Teaching Mathematics for Understanding: Approaching and Observing

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Introduction

Mathematics teaching has been the target of criticism recently (take, for example, the extensive media response to the latest PISA results). In part, these criticisms are derived from the belief that doing mathematics regardless of the nature of a learner's understanding is sufficient for schooling purposes, and that thinking mathematically is necessary only for mathematicians. These beliefs seem to be deeply rooted in our society and are difficult to change. Because of that, new approaches for teaching mathematics are being judged negatively. Sierpinska (1994) states that

> Sometimes understanding is confused (or deliberately merged) with knowing, and argued that this is perhaps not a desirable thing to do in education. Unfortunately, institutionalized education is framed to develop students' knowledge rather than thinking. This is a heritage of a long-standing tradition. (p 68)

Regardless, many different approaches to teaching mathematics for understanding have been investigated over the last few decades (Kilpatrick, Swafford and Findell 2001). In spite of positive learning outcomes demonstrated by many of the approaches, discussions continue about what it means to teach for mathematical understanding. Therefore, one purpose of this paper is to discuss teaching mathematics for understanding by considering its relevance, advantages and challenges, as well as the factors that contribute to the implementation of mathematical understanding activities in class. The second purpose is to present three theories of mathematical understanding: Pirie and Kieren's (1994) model of the growth of mathematical understanding; Tall's (2013) model of the three worlds of mathematics; and Kilpatrick, Swafford and Findell's (2001) model of mathematical proficiency, each of which can be used to observe students' mathematical understanding.

Teaching Mathematics for Understanding

As many teachers are aware, mathematical understanding can be related to more than one kind of understanding in mathematics. Skemp (2006), for instance, proposes two different meanings for the word *understanding*. He claims that understanding can be instrumental or relational. *Relational understanding* means "knowing both what to do and why" (p 89), while *instrumental understanding* is described by "rules without reasons" (p 89). This paper will refer to relational understanding when discussing teaching for understanding.

Teaching for understanding presents advantages. For students to develop understanding, the required instruction will correspond to what Ben-Hur (2006) calls concept-rich instruction—ie, instruction based on conceptual knowledge. As a consequence, the constructed knowledge should be stronger and longer lasting; hence students can draw on the meanings and understandings they have assimilated rather than depending on (perhaps long-forgotten) memorized facts and processes when they encounter new mathematical situations and problems. Kilpatrick, Swafford and Findell (2001) remind educators that if students cannot make different associations among the learned concepts, they might not be able to use them in various problem-solving situations. In this sense, the students' mathematical knowledge will be compromised because they do not understand what they are learning.

Stein, Grover and Henningsen (1996) claim that

> Complete understanding [of mathematics] ... includes the capacity to engage in the processes of mathematical thinking, in essence doing what makers and users of mathematics do: framing and solving problems, looking for patterns, making conjectures, examining constraints, making inferences from data, abstracting, inventing, explaining, justifying, challenging, and so on. (p 456)

But how does one achieve this "complete understanding" of mathematics? Involving students in high-level mathematics activities (cognitively demanding activities) seems to be an effective way to teach students for mathematical understanding; however, this is a challenging task. Henningsen and Stein (1997) argue that many factors are necessary to support engagement in cognitively high-level mathematics thinking during mathematical activities, including (1) building connections with students' background knowledge; (2) providing students with an appropriate amount of time to do the activity—not too little and not too much; (3) emphasizing meaning and requiring students to explain their understandings; (4) having students model their thinking processes and strategies; (5) providing scaffolding when necessary; (6) enabling students to self-monitor and self-question; and (7) having students draw conceptual connections. The authors point out that the activity itself will not be able to engage students in mathematical thinking if students are not properly provided with a supportive environment, including the specific assistance a teacher can provide.

If teachers are aware of these factors, why is relational understanding so difficult to achieve? The problem is not due to lack of interest or commitment by teachers. Indeed, traditional instruction is losing time to instruction that values relational understanding, rather than instrumental understanding (Silver et al 2009). So why is teaching for relational understanding so difficult to implement? In their research, Silver et al indicate that, when talking about their practice, teachers mention many different goals that they aim to achieve, of which relational understanding is but one. As the amount of necessary work to accomplish all these goals is great, teachers may need to make choices and choose some goals at the expense of others (Silver et al 2009). Henningsen and Stein (1997) highlight some issues that might hinder the engagement in the mathematical thinking process: (1) inappropriateness of the task, (2) classroom management problems, (3) inadequate amount of time spent with the task, (4) lack of accountability, (5) challenges that become nonproblems; and (6) focus on finding the right answer. Further, Henningsen and Stein explain that high-level activities require students to take risks that they might not be willing to take. This may explain why teachers feel pressured to reduce their lessons to a set of step-by-step instructions or to reduce their expectations of what learners need to do within a learning activity.

Unfortunately, simple awareness of issues does not mean that teaching for understanding is trouble-free. Indeed, recent research (Silver et al 2009) has shown that teaching can still be based on old strategies, and founded on procedural knowledge and instrumental understanding. But telling students what to do or how to do and requiring them to do only low-demanding activities will not develop their mathematical understanding. If students do not involve themselves in class activities that require them to think, reflect, try different strategies and go over the activity again and again, they will be just doing manipulations based on someone else's guidance. Hence, students will not be developing and enhancing their mathematical understanding. In this sense, it is important for teachers to observe students' understanding processes, in order to help them to benefit the most from the activities they do. The next section describes some models that might be useful in the course of observing student meaning making.

Observing Students' Mathematical Understanding

As teachers invest in teaching mathematics for understanding, it becomes necessary for teachers to have conceptual tools to observe the effectiveness of their teaching practice as it results in learner understanding. The capacity to observe student understanding is an important aspect of the whole process of instruction, because it enables improvements and encourages the ongoing promotion of mathematical understanding in lessons. By examining the following task and one possible solution for it, we can illustrate some ideas about how students' mathematical understanding is demonstrated when solving a problem. Note that this analysis is based solely on written records of one student's response to a task and the understanding displayed in the student's working papers. This partial data from the student's work most certainly means there will be incompleteness in the analysis of her mathematical understanding.

The Task

Consider the two following cell phone plans. Compare them and discuss best customer options.

1. Plan A—The customer has a total of 200 province-wide minutes of outgoing and incoming calls for $22 per month. Extra minutes will cost $0.40 each. Text messages are unlimited. As for data usage, the customer will pay per use: up to 25MB, $4; up to 100MB, $12; up to 500MB, $20; up to 1GB, $30; over 1GB, $ 0.02 per MB.
2. Plan B—The customer has unlimited Canada-wide talk, text and data usage for $42 per month.

One Possible Solution

The student recognizes that there are linear functions involved in the problem and thinks about graphing the situation as a way of comparing the different scenarios. In order to accomplish that, the student analyzes each situation and comes up with two functions for plan A and one function for plan B.

Plan A

The customer pays a minimal monthly amount no matter how low her/his cell phone call usage. If the customer uses more than 200 minutes, she/he will pay the extra cost following a proportion. This situation can be represented by the following piecewise function.

$$f(x)=\begin{cases}22 & 0\le x\le 200\\ 22+0.40(x-200) & x>200\end{cases}$$

Plan A

As for data usage, if less or equal to 1GB, the cost is constant according to the usage. If the usage is more than 1GB, the cost is a function of the usage. This situation can be represented by the following piecewise function.

$$g(t)=\begin{cases}0 & t=0\\ 4 & 0<t\le 25MB\\ 12 & 25MB<t\le 100MB\\ 20 & 100MB<t\le 500MB\\ 30 & 500MB<t\le 1GB\\ t\times 0.02 & t>1GB\end{cases}$$

Then the student realizes she can add $f(x)$ to each case of $g(t)$, in which $g(t)$ is a constant function, resulting in five different functions. She could have added $f(x)$ to the last case of $g(t)$, too. However, because the variables are different, she would have a function of two variables, and this is problematic for her. Thus, the student's choice was to graph the five different functions to illustrate at least five scenarios of monthly cost (in case the chosen plan was plan A). These five graphs can be seen in Figure 1.

Plan B

The customer pays a constant amount no matter how low or how high her/his cell phone usage. This situation can be represented by a constant function, which means the customer will always pay the same.

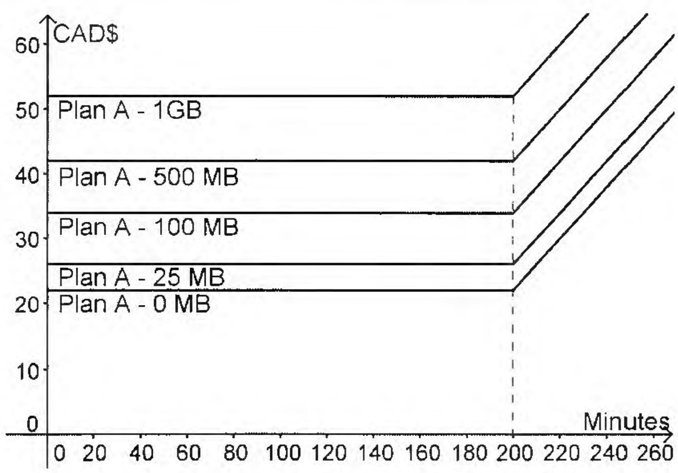
$$h(x)=42$$

The graph of the function $h(x)$ would be constant on $y=42$, which partially coincides with the plan A 500MB scenario.

Based on the described reasoning, the student was able to compare and discuss the different scenarios and come to a personal conclusion in terms of the customer's best options. It is important to notice that other aspects could be considered in this analysis as well—for instance, whether the plans are provincewide or countrywide and what is the customer need in those terms.

Different ideas and nuances can be associated with students' mathematical understanding when analyzing the above solution. In the same way, different models can be used to observe for understanding. In this paper, three contemporary models will be described for this purpose: Pirie and Kieren (1994), Tall (2013), and Kilpatrick, Swafford and Findell (2001). These models do not reflect a progression; that is, they are unrelated to each other and have different underpinnings.

Figure 1:
Monthly cost for five different scenarios



Pirie and Kieren's Model

Tom Kieren, a retired professor from the University of Alberta, and his colleague Susan Pirie developed a model for observing mathematical understanding in action (Pirie and Kieren 1994). Their model considers eight different stages (Figure 2), and students are observed in relation to the stages they demonstrate during their learning process. Although this model proposes that stages are increasingly more comprehensive, students' levels of understanding do not necessarily evolve in a linear process. Quite the opposite—students can experience a nonlinear process that might fold back to previous stages according to each student's learning constraints and affordances.

The first stage of Pirie and Kieren's (1994) model, the *primitive knowing* stage, refers to the background knowledge that the student brings with him/her to start developing other content. The *image making* stage builds on this prior knowledge as the learner makes distinctions in previous knowing and uses it in new ways (p 170). Once students can take actions without directly associating a new situation to the original one, students will have achieved the *image-having* stage. The next step, if it were linear, would be *property noticing*, when students are able to infer properties based on the images they have constructed. The *formalising* phase is accomplished when students are able to abstract "a method or common quality from the previous image-dependent knowhow which characterised her noticed properties" (p 170). After that, students are expected to come up with new understandings, in the so-called *observing* stage. After this point, once students are able to think in terms of theories, they have achieved the *structuring* stage, which "occurs when one attempts to think about one's formal observations as a theory" (p 171). Finally, when students can follow a rationale and pose reasonable questions about what they know to create new structures, new forms and new mathematics, they are at the *inventising* stage.

Analyzing the given example based on Pirie and Kieren's model, it is possible to say that the student starts on the primitive knowing stage, in that she is able to bring her knowledge into the situation. She is able to associate the task with linear and constant functions. Then the student enters the image-making stage, as she takes up graphical representations as a tool for analyzing the problem. As the student analyzes each cell phone plan and comes up with the functions for each situation and their formal equations, we observe the student at the image-having stage. Realizing that she can add two functions as a way of representing two scenarios (call usage and data usage) in only one function demonstrates property noticing. She is assimilating function properties such as adding. After this stage, the student goes back to the image-making stage, because she needs to feed her graph with the added functions she has just found out, in order to create new objects with which to make meaning. Finally, based on this image, she is able to go to the observing stage to come up with her understandings and conclusions about the task. The formalising stage, the structuring stage and the inventising stage seem not to be present. In the case of the formalising stage, there is no data to determine whether the student now understands that there is a class of piecewise functions that can be used across many situations. Further, a single episode such as the one presented does not normally lead to structuring and inventising, since it is too narrow in scope.

Following the student's path based on Pirie and Kieren's (1994) model is a nice way of acknowledging the diverse ways that a student can choose to pursue when solving a problem and also the different needs that a student might have. In this case, for example, the student needed to go back to the image-making stage once the graph became the basis of her reasoning. The teacher has a significant role in this scenario—not the role of inducing his or her students to take a specific path, but the role of encouraging the students to act with what they know and to pursue deeper understanding of the situation using mathematics.

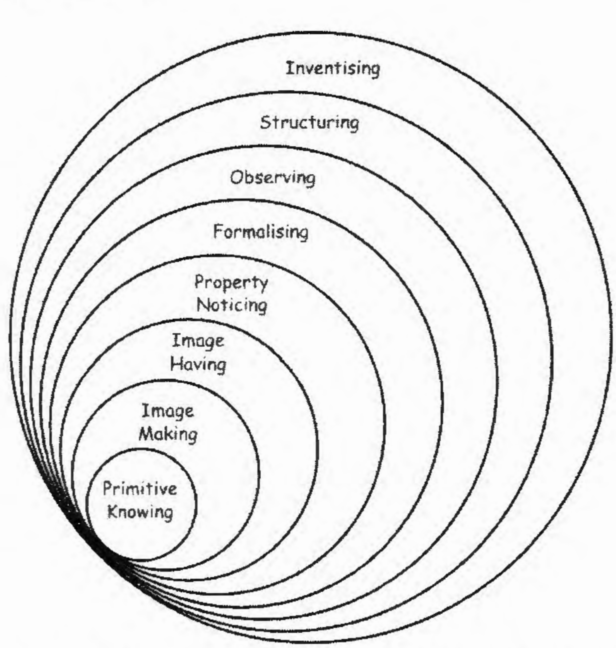


Figure 2: Pirie and Kieren's model of the growth of mathematical understanding

Tall's Model

Tall (2013), a British mathematics education researcher, also presents a model that could be used for analyzing students' mathematical understanding. Tall's perspective can be viewed as similar to Pirie and Kieren's, since both of them base their models in mathematical stages that students might demonstrate in their activity with mathematics. However, Tall contemplates only three stages, and so seems to have less detail in terms of the development of mathematical understanding.

Tall's (2013) model (Figure 3) presents what he calls the three mental worlds of mathematics: the *conceptual* world (*embodiment*), the *operational* world (*symbolism*) and the *axiomatic* world (*formalism*). According to the author, these worlds are "based on human recognition, repetition and language to evolve through perception, operation and reason" (p 153). The *conceptual* world refers to experiences that students have that enable the embodiment of mathematical concepts and, as a result, their better assimilation. These experiences emerge from students' perceptions and actions, and can be associated with concrete materials, schemas, images, gestures and so forth. Tall highlights that in this stage the focus in on objects. From there, the second world—the *operational* world—will build on the objects, and its focus will shift to actions on objects. Thus, students work on procedures related to the concepts acquired in the world of embodiment. These procedures refer to manipulations and calculations, and they might result in new understandings that are not tied to embodiments. Finally, when students achieve the third world, the world of *formalism*, they are expected to think in terms of mathematical abstraction. At this point, students will work on formal definitions and on properties derived from formal proofs. These three worlds are likely to blend, yielding combined settings for understanding. Tall calls these combined settings *embodied symbolic*, *embodied formal*, *symbolic formal* and *proof combining embodiment and symbolism*.

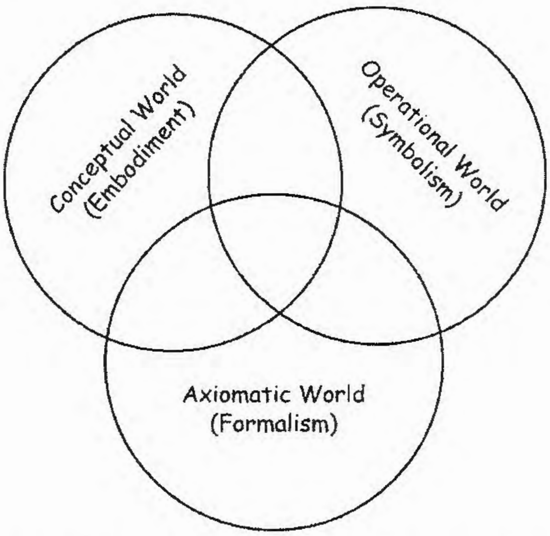


Figure 3: Tall's model of the three worlds of mathematics

Following a student's activity according to this framework can be useful in helping teachers to observe for the student's mathematical understanding and guide the student through it. In the given example, the student starts within the world of embodiment, given her necessity to illustrate the situation through a graph. The graph was the embodied way she used to understand and analyze the problem. After that, the student evolves to the world of symbolism, figuring out the functions, features and formal equations. Although she advances to the second world, she continues to draw on the embodiment world, since she still needs the graph to analyze the problem. It is reasonable to say that she ends up in the embodied symbolic combined world when she analyzes her findings and comes to a conclusion. Finally, it seems that the student does not work in the third world—the world of formalism—which might not be expected at all in this particular problem-solving activity.

With Tall's model, the teacher uses awareness of this threefold understanding process in order to help the students. The teacher might need to scaffold the student's understanding so that from the embodied idea the student can shift to the symbolic representation of this embodied idea. After this stage, the teacher's support might be even more critical in helping the student evolve to the axiomatic world by formalizing the student's ideas and understandings.

Kilpatrick, Swafford and Findell's Model

Kilpatrick, Swafford and Findell's (2001) model of five strands of mathematical proficiency (Figure 4) provides yet another observational tool for teachers. The five strands are connected as a complex whole and all of them are aspects of the development of students' mathematical proficiency. As a consequence, these strands reflect the development of students' mathematical understanding.

The first strand that Kilpatrick, Swafford and Findell describe is *conceptual understanding*, which means a connected and coherent understanding of mathematical ideas. The second strand, *procedural* *fluency*, relates to the ability to choose the right mathematical procedure and effectively perform it. It is not only about knowing what to do, it is also about knowing how to do. In this sense, it is a relevant strand; however, it is not enough, given that being procedurally fluent in mathematics does not mean understanding the concept, having a strategy to solve the task or even being able to reason. The third strand of mathematical proficiency is *strategic competence*. This refers to the ability to identify and build strategies to understand, represent and solve problems. Kilpatrick, Swafford and Findell point out that this ability is different from trying out some possibilities with the given numbers in a task, hoping to get the right answer. The fourth strand is *adaptive reasoning*, which is the ability to make connections between concepts in order to adapt and transfer relationships from one situation to another. For example, if a student has a prior knowledge about linear functions and then is introduced to arithmetic sequences, this student will be invited/required to adapt her/his reasoning to make connections between the two situations/ concepts. Finally, *productive disposition* is related to students' attitude toward mathematics. Kilpatrick, Swafford and Findell affirm that if a student perceives mathematics as worthwhile and believes that he or she is capable of doing and learning mathematics, he or she has productive disposition. In the aforementioned example, having productive disposition would mean that the student believes she is capable of coming up with a connection between linear functions and arithmetic sequences and also believes in the potential this connection might have. Although productive disposition is a personal characteristic, it is highly influenced by teachers' attitudes and teaching styles.

Kilpatrick, Swafford and Findell's (2001) model for investigating students' mathematical understanding can be a useful tool for a teacher, because it is a broad model that considers the process as a whole. In the given example, the student presents conceptual understanding, given that she can effectively connect the problem with previous conceptual knowledge about linear and constant functions. Also, she shows strategic competence when she establishes the graph as a tool to analyze the problem and looks for data to feed the graph. Procedural fluency is also present, given that she can successfully find the formal equations of each function involved in the problem. As for adaptive reasoning, it might be the case that the student connected previous knowledge in a way that would require adaptive reasoning. It might be also the case that adaptive reasoning was necessary to compare and discuss the different scenarios. None of this can be verified with the given data. However, by doing this activity in class with students, adaptive reasoning might be easily detected by the teacher. The same is valid for productive disposition.

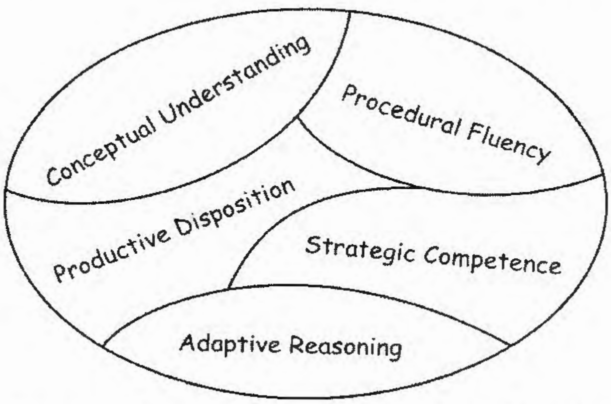


Figure 4: Kilpatrick, Swafford and Findell's model of mathematical proficiency

Once more, if the teacher is able to identify that students do not demonstrate some of the five strands, the teacher can help students develop them by prompting and guiding students through the process until they achieve proficiency in each of the five strands. For instance, if a student realizes the problem is about linear and constant functions, but is not able to come up with a strategy to solve it, the teacher may ask questions to trigger ideas of different strategies that could be used or not. Nevertheless, it is important to let the student analyze the options and choose one of them. This will allow for the student's development of strategic competence.

Each of the aforementioned models has particularities that can better fit a particular teacher's teaching style. Choosing one of these three models to observe students' mathematical understanding may be helpful in supporting teachers in the challenging role of teaching mathematics for understanding.

Final Considerations

This paper intended to address mathematical (relational) understanding as a critical issue that needs to underpin the teaching of mathematics. However, teaching mathematics for understanding is a difficult and complex task. This paper spoke to some of the challenges that are faced during this process. Because teaching for understanding is a beneficial and doable choice for mathematics teachers, three different models for observing and formatively assessing mathematics understanding were described. The models described are tools for the teacher who is seeking ways to better understand learners and who hopes to teach for relational understanding.

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