like to also add that this book is for any educator or mathematician. It will aid in understanding the mathematics that is all around us, as well as possibly inspire further learning about certain topics.

References

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YouCubed: Broadening the Conversation for Supporting Student Success in Mathematics

Holly Henderson Pinter

YouCubed is a new, nonprofit organization with a mission to revolutionize the way mathematics is taught. Spearheaded by Stanford professor of mathematics education Jo Boaler, YouCubed provides free and affordable K–12 math resources and professional development for educators and parents.

Drawing on current research and best practices, YouCubed is a collaborative effort to empower learners to find the joy of mathematics through engaging in challenging tasks. The website offers short videos appropriate for a broad audience of students, parents and teachers. Examples include short interviews with Boaler describing current research about mathematics education in the United States, including innovations in brain research and how it relates to student understanding; other video clips capture research-proven pedagogical practices in action such as the use of math talks (Richardson 2011), allowing parents to see in practice what students may be experiencing in classrooms. These videos focus on the big ideas of teaching and learning mathematics and provide support for the kinds of tasks and instructional strategies used in standards-based classrooms.

Beyond laying the foundation of information, YouCubed offers challenging mathematical tasks spanning the entire K–12 curriculum. For example, at the elementary school level, a series of activities and tasks related to geometric reasoning and patterns is posed as an engaging, challenging and yet accessible sequence to help elementary school students grapple with mathematical concepts aligned to Common Core State Standards (CCSSM) (CCSSI 2010). The tasks offered in this section are constructed with classroom implementation in mind and provide pacing and necessary materials.

YouCubed's next section targets engagement in mathematics beyond the classroom through contextual real-life examples. Providing contexts where mathematics is used in the real world has been found to be effective in terms of motivating students as well as allowing them to make important mathematical connections (Hand 2012). This section of YouCubed will provide a context for conversation about real-life mathematical connections for elementary school students.

YouCubed's last component is a resource section where parents can find ideas for working with students at home as well as Boaler's 12 steps for helping students learn mathematics, a document that includes tips on how to appropriately encourage students to stay motivated in mathematics as well as simple pedagogical strategies to use at home in helping students work through math homework.

Potentially its most significant attribute, YouCubed focuses on broadening the conversation about mathematical understanding beyond the walls of the classroom. The implementation of CCSSM across much of the nation has forced a shift in the way parents and teachers converse about mathematics and how to help students succeed. Such resources as YouCubed can be instrumental in fostering the ongoing conversation by laying a foundation of knowledge to help parents understand the how and why of CCSSM objectives.

Knowledge is power. By building the knowledge base, all stakeholders can be successful in this continuing journey.

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Diversity • Equity • Human Rights Diversity • Equity • Human Rights

Specialist councils' role in promoting diversity, equity and human rights

Alberta's rapidly changing demographics are creating an exciting cultural diversity that is reflected in the province's urban and rural classrooms. The new landscape of the school provides an ideal context in which to teach students that strength lies in diversity. The challenge that teachers face is to capitalize on the energy of today's intercultural classroom mix to lay the groundwork for all students to succeed. To support teachers in their critical roles as leaders in inclusive education, in 2000 the Alberta Teachers' Association established the Diversity, Equity and Human Rights Committee (DEHRC).

DEHRC aims to assist educators in their legal, professional and ethical responsibilities to protect all students and to maintain safe, caring and inclusive learning environments. Topics of focus for DEHRC include intercultural education, inclusive learning communities, gender equity, UNESCO Associated Schools Project Network, sexual orientation and gender variance.

Here are some activities the DEHR committee undertakes:

- Studying, advising and making recommendations on policies that reflect respect for diversity, equity and human rights
- Offering annual Inclusive Learning Communities Grants (up to $2,000) to support activities that support inclusion
- Producing *Just in Time*, an electronic newsletter that can be found at www.teachers .ab.ca; Teaching in Alberta; Diversity, Equity and Human Rights.
- Providing and creating print and web-based teacher resources
- Creating a list of presenters on DEHR topics
- Supporting the Association instructor workshops on diversity

Specialist councils are uniquely situated to learn about diversity issues directly from teachers in the field who see how diversity issues play out in subject areas. Specialist council members are encouraged to share the challenges they may be facing in terms of diversity in their own classrooms and to incorporate these discussions into specialist council activities, publications and conferences.

Diversity, equity and human rights affect the work of all members. What are you doing to make a difference?

Further information about the work of the DEHR committee can be found on the Association's website at www.teachers.ab.ca under Teaching in Alberta, Diversity, Equity and Human Rights.

Alternatively, contact Andrea Berg, executive staff officer, Professional Development, at andrea.berg@ata.ab.ca for more information.

Five great reasons to get (or update) your online ATA account now!

Please encourage teachers you know to get or update their ATA account. Here are some reasons to share with them on why they should do this.

1. Receive bargaining updates

All collective agreements between ATA bargaining units and their respective school jurisdictions expire August 31, 2016. As new negotiations get under way, regular updates and other information critical to keeping you informed of developments regarding bargaining will be posted in the Members Only section of the Association website. To gain access to the Members Only section, you must have an online ATA account.

2. Use ATA library online resources and databases

The ATA library has an extensive collection of books, periodicals, videos and other materials for teachers focusing on educational research and professional development. Library services and materials are available in both French and English. An online ATA account is necessary to log in to our catalogue to reserve books or videos.

3. Vote in ATA elections

Provincial Executive Council conducts the business of the Association and is made up of 20 members, 18 of whom are elected by you. To vote in Association elections, you need an online ATA account.

4. Get no-cost ATA specialist council memberships

As a benefit of ATA membership, active members are entitled to join one specialist council of their choice each year at no cost. Active members must have an online ATA account to select and join the specialist council of their choice.

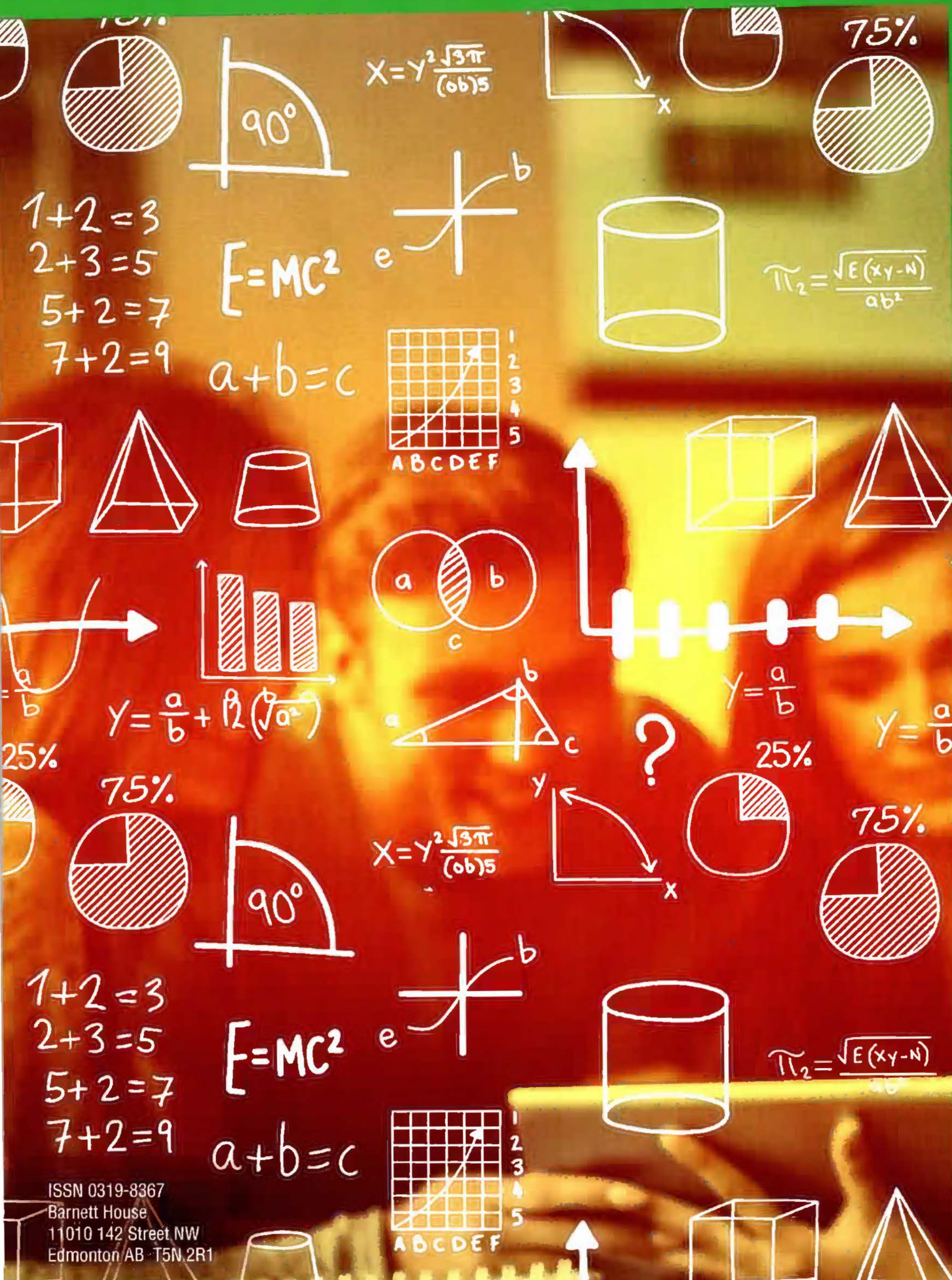
5. Print your own ATA member card

The Association is launching online self-serve membership card distribution. With an online ATA account, you will have the convenience of printing your own ATA member card from the Association website when you need it. ATA member cards are useful when your identification as a teacher or member of an association is needed to attend events, receive discounts or be eligible for other offers from retailers and service providers.

MCATA Executive 2015/16

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.



ISSN 0319-8367
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