Edmonton Junior High Mathematics Competition 2014/15

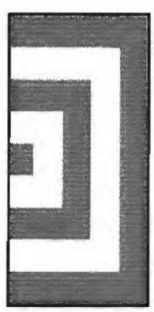
Part A: Multiple Choice

Each correct answer is worth four points. Each unanswered question is worth two points to a maximum of three unanswered questions.

1. Which of these numbers is greater than its reciprocal?
   (a) $-1.\overline{5}$
   (b) 0.995
   (c) -99.9%
   (d) $0.\overline{3}$
   (e) $\frac{2}{5}$

2. What number is doubled when $\frac{3}{4}$ of it is subtracted from 99?
   (a) 32
   (b) 36
   (c) 40
   (d) 44
   (e) 52

3. A target is made of dark and white strips of equal width as shown at the right. If a dart is thrown and lands randomly inside the target, what is the probability that it will land on white?



   (a) $\frac{2}{5}$
   (b) $\frac{3}{8}$
   (c) $\frac{4}{9}$
   (d) $\frac{1}{2}$
   (e) $\frac{1}{3}$

4. How many two-digit whole numbers less than 40 are divisible by the product of its digits?
   (a) 5
   (b) 4
   (c) 3
   (d) 2
   (e) More than 5

5. A florist has 72 roses, 90 tulips and 60 daffodils, and uses all of them to make as many identical bouquets as possible. How many flowers does the florist put in each bouquet?
   (a) 6
   (b) 18
   (c) 24
   (d) 29
   (e) 37

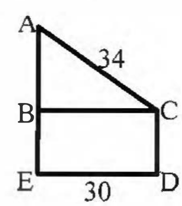
Part B: Short Answer

6. A rectangle has an area of 48 cm^2 and a perimeter of 28 cm. What is the length of the rectangle's diagonal, rounded to the nearest whole centimetre?

7. When a two-digit number is multiplied by the sum of its digits, the product is 952. What is the two-digit number?

8. Twenty-six people are seated in a circle and are lettered alphabetically from A to Z. Beginning with person A and proceeding in a clockwise direction, each alternate person leaves the circle. What is the letter of the last person to leave?

9. In the rectangle BCDE, BC = 30 cm. A is on the extension of EB, and AC=34 cm. The area of triangle ABC is 30 cm^2 less than half of the area of BCDE. What is the perimeter of the quadrilateral ACDE?



10. The age of a tortoise is 52 years more than the combined age of two elephants. In 10 years, the tortoise will be twice as old as the two elephants combined. How old is the tortoise now?

11. The angle bisectors of the two acute angles of obtuse triangle, ΔXYZ, intersect at point W. The measure of $\angle Z$ is 98°. What is the measure, in degrees, of $\angle XWY$?

12. Maria purchased a number of peaches and apples. The mean mass of the peaches is 170 g. The mean mass of the apples is 140 g. The mean mass of all the fruit is 152 g. What is the ratio of the number of peaches to apples purchased?

13. Two sides of a scalene acute triangle measure 12 cm and 13 cm. If the length of the third side is also an integer, then how many lengths are possible for the third side to be?

14. What is the largest n such that n^n is an n-digit number?

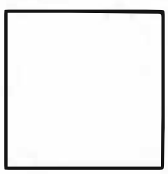
Part C: Short Answer

15. Consider the 2014 digit number consists of 2013 nines followed by 1 one.

$$\overbrace{99\ldots99}^{2013}1$$

The smallest factor is 1 and the largest factor is the number itself. Let M be the second smallest factor and N be the second largest factor. What is the sum of the digits of M and N?

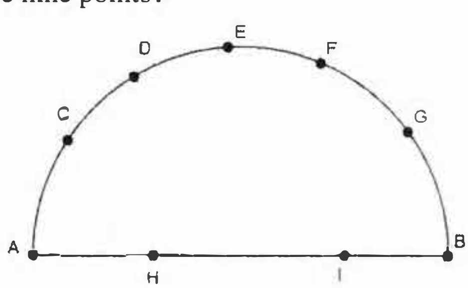
16. $ABCD$ is a square with $AC = 49.5$ cm. P is a point inside $ABCD$ such that $PB = PC$, and the area of triangle PCB is one-third of the area of $ABCD$. What is the length, in cm, of PA? Round your answer off to the nearest integer.



17 A three-digit number is equal to 17 times the product of its digits, and the hundreds digit is 1 more than the sum of the other two digits. Find all such three-digit numbers.

18. A magazine receives 32 articles, of length 1, 2, ..., 32 pages, respectively. The first article starts on page 1 and all other articles start on the page after the preceding article. The articles may be arranged in any order. What is the maximum number of articles that can start on an odd-numbered page?

19. The diagram shows nine points. How many triangles are there whose vertices are chosen from the nine points?



Solutions

1. (a) $= -\frac{14}{9} < \frac{-9}{14}$

(b) $= \frac{199}{200} < \frac{200}{199}$

(c) $= \frac{-999}{1000} > \frac{-1000}{999}$

(d) $= \frac{1}{3} < 3$

(e) $= \frac{2}{5} < \frac{5}{2}$

The correct answer is (c).

2. Let n be the number.

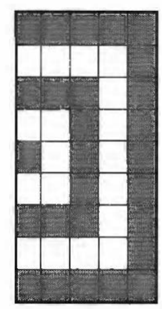
$2n = 99 - \frac{3}{4}(n)$

$\frac{11n}{4} = 99$

$n = 36$

The correct answer is (b).

3. The shape can be divided into 45 individual squares. The white squares are $\frac{18}{45} = \frac{2}{5}$ of the entire target.
The correct answer is (a).



4. There are exactly five of them: 11, 12, 15, 24 and 36.
The correct answer is (a).

5. The GCF (72, 90, 60) = 6. This gives 6 bouquets with 12 roses, 15 tulips and 10 daffodils; a total of 37 flowers in each bouquet.
The correct answer is (e).

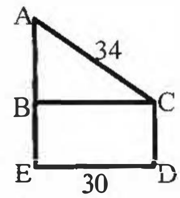
Part B: Short Answer

6. Knowing that L(W) = 48 and L+W = 14, we have L = 8 and W = 6. The diagonal is $\sqrt{8^2 + 6^2} = 10$ cm.

7. The prime factorization of 952 is $2 \times 2 \times 2 \times 7 \times 17$. Two-digit divisors are 14, 17, 28, 34, 56 and 68. Checking all cases, we have 952=68(6+8).

8. After the first round, BDFHJLNPRTVXZ are left, and the next to go is B. After the second round, DHLPTX, and the next to go is H. After the third round, DLT are left and the next to go is L. After the fourth round, DT are left and the next to go is D. After the fifth round, only T is left.

9. In the rectangle BCDE, BC=30 cm. A is on the extension of EB, and AC=34 cm. By Pythagoras' Theorem, AB=16 cm and the area of triangle ABC is 240 cm². Hence the area of BCDE is 2(240+30) or 540 cm², so that CD=18 cm. The perimeter of the trapezoid ACDE is 34+18+30+18+16=116 cm.



10. Suppose the tortoise is x years old and the two elephants together are y years old. Then $x - y = 52$. In 10 years' time; $x + 10 = 2(y + 20)$. Hence $y + 52 = 2y + 30$ so that $y = 22$ and $x = 74$.

11. In degrees, $\angle X$ plus $\angle Y$ is 82, $\angle WXY + \angle WYX$ is 41 and $\angle XWY$ is 139.

12. Let p = number of peaches and a = number of apples.

    This gives $\frac{170p+140a}{p+a} = 152$ or 18p = 12a.

    Thus the ratio of p:a = 2:3.

13. Let ABC be the triangle where AC = 13 cm and BC = 12 cm.

    When AB = 5 cm, we have a right angle triangle at $\angle ABC$.

    When $\angle ACB = 90°$, AB = 17.69.

    We now have 5 < third side < 17.69 so that it is an acute triangle. This gives 12 possible lengths for the third side from 6 to 17 cm. In order to be scalene, we need to eliminate both 12 and 13 from the list; hence, there are a total of 10 possible lengths for the third side.

14. A quick check would reveal that 10^{10} would give a total of 11 digits. In fact when n is greater than 10, the resulting power will always have more than n digits. Thus the greatest $n = 9$ giving 9^9 = 387420489 (9 digits).

Part C: Short Answer

15. First, we know the number is not divisible by 3 as it divides into all the 9s except the last digit of 1.

    As for 7, it will divide into six 9s evenly. The longest string of 9s would be 2010 digits. This leaves 9,991 which 7 do not divide evenly.

    Eleven divides into pairs of 99 but won't divide evenly into 91.

    The number 13 will go into six 9s evenly. Similar to 7, it does not divide evenly into 9,991.

    Seventeen divides evenly into a string of sixteen 9s. Leaving thirteen 9's followed by a 1, which 17 does not divide evenly.

    Nineteen divides evenly into a string of eighteen 9s. Leaving fifteen 9s followed by a 1, which 19 also does not divide evenly.

    Similarly 23 divides evenly into a string of twenty-two 9s. Leaving eleven 9s followed by a 1, which 23 does not divide evenly.

    The next prime number to try is 29. Like the previous quotients, 29 divides evenly into a number made up of a string of twenty-eight 9s. The quotient is 0344827586206896551724137931. The sum of its digits is 126. This also means there are 71 sets of this number giving a total of 1988 digits of 9s. We still have a number with 25 digits of 9s followed by a 1. A total of 26 digits left. Fortunately, 29 multiplied by the quotient less the last two digits of "31" results in exactly 25 digits of 9s followed by a 1. This concludes that 29 divides evenly into the original number.

    To recap, the second smallest factor is 29. The sum of the digits is 2 + 9 = 11.

    The second largest factor is a number of the form

    $$\overbrace{0344827586206896551724137931}^{71\ sets}$$
    $$0344827586206896551724137 9$$

    The sum of the digits is 126 × 72 – 4 = 9068.

    Therefore the total sum of the digits of M and N is 11 + 9,068 = 9,079.

Note from the Committee

Below are several related problems on the 2014-digit number n consisting of all 9s except for a 1 as its last digit.

Problem 1
Prove that n is a composite number.

Problem 2
Prove that n is not a square.
Problem 2 is needed to set up the next problem. A positive integer which is not a square has an even number of positive divisors because they form pairs whose product is n. If n is a square, which means that $\sqrt{n}$ is a positive integer, then it is paired with itself. It counts as only one divisor, making the total number of divisors odd. By problem 2, n has $2k$ positive divisors for some positive integer k, namely, $1 = d_1 < d_2 < \cdots < d_k < d_{k+1} < \cdots < d_{2k-1} < d_{2k} = n$.

Problem 3
Find the combined digit sum of d_k and d_{k+1}.

Problem 4

Find the combined digit sum of d_2 and d_{2k-1}.

Problem 1 was proposed for this year's contest. Since numerical answers were desired, it was intended to be modified as problem 3. However, the problem was worded so that problem 4 became what was actually asked.

Solution to Problem 1

Note that $n = 10^{2.014} - 9 = (10^{1.007})^2 - 3^2 = (10^{1.007} + 3)(10^{1.007} - 3)$. Since each factor is clearly greater than 1, n is a composite number.

Solution to Problem 2

Note that $n = 999... 991 = 999... 9 \times 100 + 91 \equiv 0 + 3 = 3 \pmod 4$ since $100 \equiv 0 \pmod 4$. Since all squares are congruent to 0 or 1 (mod 4), n is not a square.

Solution to Problem 3

Note that $d_k = 10^{1.007} - 3 = 999... 997$ so that its digit sum is $1,006 \times 9 + 7 = 9,061$. On the other hand, $d_{k+1} = 1,000... 003$ so that its digit sum is $1 + 3 = 4$. Hence the combined digit sum is $9,061 + 4 = 9,065$.

The rest of the article is devoted to the solution to problem 4.

Clearly, $d_2 \neq 2$ or 5. By the tests of divisibility, it is neither 3 nor 11. If $d_2 = 7$, then we must have $10^{2.014} \equiv 9 \equiv 2 \pmod 7$. Now $10 \equiv 3 \pmod 7$, $10^2 \equiv 3\times3 = 9 \pmod 7$, $10^3 \equiv 3\times9 = 27 \equiv 6 \pmod 7$, $10^4 \equiv 3 \times 6 = 18 \equiv 4 \pmod 7$, $10^5 \equiv 3 \times 4 = 12 \equiv 5 \pmod 7$ and $10^6 \equiv 3 \times 5 = 15 \equiv 1 \pmod 7$. It is not necessary to go on any further. This is because $2,014 = 335 \times 6 + 4$, so that $10^{2.014} = (10^6)^{335} \times 10^4 \equiv 1^{335} \times 4 = 4 \neq 2$. It follows that $d_2 \neq 7$.

A key step in the above argument is that $10^k \equiv 1 \pmod 7$ for some positive integer k, which happens to be 6. How do we know that such a k always exists, if we replace 7 by another prime number? Let us understand why $k = 6$ for the prime number 7. Suppose we wish to convert the fraction $\frac{1}{7}$ into a decimal. By long division, we find that $\frac{1}{7} = 0.\overline{142857}$, a decimal expansion consisting of repeating blocks of the six digits 142857.The reason that there are six digits is that when we divide by 7, the only possible remainders are 0, 1, 2, 3, 4, 5 and 6. Here 0 will not appear since no power of 10 is divisible by 7. By the time we have seen each of the non-zero remainders once, repetition must start. Thus the repeating block of decimal digits has length at most 6. In this case, it happens to be exactly 6. This means that $\frac{1}{7} = \frac{142857}{999999}$ so that 999999 is divisible by 7. It follows that $10^6 \equiv 1$.

In a similar manner, we can prove that $d_2 \neq 13$, 17, 19 or 23. We know that $10^{12} \equiv 1 \pmod{13}$, $10^{16} \equiv 1 \pmod{17}$, $10^{18} \equiv 1 \pmod{19}$ and $10^{22} \equiv 1 \pmod{23}$. As it turns out, $10^6 \equiv 1 \pmod{13}$, but the other powers, namely, 16, 18 and 22, cannot be reduced. Since $10^4 \not\equiv 9 \pmod{13}$, $d_2 \neq 13$.

Now $2,014 = 125 \times 16 + 14$ but $10^{14} \not\equiv 9 \pmod{17}$, $2,014 = 111 \times 18 + 16$ but $10^{16} \not\equiv 9 \pmod{19}$, and $2,014 = 91 \times 22 + 12$ but $10^{12} \not\equiv 9 \pmod{23}$. Hence $d_2 \neq 17$, 19 or 23.

The next candidate for d_2 is 29. We know that $10^{28} \equiv 1$, but perhaps one of 10^2, 10^4, 10^7 and 10^{14} may be too. In modulo 29, we have $10^2 = 100 \equiv 13$, $103 \equiv 10 \times 13 = 130 \equiv 14$, $10^4 \equiv 10 \times 14 = 140 \equiv 24$, $10^7 \equiv 14 \times 24 = 336 \equiv 17$ and $10^{14} \equiv 17^2 = 289 \equiv 28$. So this does not happen. Since $2,014 = 28 \times 71 + 26$, what we need is $10^{26} \equiv 9$ Now $10^5 \equiv 24 \times 10 = 240 \equiv 8$, $10^{25} \equiv 8^5 = 32,768 \equiv 27$ and $10^{26} \equiv 10 \times 27 = 270 \equiv 9$. This is exactly what we want. We are lucky that $n = 10^{2.014} - 9 = (10^{1.007} + 3)(10^{1.007} - 3)$ has a prime factor as small as 29. Each of $10^{1.007} + 3$ and $10^{1.007} - 3$ has more than 1,000 digits. Even if they were not prime numbers, they could have been products of prime numbers with over 500 digits. It would be very diffcult to find d_2 then. From $d_2 = 29$, we have $d_{2k-1} = \frac{n}{29}$. There remains only the trivial matter of determining their combined digit sums, via the following long division:

$$\frac{9999999\ 999999\ 9999999\ 9999999}{29}$$

344827586206896551724137931

The sum of the digits of the quotient is 126, and there are 71 such blocks. In the last incomplete block, the quotient is without the last 2 digits 3 and 1. It follows that the digit sum of d_{2k-1} is $126 \times 71 + 122 = 9,068$. Since the digit sum of d_2 is 11, the combined digit sum is 9,079. The solution of the following two problems are left to the readers.

Problem 5

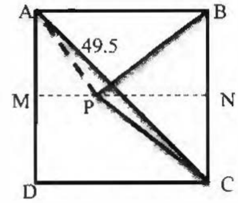
Determine which of $10^{1.007} + 3$ and $10^{1.007} - 3$ is divisible by 29.

Problem 6

For what year $y > 2,014$ would the second smallest positive divisor of $10^y - 9$ be

(a) 7 (b) 13 (c) 17 (d) 19 (e) 23?

16. Let L be the length of one side of the square.



Using Pythagorean property, we have $2L^2 = 49.5^2$. This gives $L^2 = 1225.125$, $L = 35$cm

As well, knowing that 3 times the area of ΔPCB is equal to L^2, we have $3\left(\frac{\overline{PN}\times35}{2}\right) = 35^2$, or $\overline{PN} = 23\frac{1}{3}$ and $\overline{PM} = 11\frac{2}{3}$

It follows that $\overline{PA} = \sqrt{17.5^2 + 11\frac{2}{3}^2} = 21\,\text{cm}$

17. Let a, b, c be the three digits not necessarily different. As well, we should only consider product that is less than $999 \div 17 = 59$. Since we have the hundreds digit 1 more than the sum of the other two digits, we could use the following table to sort out the three digits.

Therefore, only one such number exists and it is 816.

Alternate solution:

The number is divisible by any of its digits. Using its hundreds digit, the quotient is greater than 100 and less than 111. It is also a multiple of 17, so that it has to be 102. Now $102 = 17 \times 6$. So the last two digits are 1 and 6 or 2 and 3. It is easy to check that 861, 632 and 623 are not multiples of 17 but 816 is.

Original number	a	b	c	Product abc
	9	1	7	63
	9	2	6	108
	9	3	5	135
	9	4	4	144
17(48) = 816	8	1	6	48
	8	2	5	80
	8	3	4	96
17(35) = 595	7	1	5	35
17(56) = 952	7	2	4	56
	7	3	3	63
17(24) = 408	6	1	4	24
17(36) = 612	6	2	3	36
17(15) = 255	5	1	3	15
17(20) = 340	5	2	2	20
17(8) =136	4	1	2	8
	3	1	1	3

18. Put all 16 articles of even length first, so that they all start on odd-numbered pages. Of the other 16, half of them will start on odd-numbered pages, for a total of 24. This cannot be higher because an article of odd length changes the parity of the starting page number. This parity must change at least 15 times so that at least 8 articles must start on even-numbered pages.

19. There are six ways to choose two points from the straight line and each pair can form a triangle with each of the points on the curve; this gives $6 \times 5 = 30$ triangles.

There are 10 ways to choose two points from the curve and each pair can form a triangle with each of the points on the straight line; this gives $10 \times 4 = 40$ triangles.

Last, all three vertices can be chosen from the curve alone. There are 10 ways to do so. In total, there are $30 + 40 + 10 = 80$ triangles.