Teaching the Trigonometric Ratios Through Embodiment, Symbolism and Formalism

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Introduction

In today's classrooms, sound pedagogic practices are priceless. In the mathematics classroom, these pedagogic practices must provide students with opportunities to create a deeper understanding of mathematics. A deeper understanding of mathematics means that students understand the underpinnings of mathematical concepts; are able to represent mathematical ideas in multiple ways (concrete, numerical, graphical, geometrical, and symbolic); are able to use appropriate "mathematical language, vocabulary, and notation to represent ideas, describe relationships, and model situations"; and are able to "make meaningful connections within mathematics, to other content areas, and to real-life situations" (New Mexico State University nd, 1).

If students are to attain these standards and achieve deeper understanding of mathematics, teachers must create the pedagogic opportunities that will engage students in higher levels of thinking. D'Ambrosio, Johnson and Hobbs (1995) proposed twelve pedagogic strategies that teachers can employ to engage students in higher levels of thinking.

1. Encourage exploration and investigations: involve students in activities that will help them to construct mathematics knowledge as well as explore and investigate mathematics ideas.
2. Use students' prior knowledge: students bring to class different world knowledge and experiences that affect the way they view and solve problems.
3. Use manipulatives: the proper use of manipulatives is critical to the understanding of new mathematical ideas.
4. Use real-world problem-solving activities: link mathematics and the real world through a wide range of problem-solving activities.
5. Integrate mathematics with other content areas: this helps students to apply previously acquired knowledge to new situations.
6. Use culturally relevant materials: this helps to motivate students as the mathematics relate to students' different cultures and interests.
7. Use technology: saves time by performing complex calculations quickly and allows for drawings and demonstrations that are difficult if not impossible to achieve using a chalkboard.
8. Use oral and written expression: explaining their thinking orally and/or in writing help students to organize their thought and solution strategies.
9. Encourage collaborative problem solving: this encourages active involvement in learning by sharing and negotiating meaning, verbalizing understanding, and providing constructive criticism.
10. Use errors to enhance learning: to simply say an answer is correct or incorrect is insufficient if students are to improve their understanding of mathematics. The thinking behind students' errors must be explored if misconceptions are to be ironed out.
11. Offer an enriched curriculum and challenging activities: all students must be exposed to mathematically demanding tasks. This allows students to develop their critical thinking skills and problem-solving ability beyond routine and watered-down procedural tasks.
12. Use a variety of problem-solving experiences: use a wide variety of problems to include problems that can be solved in different ways, with more than one correct answer, and that may involve decision making and allow for different interpretations. (pp 125–35)

These twelve pedagogic strategies, however, must be used within a framework that will enhance their effectiveness whereby students and teachers will gain maximum benefit from their use. To this end, I propose Tall's framework called the "three worlds of mathematics."

In this paper I demonstrate how the sine, cosine, and tangent ratios can be introduced to secondary

(Grade 9 or 10) students using Tall's three worlds of mathematics to scaffold their learning. I begin by providing theoretical perspectives of the three worlds—embodiment, symbolism, and formalism (Tall 2013)—and draw on the work of others to further develop the ideas. I also provide a brief recap of the concepts of sine, cosine, and tangent ratios along with some areas of difficulty commonly experienced by students. Some plausible activities and relevant problems along with some guidelines are also provided. I conclude this paper with a word of caution on how these activities may be interpreted.

The Three Worlds of Mathematics

Embodiment involves the use of one or more body senses to help internalize abstract mathematical ideas through the manipulation of physical objects and/or through physical actions (Tall 2013). Tall's notion of embodiment is supported by Husserl (cited in Behnke 2011), who argued that embodiment goes beyond practical action, but is an essential factor that influences the attainment of deep understanding. Dubinsky (2000) also reflected upon the "widespread agreement that mathematical ideas begin with human activity and move from there to abstract concepts" (p 216). These perspectives are all in keeping with the proposed pedagogic strategies of D'Ambrosio, Johnson and Hobbs (1995), which encourage exploration and investigation and make use of manipulatives, but go beyond these strategies to include the use of body actions. Here, the essential point for teachers is that engaging students in actions relevant to mathematical concepts help these students to better internalize the concepts.

Symbolism, according to Tall (2013), "grows out of embodiment by focusing on the actions on objects rather than on the objects themselves" (p 141). Tall refers to this as *operational symbolism*—a notion that calls for students to perform mathematical operations on symbols. That is, while embodiment focuses on the physical properties of objects, symbolism focuses on manipulating these properties. Manipulating objects' properties includes, but is not limited to, performing calculations, writing and verbalizing mathematical symbols and notations, substituting values for variables, rearranging symbols, using relationships among properties, and connecting properties to other content areas and real-world situations. Hence, many of D'Ambrosio, Johnson and Hobbs's (1995) pedagogic strategies can be applied to Tall's notion of symbolism, thus providing rich pedagogic experiences for students.

It is important to note at this juncture that although symbolism is purported to grow out of embodiment, the transition is not always a smooth one and teachers need to afford students time and support in making that transition. D'Ambrosio, Johnson and Hobbs (1995) recommend a three-stage transition process: (1) the use of embodiment alone, followed by (2) the use of the embodiment together with symbolic representation, and then (3) the use of the symbolic representation alone. The second stage is critical, and adequate time must be spent during this stage if students are to make a successful transition from embodiment to symbolism (D'Ambrosio, Johnson and Hobbs 1995).

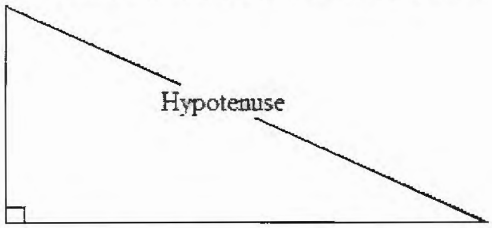
Formalism, according to Tall (2013), may take on several meanings in mathematics education. For instance, Tall refers to Piagetian formalism, when an individual reaches the formal operational stage and that person's thought process no longer needs the involvement of physical referents. Hilbert,[1] on the other hand, conceptualizes formalism as focusing on axiomatic definitions and proofs, and it is this conceptualization that Tall makes use of in his framework. Therefore, *formalism* in this paper is restricted to the use of formulas, production of images (diagrams) and basic proofs of relationships. Drawing on the strategies of D'Ambrosio, Johnson and Hobbs (1995), tasks given to students at this level should be cognitively demanding, should make use of collaborative/cooperative groupings and should make use of technological resources such as calculators, computers and the Internet.

Concepts of Sine, Cosine and Tangent Ratios

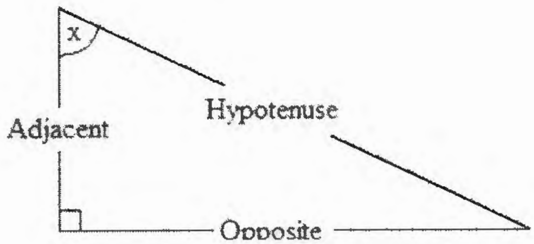
Sine, cosine and tangent, the three primary trigonometric ratios, are used in this paper to demonstrate how embodiment, symbolism and formalism, as proposed by Tall (2013), can be employed in teaching mathematics. To provide the background for this demonstration, however, a brief revision of these ratios and their underlying concepts is presented.

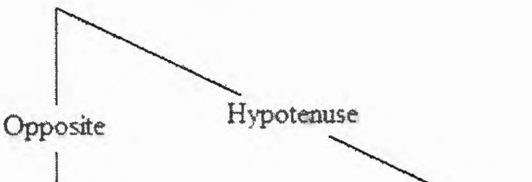
[1] Hilbert (1862–1943), one of the first formalists, believed that all theorems could be proved using only the axioms of the system.

Trigonometry is grounded in the study of triangles. The basis for the trigonometric ratios is the right-angled triangle. In a right-angled triangle, one angle—the largest—measures exactly 90° and that angle is always opposite the longest side—the hypotenuse.



This is an essential fact, and it is important that learners know that the hypotenuse is opposite the 90° angle, the angle in the above triangle marked with the conventional symbol of a right angle—a box rather than an arc. That is, students need to know that the longest side of any triangle is always opposite the biggest angle. A proof of this theorem can be found at www.youtube.com/watch?v=LeeiVVAoPUk. In relation to the ratios, the two other sides of the right-angled triangle are called the *opposite* and *adjacent* sides but, unlike the hypotenuse, their positions are not fixed but change as the angle of reference changes. That is, the opposite side and the adjacent side of a right-angled triangle switch when the reference angle—which is never the right angle—switches. This is demonstrated in the following two triangles, with x being the reference angle in both cases.





The sine, cosine and tangent ratios are formed using the sides of the right-angled triangle as follows:

- $Sine\ of\ angle\ x = \frac{\text{opposite side}}{\text{hypotenuse}}$ or $sinx = \frac{\text{opposite}}{\text{hypotenuse}}$
- $Cosine\ of\ angle\ x = \frac{\text{adjacent side}}{\text{hypotenuse}}$ or $cosx = \frac{\text{adjacent}}{\text{hypotenuse}}$
- $Tangent\ of\ angle\ x = \frac{\text{opposite side}}{\text{adjacent side}}$ or $tanx = \frac{\text{opposite}}{\text{adjacent}}$

I have taught and observed others teach the sine, cosine and tangent ratios on numerous occasions throughout my twenty-plus years as a mathematics educator. I have observed that a very common approach to teaching these concepts is to present students the above formulas and an acronym—SOH-CAH-TOA—for remembering them, followed by a few examples with explanations. Drawing on Tall's work, this approach places instruction in the world of formalism without affording students the opportunity to embody the concept and/or develop critical skills in symbolism. As a consequence, students may not develop a deep understanding of the primary trigonometric ratios, which in turn impedes their understanding of other concepts in trigonometry that make use of these ratios.

One common sign of misunderstanding is that many students struggle to identify the opposite and adjacent sides in problems where the right-angled triangle forms part of a bigger shape, where right-angled triangles are orientated in ways students are not familiar with or where the right angle is not shown but must be calculated. A second common sign of misunderstanding occurs when students struggle with identifying which ratio must be used when given a side and an angle of the right-angle triangle and asked to find another side, or when given two sides and asked to determine an angle. It is in an attempt to reduce these problems that Tall's model of the three worlds of mathematics is being proposed as an alternate approach to teaching the sine, cosine and tangent ratios.

Teaching with Embodiment

Since embodiment involves the use of the physical body—the manipulation of physical objects and/or bodily actions (Tall 2013)—pedagogic strategies employed at this stage call for students to be engaged in physical activities that will help them make sense of new concepts. Therefore, when the concepts of sine, cosine and tangent ratios are being introduced, students can be called upon to act out some of the underlying concepts, and the teacher can pose some relevant questions to help direct students' thoughts and, hence, their internalization of these concepts. Following are two activities that demonstrate how teachers can help students internalize underlying concepts of hypotenuse, opposite side and adjacent side.

Activity 1: Walking the Lines (50 minutes): The aim of this activity is to help students understand the difference in the hypotenuse, opposite and adjacent sides of a right-angled triangle.

- In an open space, draw (or use masking tape on the floor to mark off) a large right-angled triangle.

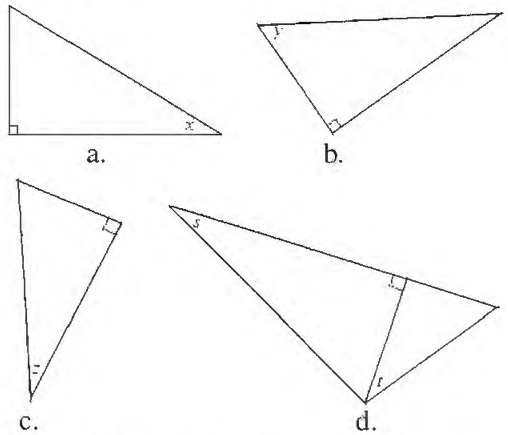
The triangle must be large enough for students to walk along its sides. The teacher stands at the right angle and asks students to walk along the hypotenuse. The teacher then changes position to stand at one of the acute angles and asks students to walk along first the side opposite to where he/she is standing and then to walk along the side adjacent to where he/she is standing. Repeat the activity with the teacher (or a student) standing at the other acute angle. This time, however, instead of saying "side opposite to where he/she is standing" or "side adjacent to where he/she is standing," use the terms *opposite side* and *adjacent side*. Repeat as needed.

Suggested key questions to pose to students:

1. In relation to the right (90°) angle, where is the hypotenuse? (Opposite to the right angle.)
2. The right angle is the largest angle in the right-angled triangle. What can you say about the hypotenuse in relation to the other sides? (The hypotenuse is the longest side of the right-angled triangle.)
3. In relation to where the teacher was standing, what do you notice when walking the opposite side versus the adjacent side? (When walking the opposite side you will never meet the teacher, but you meet the teacher when walking the adjacent side.)

Activity 2: Point and Feel (50 minutes): This is an extension of activity 1 but more individualistic in nature. It may serve as a follow-up activity for students who still have difficulty in identifying the hypotenuse, opposite and adjacent sides.

- Provide students with a sheet (or several sheets) with several right-angle triangles drawn in different orientations and with the right angle and one other angle marked.



Ask students to place one finger at the right angle of each triangle and run another finger along the hypotenuses, then label each. With one finger at the lettered angles, let students run a finger along the opposite sides and label them, then along the adjacent sides and label them.

Suggested key question:

1. What do you notice about the opposite and adjacent sides in shape **d**? (In shape **d**, the opposite side of angle s is the adjacent side of angle t. In some composite shapes, one side may serve as both opposite and adjacent.)

Teaching Through Symbolism

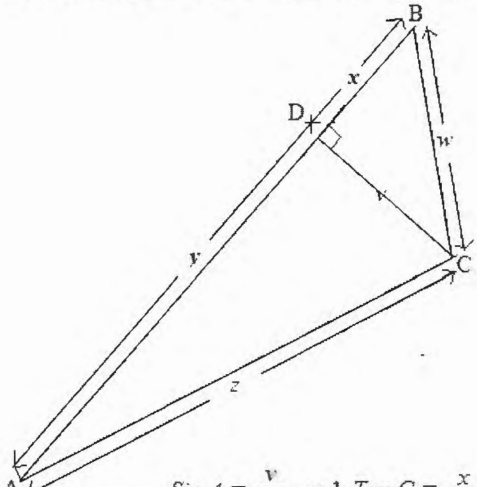
While the embodiment activities focused on the physical lines and the student's kinesthetic interaction with the lines as mathematics concepts (hypotenuse, opposite side and adjacent side), activities in this section focus on actions that can be taken on the relationships that exist among these lines. Here, students are called upon to perform calculations, write and verbalize mathematical symbols and notations, substitute numbers for symbols, rearrange symbols, and identify and use existing relationships among the ratios. These are just a few skills, taken from the world of symbolism, that are necessary if students are to successfully solve problems involving various contextualized situations. Following are three activities that demonstrate how teachers can help students develop their skills in symbolism.

Activity 3: Deck of Cards: This activity is designed to help students develop fluency in recognizing and writing the sine, cosine and tangent ratios using symbols.

- The names *sine, cosine* and *tangent* must be written on separate index cards. The ratios without the names must also be written on separate cards. The inverse of these ratios must also be written on separate cards. All cards are placed in a deck and drawn at random. If a name is drawn, students must explain/discuss briefly the ratio. If a ratio is drawn, students must give its name and say something about it. The inverse ratios must be identified and discarded, with reason. Students are encouraged to use gestures, diagrams and other available resources in their explanation and/or discussions. I encourage this as a whole-class activity, but it can also be done in small groups as needed.

Activity 4: Journal Writing: This activity is designed to help students make meaning of sine, cosine and tangent ratios as presented in problems.

- Provide students with one or more problems similar to the one following and ask them to write about the symbolic statement contained therein. Encourage them to express any difficulty experienced in understanding the symbols and if/how they were able to overcome such difficulty. Students are reminded of activity 2 and are encouraged to use it to help them identify the ratios in the diagrams. I encourage this as an individual activity with opportunities for students to share in a whole-group setting.



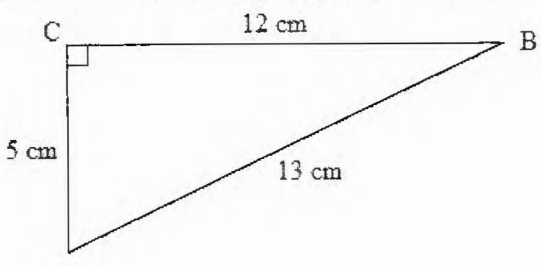
$Sin\ A = \frac{v}{z}$ and $Tan\ C = \frac{x}{v}$

Both activity 3 and activity 4 bridge the gap between the world of embodiment and the world of symbolism because they provide students with opportunities to use their bodies while performing actions in symbolism. This is important to help students make the transition from embodiment to symbolism, and time and practice must be given to individuals as needed.

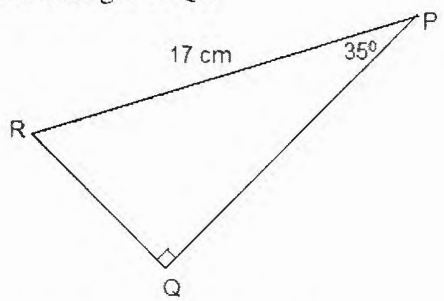
Activity 5: Multi-Levelled Computational Tasks: This activity will improve students' fluency, competence, understanding and confidence in manipulating the trig-ratios in symbolic forms.

- Teachers must engage students in problem-solving tasks in which a number of mathematics problems are solved through the manipulation of the symbolic representations of the sine, cosine and tangent ratios. The following provide the structure of the problems needed to develop what Tall (2013) calls operational symbolism.

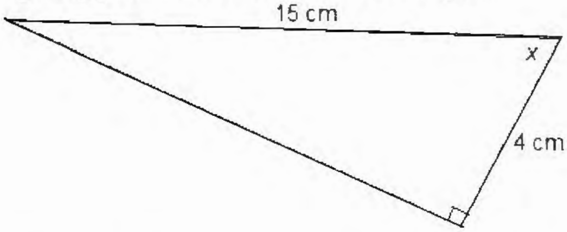
1. Calculate sin35°.
2. Given the following diagram, find cosB.



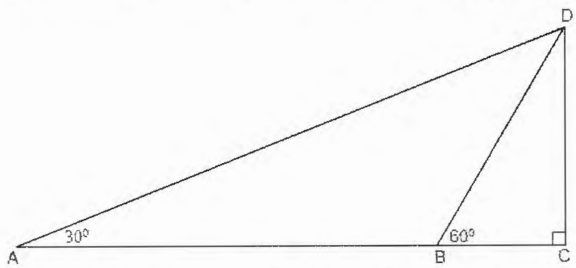
3. Find the length of QR.



4. Find angle A if Tan $A = 0.381$.
5. From the following diagram, calculate angle x, giving your answer to the nearest degree.



6. Given the following diagram, show that $\frac{Tan60}{Tan30} = \frac{AC}{BC}$.



Student textbooks such as *Foundations and Pre-Calculus: Mathematics 10* (Davis et al 2010) provide a wide range of problems that teachers may consider using with their students.

Teaching Through Formalism

Using Hilbert's conceptualization of formalism, as presented by Tall (2013), I focus this section on the use of axiomatic definitions and basic proofs. I use this restriction because the formal world of sine, cosine and tangent extends beyond the scope of this paper. Therefore, *formalism* in this context is restricted to the use of formulas, production of images (diagrams) and basic proofs of relationships among the sine, cosine and tangent ratios. I encourage the use of collaborative/cooperative groupings because this pedagogic practice encourages discussion, which helps students to refine their thinking (D'Ambrosio, Johnson and Hobbs 1995). I also encourage the use of computers and other forms of technology, which may help students save valuable time while performing complex calculations. Following is an activity that presents four situations in which students will be required to delve into the world of formalism.

Activity 6: Problems and Proofs: Given the following formulae, solve the following problems and/or derive the relevant proofs, giving meaningful explanations for each step in your solution or proof.

Formulae:

- $sinx = \frac{\text{opposite}}{\text{hypotenuse}}$
- $cosx = \frac{\text{adjacent}}{\text{hypotenuse}}$
- $tanx = \frac{\text{opposite}}{\text{adjacent}}$

1. A home owner wishes to build a ramp to his front door to make it wheelchair accessible. The door is 1.5m above ground level and the ramp must have an angle of elevation of 15°. What will be the length of the ramp?
2. A man 2m tall standing at the top of a cliff 25m high observes two ships in a straight line at sea. He observed one ship at an angle of depression of 30° and observed the other at an angle of depression of 55°. How far away from each other are the ships?
3. Prove that: $Tan\theta = \frac{Sin\theta}{Cos\theta}$
4. Given that $SinA = \frac{4}{5}$ and $Tanx = \frac{5}{12}$, prove that $TanA - Cosx = \frac{16}{39}$

Reflecting on the Instructional Progression

Learning is not a linear process, and to engage students in a few well-sequenced activities does not guarantee that they will achieve a deep understanding of the concepts being taught. To gain a deep understanding of concepts, learners need to move back and forth among the three worlds of mathematics, and need to do so at their own pace. Therefore, in teaching sine, cosine and tangent ratios using Tall's (2013) worlds of embodiment, symbolism and formalism, it should be expected that all learners will not progress at the same pace and that some learners will need to revert to the level of embodiment while working at the symbolic and/or formal levels. This is healthy and should not be discouraged. In this regard, teachers will find repeating activity 2, or variations of it, at appropriate times useful in helping students overcome difficulties such as the inability to correctly identify opposite and adjacent sides in unfamiliar shapes. I also feel compelled to state that the suggested activities are not exhaustive and neither are they cast in stone, but may be adapted to meet the pedagogical needs of teachers who choose to use them.

Conclusion

This article presents secondary school mathematics teachers with a framework for teaching mathematics in a way that will provide students with opportunities to gain a deeper understanding of mathematics concepts. It uses the concepts of sine, cosine and tangent ratios to illustrate how embodiment, symbolism and formalism can be enacted in the classroom. These illustrations are given in the form of activities that may be adapted and used by teachers. While a comprehensive list of activities is not given, it is nevertheless my hope that the activities presented will raise and/or heighten teachers' awareness of activities spanning the three stated worlds of mathematics and provide them with an entry point into acts of embodiment to complement the acts of symbolism and formalism that inform most teachers' pedagogy. Further, the tasks and questions posed, especially in the symbolism and formalism sections, are commonly found in textbooks. The difference here is that tasks are presented so as to take into account the three worlds of mathematics in an order that scaffolds understanding.

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