Addressing the Misconceptions of the Equal Sign

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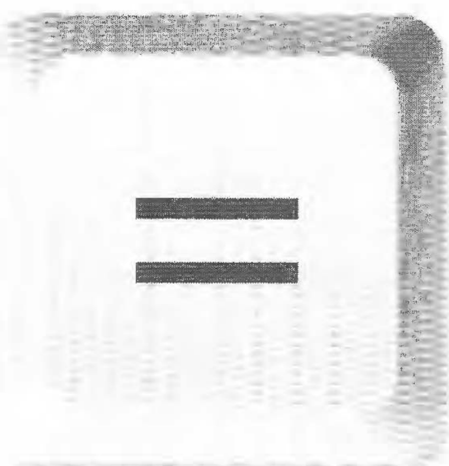

Throughout my eight years of teaching mathematics in the Grades 4 and 5 classroom, I have been consistently surprised at areas of the curriculum where my students have been challenged. Every year I would give lessons on solving equations using larger numbers in the traditional vertical or horizontal format that can be found in most classrooms. Typically after reviewing the steps for solving these equations, my students would become quite competent solving a variety of equations with 3- or 4-digit numbers. I always felt I was a very successful mathematics teacher because of the success my students achieved until we started the Patterns and Relations unit and more specifically the lessons on variables and equations. Year after year, I was caught off guard by students who could easily solve the equation 42×23, but these same students could not immediately identify the missing number in $56 = \square \times 8$. Naturally, I was concerned about the confusion my students encountered regarding basic fact equations. My curiosity prompted me to investigate the available research on what misconceptions exist when students are introduced to pre-algebraic thinking. I was surprised to find a deep body of literature that examines how students understand the equal sign.

Background

The understanding of equivalency is a key concept that is essential for all levels of mathematics from early elementary grades to university programs. Ma (1999) refers to one of her teachers who considered the equal sign "the soul of mathematical operations" (p 111). The Alberta Education program of studies also identifies the importance of understanding equality. In Grade 1, students are expected to "describe equality as a balance and inequality as an imbalance, concretely and pictorially (0 to 20)" and "record equalities, using the equal symbol" (Alberta Education 2007, 31). In spite of the early expectations for students to develop an understanding of equality, students in elementary grades continue to show a misunderstanding of how the equal sign is used. In the subject of mathematics, the equal sign as a symbol can be used in many different situations such as "a computational result, as in $2 + 2 = 4$; an identity, as in $(x + y)(x - y) = x^2 - y^2$; assignment of a rule to a function, as in ; a substitution, as in $x = ½$; and so on" (Jones and Pratt 2012, 2). All of these examples require the equal sign to show a relationship, yet students spend most of their first years studying mathematics by focusing on the computational strategy. It is generally understood that this emphasis on using the equal sign to complete equations has promoted an operational understanding of the equal sign. In the operational understanding, the equal sign is seen as a signal to find the answer to the operation that precedes it. The equal sign is used as found on a calculator where an operation is entered and the equal sign is pressed to get an answer.

Researchers emphasize that students need to develop a relational understanding of the equal sign. A relational understanding of the equal sign takes on a broader meaning that shows the relationship between the operations or numbers on the left and right side of the symbol. When the symbol is misconceived as an operator, the equal sign typically demonstrates to "get the answer" as opposed to a relational meaning of "is the same as." Small (2013) suggests that students should understand that "the equality symbol sign should be viewed as a way to say that the same number has two different names, one on either side of the equals sign" (p 625). Faulkner (2009) is even more assertive in her stance that using the phrase "the same as" is not strong enough when describing equality. She believes that students require language that is more specific because "two trucks may be equal in weight to an elephant, but they certainly aren't the same as" (p 26). When looking at $2 + 5 = 4 + 3$, students should be able to see that they are not the same, but they are "equal in value" (p 26). Mann (2004) defines the equal sign as "a symbol that indicates that a state of equality exists and that the two values on either side of the equal sign are the same. It does not mean that the answer is coming or that the answer is on the other side of the sign" (p 65). From the variety of research available on this topic, it seems clear that students and often teachers are not aware of the meaning of this symbol which is "vital to successful algebraic thinking and is one of the big ideas of algebra about which students should reason" (p 26). Mann and other researchers suggest that the proper understanding of equality and the equal sign should be taught in elementary grades in order to prevent students from developing misconceptions.

Elementary Implications

In a situation presented by Falkner, Levi and Carpenter (1999), Grade 6 teachers were asked to present their students with the problem: $8 + 4 = \square + 5$ (p 232). The results were surprising to many of the teachers because out of 145 Grade 6 students, 84 per cent of the students thought the missing number was 12, 14 per cent of the students thought the number was 17 and none of the students chose the correct value of 7. Saenz-Ludlow and Walgamuth (1998) identified similar results with Grade 3 students. The students were asked to solve the equation $246 + 14 = \_\_\_\_\_ + 246$. Although the teacher in this situation used this equation as a warm-up and expected it to be fairly simple, it led to many discussions about what the answer should be. The students did not see a relationship between the numbers on the left and right side of the equal sign. They instead focused on adding the numbers on the left to get the answer (p 167). McNeil and Alibali (2005) examined why students continue to demonstrate an operational understanding. They infer that students are exposed to the same operational patterns, and they are not able to create new strategies when they encounter different, nonstandard equations.

Another common issue that can occur with equations is using the equal sign to keep a running total (Kieran 1981). In the false equation, $16 + 4 = 20 - 5 = 15$, a person will keep a running total when solving different steps in a word problem. This procedure does not demonstrate equivalence though and can cause confusion when a student must show a relationship between two sides of the equal sign. Saenz-Ludlow and Walgamuth (1998) observed this with the Grade 3 students who were working on the equation $246 + 14 = \_\_\_\_\_ + 246$. Some students saw the solution as $246 + 14 = 260 + 246 = 506$. These children were not using the equal sign to show "quantitative sameness on both sides of it" and instead used the symbol as a "separator of their sequence of operations" (p 177). To demonstrate equivalency properly, a teacher should make sure the student writes each equation separately such as $16 + 4 = 20$ and $20 - 5 = 15$. Ma (1999) discusses how teachers she observed in the United States differed in their opinion of the running total as opposed to teachers in China. She gives the example of a teacher in the United States who often accepts a running total because "they (students) did the calculational order correctly and got the correct answer" (p 111). On the other hand, the Chinese perspective is very rigorous in regards to how mathematical symbols are used and the Chinese teacher would not accept two different values on the left and right side of the symbol (p 177). People will often use the false equation as a shortcut during calculations, but it is important for teachers to refrain from modelling this technique while teaching the meaning of equivalency.

Researchers (Hiebert 1989; McNeil and Alibali 2005) suggest that students may have too much exposure completing operations with symbols before developing a proper understanding of the symbols. In the Alberta Education program of studies (2007), understanding equality is found in the Patterns and Relations strand, but Grade 1 students are also expected to develop an understanding of addition and subtraction of numbers concretely, pictorially and symbolically. Kieran (1981) states that "many children learn fairly quickly to read and write the elementary written symbolism of simple arithmetic, but do not necessarily understand it the same way we do" (p 318). It is this early lack of attention to the foundational skills that leads to the misconceptions and issues with more complex equations later in higher grades. Falkner, Levi and Carpenter (1999) discuss an experiment where a kindergarten teacher presents her class with the problem $4 + 5 = \square + 6$. Her students unanimously thought that the number 9 should go in the box. When the teacher modelled the same equation with unifix cubes, the students were able to recognize that a grouping with a stack of four blocks and a stack of five blocks was not the same as a grouping with a stack of nine blocks and a stack of six blocks (pp 232–34). This experiment demonstrates how primary age children have an understanding of equality when using concrete objects, but they are not able to relate this to the symbolic representation. As children continue to work with equations, it appears as though they also continue to develop habits that prevent them from having a relational understanding of the equal sign. Kieran (1981) discusses how elementary school children will argue that an equation written as $\square = 4 + 3$ is actually written the wrong way (p 318). Children become so accustomed to reading equations in this order that they are conditioned to do the calculation from left to right.

The students in all grades who had a relational understanding had more success at solving the algebraic equations often because these students were more likely to use an algebraic strategy.

Hattikudur and Alibali (2010) propose that through comparison of different relational symbols, students will "recognize more abstract commonalities" (p 17) and therefore develop a better understanding of the equal sign. In their study with Grades 3 and 4 students, groups that received lessons comparing symbols such as <, > and = demonstrated more of a relational understanding of the equal sign than groups that received lessons on just the equal sign as an operational symbol. This suggests that it is useful for students to group the equal sign with relational symbols such as < or > rather than with operational symbols. Also, the researchers state, "Students in the comparing symbols group were also more likely than students in the equal sign group to correctly recognize nonstandard equations as making sense" (p 28).

Symbols are essential for mathematics, and they "offer a convenient and powerful way to represent mathematical situations and to manipulate mathematical ideas" (Hiebert 1989, 39). He goes on to suggest that students need to create "sound meanings" of symbols before they can be used and manipulated in problem situations (p 39). Similar to the experiment of the kindergarten teacher in the article by Falkner, Levi and Carpenter (1999) with the unifix cubes, Hiebert (1989) states, "We need to design instruction so that we can help students connect the new knowledge they are acquiring about written symbols with the understandings they possess from experiences outside and inside of school" (p 40). McNeil and Alibali (2005) concluded that the elementary age children in their study had become so dependent on the operational patterns they had learned that even after receiving a lesson on the meaning of the equal sign and equations, the children continued to rely on their previous operational patterns and did not solve nonstandard equations correctly. Children need opportunities to demonstrate an understanding of how to balance concrete objects on a scale, and they should be taught how to apply this understanding to the equal sign before learning to do operations.

Middle School Implications

Middle school is typically a time when a student's misconception of the equal sign can cause confusion in mathematics as more complicated algebraic thinking is introduced. Alibali et al (2007, 222) assert, "Developing an understanding of the equal sign has typically been considered mathematically straightforward—after its initial introduction during students' early elementary school, little, if any, instructional time is explicitly spent on the equal sign in later grades." Alibali et al continues by stating that research actually seems to show that students at all grade levels do not demonstrate a proficient understanding of the equal sign. Students in middle school who have had many years of experience with solving equations and using the equal sign continue to use this symbol with an operational understanding. Knuth et al (2006) studied middle school students' success with algebraic equations and if this had a connection to their understanding of the equal sign. The students in all grades who had a relational understanding had more success at solving the algebraic equations often because these students were more likely to use an algebraic strategy. The researchers conclude, "We argue that there is a clear need for continued attention to be given to the notion of equality in the middle school grades" (p 310).

Alibali et al (2007) provide a concise explanation of the importance of developing a relational understanding. "A relational view of the equal sign allows students to interpret equations appropriately, and appropriate interpretations can guide judgments about the equivalence of equations" (Alibali et al 2007, 235). Through a longitudinal study they discovered that even middle school students continue to show an operational understanding. The students' development of a relational understanding is only gradual from Grades 6 to 8 with some students still retaining the operational understanding. Alibali et al (2007, 241) identified that "students' performance varied as a function of when they had acquired a relational understanding of the equal sign."

In a study of middle school teachers, Asquith et al (2007) examined what teachers think their students know about the equal sign. The six teachers from Grade 7 predicted that 73 per cent of the students would have a relational understanding when the actual number was only 37 per cent. It was also observed in this study that many teachers did not recognize that when students have an operational view of the equal sign that it would prevent them from being successful in math. There appears to be a disconnection between how students understand the equal sign and what teachers assume that they know. Alibali et al (2007) emphasize that middle school mathematics must give attention to equivalence and the equal sign, and this attention "should include varied and regular opportunities for students to develop a relational understanding" (p 245). According to McNeil et al (2006) students were more likely to show a relational understanding when they were presented with nonstandard equations, specifically equations with operations on both sides of the equal sign.

The Equal Sign in Textbooks

Before presenting suggestions for how teachers can promote the relational understanding in the classroom, there is evidence that the typical classroom resources do not provide sufficient material for assisting teachers. The operations-equals-answer equations is traditionally the most commonly used equation and "is thought to promote an operational interpretation of the equal sign," and the operations on both sides equation is thought to be the most effective at showing a relational interpretation (McNeil et al 2006, 371). Matthews et al (2012) examined different types of equations and concluded that "not all non-standard equation formats are equally challenging" and "the more an equation varies from the standard $a + b = c$ format, the more difficult it is likely to be" (p 338). They also discovered that "equations with operations on both sides were more difficult for children to solve than those that involved operators on a single side only" (p 339). In their examination of textbooks, McNeil et al (2006) discovered that operations on both sides equations made up only 5 per cent of equations used. Powell (2012) analyzed kindergarten through Grade 5 textbooks and found that "the majority of equations across kindergarten to fifth grade fall into the standard category (operation on the right side)" (p 642). When nonstandard equations were used, they were mainly used as operation-right-side equations.

Powell (2012) took this study further by examining the teacher's manuals that are provided with each textbook series and analyzing how these manuals suggest that the equal sign should be explained to students. Throughout the teacher's manuals the definitions and explanations given for the equal sign are consistent and show a relational meaning. "No curriculum, however, provides the same definition at all grade levels, and some curricula provide different definitions across grade levels or within the same grade level" (p 642). This means that teachers in a school may not be using consistent vocabulary when instructing students, which can cause confusion as the students move to new grades.

Li et al (2008) compared US and Chinese textbooks. In general, the Chinese textbooks first provided instruction for numbers and values up to 10, then moved on to introduce the equal sign as a way to compare numbers. "After introducing the equal sign, addition and subtraction were introduced where students were provided with both standard and nonstandard forms to understand both operations and the equal sign" (p 206). Sixth grade Chinese students were almost three times more likely to provide correct answers to equations such as $6 + 9 = \_\_\_\_ + 4$ and $\_\_\_\_ + 3 = 5 + 7 = \_\_\_\_$. The information gathered from the analysis of textbooks shows that teachers will need to facilitate opportunities in the classroom to promote the understanding of the equal sign.

What Teachers Can Do

There are advantages for providing students with specific equal sign instruction. According to Mann (2004) the opportunity to introduce the concept of equivalence should not wait until middle school. She suggests that "teachers should help students in elementary school come to recognize the equals sign as a symbol that represents equivalence and balance" (p 65). Falkner, Levi and Carpenter (1999) discuss how to incorporate lessons about the equal sign in first and second grade classes. In the first grade classroom, the students will decide if various number sentences are true or false such as $8 = 8$ or $8 + 2 = 10 + 4$. These statements will lead to class discussions about what a number sentence can look like and what the equal sign means. Powell and Fuchs (2010) identified how providing word-problem tutoring combined with equal-sign instruction and practice with open equations (for example, $8 = 4 + x$) increased a Grade 3 student's ability to use a relational understanding and to solve nonstandard equations. Students will often require specific instruction about the equal sign regardless of the grade they are in, and there are a number of ways this can be achieved.

The students in all grades who had a relational understanding had more success at solving the algebraic equations often because these students were more likely to use an algebraic strategy.

Children's picture books are used as a way to introduce algebraic relationships in early elementary grades (Leavy, Hourigan and McMahon 2013; Lubinski and Otto 2002). *How Many Snails?*, by Paul Giganti (1988), presents various groups of pictures that can be categorized in different ways to show relationships between the quantities. Lubinski and Otto (2002) describe a page in the book that has yellow, pink and white flowers with black, yellow or orange centres. During the class discussion, the students are able to represent the different flowers with the equation $9 + 4 + 2$. They also create the equation $9 + 3 + 3$ to represent the different centre colours. The teacher can then discuss the relationship between these two equations. Leavy, Hourigan and McMahon (2013) use the picture book *Equal Shmequel*, by Virginia Kroll (2005), to also show relationships. This book uses wilderness animals that need to balance the two sides of a tug-of-war game in order to make the game equal. The pictorial representations can be very useful as reinforcing mathematics concepts for students.

A common tool for providing concrete examples of equality is the pan balance (Barlow and Harmon 2012; Mann 2004; Ellis and Yeh 2009). Students can easily manipulate unifix cubes or other objects on either side of a pan balance as demonstrated by Falkner, Levi and Carpenter (1999). Barlow and Harmon (2012) emphasize the importance of using pan balances as a way to provide students with the "opportunity to think about equivalence and balance without rushing toward representing the relationships with symbols" (p 98). Mann (2004) uses balance representation with see-saws to have the students create "Seesaw Rules" (p 66). These statements highlighted the essential understandings that are necessary for balancing objects on a seesaw. After creating the rules, the class then transfers to make connections between the see-saw and the equal sign.

Ellis and Yeh (2009) take the pan balance a step further with their mobile-balance puzzle. Individual numbers are further separated to keep the whole equation balanced. When shown as a balance, numbers are split to represent the different arms of the balance (that is, $12 = 12$ would be split into arms showing $8 + 4 = 6 + 6$ and this would be split into further arms showing $8 + 2 + 3 = 6 + 3 + 3$).

After students become familiar with using the equal sign symbol, they need to practise writing different equations that represent the same values as suggested by MacGregor and Stacey (1999). They give examples of a number of problems that can lead to students to create these different equations such as $24 = 2 + 10 + 2 + 10$, $24 = 4 \times 6 = 6 + 6 + 6 + 6 = 12 + 12$ and $3 + 9 + 3 + 9 = 4 + 8 + 4 + 8$. "Seeing the reasons behind such relationships requires a generalization about properties of numbers that is deeply algebraic" (p 80).

Creating opportunities for students to understand the meaning of the equal sign will be the responsibility of the mathematics teachers. Blanton and Kaput (2005) performed a case study on a Grade 3 teacher who displayed "robustness" in her ability to adapt her teaching into opportunities to explore algebraic thinking. The researchers observed that "algebraic reasoning tasks were not mathematical 'field trips' but were woven in the daily fabric of instruction" (p 440). It is the responsibility of the teacher to develop their own algebraic thinking skills in order to have this ability to incorporate it into daily mathematics practice. Blanton and Kaput conclude, "Elementary teachers must develop algebra 'eyes and ears' as a new way of both looking at the mathematics they are teaching and listening to students' thinking about it" (p 440).

Conclusion

From my experience, teachers and students generally feel as though the equal sign is a straightforward concept that does not require much attention. Research has proven overwhelmingly that this is not the case, and many students are progressing through school with only the operational understanding. This concept requires attention from mathematics teachers across all grade levels in order to ensure the success of students in developing algebraic thinking. It will be essential for schools and teachers to understand how to incorporate these concepts into the classroom mathematics practice.

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