

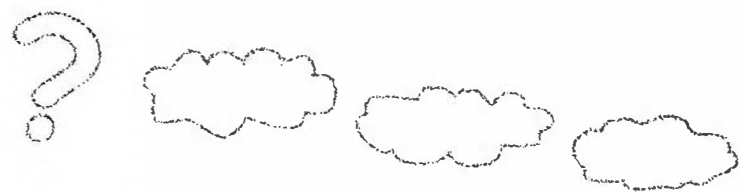


delta-k

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$\emptyset \subseteq A \cap B$ $\mathbb{Z} \times \mathbb{Z}$ $P(n) \Rightarrow P(n+1)$ \mathbb{R}

$+0=a$ $\forall \epsilon > 0 \exists \delta > 0$

$-e^x = 1$ $e^x = 1/x$

$\frac{d}{dx} = \int$ $\int \frac{1}{x} dx = \ln|x| + C$

$f(x+h) - f(x)$ $\sum f(x) dx$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$ax^2 + bx + c = 0$

$\frac{3}{6} = \frac{2}{4}$

one, two, three, four 4 $2+2$ $1+3$

$4-3=1$

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. All manuscripts should be typewritten and properly referenced. All pages should be numbered.
4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style's* author-date system or the American Psychological Association (APA) style manual.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Lorelei Boschman, c/o Medicine Hat College, Division of Arts and Education, 299 College Drive SE, Medicine Hat. AB T1A 3Y6; e-mail lboschman@mhc.ab.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

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From the Coeditors' Desks

My Thanks to You

Gladys Sterenberg

Over the past 11 years, I have had the pleasure of editing *delta-K*. This has provided me with many opportunities, the most significant of which has been to work alongside passionate mathematics educators on the executive of the Mathematics Council of the Alberta Teachers' Association. In particular, the leadership provided by members of MCATA has shaped my professional learning and, consequently, the content of this journal. I'm pleased to know many as colleagues and friends. This has been both a professional and personal journey.

I've had a great opportunity to meet and be mentored by those involved in the production of *delta-K*. The publication team at the Alberta Teachers' Association has been tremendously helpful in providing editorial, artistic and creative advice. Contributors to *delta-K* have also encouraged me through supportive e-mails and postcards (yes, some sent news of vacations) that reminded me of the importance of relationships. Past editors of *delta-K* have shared their stories and experiences with me.

The first issue that I was involved with was coedited with Dr A Craig Loewen, dean of the Faculty of Education at the University of Lethbridge. He graciously mentored me in this process and offered much of his time to help me with the many questions I had. He was instrumental in providing direction and guidance as we transitioned to a peer-reviewed journal.

Other opportunities for working alongside a coeditor emerged. Lynn McGarvey initiated the idea for a special issue (2011, volume 48, number 2) focused on early childhood and mathematics education. The special issue of March 2013 (volume 50, number 2) grew out of work with Egan Chernoff on the book celebrating 50 years of *delta-K* (see the description of the book included in this issue).

And finally, it is fitting that this issue (my last one) is coedited with Elaine Simmt. It represents all that is generative about mathematics education in our province. The collection of articles was written by mathematics teachers engaged in a graduate course. These articles show deep connections between theory and practice, a hallmark of *delta-K* since its inception. This collection demonstrates the importance of professional learning within a community. Elaine has provided an in-depth introduction in her editorial. I know that you will enjoy reading about how your colleagues are making sense of David Tall's notion of three worlds of mathematics.

It is with mixed feelings that I make a transition out of being part of MCATA, but I'm very pleased that Lorelei Boschman has volunteered to become *delta-K*'s new editor. I got to know her when she became an instructor for the mathematics curriculum and instruction course offered through the University of Alberta BED program at Medicine Hat College. She has impressed me with her enthusiasm and commitment to mathematics education in Alberta. She has already shared ideas for forthcoming issues and will bring her own leadership gifts to the MCATA team. I'm proud to have been part of this legacy and know that *delta-K* is in great hands going forward.

Editing *delta-K* has been an experience where I have stood on the shoulders of giants. To those whom I have met throughout this journey, I thank you for the pleasure of your company.

Gladys Sterenberg, PhD, was a classroom teacher in Lethbridge for 15 years before pursuing graduate studies. As a faculty member at the universities of Lethbridge and Alberta, her passion for mathematics education was supported as she was mentored by professionals and academics in the field. Currently, she is an associate professor in the Department of Education at Mount Royal University, where she continues to work alongside teachers and teacher candidates in the province to enhance mathematics learning and teaching.

From the Guest Editor

Elaine Simmt

Reading, Writing and Arithmetic: Professional Development Through Communal Reading and Reflection as Mathematics Educators

Ongoing professional growth can be achieved in many ways. As a professor of mathematics education at the University of Alberta, I meet teachers who choose graduate studies as a means to grow as educators. In the winter of 2014, a group of PhD and MEd students participated in a course on mathematics learning systems. In that course the participants read about complex learning systems from a complexity science perspective. As a class we read Professor Emeritus David Tall's new book, *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics*, published in 2013 by Cambridge University Press, and discussed each chapter. The ideas in the book came alive as the teachers interpreted his ideas in the context of their own teaching practices. In this issue we present a number of the papers that teachers wrote to reflect on Tall's work.

Tall is a British mathematics educator who studied under Richard Skemp (best known for his distinction between instrumental and relational understanding) and spent a lifetime working with learners of mathematics. His book is an ambitious account of how mathematical thinking develops from infancy through adulthood and why mathematics in one context may be quite meaningful for a person, but when taken up in another context it lacks meaning or may interfere with the new way of thinking mathematically or the new mathematics to be learned.

At the core of Tall's (2013) theory is a model of three worlds of mathematics with which humans interact: the embodied, the symbolic and the formal. This model suggests that a person's development as a mathematical thinker requires perception and action across these three worlds. The embodied mathematical world is the starting place for all humans and all mathematics. Shape and space provide the grounding on which humans generalize and from which they can symbolize. Humans act on objects in the physical world with their bodies (hence through their bodily senses) and are acted on by the physical world in which they exist (through their bodily senses). Once symbols begin to replace objects, humans begin to act on the symbols. And like other objects that act on the learner (eg, a curved surface with no edges in contrast with a flat surface with multiple edges), symbols begin to act on the learner. Traces (physical and conceptual) are left when the learner interacts with the objects (concrete or symbolic [which becomes "concrete" for the learner]). The world of mathematics also includes the formal dimension. As the reader will learn, Tall points to both mathematicians' notions of formalism (set-theoretic definition and formal proofs) and psychologists' (ie, Piaget's) notion of formal operational stage as aspects of formal mathematics.

A key concept developed by Tall (2013), the *met-before* really struck a chord with the teachers in the graduate course. Most simply stated, a met-before is a piece of mathematics or a result from some mathematics that sticks with a student and affects subsequent encounters with mathematics. A met-before can be most helpful to a student learning something new or making a connection among mathematical concepts. Consider the connections between a square number (being the result of a number multiplied by itself) and a square root (the factor that was multiplied by itself to give the radicand). Also think about the connection between integer multiplication and division. Positive numbers multiplied together will result in a positive product, and two negative numbers multiplied together will result in a positive product. Hence, the middle school teacher takes advantage of this met-before to help students understand that the radicand of a square root can be positive or

negative. However, this met-before can become problematic when the high school teacher introduces the notion of the principal root. At that point, students need to let go of the generalization they have made of roots being positive or negative and now need to distinguish the principal root.

In this set of papers we see various interpretations of Tall's (2013) theory. In sum, the papers reflect many ways in which teachers not only read theory but how they interpret it for teaching practice. Powell, Asquith and Luo explore the met-before concept most directly in their papers and illustrate from their experience how met-befores affect the development of mathematics. Powell does so in the context of graphing functions on the graphing calculator, whereas Luo explores the power of using met-befores to develop algebraic forms of quadratic equations. Asquith uses the notions of both set-before and met-before to explore possibilities for teaching ELL students mathematics. Barton and Charles discuss Tall's three worlds. Barton reflects on an experience she had doing a new piece of mathematics, from embodied pattern generating through symbolism and formalism. Through her reflection she unpacks Tall's theory and in doing so unpacks her own understanding of the mathematics. Charles's paper proposes a deliberate sequence of activities based on Tall's three worlds to scaffold learners' meaning making in trigonometry. Finally, Dias Corrêa offers Tall's model as one of three different theories for observing student meaning making in mathematics.

I hope this focus issue of *delta-K* has introduced you as a mathematics teacher to some interesting theoretical perspectives on learning and on mathematics, and to some pragmatic suggestions for teaching mathematics. At the same time I hope you might consider initiating a reading group or book club in your school or among a group of neighbouring schools as a weekly or monthly professional development opportunity. Whether you initiate a reading group or do some personal professional reading, David Tall's *How Humans Learn to Think Mathematically* might be a good selection for your next book.

Elaine Simmt, PhD, is a former secondary school teacher of mathematics, chemistry and physics. For the past 17 years she has been a professor of secondary education at the University of Alberta. Her research is focused in mathematics education. In particular, she explores teaching and learning as understood through the frame of complexity theory. A second and complementary area of study is centred in teacher education, specifically mathematics-for-teaching. In her most recent work, she has been doing international research and development projects in Tanzania to explore possibilities for mathematics teacher development in rural and remote communities. Simmt received the 2011 Friend of MCATA award from the Mathematics Council of the Alberta Teachers' Association.

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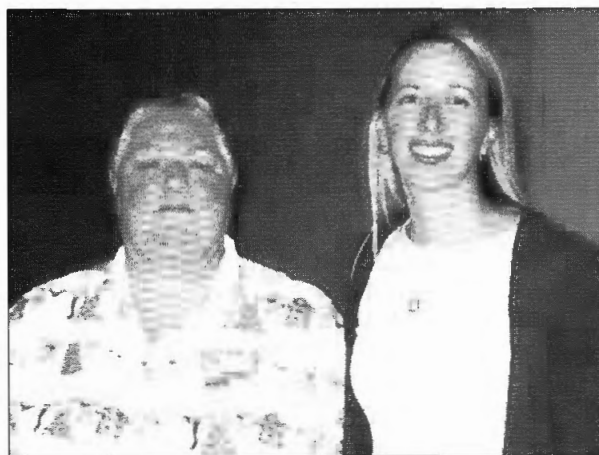
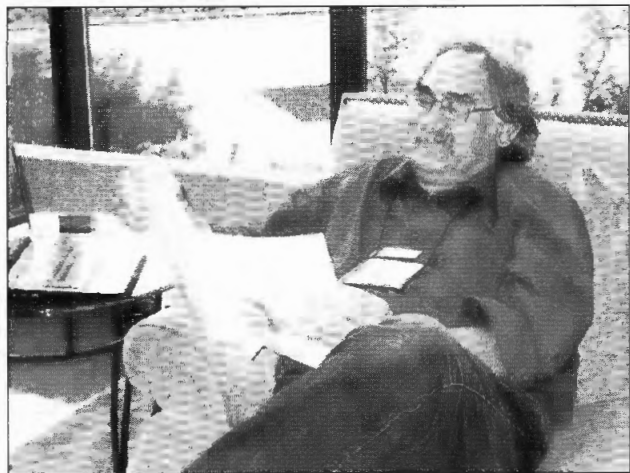
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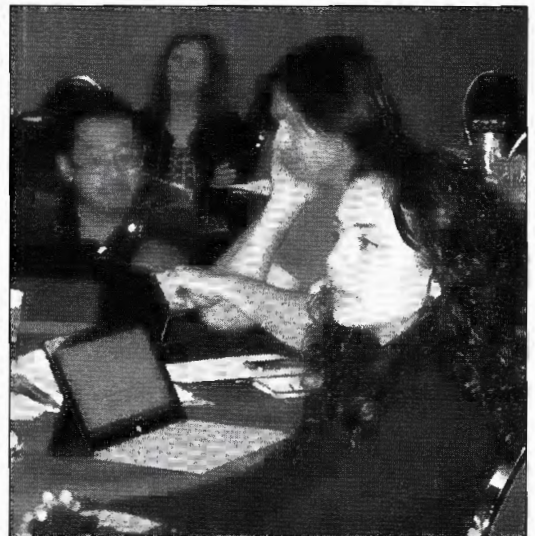
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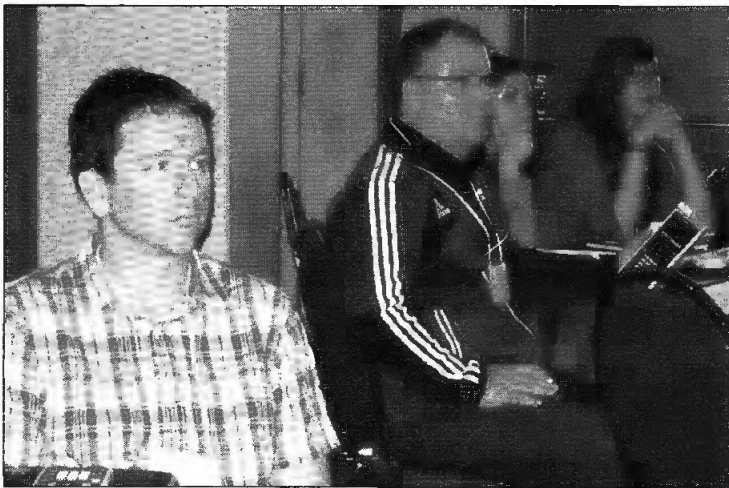
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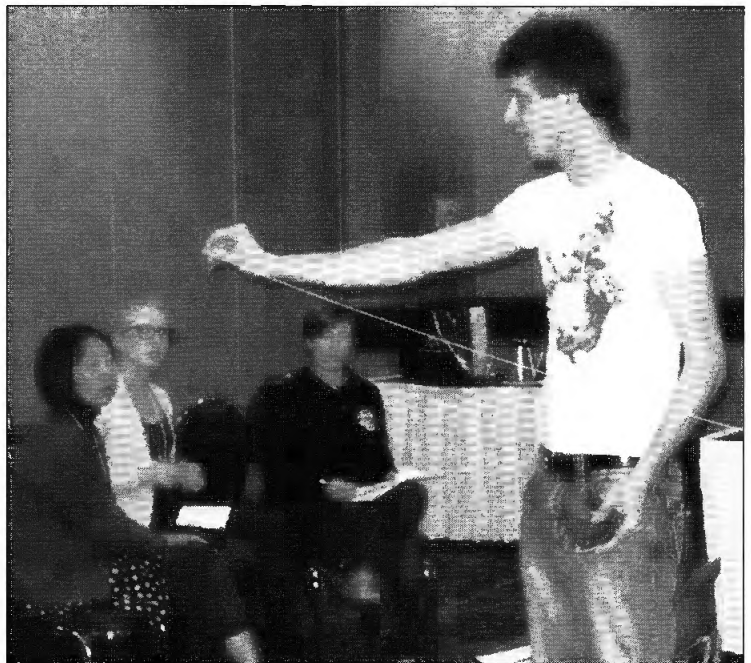


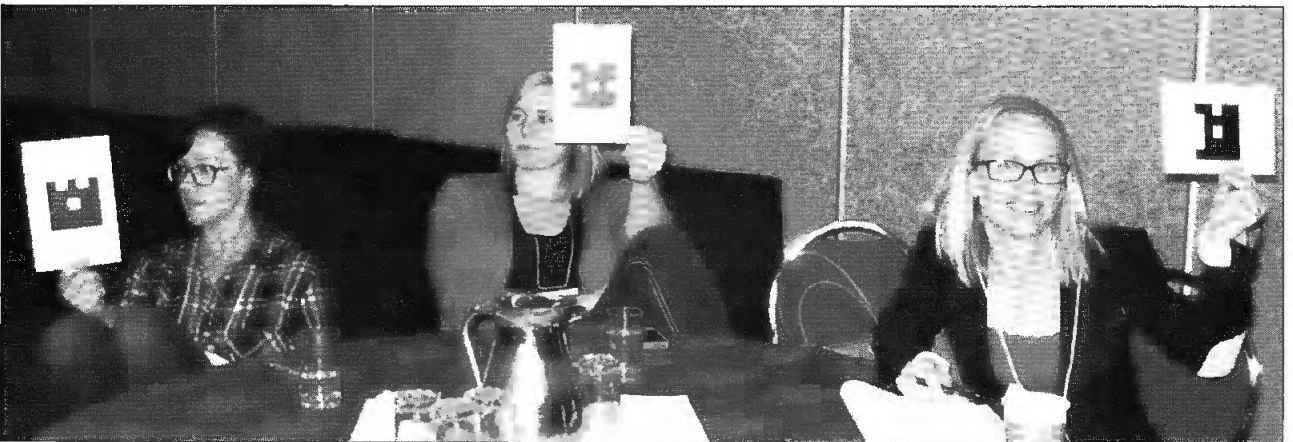
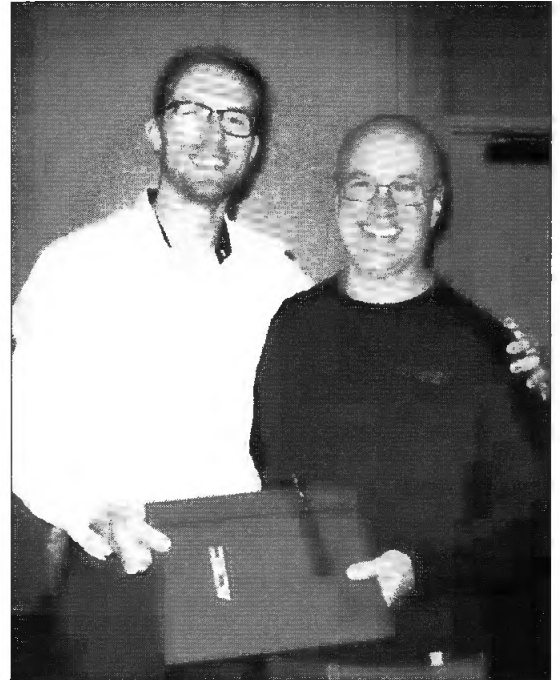
*Alberta Mathematics
Educator Award
Recipient—Leann Miller
Presented by Marj Farris*

Conference Participants









Solve the Following Equation: The Role of the Graphing Calculator in the Three Worlds of Mathematics

Jayne Powell

In 2013, David Tall published a book entitled *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics*, which tries to make sense of how mathematics is taught and learned in a world where the spectrum of positions on mathematics ranges from feelings of absolute beauty and power to anxiety and distress (p xiii). He proposes a framework of three worlds of mathematics through which learners construct mathematical meaning. As graphing calculators are now a near-ubiquitous tool in the mathematics classroom, this paper will explore how using a graphing calculator is both supportive and problematic within these three worlds of mathematics, by considering how students may come to solve quadratic functions.

The presence of the graphing calculator in the mathematics classroom has become naturalized. One does not often step back and ask how it came to be here or what it is currently doing to mathematical thinking, pedagogy and curriculum. Historically, the first device that could be considered a calculator, the abacus, began to extend mathematical thinking as early as 5,000 years ago. Then, in 1692, the French mathematician Pascal created the first mechanical calculator, which had the ability to add and subtract numbers. However, at the time Pascal concluded that it was too expensive for any practical use (Grinstein and Lipsey 2001, 87). Calculators would remain too expensive for common household use until the 1970s. Since that time, the increased use of calculators in society quickly forced educators to adapt, which gave rise to the prominent and lasting mathematics education debate about whether and how calculators should be implemented in classrooms (Banks 2008, 1–2). Then, in the early 1990s, a more powerful type of calculator—the graphing calculator—emerged on the education scene, and it was soon commonly seen in

most high school mathematics classrooms. Graphing calculators allow students to graph, analyze, calculate and solve problems graphically, numerically and algebraically. Since nongraphing calculators had become common in schools, aside from some discussions surrounding their monetary expense, the addition of graphing calculators to the classroom was less contentious. However, even if the addition of the graphing calculator was met with less resistance, it is still considered in the literature as the instigator of massive change in the high school mathematics classroom in the last 25 years. In 1992, near the beginning of the integration of the graphing calculator into the classroom, Kaput described this new technology as “a newly active volcano of the mathematical mountain ... changing before our eyes, with a myriad of forces operating on it and within it simultaneously” (p 515). Yet today, its presence goes nearly unquestioned. Learning to use a graphing calculator is merely part of the progression of learning about mathematics. The presence of the graphing calculator in education has gone from being seen as an active volcano to being naturalized. Teaching high school mathematics now implicitly includes teaching how to use a graphing calculator to aid in developing mathematical thinking and understanding.

The Three Worlds of Mathematics

Tall (2013) puts forth a framework in which to consider mathematical learning that he calls the “three worlds of mathematics”: conceptual embodiment, operational symbolism, and axiomatic formalism (p 133). Through these worlds, language, categorization and repetition produce thinkable concepts

that can be developed into *crystalline concepts*, which occur from the compression of understanding into a structure that has “inevitable properties in its given context” (p 27). The use of the word *embodiment*, in *conceptual embodiment*, can be problematic. In everyday language the word *embodiment* can mean a concrete representation of an abstract idea, or the embodiment of an idea can be linked to knowing through the body. Yet, Tall’s explanation of the first world of mathematics, *conceptual embodiment*, is more open and points to any human perceptions and actions that develop mental images that give meaning to abstract concepts, be it through the body, concrete materials or other experiences such as using a graphing calculator. The second world of mathematics, *operational symbolism*, often develops from embodied understandings and includes “symbolic procedures of calculation and manipulations that may be compressed into ... flexible operational thinking” (Tall 2013, 133). The third world of mathematics, *axiomatic formalism*, builds formal mathematical knowledge by developing definition and proof. Learners do not move through these three worlds linearly; instead they continually “fold back” (Pirie and Kieren 1994) to previous learning in order to move their understanding forward. Learners never come back to the same place in the same way, and they are always taking something different away. To think of developing understanding in this way “reveals the non-unidirectional nature of coming to understand mathematics” (Pirie and Kieren 1994, 69).

In many Alberta schools, students begin to learn to use and rely on their graphing calculators in Grade 10. Learning to use this tool develops through both formal instruction and other experiences of using the calculator, such as trial and error or play. Many students begin their formal experiences with the graphing feature by working with linear functions. When students move on to Grade 11 they will start to explicitly study nonlinear functions, often beginning with quadratic functions. The calculator then becomes more than a tool used for routine calculations and displaying the odd graph, but instead develops into an incredibly useful extension of their thinking. This extension will become as prized for its instant graphing capabilities as it is for its ability to convert rational numbers from decimal form into fractions. Yet, there are previous understandings, which Tall (2013) calls *met-befores*, that can be both supportive and problematic in developing an understanding of quadratic functions. Teachers need to be aware that “a sensible approach to learning requires not only the building towards powerful ideas that will

be encountered in the future but also addressing the problematic issues in the present that may have long term consequences” (p 116). Thus, in unpacking the graphing calculator’s role in learning about solving quadratic functions through the three worlds of mathematics, it is important to remember that both problematic and supportive met-befores are being created.

When beginning to study quadratic functions, a common starting place is to look at the features of quadratic functions and their corresponding graphs. Students place equations often given in standard, $y = ax^2 + bx + c$, vertex, $y = a(x - p)^2 + q$, and factored, $y = a(x - b)(x - c)$, forms into the [Y=] function of their calculator. They observe the U-shaped curves, opening up and down, wider and skinnier, with the vertex in multiple locations. This may further perpetuate a common met-before related to the meaning of the equals sign. For some students, an equals sign often does not represent equivalence between the two sides of an equation, but initiates a problematic “put the answer here” response.

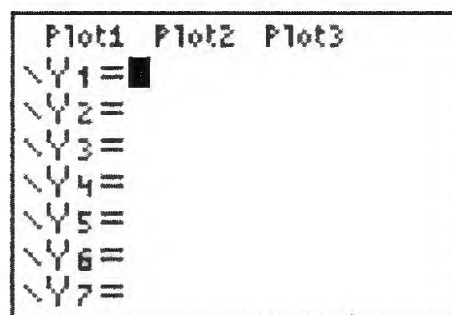


Figure 1: Home screen of graphing feature of TI 83+

Figure 1 shows how, on the main input screen for a graphing calculator, the y-variable is isolated on the left side of the screen and an equals sign indicating “put the expression here” on the right side. The y-variable becomes separated from the rest of the equation, decreasing its appearance of importance within the function. Functions may begin to lose their two-variable appearance, and importance, as the x-variable becomes the focus. It is possible that when given an equation that is not in one of the common forms, such as $y - 6 = x^2 - 5x$, to see some students reach for their calculators and enter $Y_1 = x^2 - 5x$. This response to an equals sign builds on the previous misunderstanding of the meaning of the Y_1 in their calculator’s graphing feature. This met-before is possibly perpetuated because students are often given equations with the y-variable already isolated and will then repetitively enter equations without having to enter a y or equals sign into their graphing calculator.

This met-before can be built upon further for some students when they start to solve quadratic equations, such as $x^2 + 3x - 10 = 0$, with their graphing calculators. Suddenly, the y -variable is gone, replaced with a 0, and although teachers may explicitly discuss this change some students may not build these understandings into their creation of meaning. They are now working with only a specific case of the function—when it is equal to 0. This change is made easy by the previous met-before regarding the meaning of an equals sign, for some students can ignore the 0 in the same way that they were ignoring the y -variable, the only difference being that the part being ignored is often located on the right side rather than the left. Thus, students who enter $x^2 + 3x - 10$ into $[Y_1]$ are at times completely unconcerned with the meaning of the 0. There is a lack of awareness that on their calculator screen is a representation of the function $y = x^2 + 3x - 10$, not the equation $x^2 + 3x - 10 = 0$. Students can then use the [CALC] feature to find the zeros, perhaps unmindful that the zeros are interesting because the equation is currently equal to 0. If instead $x^2 + 3x - 10 = 2$, the interest would be in the x -values of the graph when the function is at 2. Using the graphing feature of a calculator to graph the related function as a way to learn to solve equations can lead to a possible misunderstanding of the definition and meaning of a function.

When learning to solve quadratic equations by factoring, an algebraic method, the calculator can be used as a bridge between the worlds of operational symbolism and conceptual embodiment. Making the conceptual embodiment of the graph a method of developing a visual representation of the solutions arrived at algebraically in the world of operational symbolism. Students can look at the equations in factored form, such as $(x - 2)(x + 5) = 0$, and the corresponding graph, $Y_1 = (x - 2)(x + 5)$ or $Y_1 = x^2 + 3x - 10$ to recognize relationships between the two. The connection between the x -intercepts of $(2, 0)$ and $(-5, 0)$ and the numerical values of 2 and -5 in the factored form of the equation seem straightforward. This demonstrates how conceptual embodiment and operational symbolism can blend together, allowing more powerful ways of thinking mathematically (Tall 2013, 145). However, thinking about the connection between the values in factored form and the x -intercepts of the graph is not enough. In using factoring to solve a quadratic, it is difficult to develop meaningful understanding of the connection to zero. The graphing calculator reinforces the numerical values, for example the 2 and the 5, not the reason for their signs, $+2$ and -5 . Students may inappropriately generalize their own understanding to solve a

quadratic equation as a rule articulated as “just factor and take the opposite signs of the numbers.” This met-before becomes troublesome in problems such as $(2x - 3)(x + 4) = 0$, where using this “rule” often results in the incorrect solution $x = 3$ and $x = -4$. This met-before is created from experiencing many examples that have had integer solutions, which are reinforced further through seeing these integer solutions on their calculators. Although the calculator may play a role in creating this met-before, it is also incredibly supportive in considering why this understanding is incorrect. In returning to the graphical representation, students can see that the graph does not cross the x -axis at 3, but seemingly half-way between 1 and 2. As students build more and more of these experiences with conceptual embodiment and start learning the associated algebraic approach to solutions, in the world of operational symbolism, they begin to rely less and less on the calculator.

As students progress, the graphing feature of the calculator is used less to make meaning of equations or to check algebraic solutions. “The use of algebra becomes more sophisticated, and operational symbolism takes on a role of its own that no longer needs to be permanently linked to embodiment” (Tall 2013, 145). Taking these thinkable concepts and compressing their meaning, “in the symbolic world we begin to shift to a new way of making sense of the symbols themselves and the coherent ways in which they operate, without consciously referring back to their earlier meanings” (Tall 2013, 145). Yet, having created a conceptual embodiment of these concepts allows for folding back to these ideas if necessary, allowing for more flexible mathematical thinking and meaning making as teachers push their students into the world of operational symbolism.

Many students will begin to get comfortable with factoring to find the solutions to an equation, until they come to a problem where the quadratic equation is not easily factorable. Often, the first response is that “not easily factorable” means that the problem does not have a solution. If students are mathematically flexible they can fold back to the conceptual embodiment provided by the graphing calculator and are able to graph the equation to look for what they often understand to be the solutions to a quadratic equation, the x -intercepts. Some students, often through the guidance of their teacher, will come to understand polynomial solutions as being either real and unequal, real and equal, and unreal and unequal, for real solutions only. However, some students might create different meaning. Perhaps some equations might show a graph crossing the x -axis, disproving students’ previous conjecture that an equation that is not

factorable has no solutions. Students can work back and forth between the worlds of conceptual embodiment and operational symbolism to create meaning and find resolutions to their questions. However, other equations will show no x-intercepts, and students would feel more confident in their initial response of there being no solution to equations that are not easily factorable. This line of thinking could lead to a rule that there can be none, one or two solutions to a quadratic equation. When students further their mathematical learning, having a conceptual embodied understanding of the solution(s) to equations as x-intercepts of a graph creates a challenging met-before when students encounter complex roots for the first time. Students may have been told that no solutions exist when there are no x-intercepts, and yet complex roots do exist, just not in the same way. Thus, students must be flexible not only in their ability to fold back to other meanings but also to let go of constructed meanings, or “rules,” as they continue their learning.

The issue of how to find the solutions to quadratic equations when they are not factorable transitions students into the third world, *axiomatic formalism*. This is the world of formal mathematics that relies on definition and proof. Here the quadratic formula can be derived symbolically, often using the method of completing the square. Once this formula is derived in the world of axiomatic formalism, students will return to operational symbolism to work with and test this formula, often checking it against the conceptual embodied world of the calculator’s graphing feature. Again, the conceptual embodiment that the calculator creates allows for acceptance and understanding of very abstract concepts and meanings.

Tall (2013) hypothesizes “that mathematical thinking builds on ... faculties set-before birth in our genes and develops through successive experiences where new situations are interpreted using knowledge structures based on experiences that the individual has met before” (p 117). As teachers, we need to be aware of the beneficial and problematic consequences of the mathematical experiences that occur in our classrooms, especially since problematic understandings are often created accidentally and unconsciously. The

inclusion of the graphing calculator in learning to solve quadratic equations allows rapid access to the world of conceptual embodiment that just 25 years ago was not readily available for high school students. This inclusion brings deeper understanding as well as the reinforcing and creation of problematic met-befores. By moving between the worlds of conceptual embodiment on the calculator and the algebra used in operational symbolism, deeper meanings can be created. Even the world of axiomatic formalism benefits from the graphing calculator, as generalizations created here can be tested out in the world of conceptual embodiment. The graphing calculator as a tool has changed how mathematics is taken up in the classroom, allowing access in the high school classroom to the conceptual embodiment of abstract concepts that were previously considered not practical to explore.

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Repetition as a Means of Encouraging Tall's Met-Befores

Lixin Luo

In a classroom teaching and learning situation, it is common for individual students to respond differently to a new topic introduced by the teacher. While some students might be able to understand the new topic quickly, others might feel lost or confused. Students' different responses can be explained using David Tall's (2013) idea of *met-befores*. In this paper, I first interpret Tall's concept of met-before, and then I explore using repetition to help students to construct and activate met-befores in order to facilitate their mathematical growth.

Met-Befores

The term *met-before* is used to "describe how we interpret new situations in terms of experiences we have met before" (Tall 2013, 88). Tall defines a met-before as "a mental structure we have now as a result of experiences we have met before" (p.84). The term *met-before* refers not to a person's actual experience, but rather to the embodied influence of the person's previous conscious and unconscious experience. Met-befores are personal; two people who have learned the same topic might not have the same understanding of the topic. Met-befores can exist unconsciously and might not present themselves until a person is prompted by certain situations that make her met-befores problematic. For example, a student might not realize that she believes that "multiplication makes more" until she encounters fraction multiplication and the fact that multiplication makes less. Tall's met-befores are similar to presumptions, prejudices, attitudes or habitual ways of thinking formed through a person's former experience.

Of particular significance, met-befores affect how we interpret a new situation, thus influencing our learning. Some met-befores are supportive because they help learners to understand new experience, while some are problematic because they cause initial confusion (Tall 2013). For example, knowing $2x + 3x = 5x$ is helpful for one to understand $2x^2 + 3x^2 = 5x^2$, but understanding that addition makes a bigger number, based on one's experience with

positive numbers, is problematic when one first encounters adding negative numbers. Tall sees that supportive and problematic met-befores arise naturally in mathematical learning, and the development of mathematical thinking involves a change of meaning of met-befores: some supportive met-befores might continue to be helpful in a new context while some become problematic. Thus, whether a met-before is supportive or problematic is contextualized rather than a fixed attribute. For instance, a student who has calculated the square of a real number many times would find the statement "Any real number's square is positive" easy to understand, but find the idea of $i^2 = -1$ hard to grasp.

A person can have some supportive aspects of a given concept and some problematic aspects at the same time (Tall 2013). Students who can understand a new topic quickly might have sufficient supportive met-befores or they can suppress their problematic met-befores in order to move on, while students who find the topic hard to grasp might lack supportive met-befores or have problematic met-befores that they cannot resolve.

Both supportive and problematic met-befores are important for mathematical learning, yet they are not equally valued in school curriculum (Tall 2013). Supportive met-befores are commonly valued in curriculum design through the emphasis of prerequisite knowledge and skills and in teaching practices through connecting new ideas with students' experience. Problematic met-befores, however, are rarely used in mathematics classrooms as "an integral part of learning" (Tall 2013, 89). Contradictions between the new idea and one's previous understanding are not welcome because they seem to interrupt and trouble one's learning. Tall sees curriculum's focus on supportive met-befores as a problem. He argues that problematic met-befores can have "debilitating effects in long-term learning" (p 89), and the resolution of problematic met-befores is needed for confident new learning. Therefore, Tall suggests considering ways to deliberately reveal problematic met-befores so that they can be addressed. This leads us to the use of repetition.

Using Repetition to Construct and Activate Met-Befores

Repetition is one of Tall's (2013) three fundamental mental structures that humans are born with. These structures (ie, recognition, repetition, language) take time to mature as the brains make connections in early life. Tall calls these structures *set-befores*. He argues that the development of mathematical thinking is *based on set-befores* and *built on met-befores*. The importance of repetition is somehow self-evident: without our mental ability to repeat actions to form repeatable sequences, mathematical thinking is impossible. Repetition encourages generalization and abstraction. Through repetition, one can notice patterns and compress a sequence of actions into a mental object, which becomes the object for manipulations at a higher level of abstraction. While considering ways to deliberately reveal problematic met-befores, I see the possibility of using repetition to help students construct and activate both supportive and problematic met-befores.

Supportive Met-Befores

Teachers can facilitate students' construction and activation of supportive met-befores by using examples that repeat with variation. Here is a set of examples that a mathematics teacher might write one by one on the board during a lesson on solving equations:

$$\begin{aligned}x &= 0 \\x - 1 &= 0 \\2x - 1 &= 0 \\x^2 - 1 &= 0 \\2x^2 - 1 &= 0\end{aligned}$$

This set can be used at different times in a quadratic equations unit. If the students are new to solving quadratic equations, the first three linear equation examples serve as a deliberate review for students. The skills they use to solve these equations can be carried into solving the last two quadratic equations. Yet, they have to modify their skills in order to solve these quadratic equations. For example, to solve $x^2 - 1 = 0$, after students isolate the variable term, as they have done for solving $x - 1 = 0$, to obtain $x^2 = 1$, they might see x^2 somehow similar to 2 (both terms include an operation done to x), yet different (multiplying x by itself vs doubling x). Thus they have to think about a way different from dividing both sides by 2 to undo the operation in order to obtain x . The equation $x = 0$ is included as the first example because it has the form of the final stage of solving an equation.

This example illustrates a way to help students to construct and activate supportive met-befores for new learning. Each equation in the set repeats the previous

one with a subtle change. Therefore, when students move from one equation to another, they have seen part of the new equation before. The new element in each equation is noted in **bold**. The students' experience with the previous equations contributes to supportive met-befores for their encounter with a new equation. These supportive met-befores facilitate students' interpretation of the new situation and enhance students' confidence as well. The repetition in this set of equations encourages generalization, and the subtle yet salient difference between equations helps to shift student attention to the change and consequently the structure of each equation. The new equation is comparable with the old ones, yet it is not a simple extension. For instance, the change from $x^2 - 1 = 0$ to $2x^2 - 1 = 0$ can be significant from a student's perspective, as many students tend to have difficulty handling a variable term with a coefficient not equal to 1. This kind of change brings in a new structure or attribute to the new equation. Thus it is possible that after working through this set of equations, students establish sufficient met-befores, which make solving equations like $2(x - 1)^2 - 1 = 0$ or $2(\sin x)^2 - 1 = 0$ imaginable.

Problematic Met-Befores

Tall (2013) suggests that the teacher rationalize a problematic situation and make the contradiction between a met-before and a new situation obvious by deliberately having students recall situations during which the met-before works. For example, have students review a situation where "taking away makes less" works before being introduced to taking away negative numbers. Tall believes that this approach also facilitates new learning by enhancing students' confidence: "Giving confidence in an earlier situation may make it easier to see what is different in the new situation to address the issue in a position of confidence" (pp 88–89). From my point of view, Tall's approach is a form of repetition with variation. It starts with a review that activates and reinforces students' met-befores. Then, students encounter problems that resemble the old ones yet are significantly different, making students' met-befores problematic and demanding a breakthrough in students' thinking.

Similarly, teachers also can use repetition with variation to deliberately help students construct problematic met-befores. Here is a set of quadratic relations that can be used as an example of the method proposed.

$$\begin{aligned}y &= x^2 - 1 \\y &= 2x^2 - 2 \\y &= -3x^2 - 3 \\y &= -3x^2 + 1 \\y &= -3x^2 + 12 \\y &= -3x^2 - 3\end{aligned}$$

This set can be used in different grades for various purposes. Assume that this set is used in a Grade 10 mathematics lesson after students have learned finding zero(s) either by factoring or by completing a square. All the relations in this set repeat the ones that come before in some ways. While the first three relations are very much alike, the last three differ quite dramatically. After students have graphed the first three relations, they are likely to form an understanding that these graphs open upward, share the same x-intercepts (1 and -1), and cross the x-axis twice. They might not realize the met-befores' presence until they encounter the last three relations: these met-befores, one after another, are problematized (the fourth graph opens downward, the fifth graph's x-intercepts are 2 and -2, and the sixth graph does not touch the x-axis). This change from the similar examples (as represented by the first three) to not-quite-similar examples (the last three), as Watson and Mason (2006) argue, is important: it breaks the pattern perceived or conjectured by the learners to nudge learners into thinking mathematically. While working on the set of relations presented above, it is possible that students will begin to understand how the coefficient on the quadratic term of a quadratic relation affects the graph, or notice some commonalities of quadratic relations with two x-intercepts opposite to each other, or wonder about the common form of quadratic relations with no x-intercepts (and even whether $x^2 = -1$ is possible) after graphing the fourth, fifth and sixth relation respectively.

Repetition has potential for helping students construct and activate both supportive and problematic met-befores. This possibility is related to repetition's contribution in generalization when combined with variation. Through repeating with variation, students get a chance to generalize patterns, maintain enough supportive met-befores to be confident and perceive differences at the same time. Bateson's (2002) theory of mind asserts that mental activities are triggered by differences. Difference is needed for the mind to work. When the difference is small for a learner, her met-befores can be supportive enough for her new learning so she can progress in a smooth continuity. When the difference is big for the learner, her met-befores can become so problematic for her new learning that a significant change in her understanding is needed for her to move on. Such interruption of the smooth continuity of a learner's cognitive development is essential because it can break the learner's equilibrium and force her into a cycle of rebuilding equilibrium. According to Piaget (in Doll 1993), it is through the recursive cycle of equilibrium-disequilibrium-equilibrium that cognitive development

becomes possible. Clearly, with the help of variation, repetition has the capacity to both reinforce something old and generate something new.

Difference can be either a difference between two things or a change between a thing in time 1 and the same thing in time 2 (Bateson 2002). Thus, a looking-back activity, which invites students to revisit and reflect on the same topic later in time or from different perspectives, can enable them to perceive the difference between their met-befores and their current understanding of the concept. In this sense, repetition can be integrated into students' forward movement (ie, learning new knowledge) and their backward movement (ie, reviewing previously learned knowledge).

Conclusion

Tall's ideas of met-befores, although not entirely new, invite us to reconsider the balance of supportive aspects and problematic aspects in teaching and learning. Tall shows us that the change of met-befores from supportive to problematic is natural for the development of mathematical thinking. Thus, teachers need to consider both supportive and problematic met-befores of students. Repetition can be used to help students construct and activate met-befores, thus benefiting their mathematical growth. It is worth our attention to explore more ways to employ repetition to integrate met-befores, particularly the problematic ones, into teaching and learning mathematics.

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ELL Students' Set-Befores and Met-Befores in Mathematics

Tom Asquith

Recently, teachers and researchers alike have observed growing numbers of English language learning students (ELL) in American and Canadian classrooms (National Council of Teachers of Mathematics 2013). For example, Riel and Boudreau (2012) found that 15 percent of all students in Canadian classrooms do not have English as their first language. In Alberta alone, 17 per cent of all schools responding have ELL students. Of those Alberta schools, 34 per cent have at least 1 to 5 students, 39 per cent have 6 to 25 students, and 26 per cent reported more than 25 ELL students (Alberta Education 2006a, 2006b).

Not surprisingly, this demographic shift poses interesting challenges for Canadian teachers. Given that some ELL students may have received little or no formal instruction in their first language, the experience of school might be novel to them. Even for ELL students who have received prior schooling, there is the challenge of making sense of material in a language with which they are completely unfamiliar (Boaler 2008).

However, what is not obvious is that many ELL students can find a subject such as mathematics also challenging. Although mathematics is sometimes regarded as a universal language (perhaps erroneously), its structures and nuances pose a significant challenge to mathematics students—especially if they are learning mathematics in a second language. (Clark 1975; Barrow 2014). In fact, success in an English language-based mathematics classroom requires a variety of language and coding skills that go beyond merely learning mathematics (Barwell 2005, 2008; Barrow 2014).

In this paper, I aim to examine two things. First, I will look at what challenges ELL students face in terms of learning and understanding mathematics. This will be done by using some of the ideas of the respected English mathematics education researcher and theorist, David Tall, as a guide. Second, we will examine how mathematics teachers can make the task of mastering and understanding mathematics concepts and processes easier for these students.

Set-Befores and Met-Befores of English Mathematical Language

To assist in the discussion of some of the language and coding issues relating to ELL students, I will borrow some concepts and ideas from Tall's writings (Tall 2008, 2013; McGowen and Tall 2010, 2013), in particular the set-before and met-before. A *set-before* is a mental structure that humans are born with, which mature as our brains make early connections. In this category, Tall includes things like posture, identifying direction, social abilities such as gestures (eg, pointing at objects) and so on.

For math educators in particular, Tall (2008, 2013) identified the following set-befores as essential for mathematical understanding:

- The *recognition* of patterns, similarities and differences between mathematical concepts
- The *repetition* of sequences of actions until they become automatic
- The *use of language* to describe and refine the way we think about things

These three set-befores (recognition, repetition and language) form the basic skills required for learning mathematics in all of its forms. Note how the first and last in particular relate to language use. We will return to these in a moment.

In addition to the set-befores, we also need to introduce the idea of a *met-before*. For Tall, a met-before is a mental structure formed in an individual's brain based upon their previous experiences (ie, "built from experience that the individual has 'met-before'" [McGowen and Tall 2010, 169]). Though simple, the idea of a met-before can be quite helpful in dealing with mathematics, because met-befores can be supportive or problematic. A supportive met-before assists or facilitates the learning of mathematical concepts and processes; problematic met-befores, on the other hand, inhibit or make the learning of mathematics more difficult for the student (Tall 2008, 2013; McGowen and Tall 2010, 2013).

To illustrate, let us consider a phenomenon I have often seen in my junior high mathematics classes, when students first encounter the concept of multiplication of fractions. Initially, when students are first introduced to multiplication in elementary school, it becomes engrained that a small number times another small number gives a bigger number as a result (a supportive met-before). However, when students are first exposed to the multiplication of proper fractions in the junior high classroom, confusion often arises. This is because when multiplying proper fractions, the product has a much smaller value—an idea that does not appear to make sense to the students, given their previous experience (a problematic met-before).

Now, in turn, let us examine the set-befores and met-befores as they relate to ELL mathematics learning.

On the Linguistic Set-Befores and Met-Befores of ELL Students

Every student comes to class with his or her own unique experiences. But ELL students come to class with their own set-befores and met-befores that were formed prior to joining a classroom where the medium of instruction is English. This difference in background will greatly affect how the student interacts with the discourse and instruction in the mathematics classroom (Clark 1975; Cuevas 1984; Barrow 2014).

Indeed, per Tall (2013), this background, formed before entering the classroom, is important for the young student if they are to study objects correctly in a mathematical sense. Without the necessary English academic language (in an English-medium classroom), it becomes much more difficult for the students to make the steps necessary toward working in a world of conceptual embodiment, where they are able to take ideas introduced to them as they relate to the physical world and convert them into mental entities they can manipulate with their minds. Consequently, it will be very difficult for them to communicate their understandings to the teacher or to fellow students or to make sense of the materials before them.

Furthermore, in terms of language as a set-before for math instruction, it is a little more complicated, because there are two types of languages that the ELL student must master. First, there is the social language, which is the language of everyday social transactions. Luckily for the ELL student, it has been demonstrated that he or she usually has a useful working grasp of this societal language within two years (Cummins nd, 1979, 2001). However, at the same time as he or

she is working to master the language necessary to function in society, the student must be also able to concurrently learn the mathematical academic language of the classroom. This academic language can take anywhere from five to seven years to master: the rich and complex vocabulary used, the unique technical jargon and symbols, the grammatical conventions unique to mathematical discourse, and specific reading techniques required to make sense of mathematical problems (Cummins nd, 1979, 2001; Collier and Thomas 1989; Slavit and Ernst-Slavit 2007; Alberta Education 2010).

At this point, it is worth keeping in mind that while the student is receiving instruction in a new language, he or she is engaged in a task of trying to compress knowledge into thinkable concepts in mathematics. In particular, the student is placed in a situation whereby he or she must decide whether to try and process the concepts in the student's native language (by first translating it) or in the new English language, or try to make sense of these ideas by using both language systems (sometimes referred to as *code switching*). This turning of the new mathematical knowledge into thinkable concepts is an important step, as noted in Tall (2013):

Compression of knowledge enables us to think of essential ideas, without being diverted by unnecessary detail. Language facilitates this process by enabling us to name important aspects of complicated situations and talking about them to refine their meaning. This focus gives rise to a *thinkable concept*, conceived by the biological brain as a selective binding of neuronal structures, that allows us to focus our attention on it. (p 51; cf p 86)

This is particularly problematic for ELL students. Given the tug-of-war between using their original language and their new English language to categorize new ideas, encapsulate processes based on repeating actions, and define and formulate concepts for mathematical usage, it is not surprising that the research has shown that ELL students regularly create problematic met-befores as they try to make sense of the mathematics that is before them (eg, Lager 2006; Chamot et al 1992; Cuevas 1984; Bernardo and Calleja 2005).

Teachers could identify an ELL student's use and reliance on problematic met-befores by looking to see if the student is generating errors or mistakes via any one of the following pieces of evidence:

- Misusing common words or phrases in understanding word problems (Barwell 2008)—to illustrate, I have observed some of my past ELL students attempt to rephrase a given word problem in

English, but because their vocabulary is still developing, often the final question in the word problem would be read and misinterpreted by the student (for example, a question like “How many brown chickens does the farmer have *altogether*?” may lead the ELL student to try to work out the number of *all* the chickens—not just the brown chickens that the problem asked for).

- Creating and relying on faulty student-created diagrams—in the past, I have observed some of my ELL students quickly draw a figure to assist them in making sense of a question but, unfortunately, their rushed readings lead them to miss key details or words. This leads them to draw initial figures that may have the wrong dimension. For example, consider a question that asks students to find the volume of a circular swimming pool having a radius of 5 metres and a height of 2 metres, but unfortunately, in their work the students draw pictures of cylinders that have *diameters* of 5 metres. Then, after these students have found their answers, when they check their work they do not return to the original text but instead they depend solely upon their diagrams to verify their answers (see Lager 2006).
- Misinterpreting graphics—for example, a diagram of a right triangle may lead the ELL student to conclude that the base needed for an area formula is the largest side (ie, the hypotenuse), considering how it is situated on the page, when the actual base is one of the legs, even when both of the legs have provided numerical measurements (see Lowrie, Diezmann and Logan 2011).
- Not recognizing real-world constraints as they relate to word problems—for example, failing to check and notice that the answer a student provided would not be possible if they stopped and treated it as if it were a true real-world situation (Bernardo and Calleja 2005; Verschaffel, De Corte and Lasure 1994).
- Missing or neglecting semantic aspects of mathematical words, such as the difference between *divided by* and *divided into* (Lager 2006).
- Missing or misreading contextual cues that would suggest an alternative understanding of the mathematical meaning of commonly used words—for example, seeing the words *less than* in a word problem might tempt a student to leap to the conclusion that subtraction is required to find the solution when addition is actually what is called for (eg, Betne and Stanchina 2005).
- Confusing meanings for mathematical words that also have everyday meanings outside of the classroom—eg, *volume*, *table* or *power* (see Lager 2006; Moskovitch 2010).

These difficulties in understanding and using the language successfully in a mathematics classroom can lead to what Tall describes as an *epistemological anxiety*, or a “knowledge-based anxiety,” for these ELL students (Tall 2013, 127). As Tall (2013) put it,

Epistemological anxiety is a sign of inability to achieve the goal of relational understanding in mathematics. To relieve the frustration, the goal may switch to an instrumental understanding of being able to perform the requisite procedures ... with a level of success but a sense of underlying doubt. (p 127)

To clarify, Tall is suggesting that to avoid feeling uncomfortable when doing mathematics, students may be tempted to seek less cognitively demanding methods of understanding the mathematics before them. Thus, they will be enticed to focus on rote learning or on algorithms (ie, instrumental understanding) rather than choosing to build up the conceptual structure or schema needed to extend their knowledge beyond the task at hand (ie, relational understanding) in their work in class. In short, for an ELL student, although the rewards may be immediate and provide a quick and reliable method in a particular context, the success may be short-lived, in that the depth of the mathematical knowledge gained may not readily extend to future mathematics learning (Skemp 1976; Willingham 2009).

We now look at what could be done to help ELL students achieve success in our classrooms.

Designing Curricula and Assessing Progress of ELL Students

Looking at the above, it is clear that the goal of a good mathematics educator when working with ELL students is to promote the formation of supportive met-befores, while avoiding or preventing the formation and/or use of met-befores that could become problematic and thereby inhibit the progress of the student in understanding the mathematics being taught. For this, I offer four rules of thumb to guide teachers.

First, a mathematics teacher instructing ELL students must make efforts to ensure that problematic met-befores are avoided. This could be accomplished by ensuring that nonacademic mathematics language is avoided or minimized in problems, activities and instructions (Beliveau 2001; Lager 2006); ensuring that the language used in the classroom is suited to the level of ability of the ELL students—in particular, the teacher should ensure that the English used is

more likely to be encountered by students in everyday life, and the use of passive tense should be avoided in word problems (Haag et al 2013); being cautious when the mathematical words used in the classroom are polysemic (ie, words that have two different meanings, such as *plane*, *square*, *point* and *volume*)—particularly if such words may commonly be encountered outside of the mathematics classroom (see Jarrett 1999; Dale and Cuevas 1992; Beliveau 2001); and drawing on as many resources as possible to assist the students in the formation of useful, viable and accurate mental images.

For example, realia (ie, objects from real life used in the classroom, such as using a pizza or a wheel to discuss fractions or circles), manipulatives (ie, hands-on instructional tools like fraction tiles or interlocking cubes), drawings and graphics (ie, to illustrate word problems), graphs (eg, from newspapers or magazines), gestures (eg, using a hand gesture to clarify which parts of an equation or geometric shape are being worked on) and making connections to the learners' own culture and community (eg, using a First Nations folk story to assist in the teaching of surface area or volume, or a Cree bead bracelet to discuss ratio and proportion) have all been found to be helpful in guiding ELL students (Moschkovich 2012; Nguyen and Cortes 2013; Barwell 2005; Civil and Menéndez 2010; Civil 2011; Arnason et al 2001). Further, it should be noted that providing materials in the ELL students' first language, where possible, has been deemed very helpful in promoting and furthering the students' mathematical understanding (Abedi, Hofstetter and Lord 2004; Moschkovich 2002, 2012; Barwell 2005; Clarkson 2005; Civil and Menéndez 2010; Civil 2011; Civil and Planas 2012; Nguyen and Cortes 2013).

Second, the mathematics teacher must make efforts to ensure that ELL students are guided through correct problem-solving techniques to allow them to personally filter through their met-befores and understand which certain meanings and concepts are fit to use in certain contexts. It has been acknowledged that instructing students how to tackle word problems is very helpful in guiding ELL students toward the learning of the English language and simultaneously mastering the academic language of mathematics (Cuevas 1984; Moschkovich 2012). As noted in Reyhner (1994) and Jarrett (1999), modelling and guiding ELL students through a systematic approach to problem solving is most helpful. To illustrate, let's consider this word problem:

Allan's cat has a mass that is 2 kilograms less than Bert's cat. Together, their mass is 15 kilograms. How much do they each weigh?

When presenting such a word problem in the classroom, it has been found helpful to introduce ELL students to a comprehension technique, such as Polya's (1957) four-step problem-solving method (Al-Jamal and Miqdadi 2013). That is, initially the students must understand the problem (ie, the question denotes the sum of the masses of two cats, and they don't know the mass of either, only that one cat is two kilograms less than the other). Then the students must generate a plan to solve the problem (eg, assign the variables to the unknown weights of the cats, create the equation needed to solve, and solve for x where x is equal to the weight of Bert's cat); carry out the plan (ie, write out and solve the problem) and then check their work (eg, does the final answer make sense, based on how the question is worded? If one substitutes the found answer of 6.5 kg for Bert's cat into the planned equation, $x + (x + 2) = 15$, then does the equation still balance?).

Training the students to make sense of the text and carefully consider all the numbers, words and symbols present and their relations before attempting to solve the problem is invaluable (Adams 2003; Al-Jamal and Miqdadi 2013). It should be noted that the teaching of comprehension and problem-solving techniques to ELL students should not just focus on the keywords present. While teaching students to rely on the identification of keywords could help in some situations, such a technique could lead to the development of the formation of bad habits, create new problematic met-befores and hamper students' problem-solving skills (Carpenter, Hiebert and Moser 1983; Secada and Carey 1990). To draw on our example, imagine the potential confusion and difficulties a keyword-trained ELL student would have if he or she just focussed on the words, *less than* and *together* (ie, depending on how the words are used, *less than* can refer to a difference in value, an inequality or subtraction, while *together* can refer to an equality or a sum).

Third, when instructing ELL students, a teacher must remember that discussion and culture within the classroom can be quite important in the development of mathematical understanding. As such, the teacher should make efforts to create an environment that can encourage ELL students to attach their learning to their own experiences and participate in mathematical discussions as they learn English (Moschkovich 2012; Barwell 2008). A warm, friendly and tolerant classroom, where students will not be afraid to make mistakes as they explain a mathematical concept, can be extremely helpful for a student who is trying to learn not just mathematics but English as well (Kersaint, Thompson and Petkova 2013, 137–43).

Fourth and finally, care must be taken when performing assessments. Assessments for ELL students need to be continuous and ongoing. Ideally, the language of assessment should be in the language of instruction (Moschkovitch 2012; LaCelle-Peterson and Rivera 1994). This means that for written tests, the words used should be familiar to the ELL student; synonyms in the same word problems should be avoided; complex phrases should be reduced or simplified; the use of conditional clauses (eg, *if ... then*) is limited; and active verb tenses are used (Abedi and Lord 2001; Kersaint, Thompson and Petkova 2013, 127–35). Accommodations such as audiotaping questions, the use of personalized notes for use during tests, access to word walls and glossaries, and the ability to use concrete materials such as manipulatives during tests have been shown to be effective (Kersaint, Thompson and Petkova 2013, 127–35). Further, mathematics teachers of ELL students should avail themselves of more than one type of assessment to provide a more accurate view of the students' mathematical understanding and where they may need assistance (Jarrett 1999; Buchanan and Helman 1997). These other modes of assessments could include performance assessments, project-based assessments, personal interviews and examining written responses (Kersaint, Thompson and Petkova 2013, 127–35; Moschkovich, 2010, 164). It would behoove the teacher when creating assessments to bear in mind a student's set-befores and met-befores—in particular, the concepts, knowledge, skills and applications required for the student to complete the challenge or problem presented (Jarrett 1999).

Concluding Remarks

In sum, care must be taken when working with ELL students in the mathematics classroom to avoid the raising of problematic met-befores. The mathematics teacher faced with the challenge of teaching ELL students must keep in mind the set-befores and met-befores facing his or her students. The teacher must aim to promote relational understandings in his or her students to assist them in the long term to find success, although the students may attempt to head to a weaker instructional understanding. Further, the teacher must remember to take precautionary steps to avoid the problems relating to faulty met-befores, make efforts to develop proper problem-solving techniques, promote a positive culture in his or her classroom to make the learning of mathematics welcoming and, finally, he or she must ensure that the assessments used in the classroom do not place the

ELL students at a disadvantage as they try to master the challenges of two new languages: the language of English and the language of mathematics.

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Mathematical Thinking: An Argument for Not Defining Your Terms

Shelley Barton

A hum of activity ebbs and flows in the room. Seated at round tables, the participants are all engaged, although not all in the same fashion. Some are noisily working in pairs, meticulously laying out rows of neatly organized dominos, row upon row. Some are working independently, slowly, thoughtfully, rearranging the dominos in front of them. As progress is made, the ideas flow through the room, rushing by those who already *know*, and forcing others to pull their attention from their own thoughts and attend to the ideas in the room. This is the picture of a room learning—as Doll (1989) writes, a room that is doing “more dancing and less marching” (p 67). This productive hive of activity is the outcome of a good mathematics problem. However the participants are not students—they are teachers.

This is not a unique occurrence. Put any group of mathematics teachers together with a good problem and the hive will spontaneously erupt. The definition of a *good* problem lurks just out of reach, like an idea from a dream you cannot quite remember. Some mathematics teachers have a good intuition when it comes to judging a problem as good; a select few can even produce good problems effortlessly. All mathematics teachers know a problem is good by the response of their class. It may not even be the problem alone. Instead it may be a perfect storm coming together, out of unidentifiable elements like day of the week, time of the day, the past of the participants, the safety of the learning atmosphere and more. However, like the good problem, the perfect storm is recognizable when it rains down.

As a participant in this particular hive, I discerned new ideas about mathematical thinking as I worked on the mathematics. The problem was a tiling activity with dominos that ended up generating the Fibonacci sequence. I started with the dominos but quickly moved to paper, developing a symbolic representation for the problem so that I could organize the arrangements into types and count using combinatorics. Others in the group were using the language of transformational geometry. This did not occur to me. Some had completely abandoned the dominos and were

working exclusively on paper. Still others were working solely with the dominos.

After the emergence of the Fibonacci sequence was discovered and agreed upon, the group moved on to something else, but I stayed with this problem. I found myself listing the terms of the sequence and the symbolic pattern until the 11th iteration, then looking for a formula that would generate the sequence using sigma notation. There is something about this experience that is deeply connected to the kind of mathematical thinking I would like to support students in developing.

Tall (2013) has two ideas related to mathematical thinking that are connected to this experience with a good problem. The first is the concept of the *met-before*, which Tall initially describes as “a structure we have in our brains *now* as a result of experiences we have met before” (p 23). Later Tall writes that “‘met-before’ refers not to the actual experience itself, but to the trace that it leaves in the mind that affects our current thinking” (p 88). Both of these descriptions create a picture of something left behind in the mind as a result of a mathematical experience that may or may not be a complete object. The decision to use combinatorics to approach the problem was not a conscious one. I did not have the thought “I will use combinatorics,” nor did I *decide* to stop using the dominos and start using a symbolic representation. These approaches seemed to evolve organically, just as equally valid approaches evolved organically in other members of the group (this may point to one of the qualities of a good problem). This could be similar to the experience of a met-before, a residual experience with a mathematical idea that unconsciously appeared in my work and influenced my thinking. A met-before, like *take it to the other side and change the sign*, is supportive for a student in solving $2x - 6 = 10$. When the same student is faced with $2x + 5 = 6x - 10$, then the met-before can become problematic. *What should I move and which side should I take it to?* This residual left behind in the mind can lead students to productive approaches or stop them in their tracks, depending on the situation.

Tall's (2013) idea of "three mental worlds of mathematics" (p 133) forced me to reflect on teaching and doing mathematics differently. The three worlds of mathematics are *conceptual embodiment*, *operational symbolism* and *axiomatic formalism*. For Tall, *conceptual embodiment* occurs when "human perception and action" (p 133) lead to the development of mental images that grow into "perfect mental entities in our imagination" (p 133). Tall uses conceptual embodiment to refer to the initial formulation of thinkable concepts "through recognition and categorization" (p 133). Conceptual embodiment is a "compression from procedure to process that can be seen by shifting the focus of attention from the *steps* of a procedure to the *effect* of the procedure" (Tall 2008, 12), where "compression is seen as a general cognitive process that compresses situations in time and space into events that can be comprehended in a single structure by the human brain" (Tall 2008, 13) that involves both conceptual embodiment and operational symbolism. Compression is not a linear progression through stages. In itself, compression is a process that moves backward and forward as new situations confront met-befores and inconsistencies are resolved. Conceptual embodiment is a process that begins when repeated actions become embodied procedures. Then the procedures come to be understood by their effect and, finally, this effect becomes an embodied concept in the mind.

Operational symbolism occurs when "embodied human actions" (Tall 2013, 133) develop into "symbolic procedures of calculation and manipulation that may be compressed into procepts to enable flexible thinking" (p 133). A *procept* is a mathematical idea that is both a process and a concept (object). A student learning mathematics typically learns one first, then becomes aware of the other and then, through the process of compression, gains understanding of and the ability to utilize both flexibly, as required. A symbol can suggest "a *process* that produces a mathematical *object*" (Gray and Tall 1994, 121). Thus, the symbol for the minus sign has three components embedded into it: it represents the process of subtraction, it represents the concept of difference and it is a symbol with its own meaning. When a child learns to count, *four* is a process. Later when the child adds two to four by counting on from four, *four* has become a mathematical object.

According to Tall (2013), conceptual embodiment and operational symbolism are intertwined and occur together in overlapping layers through compression. However, there is a key distinction. Conceptual embodiment focuses on objects (and actions on the objects) and operational symbolism focuses on symbols

(and the manipulation of symbols) (p 155). By focusing on objects, conceptual embodiment offers the possibility of sensing what happens as a consequence of the operation. "It has an effect that can be seen" (p 155). For example, a student is asked to compare the graph of $f(x) = \frac{(2x-1)(x+3)}{x+3}$ with the graph of $g(x) = 2x - 1$, and decides to use the zoom feature on a graphic display calculator (GDC) in the neighbourhood of $x = -3$. When the hole in the graph of $f(x)$ at $x = -3$ appears, it allows the student to see the effect the $(x - 3)$ factor has in the denominator. The *procedure* of using the zoom function on a GDC shows (perceived through sight) the *effect* of the $(x - 3)$ factor. The conceptual embodiment of this effect represents the visual difference and similarities of the graphs of $f(x)$ and $g(x)$. Manipulating the symbolic representation of by dividing out the common factor $(x - 3)$, is a symbolic *procedure*. This is part of the *process* of simplifying rational expressions to demonstrate the concept that $f(x)$ is equal to $g(x)$, everywhere except at $x = 3$. The compression of the process of simplifying rational expressions with the concept that the original and the simplified functions name the same thing (except at the restrictions) is a *procept*.

Finally, when Tall (2013) uses the term *axiomatic formalism*, he is referring to "building formal knowledge in axiomatic systems specified by set-theoretical definitions, whose properties are deduced by mathematical proof" (p 133). This world is a different world all together. Axiomatic formalism turns the processes discussed thus far upside down. "Instead of studying objects or operations that *have* (natural) properties, the chosen properties (axioms) are specified first and the structure is shown to have other properties that can be *deduced* from the axioms" (p 149). Consider the axiom "If x and y are sets, then the set of pairs (x, y) or the set of pairs (y, x) exists" (Wells 2006, 2137). This is the axiom that allows for the creation of new sets from existing sets and thus forms the basis for the definitions of a relation and a function. Note that this axiom does not require a list of the elements of a set to make the claim the set exists. Axiomatic formalism sits atop the intertwined conceptual embodiment and operational symbolism. Tall's discussion (2013, 2008) of his model of these three worlds is summarized in Figure 1. This model is at the root of my reflection on my teaching in light of the domino tiling. If axiomatic formalism comes after the intertwined conceptual embodiment and operational symbolism, why do I consider it important to start with a formal proof? Is there a better way to help students develop mathematical thinking?

Figure 1

Formal	Axiomatic Formal		
	Conceptual embodiment of this effect	Procept (as both process and thinkable concept)	Object
	Effect (of procedure or action)	Process (seen as a whole)	Process
	Procedure (through perception or action)	Procedure (expressed symbolically step by step)	Action
	EMBODIED	SYMBOLIC	

Returning to my experience with the domino activity, and considering this experience in the light of Tall's (2013) three mathematical worlds, there are some interesting observations. To explain the activity, Figure 2 illustrates the possible tilings for 2×1 space, 2×2 space and 2×3 space respectively.

I moved away from physically manipulating the dominos rather quickly; however, I stayed in the conceptual embodiment world throughout the activity, continuing to draw possible tilings for eleven iterations. The *drawing* of the three possible 2×3 tilings seen in Figure 2 looked similar to $\{III, I=, =I\}$. I continued to draw iterations of the tilings long after I had established the numerical sequence and symbolic representations. I moved quickly into the operational symbolism world, creating symbolic representations for each specific group of tilings. However, I did not leave the conceptual embodied world behind. Beside each of the drawings depicting each iteration was a combination formula equivalent to the numerical values of the number of arrangements of each configuration. These formulas were symbolically manipulated to highlight the various components.

Figure 3 is a sample row from the table I constructed showing how I recorded the information for a 2×8 space. Changing the number of vertical and the number of horizontal dominos generated different arrangements. I did not physically create all of these arrangements. My conceptual embodiment continued as I continued to draw arrangements; however, I did not draw dominos. Thus, the symbolic representation took two different forms. For the first I used symbols to represent the dominos in each tiling. For example, one of the possible 2×8 tilings was depicted by $IIIIII=$, which represented six vertical dominos and two horizontal dominos. The second symbolic representation, combinatorics notation, was used to count the arrangements of each iteration. The six vertical and two horizontal tiling can be arranged in seven different ways, as seen in Figure 3. However, I counted these arrangements ways using combinatorics: of the seven possible places for the two horizontal dominos select one. All the possible tilings for this two-by-eight example are summarized in Figure 3 using my conceptual embodiment (the drawings) and my operational symbolism (combinatorics).

Figure 2

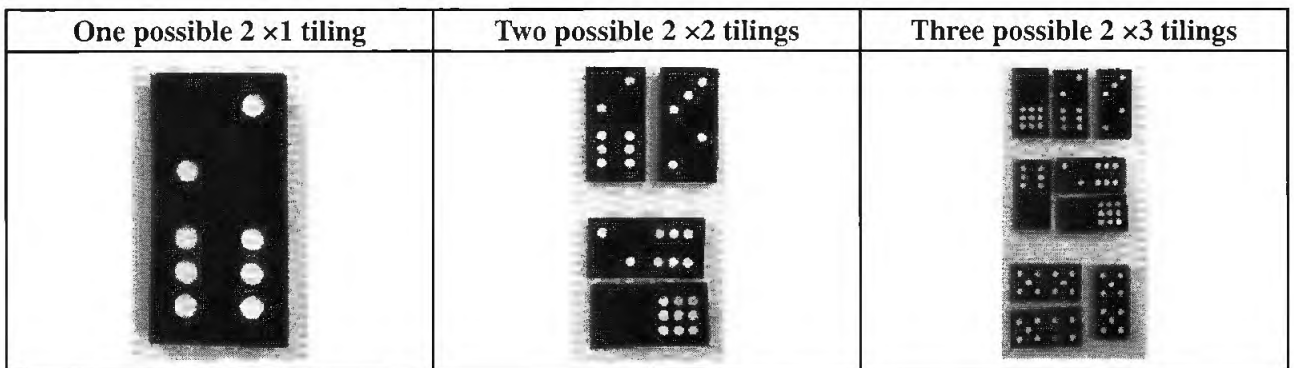


Figure 3

Iteration	Size	Conceptual Embodiment	Operations Symbolism	Term in the Sequence
Number of Dominos	Area (in "½ domino units")	Drawing with the number of possible iterations	Combination formulas	Total number of arrangements
n = 8	2 × 8 rectangle	IIIIII (1) IIIII= (7) IIII== (15) II=== (10) ===== (1)	$\binom{8}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{4}{4}$	34

In the end, I was able to generalize the deconstructed pattern and derive a function that, given the size of the space to be tiled, returns the number of possible tilings, accounting for all possible arrangements of horizontal and vertical arrangements.

In truth, while I have read about tiling activities, this was my first experience actually working on a tiling problem and the first time in a long time I sat down to do mathematics with which I am unfamiliar. Reflecting on this work, it is clear to me that there were occasions I was working in the conceptual embodiment world, which is in some cases distinct from and in others interrelated with occasions when I was creating an operational symbolism. It is equally clear how these two worlds are intertwined and how they worked together to create, in my mind, a procept that represents this problem. Now the physical model, the symbolic representation of this model, the processes used to generate the symbolic iterations and the generalized formula sit together in my mind and my attention can float between them. What is most interesting is that I never moved into the axiomatic formalized world. For years, I have taught students to use the method of proof by mathematical induction, yet I did not produce a formal proof of my generalized formula, which normally I would have. I was satisfied that my formula was valid because my formula was able to generate the numerical values I was expecting for the first three iterations. This was all the *proof* I needed. The compression process of the two worlds—conceptual embodiment and operational symbolism—creates the flexible procept.

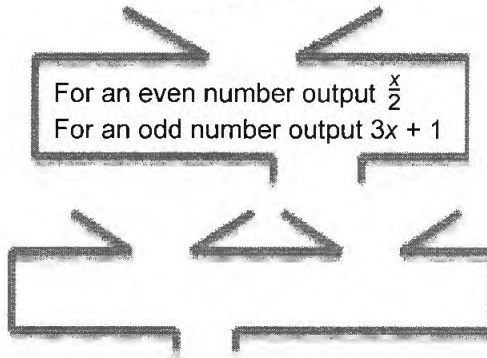
This is the point where I can now explain why Tall's three mathematical worlds absorbed my attention. I teach Grade 10, 11 and 12 mathematics. Functions are a part of each of these courses, and I use functions as the theme for teaching each of these mathematics courses, introducing each topic as a new function.

Volume, surface area and perimeter become functions of their linear measure(s). Quadratics and lines are functions. Simplifying rational expressions becomes investigating rational functions. The logarithmic function is introduced as the inverse of the exponential function. Sequences and series are discrete functions on the positive integers. Probability becomes functions on random variables. Trigonometry begins with trigonometric functions then moves into triangle trigonometry. Using the theme of *function* has allowed me to circle back to topics to create deeper understanding. Yet I always begin with formal definitions like those of relation, function, domain and range. I always begin in the world of axiomatic formalism.

My absorbing thought is why start the topic of functions in the axiomatic formalism world? The answer may be that this approach is my own met-before that is problematic for students. These worlds of conceptual embodiment and operational symbolism lead me to rethink the way I introduce functions. Creating situations in which students can move between these two worlds and up and down the compression continuum may help students build flexible mathematical procepts as opposed to problematic met-befores. One such situation is the function machine. "A function may be represented in a more concrete manner as a function machine (or function box) that has the property that can be imagined both as a process (as a machine taking an input and producing an output) or as an object (a box that contains the machine performing the operations)" (McGowen and Tall 2013, 531). The use of a function machine can allow students to work in the conceptual embodiment world at the same time as they work in the operational symbolism world, which in turn will help students build a flexible function procept. As seen in Figure 4, functions can have rules that vary depending on the input, have more than one input, work in multiple discrete stages or follow other rules in

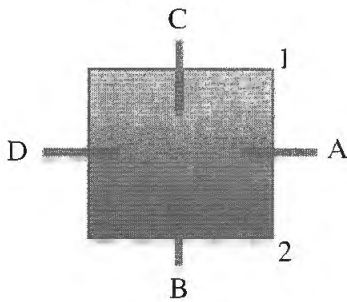
addition to algebraic rules. All of this helps students build a more flexible precept for function at the same time as it intertwines the conceptual embodiment and operational symbolism worlds, which in turn will create a solid platform for the movement into the axiomatic formalism world.

Figure 4



Another example that can help students build flexible precepts is the wrapping function. Consider the image below, Figure 5, a square with sides of length two. Starting at any vertex or midpoint, a path can be traced around the perimeter of the square. This simple process can create many functions.

Figure 5

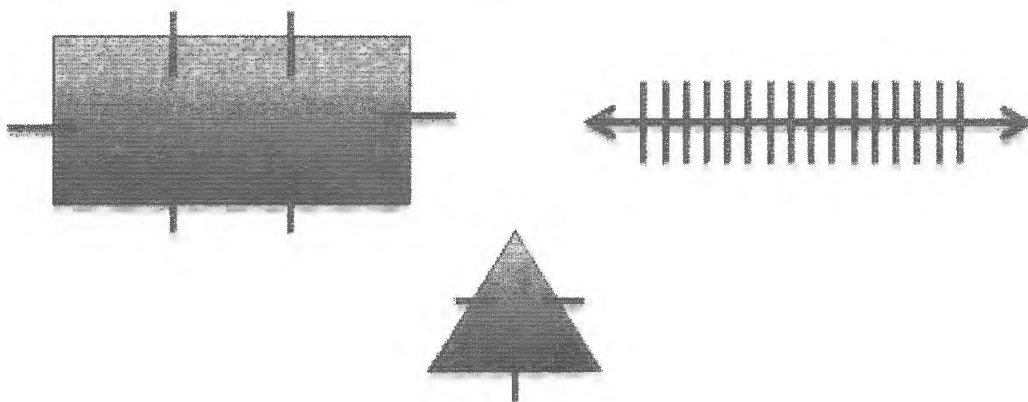


For example, starting at A and travelling in a counter-clockwise direction for a length of two units, the path ends at C. This instruction could be represented as the $(A, 2)$, both of which are inputs. The output would be C. However, this is only one of the many functions that can be generated from the model in Figure 5. The input $(A, 1)$ could be linked to an output of 2 (vertical height), or 0 (horizontal distance from start) or both $(2, 0)$. Cartesian graphs can be generated from a variety of mappings from one set onto another set. An input of $(A, -2)$ would have an output of B, the path length of negative 2 being a path in a clockwise direction. Figure 6 shows examples of different shapes that can be used as the wrapping function and the Cartesian graphs of different mappings that can be drawn.

The periodic nature can be explored in both the conceptual embodied world and the operational symbolism world. Not only does an activity like this lead into the generating idea for sine and cosine functions, it also provides a conceptual embodiment experience that relates to other mathematical concepts such as vectors.

The concept of a function is central to a significant part of the high school mathematics program; therefore, devoting more time to encouraging students to create a flexible precept based on the worlds of conceptual embodiment and operational symbolism may be more beneficial to them than spending time in the axiomatic formalism world. Creating problems linked to mathematics curriculum that allow students to move between the conceptual embodiment and operational symbolism worlds as they feel comfortable will help students create flexible precepts. Additionally, the potential to work in the two worlds may be another quality of a *good* problem.

Figure 6



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Teaching the Trigonometric Ratios Through Embodiment, Symbolism and Formalism

Christopher Charles

Introduction

In today's classrooms, sound pedagogic practices are priceless. In the mathematics classroom, these pedagogic practices must provide students with opportunities to create a deeper understanding of mathematics. A deeper understanding of mathematics means that students understand the underpinnings of mathematical concepts; are able to represent mathematical ideas in multiple ways (concrete, numerical, graphical, geometrical, and symbolic); are able to use appropriate "mathematical language, vocabulary, and notation to represent ideas, describe relationships, and model situations"; and are able to "make meaningful connections within mathematics, to other content areas, and to real-life situations" (New Mexico State University nd, 1).

If students are to attain these standards and achieve deeper understanding of mathematics, teachers must create the pedagogic opportunities that will engage students in higher levels of thinking. D'Ambrosio, Johnson and Hobbs (1995) proposed twelve pedagogic strategies that teachers can employ to engage students in higher levels of thinking.

1. Encourage exploration and investigations: involve students in activities that will help them to construct mathematics knowledge as well as explore and investigate mathematics ideas.
2. Use students' prior knowledge: students bring to class different world knowledge and experiences that affect the way they view and solve problems.
3. Use manipulatives: the proper use of manipulatives is critical to the understanding of new mathematical ideas.
4. Use real-world problem-solving activities: link mathematics and the real world through a wide range of problem-solving activities.
5. Integrate mathematics with other content areas: this helps students to apply previously acquired knowledge to new situations.
6. Use culturally relevant materials: this helps to motivate students as the mathematics relate to students' different cultures and interests.
7. Use technology: saves time by performing complex calculations quickly and allows for drawings and demonstrations that are difficult if not impossible to achieve using a chalkboard.
8. Use oral and written expression: explaining their thinking orally and/or in writing help students to organize their thought and solution strategies.
9. Encourage collaborative problem solving: this encourages active involvement in learning by sharing and negotiating meaning, verbalizing understanding, and providing constructive criticism.
10. Use errors to enhance learning: to simply say an answer is correct or incorrect is insufficient if students are to improve their understanding of mathematics. The thinking behind students' errors must be explored if misconceptions are to be ironed out.
11. Offer an enriched curriculum and challenging activities: all students must be exposed to mathematically demanding tasks. This allows students to develop their critical thinking skills and problem-solving ability beyond routine and watered-down procedural tasks.
12. Use a variety of problem-solving experiences: use a wide variety of problems to include problems that can be solved in different ways, with more than one correct answer, and that may involve decision making and allow for different interpretations. (pp 125–35)

These twelve pedagogic strategies, however, must be used within a framework that will enhance their effectiveness whereby students and teachers will gain maximum benefit from their use. To this end, I propose Tall's framework called the "three worlds of mathematics."

In this paper I demonstrate how the sine, cosine, and tangent ratios can be introduced to secondary

(Grade 9 or 10) students using Tall's three worlds of mathematics to scaffold their learning. I begin by providing theoretical perspectives of the three worlds—embodiment, symbolism, and formalism (Tall 2013)—and draw on the work of others to further develop the ideas. I also provide a brief recap of the concepts of sine, cosine, and tangent ratios along with some areas of difficulty commonly experienced by students. Some plausible activities and relevant problems along with some guidelines are also provided. I conclude this paper with a word of caution on how these activities may be interpreted.

The Three Worlds of Mathematics

Embodiment involves the use of one or more body senses to help internalize abstract mathematical ideas through the manipulation of physical objects and/or through physical actions (Tall 2013). Tall's notion of embodiment is supported by Husserl (cited in Behnke 2011), who argued that embodiment goes beyond practical action, but is an essential factor that influences the attainment of deep understanding. Dubinsky (2000) also reflected upon the "widespread agreement that mathematical ideas begin with human activity and move from there to abstract concepts" (p 216). These perspectives are all in keeping with the proposed pedagogic strategies of D'Ambrosio, Johnson and Hobbs (1995), which encourage exploration and investigation and make use of manipulatives, but go beyond these strategies to include the use of body actions. Here, the essential point for teachers is that engaging students in actions relevant to mathematical concepts help these students to better internalize the concepts.

Symbolism, according to Tall (2013), "grows out of embodiment by focusing on the actions on objects rather than on the objects themselves" (p 141). Tall refers to this as *operational symbolism*—a notion that calls for students to perform mathematical operations on symbols. That is, while embodiment focuses on the physical properties of objects, symbolism focuses on manipulating these properties. Manipulating objects' properties includes, but is not limited to, performing calculations, writing and verbalizing mathematical symbols and notations, substituting values for variables, rearranging symbols, using relationships among properties, and connecting properties to

other content areas and real-world situations. Hence, many of D'Ambrosio, Johnson and Hobbs's (1995) pedagogic strategies can be applied to Tall's notion of symbolism, thus providing rich pedagogic experiences for students.

It is important to note at this juncture that although symbolism is purported to grow out of embodiment, the transition is not always a smooth one and teachers need to afford students time and support in making that transition. D'Ambrosio, Johnson and Hobbs (1995) recommend a three-stage transition process: (1) the use of embodiment alone, followed by (2) the use of the embodiment together with symbolic representation, and then (3) the use of the symbolic representation alone. The second stage is critical, and adequate time must be spent during this stage if students are to make a successful transition from embodiment to symbolism (D'Ambrosio, Johnson and Hobbs 1995).

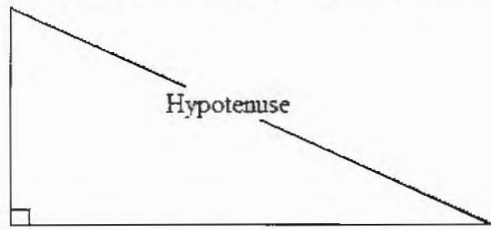
Formalism, according to Tall (2013), may take on several meanings in mathematics education. For instance, Tall refers to Piagetian formalism, when an individual reaches the formal operational stage and that person's thought process no longer needs the involvement of physical referents. Hilbert,¹ on the other hand, conceptualizes formalism as focusing on axiomatic definitions and proofs, and it is this conceptualization that Tall makes use of in his framework. Therefore, *formalism* in this paper is restricted to the use of formulas, production of images (diagrams) and basic proofs of relationships. Drawing on the strategies of D'Ambrosio, Johnson and Hobbs (1995), tasks given to students at this level should be cognitively demanding, should make use of collaborative/cooperative groupings and should make use of technological resources such as calculators, computers and the Internet.

Concepts of Sine, Cosine and Tangent Ratios

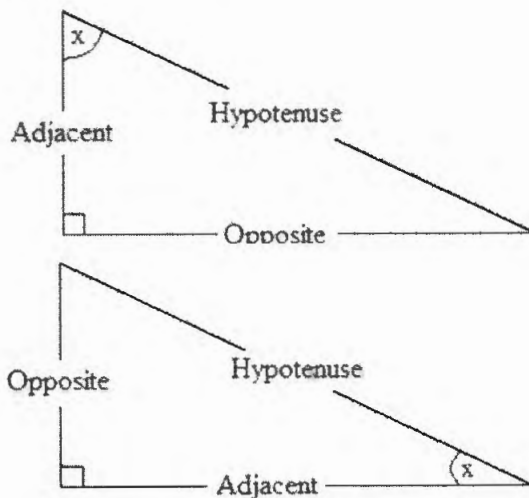
Sine, cosine and tangent, the three primary trigonometric ratios, are used in this paper to demonstrate how embodiment, symbolism and formalism, as proposed by Tall (2013), can be employed in teaching mathematics. To provide the background for this demonstration, however, a brief revision of these ratios and their underlying concepts is presented.

¹ Hilbert (1862–1943), one of the first formalists, believed that all theorems could be proved using only the axioms of the system.

Trigonometry is grounded in the study of triangles. The basis for the trigonometric ratios is the right-angled triangle. In a right-angled triangle, one angle—the largest—measures exactly 90° and that angle is always opposite the longest side—the hypotenuse.



This is an essential fact, and it is important that learners know that the hypotenuse is opposite the 90° angle, the angle in the above triangle marked with the conventional symbol of a right angle—a box rather than an arc. That is, students need to know that the longest side of any triangle is always opposite the biggest angle. A proof of this theorem can be found at www.youtube.com/watch?v=LeeiVVAoPUk. In relation to the ratios, the two other sides of the right-angled triangle are called the *opposite* and *adjacent* sides but, unlike the hypotenuse, their positions are not fixed but change as the angle of reference changes. That is, the opposite side and the adjacent side of a right-angled triangle switch when the reference angle—which is never the right angle—switches. This is demonstrated in the following two triangles, with x being the reference angle in both cases.



The sine, cosine and tangent ratios are formed using the sides of the right-angled triangle as follows:

- Sine of angle $x = \frac{\text{opposite side}}{\text{hypotenuse}}$ or $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$
- Cosine of angle $x = \frac{\text{adjacent side}}{\text{hypotenuse}}$ or $\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$
- Tangent of angle $x = \frac{\text{opposite side}}{\text{adjacent side}}$ or $\tan x = \frac{\text{opposite}}{\text{adjacent}}$

I have taught and observed others teach the sine, cosine and tangent ratios on numerous occasions throughout my twenty-plus years as a mathematics educator. I have observed that a very common approach to teaching these concepts is to present students the above formulas and an acronym—SOH-CAH-TOA—for remembering them, followed by a few examples with explanations. Drawing on Tall's work, this approach places instruction in the world of formalism without affording students the opportunity to embody the concept and/or develop critical skills in symbolism. As a consequence, students may not develop a deep understanding of the primary trigonometric ratios, which in turn impedes their understanding of other concepts in trigonometry that make use of these ratios.

One common sign of misunderstanding is that many students struggle to identify the opposite and adjacent sides in problems where the right-angled triangle forms part of a bigger shape, where right-angled triangles are orientated in ways students are not familiar with or where the right angle is not shown but must be calculated. A second common sign of misunderstanding occurs when students struggle with identifying which ratio must be used when given a side and an angle of the right-angle triangle and asked to find another side, or when given two sides and asked to determine an angle. It is in an attempt to reduce these problems that Tall's model of the three worlds of mathematics is being proposed as an alternate approach to teaching the sine, cosine and tangent ratios.

Teaching with Embodiment

Since embodiment involves the use of the physical body—the manipulation of physical objects and/or bodily actions (Tall 2013)—pedagogic strategies employed at this stage call for students to be engaged in physical activities that will help them make sense of new concepts. Therefore, when the concepts of sine, cosine and tangent ratios are being introduced, students can be called upon to act out some of the underlying concepts, and the teacher can pose some relevant questions to help direct students' thoughts and, hence, their internalization of these concepts. Following are two activities that demonstrate how teachers can help students internalize underlying concepts of hypotenuse, opposite side and adjacent side.

Activity 1: Walking the Lines (50 minutes): The aim of this activity is to help students understand the difference in the hypotenuse, opposite and adjacent sides of a right-angled triangle.

- In an open space, draw (or use masking tape on the floor to mark off) a large right-angled triangle.

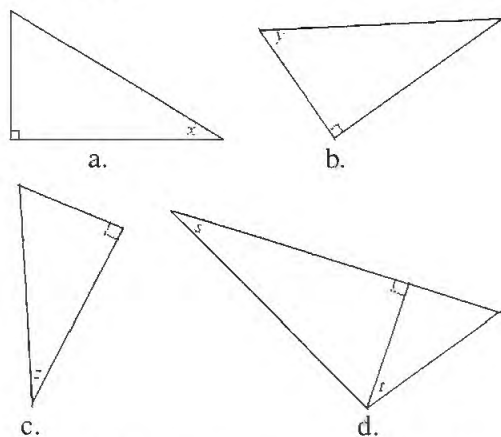
The triangle must be large enough for students to walk along its sides. The teacher stands at the right angle and asks students to walk along the hypotenuse. The teacher then changes position to stand at one of the acute angles and asks students to walk along first the side opposite to where he/she is standing and then to walk along the side adjacent to where he/she is standing. Repeat the activity with the teacher (or a student) standing at the other acute angle. This time, however, instead of saying "side opposite to where he/she is standing" or "side adjacent to where he/she is standing," use the terms *opposite side* and *adjacent side*. Repeat as needed.

Suggested key questions to pose to students:

1. In relation to the right (90°) angle, where is the hypotenuse? (Opposite to the right angle.)
2. The right angle is the largest angle in the right-angled triangle. What can you say about the hypotenuse in relation to the other sides? (The hypotenuse is the longest side of the right-angled triangle.)
3. In relation to where the teacher was standing, what do you notice when walking the opposite side versus the adjacent side? (When walking the opposite side you will never meet the teacher, but you meet the teacher when walking the adjacent side.)

Activity 2: Point and Feel (50 minutes): This is an extension of activity 1 but more individualistic in nature. It may serve as a follow-up activity for students who still have difficulty in identifying the hypotenuse, opposite and adjacent sides.

- Provide students with a sheet (or several sheets) with several right-angle triangles drawn in different orientations and with the right angle and one other angle marked.



Ask students to place one finger at the right angle of each triangle and run another finger along the hypotenuses, then label each. With

one finger at the lettered angles, let students run a finger along the opposite sides and label them, then along the adjacent sides and label them.

Suggested key question:

1. What do you notice about the opposite and adjacent sides in shape **d**? (In shape **d**, the opposite side of angle *s* is the adjacent side of angle *t*. In some composite shapes, one side may serve as both opposite and adjacent.)

Teaching Through Symbolism

While the embodiment activities focused on the physical lines and the student's kinesthetic interaction with the lines as mathematics concepts (hypotenuse, opposite side and adjacent side), activities in this section focus on actions that can be taken on the relationships that exist among these lines. Here, students are called upon to perform calculations, write and verbalize mathematical symbols and notations, substitute numbers for symbols, rearrange symbols, and identify and use existing relationships among the ratios. These are just a few skills, taken from the world of symbolism, that are necessary if students are to successfully solve problems involving various contextualized situations. Following are three activities that demonstrate how teachers can help students develop their skills in symbolism.

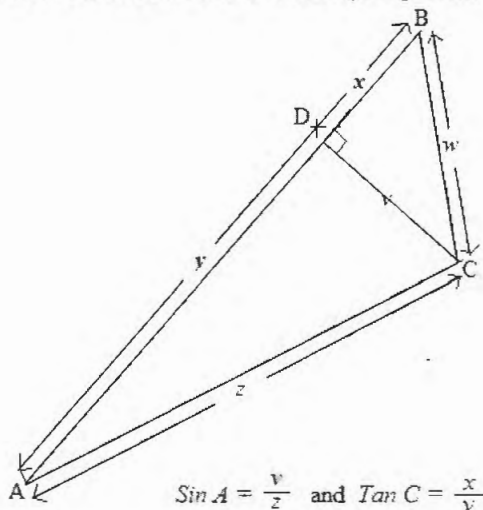
Activity 3: Deck of Cards: This activity is designed to help students develop fluency in recognizing and writing the sine, cosine and tangent ratios using symbols.

- The names *sine*, *cosine* and *tangent* must be written on separate index cards. The ratios without the names must also be written on separate cards. The inverse of these ratios must also be written on separate cards. All cards are placed in a deck and drawn at random. If a name is drawn, students must explain/discuss briefly the ratio. If a ratio is drawn, students must give its name and say something about it. The inverse ratios must be identified and discarded, with reason. Students are encouraged to use gestures, diagrams and other available resources in their explanation and/or discussions. I encourage this as a whole-class activity, but it can also be done in small groups as needed.

Activity 4: Journal Writing: This activity is designed to help students make meaning of sine, cosine and tangent ratios as presented in problems.

- Provide students with one or more problems similar to the one following and ask them to write about the symbolic statement contained therein. Encourage

them to express any difficulty experienced in understanding the symbols and if/how they were able to overcome such difficulty. Students are reminded of activity 2 and are encouraged to use it to help them identify the ratios in the diagrams. I encourage this as an individual activity with opportunities for students to share in a whole-group setting.



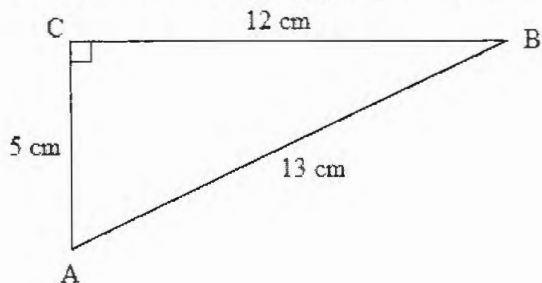
Both activity 3 and activity 4 bridge the gap between the world of embodiment and the world of symbolism because they provide students with opportunities to use their bodies while performing actions in symbolism. This is important to help students make the transition from embodiment to symbolism, and time and practice must be given to individuals as needed.

Activity 5: Multi-Levelled Computational Tasks:

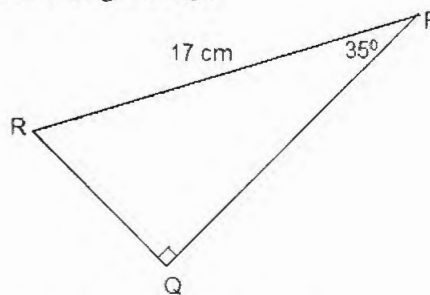
This activity will improve students' fluency, competence, understanding and confidence in manipulating the trig-ratios in symbolic forms.

- Teachers must engage students in problem-solving tasks in which a number of mathematics problems are solved through the manipulation of the symbolic representations of the sine, cosine and tangent ratios. The following provide the structure of the problems needed to develop what Tall (2013) calls operational symbolism.

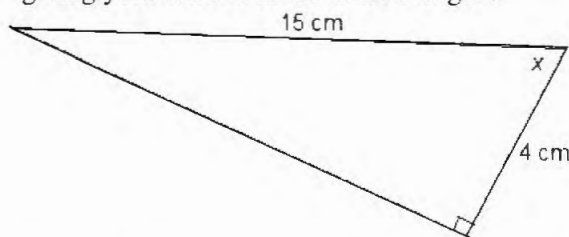
- Calculate $\sin 35^\circ$.
- Given the following diagram, find $\cos B$.



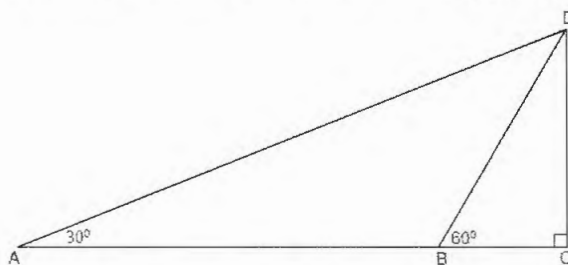
- Find the length of QR.



- Find angle A if $\tan A = 0.381$.
- From the following diagram, calculate angle x, giving your answer to the nearest degree.



- Given the following diagram, show that $\frac{\tan 60^\circ}{\tan 30^\circ} = \frac{AC}{BC}$.



Student textbooks such as *Foundations and Pre-Calculus: Mathematics 10* (Davis et al 2010) provide a wide range of problems that teachers may consider using with their students.

Teaching Through Formalism

Using Hilbert's conceptualization of formalism, as presented by Tall (2013), I focus this section on the use of axiomatic definitions and basic proofs. I use this restriction because the formal world of sine, cosine and tangent extends beyond the scope of this paper. Therefore, *formalism* in this context is restricted to the use of formulas, production of images (diagrams) and basic proofs of relationships among the sine, cosine and tangent ratios. I encourage the use of collaborative/cooperative groupings because this pedagogic practice encourages discussion, which helps students to refine their thinking (D'Ambrosio, Johnson and Hobbs 1995). I also encourage the use of computers and other forms of technology, which may help students save valuable time while performing

complex calculations. Following is an activity that presents four situations in which students will be required to delve into the world of formalism.

Activity 6: Problems and Proofs: Given the following formulae, solve the following problems and/or derive the relevant proofs, giving meaningful explanations for each step in your solution or proof.

Formulae:

- $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$
- $\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$
- $\tan x = \frac{\text{opposite}}{\text{adjacent}}$

1. A home owner wishes to build a ramp to his front door to make it wheelchair accessible. The door is 1.5m above ground level and the ramp must have an angle of elevation of 15° . What will be the length of the ramp?
2. A man 2m tall standing at the top of a cliff 25m high observes two ships in a straight line at sea. He observed one ship at an angle of depression of 30° and observed the other at an angle of depression of 55° . How far away from each other are the ships?
3. Prove that: $\tan \theta = \frac{\sin \theta}{\cos \theta}$
4. Given that $\sin A = \frac{4}{5}$ and $\tan x = \frac{5}{12}$, prove that $\tan A - \cos x = \frac{16}{39}$

Reflecting on the Instructional Progression

Learning is not a linear process, and to engage students in a few well-sequenced activities does not guarantee that they will achieve a deep understanding of the concepts being taught. To gain a deep understanding of concepts, learners need to move back and forth among the three worlds of mathematics, and need to do so at their own pace. Therefore, in teaching sine, cosine and tangent ratios using Tall's (2013) worlds of embodiment, symbolism and formalism, it should be expected that all learners will not progress at the same pace and that some learners will need to revert to the level of embodiment while working at the symbolic and/or formal levels. This is healthy and should not be discouraged. In this regard, teachers will find repeating activity 2, or variations of it, at appropriate times useful in helping students overcome difficulties such as the inability to correctly identify opposite and adjacent sides in unfamiliar shapes. I also feel compelled to state that the suggested activities are not exhaustive and neither are they cast in stone, but may be adapted to meet the pedagogical needs of teachers who choose to use them.

Conclusion

This article presents secondary school mathematics teachers with a framework for teaching mathematics in a way that will provide students with opportunities to gain a deeper understanding of mathematics concepts. It uses the concepts of sine, cosine and tangent ratios to illustrate how embodiment, symbolism and formalism can be enacted in the classroom. These illustrations are given in the form of activities that may be adapted and used by teachers. While a comprehensive list of activities is not given, it is nevertheless my hope that the activities presented will raise and/or heighten teachers' awareness of activities spanning the three stated worlds of mathematics and provide them with an entry point into acts of embodiment to complement the acts of symbolism and formalism that inform most teachers' pedagogy. Further, the tasks and questions posed, especially in the symbolism and formalism sections, are commonly found in textbooks. The difference here is that tasks are presented so as to take into account the three worlds of mathematics in an order that scaffolds understanding.

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Teaching Mathematics for Understanding: Approaching and Observing

Priscila Dias Corrêa

Introduction

Mathematics teaching has been the target of criticism recently (take, for example, the extensive media response to the latest PISA results). In part, these criticisms are derived from the belief that doing mathematics regardless of the nature of a learner's understanding is sufficient for schooling purposes, and that thinking mathematically is necessary only for mathematicians. These beliefs seem to be deeply rooted in our society and are difficult to change. Because of that, new approaches for teaching mathematics are being judged negatively. Sierpinska (1994) states that

Sometimes understanding is confused (or deliberately merged) with knowing, and argued that this is perhaps not a desirable thing to do in education. Unfortunately, institutionalized education is framed to develop students' knowledge rather than thinking. This is a heritage of a long-standing tradition. (p 68)

Regardless, many different approaches to teaching mathematics for understanding have been investigated over the last few decades (Kilpatrick, Swafford and Findell 2001). In spite of positive learning outcomes demonstrated by many of the approaches, discussions continue about what it means to teach for mathematical understanding. Therefore, one purpose of this paper is to discuss teaching mathematics for understanding by considering its relevance, advantages and challenges, as well as the factors that contribute to the implementation of mathematical understanding activities in class. The second purpose is to present three theories of mathematical understanding: Pirie and Kieren's (1994) model of the growth of mathematical understanding; Tall's (2013) model of the three worlds of mathematics; and Kilpatrick, Swafford and Findell's (2001) model of mathematical proficiency, each of which can be used to observe students' mathematical understanding.

Teaching Mathematics for Understanding

As many teachers are aware, mathematical understanding can be related to more than one kind of understanding in mathematics. Skemp (2006), for instance, proposes two different meanings for the word *understanding*. He claims that understanding can be instrumental or relational. *Relational understanding* means "knowing both what to do and why" (p 89), while *instrumental understanding* is described by "rules without reasons" (p 89). This paper will refer to relational understanding when discussing teaching for understanding.

Teaching for understanding presents advantages. For students to develop understanding, the required instruction will correspond to what Ben-Hur (2006) calls concept-rich instruction—ie, instruction based on conceptual knowledge. As a consequence, the constructed knowledge should be stronger and longer lasting; hence students can draw on the meanings and understandings they have assimilated rather than depending on (perhaps long-forgotten) memorized facts and processes when they encounter new mathematical situations and problems. Kilpatrick, Swafford and Findell (2001) remind educators that if students cannot make different associations among the learned concepts, they might not be able to use them in various problem-solving situations. In this sense, the students' mathematical knowledge will be compromised because they do not understand what they are learning.

Stein, Grover and Henningsen (1996) claim that Complete understanding [of mathematics] ... includes the capacity to engage in the processes of mathematical thinking, in essence doing what makers and users of mathematics do: framing and solving problems, looking for patterns, making conjectures, examining constraints, making inferences from data, abstracting, inventing, explaining, justifying, challenging, and so on. (p 456)

But how does one achieve this “complete understanding” of mathematics? Involving students in high-level mathematics activities (cognitively demanding activities) seems to be an effective way to teach students for mathematical understanding; however, this is a challenging task. Henningsen and Stein (1997) argue that many factors are necessary to support engagement in cognitively high-level mathematics thinking during mathematical activities, including (1) building connections with students’ background knowledge; (2) providing students with an appropriate amount of time to do the activity—not too little and not too much; (3) emphasizing meaning and requiring students to explain their understandings; (4) having students model their thinking processes and strategies; (5) providing scaffolding when necessary; (6) enabling students to self-monitor and self-question; and (7) having students draw conceptual connections. The authors point out that the activity itself will not be able to engage students in mathematical thinking if students are not properly provided with a supportive environment, including the specific assistance a teacher can provide.

If teachers are aware of these factors, why is relational understanding so difficult to achieve? The problem is not due to lack of interest or commitment by teachers. Indeed, traditional instruction is losing time to instruction that values relational understanding, rather than instrumental understanding (Silver et al 2009). So why is teaching for relational understanding so difficult to implement? In their research, Silver et al indicate that, when talking about their practice, teachers mention many different goals that they aim to achieve, of which relational understanding is but one. As the amount of necessary work to accomplish all these goals is great, teachers may need to make choices and choose some goals at the expense of others (Silver et al 2009). Henningsen and Stein (1997) highlight some issues that might hinder the engagement in the mathematical thinking process: (1) inappropriateness of the task, (2) classroom management problems, (3) inadequate amount of time spent with the task, (4) lack of accountability, (5) challenges that become nonproblems; and (6) focus on finding the right answer. Further, Henningsen and Stein explain that high-level activities require students to take risks that they might not be willing to take. This may explain why teachers feel pressured to reduce their lessons to a set of step-by-step instructions or to reduce their expectations of what learners need to do within a learning activity.

Unfortunately, simple awareness of issues does not mean that teaching for understanding is trouble-free. Indeed, recent research (Silver et al 2009) has shown

that teaching can still be based on old strategies, and founded on procedural knowledge and instrumental understanding. But telling students what to do or how to do and requiring them to do only low-demanding activities will not develop their mathematical understanding. If students do not involve themselves in class activities that require them to think, reflect, try different strategies and go over the activity again and again, they will be just doing manipulations based on someone else’s guidance. Hence, students will not be developing and enhancing their mathematical understanding. In this sense, it is important for teachers to observe students’ understanding processes, in order to help them to benefit the most from the activities they do. The next section describes some models that might be useful in the course of observing student meaning making.

Observing Students’ Mathematical Understanding

As teachers invest in teaching mathematics for understanding, it becomes necessary for teachers to have conceptual tools to observe the effectiveness of their teaching practice as it results in learner understanding. The capacity to observe student understanding is an important aspect of the whole process of instruction, because it enables improvements and encourages the ongoing promotion of mathematical understanding in lessons. By examining the following task and one possible solution for it, we can illustrate some ideas about how students’ mathematical understanding is demonstrated when solving a problem. Note that this analysis is based solely on written records of one student’s response to a task and the understanding displayed in the student’s working papers. This partial data from the student’s work most certainly means there will be incompleteness in the analysis of her mathematical understanding.

The Task

Consider the two following cell phone plans. Compare them and discuss best customer options.

1. Plan A—The customer has a total of 200 province-wide minutes of outgoing and incoming calls for \$22 per month. Extra minutes will cost \$0.40 each. Text messages are unlimited. As for data usage, the customer will pay per use: up to 25MB, \$4; up to 100MB, \$12; up to 500MB, \$20; up to 1GB, \$30; over 1GB, \$ 0.02 per MB.
2. Plan B—The customer has unlimited Canada-wide talk, text and data usage for \$42 per month.

One Possible Solution

The student recognizes that there are linear functions involved in the problem and thinks about graphing the situation as a way of comparing the different scenarios. In order to accomplish that, the student analyzes each situation and comes up with two functions for plan A and one function for plan B.

Plan A

The customer pays a minimal monthly amount no matter how low her/his cell phone call usage. If the customer uses more than 200 minutes, she/he will pay the extra cost following a proportion. This situation can be represented by the following piecewise function.

$$f(x) = \begin{cases} 22 & 0 \leq x \leq 200 \\ 22 + 0.40(x - 200) & x > 200 \end{cases}$$

Plan A

As for data usage, if less or equal to 1GB, the cost is constant according to the usage. If the usage is more than 1GB, the cost is a function of the usage. This situation can be represented by the following piecewise function.

$$g(t) = \begin{cases} 0 & t = 0 \\ 4 & 0 < t \leq 25MB \\ 12 & 25MB < t \leq 100MB \\ 20 & 100MB < t \leq 500MB \\ 30 & 500MB < t \leq 1GB \\ t \times 0.02 & t > 1GB \end{cases}$$

Then the student realizes she can add $f(x)$ to each case of $g(t)$, in which $g(t)$ is a constant function, resulting in five different functions. She could have added $f(x)$ to the last case of $g(t)$, too. However, because the variables are different, she would have a function of two variables, and this is problematic for her. Thus, the student's choice was to graph the five different functions to illustrate at least five scenarios of monthly cost (in case the chosen plan was plan A). These five graphs can be seen in Figure 1.

Plan B

The customer pays a constant amount no matter how low or how high her/his cell phone usage. This situation can be represented by a constant function, which means the customer will always pay the same.

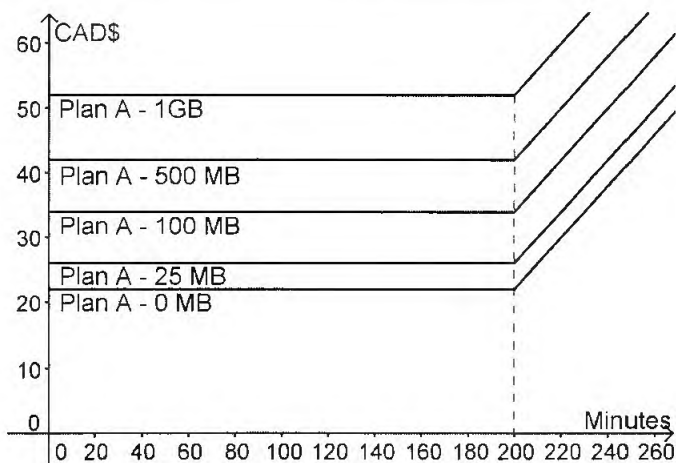
$$h(x) = 42$$

The graph of the function $h(x)$ would be constant on $y = 42$, which partially coincides with the plan A 500MB scenario.

Based on the described reasoning, the student was able to compare and discuss the different scenarios and come to a personal conclusion in terms of the customer's best options. It is important to notice that other aspects could be considered in this analysis as well—for instance, whether the plans are provincewide or countrywide and what is the customer need in those terms.

Different ideas and nuances can be associated with students' mathematical understanding when analyzing the above solution. In the same way, different models can be used to observe for understanding. In this paper, three contemporary models will be described for this purpose: Pirie and Kieren (1994), Tall (2013), and Kilpatrick, Swafford and Findell (2001). These models do not reflect a progression; that is, they are unrelated to each other and have different underpinnings.

Figure 1:
Monthly cost for five different scenarios



Pirie and Kieren's Model

Tom Kieren, a retired professor from the University of Alberta, and his colleague Susan Pirie developed a model for observing mathematical understanding in action (Pirie and Kieren 1994). Their model considers eight different stages (Figure 2), and students are observed in relation to the stages they demonstrate during their learning process. Although this model proposes that stages are increasingly more comprehensive, students' levels of understanding do not necessarily evolve in a linear process. Quite the opposite—students can experience a nonlinear process that might fold back to previous stages according to each student's learning constraints and affordances.

The first stage of Pirie and Kieren's (1994) model, the *primitive knowing* stage, refers to the background knowledge that the student brings with him/her to start developing other content. The *image making* stage builds on this prior knowledge as the learner makes distinctions in previous knowing and uses it in new ways (p 170). Once students can take actions without directly associating a new situation to the original one, students will have achieved the *image-having* stage. The next step, if it were linear, would be *property noticing*, when students are able to infer properties based on the images they have constructed. The *formalising* phase is accomplished when students are able to abstract "a method or common quality from the previous image-dependent knowhow which characterised her noticed properties" (p 170). After that, students are expected to come up with new understandings, in the so-called *observing* stage. After this point, once students are able to think in terms of theories, they have achieved the *structuring* stage, which "occurs when one attempts to think about one's formal observations as a theory" (p 171). Finally, when students can follow a rationale and pose reasonable questions about what they know to create new structures, new forms and new mathematics, they are at the *inventising* stage.

Analyzing the given example based on Pirie and Kieren's model, it is possible to say that the student starts on the primitive knowing stage, in that she is able to bring her knowledge into the situation. She is able to associate the task with linear and constant functions. Then the student enters the image-making stage, as she takes up graphical representations as a tool for analyzing the problem. As the student analyzes each cell phone plan and comes up with the functions for each situation and their formal equations, we observe the student at the image-having stage. Realizing that she can add two functions as a

way of representing two scenarios (call usage and data usage) in only one function demonstrates property noticing. She is assimilating function properties such as adding. After this stage, the student goes back to the image-making stage, because she needs to feed her graph with the added functions she has just found out, in order to create new objects with which to make meaning. Finally, based on this image, she is able to go to the observing stage to come up with her understandings and conclusions about the task. The formalising stage, the structuring stage and the inventising stage seem not to be present. In the case of the formalising stage, there is no data to determine whether the student now understands that there is a class of piecewise functions that can be used across many situations. Further, a single episode such as the one presented does not normally lead to structuring and inventising, since it is too narrow in scope.

Following the student's path based on Pirie and Kieren's (1994) model is a nice way of acknowledging the diverse ways that a student can choose to pursue when solving a problem and also the different needs that a student might have. In this case, for example, the student needed to go back to the image-making stage once the graph became the basis of her reasoning. The teacher has a significant role in this scenario—not the role of inducing his or her students to take a specific path, but the role of encouraging the students to act with what they know and to pursue deeper understanding of the situation using mathematics.

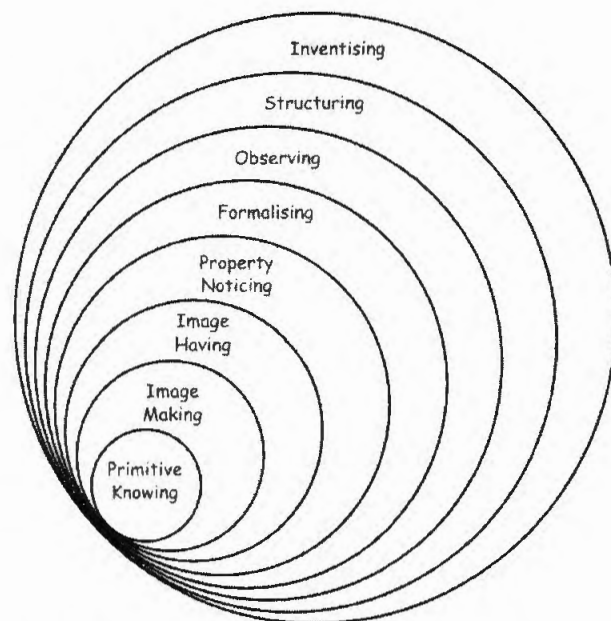


Figure 2: Pirie and Kieren's model of the growth of mathematical understanding

Tall's Model

Tall (2013), a British mathematics education researcher, also presents a model that could be used for analyzing students' mathematical understanding. Tall's perspective can be viewed as similar to Pirie and Kieren's, since both of them base their models in mathematical stages that students might demonstrate in their activity with mathematics. However, Tall contemplates only three stages, and so seems to have less detail in terms of the development of mathematical understanding.

Tall's (2013) model (Figure 3) presents what he calls the three mental worlds of mathematics: the *conceptual world* (*embodiment*), the *operational world* (*symbolism*) and the *axiomatic world* (*formalism*). According to the author, these worlds are "based on human recognition, repetition and language to evolve through perception, operation and reason" (p 153). The *conceptual world* refers to experiences that students have that enable the embodiment of mathematical concepts and, as a result, their better assimilation. These experiences emerge from students' perceptions and actions, and can be associated with concrete materials, schemas, images, gestures and so forth. Tall highlights that in this stage the focus is on objects. From there, the second world—the *operational world*—will build on the objects, and its focus will shift to actions on objects. Thus, students work on procedures related to the concepts acquired in the world of embodiment. These procedures refer to manipulations and calculations, and they might result in new understandings that are not tied to embodiments. Finally, when students achieve the third

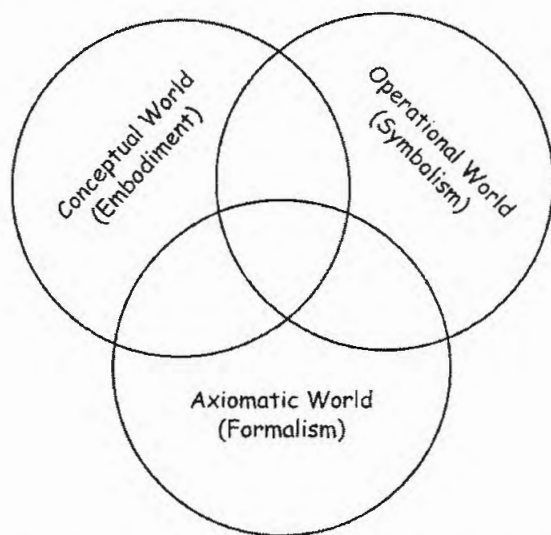


Figure 3: Tall's model of the three worlds of mathematics

world, the world of *formalism*, they are expected to think in terms of mathematical abstraction. At this point, students will work on formal definitions and on properties derived from formal proofs. These three worlds are likely to blend, yielding combined settings for understanding. Tall calls these combined settings *embodied symbolic*, *embodied formal*, *symbolic formal* and *proof combining embodiment and symbolism*.

Following a student's activity according to this framework can be useful in helping teachers to observe for the student's mathematical understanding and guide the student through it. In the given example, the student starts within the world of embodiment, given her necessity to illustrate the situation through a graph. The graph was the embodied way she used to understand and analyze the problem. After that, the student evolves to the world of symbolism, figuring out the functions, features and formal equations. Although she advances to the second world, she continues to draw on the embodiment world, since she still needs the graph to analyze the problem. It is reasonable to say that she ends up in the embodied symbolic combined world when she analyzes her findings and comes to a conclusion. Finally, it seems that the student does not work in the third world—the world of formalism—which might not be expected at all in this particular problem-solving activity.

With Tall's model, the teacher uses awareness of this threefold understanding process in order to help the students. The teacher might need to scaffold the student's understanding so that from the embodied idea the student can shift to the symbolic representation of this embodied idea. After this stage, the teacher's support might be even more critical in helping the student evolve to the axiomatic world by formalizing the student's ideas and understandings.

Kilpatrick, Swafford and Findell's Model

Kilpatrick, Swafford and Findell's (2001) model of five strands of mathematical proficiency (Figure 4) provides yet another observational tool for teachers. The five strands are connected as a complex whole and all of them are aspects of the development of students' mathematical proficiency. As a consequence, these strands reflect the development of students' mathematical understanding.

The first strand that Kilpatrick, Swafford and Findell describe is *conceptual understanding*, which means a connected and coherent understanding of mathematical ideas. The second strand, *procedural*

fluency, relates to the ability to choose the right mathematical procedure and effectively perform it. It is not only about knowing what to do, it is also about knowing how to do. In this sense, it is a relevant strand; however, it is not enough, given that being procedurally fluent in mathematics does not mean understanding the concept, having a strategy to solve the task or even being able to reason. The third strand of mathematical proficiency is *strategic competence*. This refers to the ability to identify and build strategies to understand, represent and solve problems. Kilpatrick, Swafford and Findell point out that this ability is different from trying out some possibilities with the given numbers in a task, hoping to get the right answer. The fourth strand is *adaptive reasoning*, which is the ability to make connections between concepts in order to adapt and transfer relationships from one situation to another. For example, if a student has a prior knowledge about linear functions and then is introduced to arithmetic sequences, this student will be invited/required to adapt her/his reasoning to make connections between the two situations/concepts. Finally, *productive disposition* is related to students' attitude toward mathematics. Kilpatrick, Swafford and Findell affirm that if a student perceives mathematics as worthwhile and believes that he or she is capable of doing and learning mathematics, he or she has productive disposition. In the aforementioned example, having productive disposition would mean that the student believes she is capable of coming up with a connection between linear functions and arithmetic sequences and also believes in the potential this connection might have. Although productive disposition is a personal characteristic, it is highly influenced by teachers' attitudes and teaching styles.

Kilpatrick, Swafford and Findell's (2001) model for investigating students' mathematical understanding can be a useful tool for a teacher, because it is a

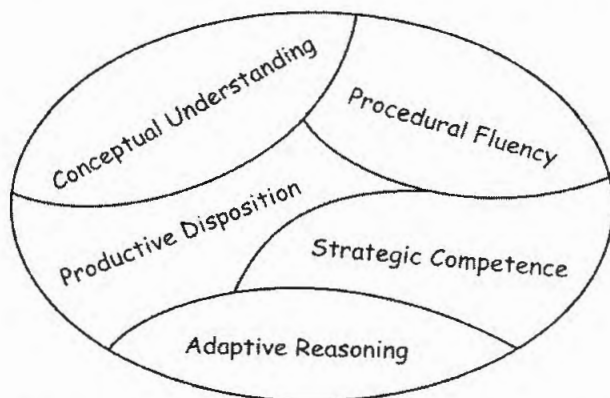


Figure 4: Kilpatrick, Swafford and Findell's model of mathematical proficiency

broad model that considers the process as a whole. In the given example, the student presents conceptual understanding, given that she can effectively connect the problem with previous conceptual knowledge about linear and constant functions. Also, she shows strategic competence when she establishes the graph as a tool to analyze the problem and looks for data to feed the graph. Procedural fluency is also present, given that she can successfully find the formal equations of each function involved in the problem. As for adaptive reasoning, it might be the case that the student connected previous knowledge in a way that would require adaptive reasoning. It might be also the case that adaptive reasoning was necessary to compare and discuss the different scenarios. None of this can be verified with the given data. However, by doing this activity in class with students, adaptive reasoning might be easily detected by the teacher. The same is valid for productive disposition.

Once more, if the teacher is able to identify that students do not demonstrate some of the five strands, the teacher can help students develop them by prompting and guiding students through the process until they achieve proficiency in each of the five strands. For instance, if a student realizes the problem is about linear and constant functions, but is not able to come up with a strategy to solve it, the teacher may ask questions to trigger ideas of different strategies that could be used or not. Nevertheless, it is important to let the student analyze the options and choose one of them. This will allow for the student's development of strategic competence.

Each of the aforementioned models has particularities that can better fit a particular teacher's teaching style. Choosing one of these three models to observe students' mathematical understanding may be helpful in supporting teachers in the challenging role of teaching mathematics for understanding.

Final Considerations

This paper intended to address mathematical (relational) understanding as a critical issue that needs to underpin the teaching of mathematics. However, teaching mathematics for understanding is a difficult and complex task. This paper spoke to some of the challenges that are faced during this process. Because teaching for understanding is a beneficial and doable choice for mathematics teachers, three different models for observing and formatively assessing mathematics understanding were described. The models described are tools for the teacher who is seeking ways to better understand learners and who hopes to teach for relational understanding.

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Selected Writings from the Journal of the Mathematics Council of the Alberta Teachers' Association: Celebrating 50 years (1962–2012) of delta-K

Edited by **Egan J Chernoff**, University of Saskatchewan
and **Gladys Sterenberg**, Mount Royal University

A volume in the series The Montana Mathematics Enthusiast: Monograph Series in Mathematics Education. Series Editor: Bharath Sriraman, The University of Montana

The teaching and learning of mathematics in Alberta—one of three Canadian provinces sharing a border with Montana—has a long and storied history. An integral part of the past 50 years (1962–2012) of this history has been *delta-K: Journal of the Mathematics Council of the Alberta Teachers' Association*. This volume, which presents ten memorable articles from each of the past five decades, that is, 50 articles from the past 50 years of the journal, provides an opportunity to share this rich history with a wide range of individuals interested in the teaching and learning of mathematics and mathematics education. Each decade begins with an introduction, providing a historical context, and concludes with a commentary from a prominent member of the Alberta mathematics education community. As a result, this monograph provides a historical account as well as a contemporary view of many of the trends and issues in the teaching and learning of mathematics. This volume is meant to serve as a resource for a variety of individuals, including teachers of mathematics, mathematics teacher educators, mathematics education researchers, historians, and undergraduate and graduate students. Most important, this volume is a celebratory retrospective on the work of the Mathematics Council of the Alberta Teachers' Association.

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Specialist councils' role in promoting diversity, equity and human rights

Alberta's rapidly changing demographics are creating an exciting cultural diversity that is reflected in the province's urban and rural classrooms. The new landscape of the school provides an ideal context in which to teach students that strength lies in diversity. The challenge that teachers face is to capitalize on the energy of today's intercultural classroom mix to lay the groundwork for all students to succeed. To support teachers in their critical roles as leaders in inclusive education, in 2000 the Alberta Teachers' Association established the Diversity, Equity and Human Rights Committee (DEHRC).

DEHRC aims to assist educators in their legal, professional and ethical responsibilities to protect all students and to maintain safe, caring and inclusive learning environments. Topics of focus for DEHRC include intercultural education, inclusive learning communities, gender equity, UNESCO Associated Schools Project Network, sexual orientation and gender variance.

Here are some activities the DEHR committee undertakes:

- Studying, advising and making recommendations on policies that reflect respect for diversity, equity and human rights
- Offering annual Inclusive Learning Communities Grants (up to \$2,000) to support activities that support inclusion
- Producing *Just in Time*, an electronic newsletter that can be found at www.teachers.ab.ca; Teaching in Alberta; Diversity, Equity and Human Rights.
- Providing and creating print and web-based teacher resources
- Creating a list of presenters on DEHR topics
- Supporting the Association instructor workshops on diversity

Specialist councils are uniquely situated to learn about diversity issues directly from teachers in the field who see how diversity issues play out in subject areas. Specialist council members are encouraged to share the challenges they may be facing in terms of diversity in their own classrooms and to incorporate these discussions into specialist council activities, publications and conferences.

Diversity, equity and human rights affect the work of all members. What are you doing to make a difference?

Further information about the work of the DEHR committee can be found on the Association's website at www.teachers.ab.ca under Teaching in Alberta; Diversity, Equity and Human Rights.

Alternatively, contact Andrea Berg, executive staff officer, Professional Development, at andrea.berg@ata.ab.ca for more information.

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