Solve the Following Equation: The Role of the Graphing Calculator in the Three Worlds of Mathematics

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In 2013, David Tall published a book entitled *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics*, which tries to make sense of how mathematics is taught and learned in a world where the spectrum of positions on mathematics ranges from feelings of absolute beauty and power to anxiety and distress (p xiii). He proposes a framework of three worlds of mathematics through which learners construct mathematical meaning. As graphing calculators are now a near-ubiquitous tool in the mathematics classroom, this paper will explore how using a graphing calculator is both supportive and problematic within these three worlds of mathematics, by considering how students may come to solve quadratic functions.

The presence of the graphing calculator in the mathematics classroom has become naturalized. One does not often step back and ask how it came to be here or what it is currently doing to mathematical thinking, pedagogy and curriculum. Historically, the first device that could be considered a calculator, the abacus, began to extend mathematical thinking as early as 5,000 years ago. Then, in 1692, the French mathematician Pascal created the first mechanical calculator, which had the ability to add and subtract numbers. However, at the time Pascal concluded that it was too expensive for any practical use (Grinstein and Lipsey 2001, 87). Calculators would remain too expensive for common household use until the 1970s. Since that time, the increased use of calculators in society quickly forced educators to adapt, which gave rise to the prominent and lasting mathematics education debate about whether and how calculators should be implemented in classrooms (Banks 2008, 1–2). Then, in the early 1990s, a more powerful type of calculator—the graphing calculator—emerged on the education scene, and it was soon commonly seen in most high school mathematics classrooms. Graphing calculators allow students to graph, analyze, calculate and solve problems graphically, numerically and algebraically. Since nongraphing calculators had become common in schools, aside from some discussions surrounding their monetary expense, the addition of graphing calculators to the classroom was less contentious. However, even if the addition of the graphing calculator was met with less resistance, it is still considered in the literature as the instigator of massive change in the high school mathematics classroom in the last 25 years. In 1992, near the beginning of the integration of the graphing calculator into the classroom, Kaput described this new technology as "a newly active volcano of the mathematical mountain … changing before our eyes, with a myriad of forces operating on it and within it simultaneously" (p 515). Yet today, its presence goes nearly unquestioned. Learning to use a graphing calculator is merely part of the progression of learning about mathematics. The presence of the graphing calculator in education has gone from being seen as an active volcano to being naturalized. Teaching high school mathematics now implicitly includes teaching how to use a graphing calculator to aid in developing mathematical thinking and understanding.

The Three Worlds of Mathematics

Tall (2013) puts forth a framework in which to consider mathematical learning that he calls the "three worlds of mathematics": conceptual embodiment, operational symbolism, and axiomatic formalism (p 133). Through these worlds, language, categorization and repetition produce thinkable concepts that can be developed into *crystalline concepts*, which occur from the compression of understanding into a structure that has "inevitable properties in its given context" (p 27). The use of the word *embodiment*, in *conceptual embodiment*, can be problematic. In everyday language the word *embodiment* can mean a concrete representation of an abstract idea, or the embodiment of an idea can be linked to knowing through the body. Yet, Tall's explanation of the first world of mathematics, *conceptual embodiment*, is more open and points to any human perceptions and actions that develop mental images that give meaning to abstract concepts, be it through the body, concrete materials or other experiences such as using a graphing calculator. The second world of mathematics, *operational symbolism*, often develops from embodied understandings and includes "symbolic procedures of calculation and manipulations that may be compressed into … flexible operational thinking" (Tall 2013, 133). The third world of mathematics, *axiomatic formalism*, builds formal mathematical knowledge by developing definition and proof. Learners do not move through these three worlds linearly; instead they continually "fold back" (Pirie and Kieren 1994) to previous learning in order to move their understanding forward. Learners never come back to the same place in the same way, and they are always taking something different away. To think of developing understanding in this way "reveals the non-unidirectional nature of coming to understand mathematics" (Pirie and Kieren 1994, 69).

In many Alberta schools, students begin to learn to use and rely on their graphing calculators in Grade 10. Learning to use this tool develops through both formal instruction and other experiences of using the calculator, such as trial and error or play. Many students begin their formal experiences with the graphing feature by working with linear functions. When students move on to Grade 11 they will start to explicitly study nonlinear functions, often beginning with quadratic functions. The calculator then becomes more than a tool used for routine calculations and displaying the odd graph, but instead develops into an incredibly useful extension of their thinking. This extension will become as prized for its instant graphing capabilities as it is for its ability to convert rational numbers from decimal form into fractions. Yet, there are previous understandings, which Tall (2013) calls *met-befores*, that can be both supportive and problematic in developing an understanding of quadratic functions. Teachers need to be aware that "a sensible approach to learning requires not only the building towards powerful ideas that will be encountered in the future but also addressing the problematic issues in the present that may have long term consequences" (p 116). Thus, in unpacking the graphing calculator's role in learning about solving quadratic functions through the three worlds of mathematics, it is important to remember that both problematic and supportive met-befores are being created.

When beginning to study quadratic functions, a common starting place is to look at the features of quadratic functions and their corresponding graphs. Students place equations often given in standard, $y = ax^2 + bx + c$, vertex, $y = a(x - p)^2 + q$, and factored, $y = a(x - b)(x - c)$, forms into the [Y=] function of their calculator. They observe the U-shaped curves, opening up and down, wider and skinnier, with the vertex in multiple locations. This may further perpetuate a common met-before related to the meaning of the equals sign. For some students, an equals sign often does not represent equivalence between the two sides of an equation, but initiates a problematic "put the answer here" response.

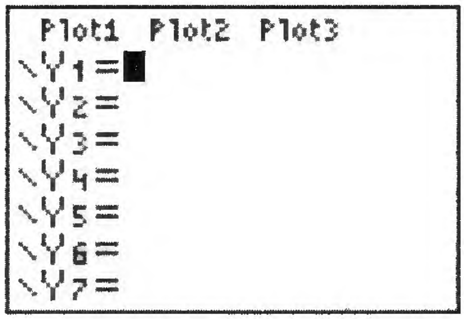

Figure 1: Home screen of graphing feature of TI 83+

Figure 1 shows how, on the main input screen for a graphing calculator, the y-variable is isolated on the left side of the screen and an equals sign indicating "put the expression here" on the right side. The y-variable becomes separated from the rest of the equation, decreasing its appearance of importance within the function. Functions may begin to lose their two-variable appearance, and importance, as the x-variable becomes the focus. It is possible that when given an equation that is not in one of the common forms, such as $y - 6 = x^2 - 5x$, to see some students reach for their calculators and enter $Y_1 = x^2 - 5x$. This response to an equals sign builds on the previous misunderstanding of the meaning of the Y_1 in their calculator's graphing feature. This met-before is possibly perpetuated because students are often given equations with the y-variable already isolated and will then repetitively enter equations without having to enter a y or equals sign into their graphing calculator.

This met-before can be built upon further for some students when they start to solve quadratic equations, such as $x^2 + 3x - 10 = 0$, with their graphing calculators. Suddenly, the y-variable is gone, replaced with a 0, and although teachers may explicitly discuss this change some students may not build these understandings into their creation of meaning. They are now working with only a specific case of the function—when it is equal to 0. This change is made easy by the previous met-before regarding the meaning of an equals sign, for some students can ignore the 0 in the same way that they were ignoring the y-variable, the only difference being that the part being ignored is often located on the right side rather than the left. Thus, students who enter $x^2 + 3x - 10$ into [Y_1] are at times completely unconcerned with the meaning of the 0. There is a lack of awareness that on their calculator screen is a representation of the function $y = x^2 + 3x - 10$, not the equation $x^2 + 3x - 10 = 0$. Students can then use the [CALC] feature to find the zeros, perhaps unmindful that the zeros are interesting because the equation is currently equal to 0. If instead $x^2 + 3x - 10 = 2$, the interest would be in the x-values of the graph when the function is at 2. Using the graphing feature of a calculator to graph the related function as a way to learn to solve equations can lead to a possible misunderstanding of the definition and meaning of a function.

When learning to solve quadratic equations by factoring, an algebraic method, the calculator can be used as a bridge between the worlds of operational symbolism and conceptual embodiment. Making the conceptual embodiment of the graph a method of developing a visual representation of the solutions arrived at algebraically in the world of operational symbolism. Students can look at the equations in factored form, such as $(x - 2)(x + 5) = 0$, and the corresponding graph, $Y_1 = (x - 2)(x + 5)$ or $Y_1 = x^2 + 3x - 10$ to recognize relationships between the two. The connection between the x-intercepts of (2, 0) and (-5, 0) and the numerical values of 2 and –5 in the factored form of the equation seem straightforward. This demonstrates how conceptual embodiment and operational symbolism can blend together, allowing more powerful ways of thinking mathematically (Tall 2013, 145). However, thinking about the connection between the values in factored form and the x-intercepts of the graph is not enough. In using factoring to solve a quadratic, it is difficult to develop meaningful understanding of the connection to zero. The graphing calculator reinforces the numerical values, for example the 2 and the 5, not the reason for their signs, +2 and –5. Students may inappropriately generalize their own understanding to solve a quadratic equation as a rule articulated as "just factor and take the opposite signs of the numbers." This met-before becomes troublesome in problems such as $(2x - 3)(x + 4) = 0$, where using this "rule" often results in the incorrect solution $x = 3$ and $x = -4$. This met-before is created from experiencing many examples that have had integer solutions, which are reinforced further through seeing these integer solutions on their calculators. Although the calculator may play a role in creating this met-before, it is also incredibly supportive in considering why this understanding is incorrect. In returning to the graphical representation, students can see that the graph does not cross the x-axis at 3, but seemingly half-way between 1 and 2. As students build more and more of these experiences with conceptual embodiment and start learning the associated algebraic approach to solutions, in the world of operational symbolism, they begin to rely less and less on the calculator.

As students progress, the graphing feature of the calculator is used less to make meaning of equations or to check algebraic solutions. "The use of algebra becomes more sophisticated, and operational symbolism takes on a role of its own that no longer needs to be permanently linked to embodiment" (Tall 2013, 145). Taking these thinkable concepts and compressing their meaning, "in the symbolic world we begin to shift to a new way of making sense of the symbols themselves and the coherent ways in which they operate, without consciously referring back to their earlier meanings" (Tall 2013, 145). Yet, having created a conceptual embodiment of these concepts allows for folding back to these ideas if necessary, allowing for more flexible mathematical thinking and meaning making as teachers push their students into the world of operational symbolism.

Many students will begin to get comfortable with factoring to find the solutions to an equation, until they come to a problem where the quadratic equation is not easily factorable. Often, the first response is that "not easily factorable" means that the problem does not have a solution. If students are mathematically flexible they can fold back to the conceptual embodiment provided by the graphing calculator and are able to graph the equation to look for what they often understand to be the solutions to a quadratic equation, the x-intercepts. Some students, often through the guidance of their teacher, will come to understand polynomial solutions as being either real and unequal, real and equal, and unreal and unequal, for real solutions only. However, some students might create different meaning. Perhaps some equations might show a graph crossing the x-axis, disproving students' previous conjecture that an equation that is not factorable has no solutions. Students can work back and forth between the worlds of conceptual embodiment and operational symbolism to create meaning and find resolutions to their questions. However, other equations will show no x-intercepts, and students would feel more confident in their initial response of there being no solution to equations that are not easily factorable. This line of thinking could lead to a rule that there can be none, one or two solutions to a quadratic equation. When students further their mathematical learning, having a conceptual embodied understanding of the solution(s) to equations as x-intercepts of a graph creates a challenging met-before when students encounter complex roots for the first time. Students may have been told that no solutions exist when there are no x-intercepts, and yet complex roots do exist, just not in the same way. Thus, students must be flexible not only in their ability to fold back to other meanings but also to let go of constructed meanings, or "rules," as they continue their learning.

The issue of how to find the solutions to quadratic equations when they are not factorable transitions students into the third world, *axiomatic formalism*. This is the world of formal mathematics that relies on definition and proof. Here the quadratic formula can be derived symbolically, often using the method of completing the square. Once this formula is derived in the world of axiomatic formalism, students will return to operational symbolism to work with and test this formula, often checking it against the conceptual embodied world of the calculator's graphing feature. Again, the conceptual embodiment that the calculator creates allows for acceptance and understanding of very abstract concepts and meanings.

Tall (2013) hypothesizes "that mathematical thinking builds on ... faculties set-before birth in our genes and develops through successive experiences where new situations are interpreted using knowledge structures based on experiences that the individual has met before" (p 117). As teachers, we need to be aware of the beneficial and problematic consequences of the mathematical experiences that occur in our classrooms, especially since problematic understandings are often created accidentally and unconsciously. The inclusion of the graphing calculator in learning to solve quadratic equations allows rapid access to the world of conceptual embodiment that just 25 years ago was not readily available for high school students. This inclusion brings deeper understanding as well as the reinforcing and creation of problematic met-befores. By moving between the worlds of conceptual embodiment on the calculator and the algebra used in operational symbolism, deeper meanings can be created. Even the world of axiomatic formalism benefits from the graphing calculator, as generalizations created here can be tested out in the world of conceptual embodiment. The graphing calculator as a tool has changed how mathematics is taken up in the classroom, allowing access in the high school classroom to the conceptual embodiment of abstract concepts that were previously considered not practical to explore.

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