Mathematical Thinking: An Argument for Not Defining Your Terms

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A hum of activity ebbs and flows in the room. Seated at round tables, the participants are all engaged, although not all in the same fashion. Some are noisily working in pairs, meticulously laying out rows of neatly organized dominos, row upon row. Some are working independently, slowly, thoughtfully, rearranging the dominos in front of them. As progress is made, the ideas flow through the room, rushing by those who already *know*, and forcing others to pull their attention from their own thoughts and attend to the ideas in the room. This is the picture of a room learning—as Doll (1989) writes, a room that is doing "more dancing and less marching" (p 67). This productive hive of activity is the outcome of a good mathematics problem. However the participants are not students—they are teachers.

This is not a unique occurrence. Put any group of mathematics teachers together with a good problem and the hive will spontaneously erupt. The definition of a *good* problem lurks just out of reach, like an idea from a dream you cannot quite remember. Some mathematics teachers have a good intuition when it comes to judging a problem as good; a select few can even produce good problems effortlessly. All mathematics teachers know a problem is good by the response of their class. It may not even be the problem alone. Instead it may be a perfect storm coming together, out of unidentifiable elements like day of the week, time of the day, the past of the participants, the safety of the learning atmosphere and more. However, like the good problem, the perfect storm is recognizable when it rains down.

As a participant in this particular hive, I discerned new ideas about mathematical thinking as I worked on the mathematics. The problem was a tiling activity with dominos that ended up generating the Fibonacci sequence. I started with the dominos but quickly moved to paper, developing a symbolic representation for the problem so that I could organize the arrangements into types and count using combinatorics. Others in the group were using the language of transformational geometry. This did not occur to me. Some had completely abandoned the dominos and were working exclusively on paper. Still others were working solely with the dominos.

After the emergence of the Fibonacci sequence was discovered and agreed upon, the group moved on to something else, but I stayed with this problem. I found myself listing the terms of the sequence and the symbolic pattern until the 11th iteration, then looking for a formula that would generate the sequence using sigma notation. There is something about this experience that is deeply connected to the kind of mathematical thinking I would like to support students in developing.

Tall (2013) has two ideas related to mathematical thinking that are connected to this experience with a good problem. The first is the concept of the *met-before*, which Tall initially describes as "a structure we have in our brains *now* as a result of experiences we have met before" (p 23). Later Tall writes that "'met-before' refers not to the actual experience itself, but to the trace that it leaves in the mind that affects our current thinking" (p 88). Both of these descriptions create a picture of something left behind in the mind as a result of a mathematical experience that may or may not be a complete object. The decision to use combinatorics to approach the problem was not a conscious one. I did not have the thought "I will use combinatorics," nor did I *decide* to stop using the dominos and start using a symbolic representation. These approaches seemed to evolve organically, just as equally valid approaches evolved organically in other members of the group (this may point to one of the qualities of a good problem). This could be similar to the experience of a met-before, a residual experience with a mathematical idea that unconsciously appeared in my work and influenced my thinking. A met-before, like *take it to the other side and change the sign*, is supportive for a student in solving $2x - 6 = 10$. When the same student is faced with $2x + 5 = 6x - 10$, then the met-before can become problematic. *What should I move and which side should I take it to?* This residual left behind in the mind can lead students to productive approaches or stop them in their tracks, depending on the situation.

Tall's (2013) idea of "three mental worlds of mathematics" (p 133) forced me to reflect on teaching and doing mathematics differently. The three worlds of mathematics are *conceptual embodiment*, *operational symbolism* and *axiomatic formalism*. For Tall, *conceptual embodiment* occurs when "human perception and action" (p 133) lead to the development of mental images that grow into "perfect mental entities in our imagination" (p 133). Tall uses conceptual embodiment to refer to the initial formulation of thinkable concepts "through recognition and categorization" (p 133). Conceptual embodiment is a "compression from procedure to process that can be seen by shifting the focus of attention from the *steps* of a procedure to the *effect* of the procedure" (Tall 2008, 12), where "compression is seen as a general cognitive process that compresses situations in time and space into events that can be comprehended in a single structure by the human brain" (Tall 2008, 13) that involves both conceptual embodiment and operational symbolism. Compression is not a linear progression through stages. In itself, compression is a process that moves backward and forward as new situations confront met-befores and inconsistencies are resolved. Conceptual embodiment is a process that begins when repeated actions become embodied procedures. Then the procedures come to be understood by their effect and, finally, this effect becomes an embodied concept in the mind.

Operational symbolism occurs when "embodied human actions" (Tall 2013, 133) develop into "symbolic procedures of calculation and manipulation that may be compressed into procepts to enable flexible thinking" (p 133). A *procept* is a mathematical idea that is both a process and a concept (object). A student learning mathematics typically learns one first, then becomes aware of the other and then, through the process of compression, gains understanding of and the ability to utilize both flexibly, as required. A symbol can suggest "a *process* that produces a mathematical *object*" (Gray and Tall 1994, 121). Thus, the symbol for the minus sign has three components embedded into it: it represents the process of subtraction, it represents the concept of difference and it is a symbol with its own meaning. When a child learns to count, *four* is a process. Later when the child adds two to four by counting on from four, *four* has become a mathematical object.

According to Tall (2013), conceptual embodiment and operational symbolism are intertwined and occur together in overlapping layers through compression. However, there is a key distinction. Conceptual embodiment focuses on objects (and actions on the objects) and operational symbolism focuses on symbols (and the manipulation of symbols) (p 155). By focusing on objects, conceptual embodiment offers the possibility of sensing what happens as a consequence of the operation. "It has an effect that can be seen" (p 155). For example, a student is asked to compare the graph of $f(x) = \frac{(2x-1)(x+3)}{x+3}$ with the graph of $g(x) = 2x - 1$, and decides to use the zoom feature on a graphic display calculator (GDC) in the neighbourhood of $x = -3$. When the hole in the graph of $f(x)$ at $x = -3$ appears, it allows the student to see the effect the $(x - 3)$ factor has in the denominator. The *procedure* of using the zoom function on a GDC shows (perceived through sight) the *effect* of the $(x - 3)$ factor. The conceptual embodiment of this effect represents the visual difference and similarities of the graphs of $f(x)$ and $g(x)$ Manipulating the symbolic representation of by dividing out the common factor $(x - 3)$, is a symbolic *procedure*. This is part of the *process* of simplifying rational expressions to demonstrate the concept that $f(x)$ is equal to $g(x)$, everywhere except at $x = 3$. The compression of the process of simplifying rational expressions with the concept that the original and the simplified functions name the same thing (except at the restrictions) is a *procept*.

Finally, when Tall (2013) uses the term *axiomatic formalism*, he is referring to "building formal knowledge in axiomatic systems specified by set-theoretical definitions, whose properties are deduced by mathematical proof" (p 133). This world is a different world all together. Axiomatic formalism turns the processes discussed thus far upside down. "Instead of studying objects or operations that *have* (natural) properties, the chosen properties (axioms) are specified first and the structure is shown to have other properties that can be *deduced* from the axioms" (p 149). Consider the axiom "If x and y are sets, then the set of pairs (x, y) or the set of pairs (y, x) exists" (Wells 2006, 2137). This is the axiom that allows for the creation of new sets from existing sets and thus forms the basis for the definitions of a relation and a function. Note that this axiom does not require a list of the elements of a set to make the claim the set exists. Axiomatic formalism sits atop the intertwined conceptual embodiment and operational symbolism. Tall's discussion (2013, 2008) of his model of these three worlds is summarized in Figure 1. This model is at the root of my reflection on my teaching in light of the domino tiling. If axiomatic formalism comes after the intertwined conceptual embodiment and operational symbolism, why do I consider it important to start with a formal proof? Is there a better way to help students develop mathematical thinking?

Figure 1

Formal	Axiomatic Formal		
	Conceptual embodiment of this effect	Procept (as both process and thinkable concept)	Object
	Effect (of procedure or action)	Process (seen as a whole)	Process
	Procedure (through perception or action)	Procedure (expressed symbolically step by step)	Action
	EMBODIED	**SYMBOLIC**	

Returning to my experience with the domino activity, and considering this experience in the light of Tall's (2013) three mathematical worlds, there are some interesting observations. To explain the activity, Figure 2 illustrates the possible tilings for 2 × 1 space, 2 × 2 space and 2 × 3 space respectively.

I moved away from physically manipulating the dominos rather quickly; however, I stayed in the conceptual embodiment world throughout the activity, continuing to draw possible tilings for eleven iterations. The *drawing* of the three possible 2 × 3 tilings seen in Figure 2 looked similar to {III, I=, =I}. I continued to draw iterations of the tilings long after I had established the numerical sequence and symbolic representations. I moved quickly into the operational symbolism world, creating symbolic representations for each specific group of tilings. However, I did not leave the conceptual embodied world behind. Beside each of the drawings depicting each iteration was a combination formula equivalent to the numerical values of the number of arrangements of each configuration. These formulas were symbolically manipulated to highlight the various components.

Figure 3 is a sample row from the table I constructed showing how I recorded the information for a 2 × 8 space. Changing the number of vertical and the number of horizontal dominos generated different arrangements. I did not physically create all of these arrangements. My conceptual embodiment continued as I continued to draw arrangements; however, I did not draw dominos. Thus, the symbolic representation took two different forms. For the first I used symbols to represent the dominos in each tiling. For example, one of the possible 2 × 8 tilings was depicted by IIIIII=, which represented six vertical dominos and two horizontal dominos. The second symbolic representation, combinatorics notation, was used to count the arrangements of each iteration. The six vertical and two horizontal tiling can be arranged in seven different ways, as seen in Figure 3. However, I counted these arrangements ways using combinatorics: of the seven possible places for the two horizontal dominos select one. All the possible tilings for this two-by-eight example are summarized in Figure 3 using my conceptual embodiment (the drawings) and my operational symbolism (combinatorics).

Figure 2

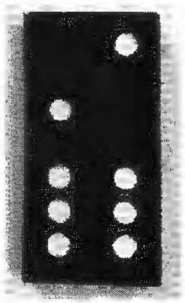
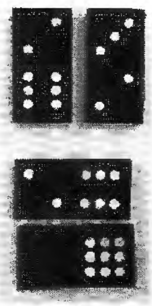
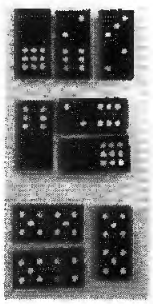
One possible 2 ×1 tiling	**Two possible 2 ×2 tilings**	**Three possible 2 ×3 tilings**
		

Figure 3

Iteration	Size	Conceptual Embodiment	Operations Symbolism	Term in the Sequence
Number of Dominos	Area (in "½ domino units")	Drawing with the number of possible iterations	Combination formulas	Total number of arrangements
n = 8	2 × 8 rectangle	llllllll (1) llllll= (7) llll== (15) ll=== (10) ==== (1)	$\binom{8}{0}+\binom{7}{1}+\binom{6}{2}+\binom{5}{3}+\binom{4}{4}$	34

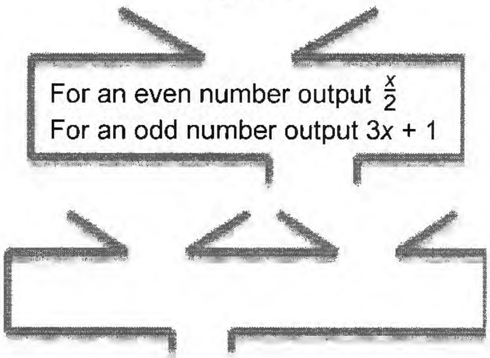
In the end, I was able to generalize the deconstructed pattern and derive a function that, given the size of the space to be tiled, returns the number of possible tilings, accounting for all possible arrangements of horizontal and vertical arrangements.

In truth, while I have read about tiling activities, this was my first experience actually working on a tiling problem and the first time in a long time I sat down to do mathematics with which I am unfamiliar. Reflecting on this work, it is clear to me that there were occasions I was working in the conceptual embodiment world, which is in some cases distinct from and in others interrelated with occasions when I was creating an operational symbolism. It is equally clear how these two worlds are intertwined and how they worked together to create, in my mind, a procept that represents this problem. Now the physical model, the symbolic representation of this model, the processes used to generate the symbolic iterations and the generalized formula sit together in my mind and my attention can float between them. What is most interesting is that I never moved into the axiomatic formalized world. For years, I have taught students to use the method of proof by mathematical induction, yet I did not produce a formal proof of my generalized formula, which normally I would have. I was satisfied that my formula was valid because my formula was able to generate the numerical values I was expecting for the first three iterations. This was all the *proof* I needed. The compression process of the two worlds—conceptual embodiment and operational symbolism—creates the flexible procept.

This is the point where I can now explain why Tall's three mathematical worlds absorbed my attention. I teach Grade 10, 11 and 12 mathematics. Functions are a part of each of these courses, and I use functions as the theme for teaching each of these mathematics courses, introducing each topic as a new function. Volume, surface area and perimeter become functions of their linear measure(s). Quadratics and lines are functions. Simplifying rational expressions becomes investigating rational functions. The logarithmic function is introduced as the inverse of the exponential function. Sequences and series are discrete functions on the positive integers. Probability becomes functions on random variables. Trigonometry begins with trigonometric functions then moves into triangle trigonometry. Using the theme of *function* has allowed me to circle back to topics to create deeper understanding. Yet I always begin with formal definitions like those of relation, function, domain and range. I always begin in the world of axiomatic formalism.

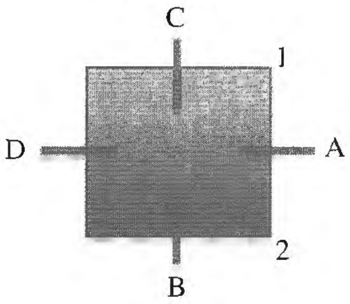
My absorbing thought is why start the topic of functions in the axiomatic formalism world? The answer may be that this approach is my own met-before that is problematic for students. These worlds of conceptual embodiment and operational symbolism lead me to rethink the way I introduce functions. Creating situations in which students can move between these two worlds and up and down the compression continuum may help students build flexible mathematical procepts as opposed to problematic met-befores. One such situation is the function machine. "A function may be represented in a more concrete manner as a function machine (or function box) that has the property that can be imagined both as a process (as a machine taking an input and producing an output) or as an object (a box that contains the machine preforming the operations)" (McGowen and Tall 2013, 531). The use of a function machine can allow students to work in the conceptual embodiment world at the same time as they work in the operational symbolism world, which in turn will help students build a flexible function procept. As seen in Figure 4, functions can have rules that vary depending on the input, have more than one input, work in multiple discrete stages or follow other rules in addition to algebraic rules. All of this helps students build a more flexible procept for function at the same time as it intertwines the conceptual embodiment and operational symbolism worlds, which in turn will create a solid platform for the movement into the axiomatic formalism world.

Figure 4



Another example that can help students build flexible procepts is the wrapping function. Consider the image below, Figure 5, a square with sides of length two. Starting at any vertex or midpoint, a path can be traced around the perimeter of the square. This simple process can create many functions.

Figure 5

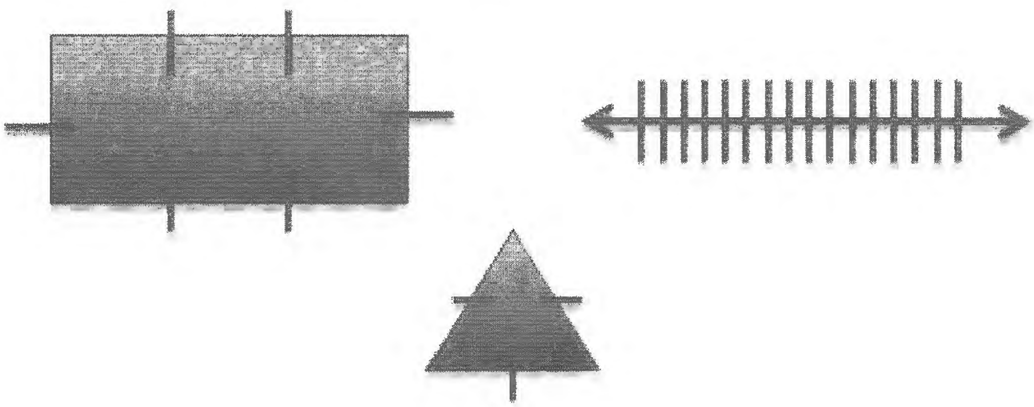


For example, starting at A and travelling in a counter-clockwise direction for a length of two units, the path ends at C. This instruction could be represented as the (A, 2), both of which are inputs. The output would be C. However, this is only one of the many functions that can be generated from the model in Figure 5. The input (A, 1) could be linked to an output of 2 (vertical height), or 0 (horizontal distance from start) or both (2, 0). Cartesian graphs can be generated from a variety of mappings from one set onto another set. An input of (A, –2) would have an output of B, the path length of negative 2 being a path in a clockwise direction. Figure 6 shows examples of different shapes that can be used as the wrapping function and the Cartesian graphs of different mappings that can be drawn.

The periodic nature can be explored in both the conceptual embodied world and the operational symbolism world. Not only does an activity like this lead into the generating idea for sine and cosine functions, it also provides a conceptual embodiment experience that relates to other mathematical concepts such as vectors.

The concept of a function is central to a significant part of the high school mathematics program; therefore, devoting more time to encouraging students to create a flexible procept based on the worlds of conceptual embodiment and operational symbolism may be more beneficial to them than spending time in the axiomatic formalism world. Creating problems linked to mathematics curriculum that allow students to move between the conceptual embodiment and operational symbolism worlds as they feel comfortable will help students create flexible procepts. Additionally, the potential to work in the two worlds may be another quality of a *good* problem.

Figure 6



References

Doll, W E, Jr. 1989. "Complexity in the Classroom." *Educational Leadership* 7, no 1: 65–70.

Gray, E M, and D O Tall. 1994. "Duality, Ambiguity, and Flexibility: A 'Proceptual' View of Simple Arithmetic." *Journal for Research in Mathematics Education* 25, no 2: 116–40.

McGowen, M A, and D O Tall. 2013. "Flexible Thinking and Met-Befores: Impact on Learning Mathematics." *The Journal of Mathematical Behavior* 32, no 3: 527–37.

Tall, D. 2008. "The Transition to Formal Thinking in Mathematics." *Mathematics Education Research Journal* 20, no 2: 5–24.

———. 2013. *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics*. New York: Cambridge University Press.

Wells, R. 2006. "The Critical Philosophy and the Phenomenon of Mind." Available at www.mrc.uidaho.edu/mrc/team/printPeople.php?ID=21 (accessed January 23, 2015).

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