delta-k

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Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. All manuscripts should be typewritten and properly referenced. All pages should be numbered.
4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author-date system or the American Psychological Association (APA) style manual.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB, T1S 2L4; e-mail gsterenberg@mtroyal.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

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From the Editor's Desk

Gladys Sterenberg

This issue of *delta-K* showcases writing by teacher educators, mathematicians, and graduate and undergraduate education students. This collection of articles is a reminder of the rich heritage of researching and teaching mathematics that we share in Alberta. Despite challenges related to grading policies and back-to-the-basics movements that have been highlighted in the media recently, teachers are engaged in creating learning opportunities for their students that embrace fluency and conceptual development of mathematics.

Ronald Persky provides a glimpse into retention and grading policies at the postsecondary level. These mirror much of the Alberta scene, especially in light of the grading controversy in Edmonton.[1] As we enact the ministerial order on student learning, which requires teachers to attend to the development of engaged thinkers, ethical citizens and an entrepreneurial spirit, I think our understanding of assessment will continue to deepen.[2]

Lisa Sauder and Marina Spreen each present current research on topics emerging from their teaching practice. Their work as graduate students demonstrates the strong presence of teacher researchers in our schools.

Nico Higgs and Egan Chernoff interrogate the importance of pedagogical content knowledge for math teachers, from a Saskatchewan perspective. Their review of the literature and programs has similar implications for teaching mathematics in Alberta. In light of the explicit inclusion of basic facts in the K–6 mathematics program of studies, these authors remind us that a robust mathematical background in professional knowledge is necessary.

Jessica Shaw and Maxine Elter offer an idea for integrating Aboriginal perspectives into math. The idea emerged from their practice and from Maxine's experiences in an undergraduate teacher education program.

Finally, I review *Mathematical Models for Teaching: Reasoning Without Memorization*, by Ann Kajander and Tom Boland. This book presents teaching ideas that are timely for teacher educators, teachers and education students as we navigate the changing landscapes of assessment, fluency and conceptual understandings, as well as the professional learning of math teachers.

I am encouraged by the professional learning demonstrated in these articles, and I appreciate the consistent focus on teaching and learning mathematics. I am proud to be a teacher in Alberta during this time of transition, and I look forward to hearing from you as you share your readers' responses, book reviews, student work, and research and teaching articles.

Notes

1. For a reminder of what was in the news in early September, see www.cbc.ca/news/canada/edmonton/lynden-dorval-fired-for-giving-zeros-treated-unfairly-appeal-board-rules-1.2751007 (accessed November 24, 2014).

2. The ministerial order on student learning can be found at http://education.alberta.ca/department/policy/standards/goals.aspx (accessed November 24, 2014).

Are There Any Questions?

Ronald Persky

Although I have spent my career teaching mathematics, this article has nothing to do with mathematics. It is about teaching and learning, and it can be applied to an English, history or math class.

It should be said at the beginning that I have done no extensive survey. The observations herein are based on recent experiences at three universities, and these observations do no more than invite a response from anyone reading this.

Start at the top and call it trickle-down economics. At University X, Y or Z, the administration (president, vice-president, provost) receives statistics that generate frowns. Our students are dropping out or transferring, and we need to act. The initiative will always come from the administration. For the faculty, attrition is normal—nothing unusual, nothing noteworthy. For the administration, it means losing dollars, and that is always noteworthy, even if the current statistics are in line with the average over the past 10 years.

What to do? The provost calls a meeting of the deans. The deans call meetings with the department chairs. The chairs call departmental meetings. In the meantime, a new office has been created: the Office of Retention.

Back to the departmental meetings. The discussion goes as follows:

CHAIR. We have a problem. It's called retention.

FACULTY MEMBER. Over the last 10 years, my grade distribution has been pretty much the same.

CHAIR. That doesn't make it good.

FACULTY MEMBER. Do you mean all my Fs and Ds should have been Bs and As?

CHAIR. Somewhat.

FACULTY MEMBER. What have I been doing wrong?

CHAIR. You need to be more responsible for the performance of your students.

FACULTY MEMBER. Do you mean that if a student really doesn't want to be in this class, or even at this university, it is my responsibility to convince him otherwise?

CHAIR. At least try to increase his motivation in order to give your class a chance for success.

And so it goes. There are lots of reasons a student does poorly. Here are a few:

- "I never was interested in academics, but I'm not ready to be independent of Mom and Dad. It's a scary thing leaving home and being on my own."
- "Peer pressure is what put me in university."
- "My parents would only support me if I went to university."
- "I'm here, but I really don't know what I want to do."
- "I have a positive attitude about every form of entertainment you can imagine."

From my tone, you may think that I disagree with the thinking of the administration. To the contrary, I have nothing against a professor trying to give students more motivation to do well and remain at the university. If a student's grade is A, B or C, the student leaves the class with a better attitude than if it were F or D.

So what are professors to do in order to increase interest and motivation that they haven't already been doing over the past 15 years? That brings us to the main object of this note. I have seen an increase in making homework part of the student's grade. The professor collects homework (say, once a week) and uses it to determine a percentage of the grade. There's nothing wrong with this. But what I have seen is a weekly assignment being given and then collected one week later. There is no work done to cover that day's lecture and, consequently, the professor never opens the next class asking, "Are there any questions?" Even on the day the assignments are collected, the professor does not ask for questions.

What is wrong with this picture? From the student's point of view, it is the lack of daily motivation—that is, "Give me something that will motivate me to think about today's lecture. And when I return, ask me if I have questions. I almost certainly will."

And so this note ends with an appeal to professors who use homework as part of the grade: do a mixture. Provide work to be done after today's lecture and discussed at the next class, as well as work to be collected over some time period.

Seven Pointers for Teaching Mathematics

Marlow Ediger

Mathematics has its roots deep in the soil of everyday life, and it is basic to our highest technological achievements. Even though almost everything of a concrete character is mathematics, it is the most abstract and the most hypothetical of the sciences. In fact, mathematics is a human science. It is the numerical and calculation part of a person's life and knowledge. It helps provide exact interpretation of ideas and conclusions. It deals with quantitative facts and relationships, as well as with problems involving space and form. It also deals with relationships between magnitudes. Mathematics has always held a key position in the school curriculum, because it has been held to be indispensable to the educated (Savithiri 2006).

Since mathematics is a vital academic discipline, the teacher must study and analyze difficulties faced by students in mathematics. The problem areas must be remedied using the best methods of teaching possible. In an exact field of endeavour, it is important to emphasize accuracy.

Having supervised student teachers for 30 years, I have identified seven major problem areas related to student achievement and progress in learning mathematics.

1. *Students do not understand what is taught.*

For example, students fail to attach meaning to reducing fractions to lowest terms. With teacher assistance, the student may show concretely that two-thirds is equal to six-ninths by placing a large congruent circle divided into thirds over a circle divided into ninths. Thus, the learner comes to understand that when we multiply the numerator and denominator by the same number, the value of the fraction does not change.

2. *Students fail to read story problems correctly.*

If students are not able to read story problem content correctly, there are several ways to help, including

- reading the problem aloud with those who are experiencing difficulty (students should follow the printed text during the read-aloud),
- assisting the student with phonetic analysis and context clues in order to identify unknown words, and
- pronouncing individual words as needed for students as they are reading silently.

3. *Students copy numerals incorrectly from the textbook when engaged in addition, subtraction, multiplication or division, as well as when solving a story problem.*

This miscopying may include making reversals, as in the case of dyslectic children. Teacher assistance and peer teaching, in an atmosphere of respect, should gradually help learners achieve more optimally.

4. *Students hurry to complete an assignment, thus making various kinds of errors.*

Here, students need help with proofreading their completed work. Peer editing or help from a teacher's aide is beneficial.

5. *Students fail to reflect on the kinds of errors made.*

Students must analyze why they are having difficulty. This emphasis on *why* helps them refrain from making the same errors. If a concept is not meaningful to students, it should be retaught. Concepts form generalizations, and generalizations must be meaningful in order to be useful in problem solving. Students, upon reflecting and with teacher guidance, might well determine effective approaches for doing accurate work in mathematics.

6. *Students do not use logical thinking.*

Correct reasoning must be emphasized. Reasoning allows for quality sequencing in doing mathematics. There is order, then, in pursuing answers when using the commutative and associative properties of addition and multiplication, or the distributive property of multiplication over addition. Logical thinking is inherent, here, and in the solving of problems involving in-depth thinking. Logical thinking should be emphasized from kindergarten, such as with *greater than, lesser than* and *equal to*. With this foundation, more complex ideas should accrue sequentially in succeeding grade levels.

7. *Students must experience challenge in ongoing lessons and units of study, yet feel successful in mathematics.*

Wholesome attitudes toward mathematics are developed under these circumstances. Optimal progress should be a result.

Reference

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Mathematical Discourse in the French Immersion Classroom

Lisa Sauder

Math class begins, and I ask my students questions, trying to encourage dialogue or discourse as we attempt to solve a problem. "Je ne comprends pas" ("I don't understand") and "Je ne sais pas comment dire" ("I don't know how to say") are often the first comments out of my students' mouths.

I'm frustrated. Mathematical discourse helps students extend their thinking, link mathematical concepts, and develop language skills and vocabulary (Alberta Education 2007). However, as an elementary French immersion teacher, I know that creating an atmosphere that encourages mathematical discourse can be challenging.

I often find myself rushing to the aid of my students, giving them not only vocabulary but also strategies. I habitually end up doing much of the work for them, which has driven me to search for strategies that will help me facilitate mathematical discourse in my elementary French immersion classroom. Finding strategies specific to French immersion is no easy task, but much research has been done on the importance of mathematical discourse and on its characteristics, as well as on second language acquisition (SLA) (Hufferd-Ackles, Fuson and Sherin 2004; Moschkovich 2012; Yackel and Cobb 1996). We will examine these aspects as we consider how French immersion teachers can promote mathematical discourse in the classroom.

According to the National Council of Teachers of Mathematics (NCTM 1991, 34), *discourse* refers to "the ways of representing, thinking, talking, agreeing and disagreeing." The NCTM's communication standards place importance on students being able to organize their thinking through communication, to communicate clearly and coherently (with the teacher and with peers), and to evaluate the thinking and strategies of their peers.[1]

Creating effective mathematical discourse in a classroom setting requires skills and knowledge related to student learning and how to effectively teach mathematics. This task can be complicated by the fact that students in French immersion are learning a second language and often lack the vocabulary and confidence needed to participate in discourse.

Alberta Education's (2014) *Handbook for French Immersion Administrators* states that a successful French immersion program includes ongoing development of French-language skills, such as speaking and listening, in all subject areas. Developing these language skills in mathematics is not always a priority for teachers because of the challenges it presents. Discourse, however, is clearly an important part of Alberta's mathematics program of studies (Alberta Education 2007). Three of the seven mathematical processes outlined in the program of studies expect students to "communicate in order to learn and express their understanding," "connect mathematical ideas to other concepts in mathematics" and "develop and apply new mathematical knowledge through problem solving" (p 4). While French immersion teachers may see the value of using discourse in mathematics, it is not always something that comes naturally to teachers and students.

The NCTM process standards recommend that students communicate with each other, not just with the teacher. Communicating with peers allows them to organize and justify their thinking. The teacher's role as a facilitator means that he or she must guide discourse and create meaningful opportunities for students to share and justify their mathematical thinking, and students need to actively listen and respond not only to the teacher but also to their peers.

There are, in fact, many strategies French immersion teachers can use to promote mathematical discourse. First, they can help students connect mathematical understanding to language, especially in the SLA classroom. Second, it is imperative that social and sociomathematical norms be established from the beginning of the school year, in order to enrich mathematical discourse. French immersion teachers can also use strategies such as creating activities that require output from students and using revoicing to help students share their understanding. After discussing these strategies, I will share tools I have used in my own elementary French immersion mathematics classroom as I strive to promote and increase mathematical discourse.

Connecting Mathematical Understanding to Language

A significant dilemma in creating mathematical discourse in the French immersion classroom is that students come to class with varying levels of language skills and prior knowledge. Adler (1997) discusses the participatory-inquiry approach, in which students are expected to take responsibility for their learning by working together in small groups to solve engaging mathematical tasks. According to Adler, it is important that teachers withdraw from helping students too much and that they use mediation, which is "essential to improving the substance of communication about mathematics and the development of scientific concepts" (p 255). French immersion teachers struggle to find a balance between withdrawing and mediation, but it is important to find ways to link mathematical understanding to language as students participate with each other through discourse to help support their communication.

Moschkovich (2012) makes five recommendations for linking mathematical understanding to language. These recommendations are particularly helpful for teachers of students learning a second language, and they can help French immersion teachers develop strategies for supporting students' mathematical thinking, as well as facilitating discourse in the SLA classroom. It is important that French immersion teachers understand that although language can be a barrier to sharing understanding, it can also be an effective tool to help students make connections and justify their thinking and understanding. Moschkovich's five recommendations are as follows.

First, teachers should focus on the mathematical content of what students are saying, not the accuracy of their language. Teachers should try to understand students' mathematical thinking by asking questions and rewording what students have said.

Second, teachers need to shift their focus to mathematical discourse practices and move away from simplified views of language. During mathematical discourse, the focus should be on explaining, justifying and expanding ideas, rather than on simple vocabulary and definitions. Teachers can use a variety of tools to help students share their understanding, such as asking students diverse questions that require different levels of thinking, modelling brainstorming for students and allowing students to work in groups.

Third, teachers should recognize and support students as the complexity of language increases. Discourse should incorporate

- a variety of modes that will meet the different needs of students, such as oral and written communication, as well as the use of rewording and gestures;
- numerous representations, such as graphs, symbols, pictures and words;
- a variety of written texts, including word problems and student and teacher explanations;
- exploratory and expository talking; and
- various audiences, including the teacher and other students.

Fourth, teachers should see students and their growing language skills as resources. It is important for teachers to remember that as students begin to make sense of mathematical concepts, they will begin to make connections between their understanding and their language.

Fifth, it is crucial that teachers uncover the mathematics in what students say and do. Teachers should use a variety of strategies to understand how their students are thinking mathematically, which may include rewording what students have said, conferencing with students one-on-one, and using scaffolding and other supports as means of differentiation.

Establishing Norms

Establishing norms in the mathematics classroom is an important part of discourse (Cobb 1999; Cobb et al 1992; Yackel and Cobb 1996). This is especially true in a French immersion setting as students learn what is expected of them, as well as how they can respond to the teacher and to each other. Teachers must monitor student engagement to ensure that all students are actively participating. They must also be patient and have effective classroom-management skills if they are to be successful in creating mathematical discourse in which all students are active participants and listeners (Fraivillig, Murphy and Fuson 1999).

When establishing social norms for mathematical discourse in the classroom, it is important to distinguish between social norms and sociomathematical norms:

> The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of

what counts as an acceptable mathematical explanation is a sociomathematical norm. Likewise, the understanding that when discussing a problem students should offer solutions different from those already contributed is a social norm, whereas the understanding of what constitutes mathematical difference is a sociomathematical norm. (Yackel and Cobb 1996, 461)

In mathematical discourse, students are required to give different solutions for the same problem. Yackel and Cobb (1996) refer to this as mathematical difference. If teachers require their students to share different methods for solving problems, it is important that they first establish what is meant by *mathematical difference*. Through discourse and the exchange of ideas, students begin to see how their solutions are mathematically different, and the teacher begins to understand how to guide students to offer mathematically different solutions. In a French immersion setting, it is especially important that students share mathematical differences for problems, because this task not only extends their understanding but also exposes them to a variety of vocabulary and language models.

One of the teacher's most important roles during mathematical discourse is to facilitate rather than lead (by avoiding doing all the talking and presenting all the strategies and solutions); however, teachers are also participants who can help students decide if strategies and solutions are valid (Yackel and Cobb 1996). Hufferd-Ackles, Fuson and Sherin (2004) found that teachers must help students move through various trajectories, from being primarily teacher-led to student-led in the areas of questioning, explaining mathematical thinking, source of mathematical ideas and responsibility for learning. Teachers should ask open-ended, high-level questions, and students should be encouraged to share their ideas, even after the correct answer has been provided. This allows students to share their strategies with the entire class, thereby allowing all students to learn new strategies from each other, as well as to increase their language skills.

Teachers should not assume a passive role during discourse, especially in a French immersion setting, where students may not have the necessary vocabulary to participate (Moschkovich 2012). Rather, teachers should assume an active role. According to Yackel and Cobb (1996, 466), "The increasingly sophisticated way [teachers] select tasks and respond to children's solutions shows their own developing understanding of the students' mathematical activity and conceptual development." This is an important characteristic of the role of the French immersion teacher. Teachers must guide their students to deeper mathematical understanding by using a variety of tools and strategies that will meet students' needs as their mathematical understanding grows and develops.

Teachers must also create a safe environment for their students, enabling them to respectfully argue and justify their mathematical thinking and understanding (Cobb et al 1992). Teachers must make it clear to students that it is safe to make mistakes and that mistakes allow people to learn (McCrone 2005; Wood 1999). This is especially important in a French immersion setting, where students need to feel safe as they share their thinking, even if their language use is not completely accurate. The more that students see the classroom as a safe place to learn and share, the more they will be willing to take risks by actively participating in mathematical discourse.

Krussel, Edwards and Springer (2004, 308) contend that "traditionally, discourse analysis has been concerned with the rules and norms in the classroom, but little attention has been paid to the relation of discourse to the development of mathematical objectives themselves." Teachers should focus not only on how students share during discourse but also on the "nuances and subtleties of the discourse" (p 308) in relation to mathematical objectives. Establishing norms that tie mathematical objectives to discourse is an important element that is often overlooked in the French immersion classroom.

Creating Activities That Require Multiple Representations

The goal of French immersion is not only to help students acquire a second language but also to ensure that they can use their new language to explain their thinking and understanding in all subject areas, including mathematics. Students learning a second language must participate in extended discourse in order to process the language (Swain 2000). During discourse, students must be required to somehow participate, such as orally or in writing. This is especially true in the French immersion classroom, and teachers must ensure that they create activities that require students to represent their thinking in many different ways. As students share their understanding, teachers can help them increase their vocabulary and check for understanding. As well, when students speak, other students are given an opportunity to respond through the use of collaborative dialogue.

Collaborative dialogue is "dialogue in which speakers are engaged in problem solving and knowledge building" (Swain 2000, 102). It allows students to explore their own thinking and their peers' ideas as they work together to solve problems. As students participate in discourse, they begin to internalize their learning, and their language skills develop and grow. As students actively work together and use collaborative dialogue to solve problems, they use language to explain and clarify their mathematical thinking.

Output is an important part of mathematical discourse, both in small groups and as a class, especially as students are learning a second language (Swain 2000). It is important that students feel comfortable, respected and supported as they actively share their mathematical understanding and strategies. They should also feel comfortable with saying that they do not know the answer (Manouchehri and Enderson 1999). In a French immersion setting, students must be allowed to say that they are not sure how to explain their thinking, and they must be given opportunities to help each other and to get help from the teacher as they expand their language use and vocabulary.

It is vital that all students understand that they are part of a mathematics community in which their knowledge extends through sharing their understanding by using output. This type of discourse creates "meaningful collaborative math-talk" (Hufferd-Ackles, Fuson and Sherin 2004, 91). In a math-talk community, it is the role of students to actively direct their questions and ideas to each other, not only to the teacher. Teachers must set clear expectations for students to be active participants. French immersion students should understand that they have valuable information to share that can help their peers learn, even if they have not yet acquired precise vocabulary. As French immersion teachers, we must develop and encourage mathematical activities and discourse that require output from students in order to allow them to share their understanding, as well as to help them increase their vocabulary.

Using Revoicing

Teachers will often repeat, or revoice, what students have said in order to clarify students' thinking and understanding. *Revoicing* is defined as a linguistic structure that "affords the teacher the tools to coordinate the elements of academic task structure and social participation structure, while simultaneously bringing students into the process of intellectual socialization" (O'Connor and Michaels 1993, 319). Revoicing gives French immersion students a model they can follow for sharing their own mathematical thoughts and ideas. By restating what students have said, teachers give them the opportunity to explain and justify their thinking. Teachers can also ask students to summarize or restate what another student has shared. Revoicing is an effective tool for mathematical discourse in the French immersion classroom because it allows students to share their mathematical thinking while being supported by the teacher or their peers.

As the teacher uses revoicing to clarify or restate student thinking, other students have the opportunity to clarify their own thinking, and they are then more likely to use argumentation and debate, which helps them make mathematical connections and extend their thinking. Revoicing allows the teacher to point out successful participation by students, which encourages other students to participate. Revoicing also models different ways students can share their mathematical understanding.

Recommendations

Teachers can do many things to promote mathematical discourse in the French immersion classroom. Activities should be rich in vocabulary and must actively involve student participation and language use through sharing ideas and thinking, as well as using revoicing to restate the ideas of peers (Moschkovich 2012; O'Connor and Michaels 1993). Students need to use vocabulary in a variety of ways over extended periods of time, and they need to interact with each other in a collaborative manner (Savignon 1991). Tasks and activities must require student output in order for students to not only show their mathematical understanding but also build their language use and vocabulary (Swain 2000). Teachers must ensure that mathematical activities are varied and that they allow students to share their thinking in different ways. I often use games and math stations, even with my older students, to encourage them to use their language as they share their understanding with each other.

Teachers make specific pedagogical choices as they teach, including in the French immersion classroom during mathematical discourse. The types of questions asked by the teacher can encourage student thinking. The teacher has an important role in helping students learn vocabulary and terminology that will allow them to become competent communicators during mathematical discourse (McCrone 2005). Walsh and Sattes (2005) propose six norms that shape learning experiences:

- Students need time to reflect on past experiences in order to create new understandings.
- Students need time to reflect before sharing their thinking.
- Students need time to think out loud.

- Students learn best when they formulate and answer their own questions.
- Students learn best when they respectfully listen to each other.
- When students share talk time, they are showing respect for each other and their ideas.

Teachers must be thoughtful about how they ask questions, how much time they give students to think and how they ensure that all students are participating. According to Walsh and Sattes, a thoughtful teacher "creates structures that engage *all* students in thinking and responding to *all* questions" (p 5). An important shared belief is that quality questions help students learn and that all students can respond to all questions. These norms are especially important in a French immersion setting, as students must be given time to reflect on their thinking before they can formulate a new understanding. Additionally, students must understand that everyone has important ideas to add, even if they cannot share their thinking using perfect or precise language.

French immersion teachers also model different ways of demonstrating understanding and thinking, through the use of gestures, words and even writing (Hintz 2011). Writing in a journal can be a powerful tool for students to record their mathematical thinking. Students can use prompts such as the following:

- What do I already know about this topic?
- What does this question mean in my own words?
- How did I solve the problem?
- How do I know the solution is correct?

Teachers can then use the journals to get a better understanding of students' knowledge of the concepts.

Graphic organizers (such as Venn diagrams, mind maps, lists and charts) also allow French immersion teachers to highlight important mathematical vocabulary.

Conclusion

Mathematical discourse could be compared to an intricate dance, in which all participants must be actively engaged in order for the dance to be its most beautiful and effective. In the past, it was acceptable for the teacher to lead mathematical discourse, but research shows that this role should be shared with students, especially in the French immersion classroom as students learn the new language. Communication standards for mathematics continue to be updated, giving teachers the tools and strategies they will need in order to adapt to the ever-changing dance of mathematical discourse. As students build their confidence and their language skills, they will become better equipped to share and justify their mathematical thinking, and the intricate dance will begin.

Note

1. See www.nctm.org/standards/content.aspx?id=322 (accessed October 23, 2014).

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Gender and Mathematics: How to Control an Uncontrollable Variable

Marina Spreen

Many education researchers have pursued the topic of gender, especially in the field of mathematics. Gender is one of the few variables policy-makers, schools and teachers cannot control or change. However, as analyzing standardized test results has repeatedly shown, not only do boys and girls perform differently when completing mathematical questions but most boys continue to outperform girls in a variety of mathematical areas (Aunola et al 2004; Carr et al 2008; Council of Ministers of Education, Canada [CMEC] 2012; Preckel et al 2008).

As I contemplated gender in mathematics and reflected on the research studies and assessment results, I began to wonder why girls get lower marks in mathematics than boys do. I thought about my Grade 3 class, and I considered the possibility that researchers would find similar results with my students. As quickly as I conceived the thought, I felt appalled. Because I strongly believe that boys and girls are equal in their ability to learn any given subject, the only explanation for such assessment results would therefore be that I do not support my female students and my male students equally in their quest to discover the world of mathematics. Since I am passionate about learning and teaching mathematics, this thought saddened me. Unfortunately, many young girls across the country seem to be receiving insufficient support from their math teachers, as the latest results of the Programme for International Student Assessment (PISA) indicate (CMEC 2012). As an education professional, I find this situation unacceptable. For the sake of not only the girls in my classroom but also my own two daughters, I have undertaken a journey of inquiry into the issue.

In my quest, I found that education researchers have revealed four possible reasons for gender differences in mathematics performance in the elementary years:

- Gender-specific learning approaches and strategy preferences
- Gender differences related to motivation and competence belief
- Gender-biased assessment procedures and tools
- Gender-biased teacher beliefs and conduct

These factors may at first glance appear to be separate and unrelated, but they are in fact closely linked, and they create a complex and compelling argument for why gender is such a powerful variable affecting students' ability to learn mathematics, from as early as Grade 1. Here, I examine these underlying causes of gender differences in mathematics performance while exploring practical classroom strategies for limiting or avoiding the impact of gender, specifically in the elementary mathematics classroom.

Gender-Specific Learning Approaches and Strategy Preferences

Educators, regardless of which age group they work with, have long known that boys and girls learn and play differently (Cherney and London 2006; Gurian 2011; Gurian and Stevens 2006; James 2009). Generally, boys tend to thrive in a competitive environment in which their peer group drives them to do their best, whereas girls are much more interested in taking a cooperative approach and are often more concerned about the well-being and success of others (Bonomo 2010). In their review of the assessment data, literature and research done on gender differences in mathematics, Geist and King (2008, 46) point out that "girls use cooperation more than a competitive approach and are less concerned with being 'first' or 'best' and more with being sure the needs of their close friends are met as well as their own," whereas boys "function better in a competitive environment (such as number grades on worksheets and tests and teacher recognition)" and therefore "have the advantage in a traditional classroom."

However, the differences between boys and girls do not end there. An overwhelming amount of evidence points to a clear distinction between boys' and girls' preferred use of strategies when completing mathematical tasks (Bailey, Littlefield and Geary 2012; Hickendorff et al 2010; Imbo and Vandierendonck 2007). For example, Carr et al (2008) discovered that girls favour the use of manipulatives when solving arithmetic problems, and this hands-on strategy choice "may eventually constrain [their] development of mathematical knowledge and skills" (p 72). In contrast, the researchers showed that although boys prefer the use of cognitive strategies and retrieval, they are equally capable of reverting to the use of manipulatives when needed. Therefore, boys can switch from using retrieval to using manipulatives when dealing with more difficult problems, while the mathematical approach of girls is much more restricted (Carr and Davis 2001).

This gender-specific strategy choice is not limited to arithmetic problems. Fennema et al (1998) had previously shown that boys and girls also differ in their approaches to problem solving. In their study, the researchers established that girls tend to use modelling and counting strategies, while boys prefer more abstract strategies (such as derived facts and invented algorithms). Although researchers have been unable to establish why girls do not use invented algorithms as often as boys do, as educators we should be concerned that, without attempting to extend their knowledge through the invention of new arithmetic strategies, girls are limiting the extent of their mathematical learning to the simple recall of procedures and algorithms.

At this point in my quest, I began to wonder how I could assist young girls with learning mathematical concepts and skills in my classroom when their approaches to learning and their strategy preferences are clearly so different and distinct. After considering the research, I came to realize that many traditional classroom activities and instructional approaches create a clear disadvantage for the female mathematicians in our classrooms. However, in my effort to find concrete examples of how to adjust instructional strategies and create activities that support girls and boys equally in their mathematical learning, I found only a handful of articles (Gavin and Reis 2003; Karp et al 1998; King and Gurian 2006; Levi 2000).

As the research indicates, the traditional classroom environment, where students work individually and where competition is frequently used as a motivational strategy, is not conducive to girls' way of learning mathematics. Instead, as Gavin and Reis (2003, 38) point out, mathematics teachers should strive for a more balanced instructional approach in which they "provide some competitive, some cooperative, and some individual learning situations and allow choice whenever possible." Additionally, teachers should provide opportunities for students to work in same-sex groups, allowing girls to discover mathematics without the competitive edge boys might bring to such activities. Furthermore, to promote the use of a variety of mathematical strategies, teachers should carefully observe their students in order to determine when they are developmentally ready to move beyond using only manipulatives to including cognitive strategies and retrieval in their mathematical repertoire. At the appropriate time, elementary teachers can then offer many opportunities for students to solve mathematical problems without the use of manipulatives. In this way, teachers will encourage the use of cognitive strategies and retrieval, which is essential for the successful mathematical learning of female students in particular.

Gender Differences Related to Motivation and Competence Belief

The emotional and psychological state of mind of students greatly affects their ability to successfully complete tasks or solve problems. Educators know that students whose emotional needs are not being met, students who lack confidence or students who are unmotivated will experience more difficulties in learning and understanding new skills and concepts. This also holds true when students are completing mathematical tasks (Bouffard et al 2003; Gottfried 1990; Wigfield and Eccles 2000). Studies have consistently supported the idea that boys and girls develop different beliefs about their confidence and competence in the area of mathematics, and that they do not experience the same level of motivation when engaged in mathematical tasks.

Eccles et al (1993) showed that not only do boys value mathematical activities more than girls do and, therefore, are more motivated to engage in such activities but boys also consistently report higher levels of confidence than girls of their age when considering their mathematical abilities. Additionally, Lindberg et al (2013) observed an increasing gap between the math self-concept of male students and that of female students during the early elementary years. They point out that "this increasing gender gap in math self-concept may later on lead to actual gender differences in math achievement" (p 4). Unfortunately, even when academic grades indicated that girls had performed as well as their male counterparts had, girls reportedly experienced significantly less enjoyment and pride (Frenzel, Pekrun and Goetz 2007). Additionally, as the researchers point out, this low competence belief not only negatively influences girls' perception of the value and enjoyment of learning mathematics but will once again negatively affect their motivation to engage in mathematical activities in the future. This research indicates that limited motivation, low confidence, and inadequate experience of enjoyment and pride even when successful create a vicious cycle that hinders girls in learning mathematical concepts as easily as boys.

Most elementary mathematics teachers face this challenge when trying to encourage their female students to embrace the world of mathematics. As classroom teachers, we might feel overwhelmed when trying to come up with creative solutions to counteract the negative emotions girls have toward the value of mathematics and their engagement with the subject. Once again, the existing literature offers little guidance and few suggestions for elementary teachers about how to adjust current classroom strategies in order to sufficiently support girls in their mathematical learning. Yet, the solution might be easier than we expect.

To help female students gain confidence and take more pride in their mathematical accomplishments, our classrooms need to become safe, caring and supportive learning environments. Teachers should be a source of encouragement and support for all students, but particularly for female students. Above all, letting our female students know that all great mathematicians struggle and encounter difficulties is essential when trying to help them deal with the discomfort associated with such experiences (Gavin and Reis 2003). Furthermore, allowing students to share their mathematical ideas in a variety of ways will allow even the most self-conscious students to receive positive feedback. Classroom displays, journal writing, and the discussion of mathematical ideas with a partner or in small groups allow students to experience the mathematical world in a nonthreatening way.

To help girls recognize the value of mathematics and motivate them to participate, ask students to work on real-life math problems that touch on the interests of girls as well as those of boys. Whenever possible, give students a choice about what they will work on and what mathematical questions they will pursue in order to motivate all students to be actively engaged in the lesson. As most girls are not aware of the significant contributions women have made to mathematics and science, elementary teachers should present such information in the form of classroom discussions and displays. As Gilbert and Gilbert (2002, 526) point out, "Few students anticipate excelling in a field in which they think no one who looks like them has excelled before." This particularly holds true for our female mathematics students. The National Women's History Project website (www.nwhp.org) contains resources for classroom teachers who want to encourage female students in this way.

Gender-Biased Assessment Procedures and Tools

As previously indicated, existing large-scale assessment results consistently point toward gender-specific differences in mathematical ability as early as ages five and six (Voyer, Voyer and Bryden 1995). This trend continues today as boys consistently outperform girls on standardized achievement tests (CMEC 2012; Kenney-Benson et al 2006). However, as accountability in education continues to gain importance and the use of standardized assessment tools becomes inevitable, educators must take a closer look at the assessment procedures and tools they rely on, particularly if the results are used as evidence that girls are less capable in mathematics.

As previously established, girls prefer playing and learning cooperatively (Barnes 2011; Diamond 1994; Schwartz and Hanson 1992). Therefore, standardized assessment tools, regardless of whether the emphasis is on competition or on students' individual achievements, do not support girls' way of learning. Additionally, if such assessment tools include timed tests, girls will, more often than boys, experience unnecessary stress and, consequently, will be unable to do their best mathematical thinking (Gavin and Reis 2003).

As researchers have established, the problems do not end there. Procedural factors can also greatly influence students' performance results. As Voyer, Voyer and Bryden (1995, 263) point out,

> Larger [differences in achievement results] were obtained when the test was administered individually than when it was given in a group. This would suggest that there are meaningful sex differences in the way [participants] respond to the differences between these two testing situations.

The researchers explain that scoring procedures also appear to have an impact on the magnitude of gender differences in test results. Finally, the test questions themselves can give boys an unfair advantage, particularly if the questions are based on prior knowledge that girls might not have, such as knowledge of sports (Duffy, Gunther and Walters 1997; Zumbo and Gelin 2005).

Even though the use of standardized assessment tools is problematic, especially when considering gender-specific learning styles and psychological factors, this form of assessment has become an integral part of today's educational world, even during the elementary years. Therefore, our challenge is to find new and innovative ways to carefully incorporate such mandatory standardized assessment tools in order to minimize their negative effects on female students. It is critical here to note the distinction between incorporating standardized assessment and relying on it. Informal assessment, such as observing students while they are engaged in mathematical tasks or asking them to share their thoughts as they attempt to problem solve, not only allows students to reveal their learning in a nonthreatening way but also provides teachers with a rich source of information to gauge the quality of their instruction. Allowing students to demonstrate their knowledge and skills through mathematics portfolios, creative projects or small group assignments, for example, will nurture creative thinking and encourage risk taking, and will therefore elicit a more accurate picture of girls' mathematical abilities (Gavin and Reis 2003).

Gender-Biased Teacher Beliefs and Conduct

While problems with test questions, assessment procedures and scoring practices may explain existing differences between the test results of boys and girls in mathematics, my inquiry indicates that teacher–student interaction and, more specifically, teacher conduct may also affect students' mathematical performance. After careful review of the literature on the topic, I noted that teachers can negatively affect girls' mathematical learning in three ways:

- Through gender-biased beliefs and the application of gender stereotypes (Garrahy 2001; Gilbert and Gilbert 2002; Tiedemann 2002)
- Through the use of gendered language (Damarin 1990; Gavin and Reis 2003; Gilbert and Gilbert 2002)
- By focusing on boys more than girls during instructional time and classroom discourse (Wimer et al 2001)

As I contemplated the powerful implications of these disturbing research results, I came to realize the importance of unravelling my own beliefs about gender and understanding how those views affect my interactions with students. It worried me that "the magnitude of gender imbalance and gender bias [in classrooms] could be enormous and . . . detrimental to the education of girls and boys" (Garrahy 2001, 93). Once again I felt compelled to continue my quest to find solutions to eliminate this problem in elementary mathematics.

Classroom teachers frequently hold on to a desire to see all their students as the same and remain blind to obvious differences between boys and girls. However, as Garrahy (2001) points out, this view gives teachers a false sense of objectivity and impartiality, because the generic child does not exist. In reality, teachers often "unconsciously [apply] gender stereotypes by assuming that girls should use their abilities to help and empower other students, whereas boys should use their abilities to further excel in mathematics and empower themselves" (Gilbert and Gilbert 2002, 526), thus threatening gender equity in the classroom. Instead, as Levi (2000) establishes, teachers can take on one of three roles in order to address gender differences:

- Focus on providing equal opportunities, and respect the gender differences between their students
- Ensure that girls and boys have the same experiences
- Attempt to compensate for gender differences in society

Unfortunately, research does not offer a conclusive answer as to which role would address students' gender differences most efficiently. Therefore, educators should make pedagogical decisions grounded in the particularities of each class and each student in order to adequately support the mathematical learning of boys and girls (Levi 2000).

The language that teachers use can greatly hinder efforts toward gender equity in the classroom. Gilbert and Gilbert (2002) indicate that teachers' repeated use of the generic *he*, as well as addressing the entire class as *guys*, is both ambiguous and discriminatory toward female students. Additionally, teachers usually assume that it is a student's mother who should be contacted when issues arise. The repeated use of "I guess I will have to call your mother" sends a strong message to students about expected gender roles and fosters a gender-biased classroom environment. Gavin and Reis (2003) point to an analysis by Damarin (1990) of the traditional vocabulary used in mathematics that indicated a strong male influence on the type of words that are part of the daily instructional discourse. The goals of *mastery* and mathematical *power*, the strategy of *attacking* problems, and the use of *drills* and *competitions* may support boys' way of learning while leaving girls struggling in a seemingly male-dominated mathematical world. Therefore, the use and modelling of nongendered language should be essential to teachers' efforts to combat gender stereotypes in the mathematics classroom.

The prevalence of gender-biased beliefs among teachers and the use of sexist language during mathematical discourse are not the only ways teachers inhibit gender equity in their classrooms. In their study on teachers' questioning in elementary mathematics, Wimer et al (2001) discovered that although teachers directed their questions equally toward girls and boys, they would call on boys more frequently than girls when no student volunteered an answer. This observation is supported by Gavin and Reis (2003), who point out that in order to promote girls' mathematical learning, teachers need to give equal attention to their female students. The researchers go on to suggest that peer observation between colleagues can help teachers establish whether they are, in fact, dividing their attention equally among their students. Additionally, shifting classroom discourse away from an argumentative approach to a much more supportive learning activity in which students discuss concepts and practise their reasoning skills to help each other gain mathematical understanding might very well encourage girls to participate in discussions more easily (Morrow 1996), as well as increase the amount of teacher attention they receive. Additionally, educators should consider reducing the amount of teacher talk and instead offer students more time to engage in class discussion and work cooperatively (Becker 2003), while supporting their efforts to develop and share their mathematical thinking.

Conclusion

My inquiry made it clear to me that if I wanted to change the mathematical fate of my current and future female students, I needed to make a variety of critical changes to my professional practice. Consequently, I began to carefully reflect on and change the daily activities I planned for my class, the assessment procedures and tools I had been confidently relying upon, and the language I used when interacting with my students. I now give my students more opportunities to work cooperatively and support each other in their mathematical learning. Additionally, I have found a new appreciation for the use of a broad spectrum of assessment tools, and I now more frequently incorporate informal assessment during instructional time. Finally, when talking to my class or to individual students, I select my words more carefully and strive to avoid the use of gender-biased language. While I cannot claim to have completely levelled the mathematical playing field for my female students, I continue to adjust my professional practice according to the information I have discovered through my research.

As my journey of inquiry has revealed, elementary school girls face a great many disadvantages in their quest to acquire mathematical skills and knowledge. Not only do traditional teaching approaches and classroom activities often fail to support girls' ways of learning, but teachers' gender-biased beliefs and conduct continue to undermine the motivation and confidence of their female students. Therefore, I no longer find it surprising that many girls do not perform as well as their male classmates on large-scale standardized assessments. However, after carefully reviewing the existing literature on this topic, identifying the causes of the existing gender gap in mathematical achievements, and exploring classroom strategies for limiting the impact of gender, I am convinced that the goal of gender equity in the elementary mathematics classroom is attainable and, therefore, can be incorporated into our professional practice—not just for the sake of our students but also for our daughters.

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Content Knowledge for Teaching Mathematics: How Much Is Needed and Are Saskatchewan Teacher Candidates Getting Enough?

Nico Higgs and Egan J Chernoff

What makes a good math teacher? A dichotomy often emerges when this question is asked—one that pits two hypothetical math teachers against each other. Is the teacher who is an expert in math but not very skilled in pedagogy better than the teacher who knows very little about math but is highly skilled in pedagogy? Various arguments are thrown around, usually with both sides eventually conceding that an effective math teacher needs at least a decent understanding of both math and pedagogy.

Yet the debate continues. In this article, we consider the research on the topic of how much math knowledge teachers and teacher candidates need in order to effectively teach math. We begin with a review of the research and theories on the importance of mathematical knowledge for teacher candidates. Then, we analyze how they fit with the current education that teacher candidates are receiving (with a special focus on the University of Saskatchewan and local school divisions). We conclude with a discussion of the implications for aspiring math teachers.

A Brief Look at the Literature

It may seem that if teachers have greater mathematical knowledge, they will be more effective math teachers and their students will be more successful in mathematics; however, research has shown that this is not the case. Begle (1979) measured teachers' mathematical knowledge (determined by the number of postsecondary courses taken) against student achievement, and found no positive correlation. The Third International Mathematics and Science Study (TIMSS) reported that American students in Grade 4 were adequate and those in Grade 12 were poor at mathematics (US Department of Education 1997, 1998). To see if teacher training had an effect on these results, the US Department of Education (1996, 1997, 1998) conducted a study that found that the training and knowledge of teachers in the United States were comparable to those of their counterparts in other countries, which seems to suggest that teachers' mathematical knowledge has no effect on student achievement (Howe 1999).

In institutions of higher learning, teachers learn increasingly abstract concepts and are able to pass math courses without learning how to increase understanding of more elementary material (Fi 2003). Researchers have investigated teachers' mathematical knowledge and have found evidence that mathematical knowledge does, in fact, play a vital role in students' learning of math (Ball 1990, 1991; Ball, Hill and Bass 2005; Ball, Lubienski and Mewborn 2001; Ball, Thames and Phelps 2008; Conference Board of the Mathematical Sciences 2001; Hill, Rowan and Ball 2005; Ma 1999). However, teachers require a different type of understanding—an understanding that they do not necessarily obtain in postsecondary math courses.

Much of Ball's research, and that of Hill and Ma, is built upon Shulman's (1986, 9) definition of *pedagogical content knowledge:*

> The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others.... Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons.

Shulman defines, in essence, a discipline of study for teachers different from their subject area and general pedagogy—one that sees teaching as professional work with its own unique professional knowledge base for each subject area, math especially (Ball, Thames and Phelps 2008). It is not only knowledge of content or knowledge of pedagogy but, rather, an amalgam of the two that is central to the knowledge needed for teaching.

From this perspective, recent experience with advanced math courses should make teacher candidates experts, but it does not (Fi 2003). Research into this new understanding of the mathematical knowledge that teachers need shows that mathematical knowledge correlates with student success, which leads to the conclusion that teacher candidates need a profound understanding of fundamental mathematics in order to teach effectively (Ma 1999).

Further research on the mathematical knowledge teachers need has led to many theories, which have been summarized elsewhere (Dossey 1992; Fennema and Franke 1992; Hiebert and Carpenter 1992; Hiebert et al 1997). These theories help mathematics educators better understand teacher knowledge through the connections between beliefs, the affective domain, pedagogical content knowledge, subjectivity of knowing (von Glasersfeld 1996) and teachers' subject matter (Fi 2003)—as opposed to teachers' subject matter knowledge alone. Reviews of such literature show that teacher candidates ought to be conversant with the subject matter they intend to teach (Ball 1988, 1991; Conference Board of the Mathematical Sciences 2001; Fi 2003; Ma 1999). A more recent study, by Ball, Hill and Bass (2005), shows that student achievement with a teacher who ranks in the top quartile of teacher knowledge is the equivalent of two to three weeks of instruction ahead of students who have a teacher with average teacher knowledge. Another interesting finding is that the size of the effect of teachers' mathematical knowledge for teaching is comparable to the size of the effect of socioeconomic status on student gains. This is significant because it demonstrates that teachers' content knowledge can help create equity for all students and may help combat the frequently reported widening of the achievement gap.

Based on the research and theories discussed, content knowledge and pedagogical content knowledge are significant and necessary components for teacher candidates to become effective math teachers (Kilpatrick, Swafford and Findell 2001). However, mathematical knowledge, both content and pedagogical, among teacher candidates is lacking and needs to be addressed by teacher education programs (Cooney 1999).

The Mathematical Knowledge of Teacher Candidates

A study of teacher candidates' content knowledge, pedagogical content knowledge and envisioned pedagogy of trigonometry found teacher candidates' mathematical knowledge lacking (Fi 2003). Fi's study investigated similar studies that also found teacher candidates' mathematical knowledge inadequate (Ball, Lubienski and Mewborn 2001; Howald 1998; Ma 1999). The study revealed that teacher candidates had not seriously revisited since high school the basic notions and conceptual understandings of the math they were to teach, and that they themselves had never adequately learned the concepts. The teacher candidates also claimed to have not been exposed to foundational ideas in their postsecondary mathematics courses, which is perhaps why teacher education should address these particular issues (Cooney 1999). The research reveals the need to reacquaint teacher candidates with the fundamental mathematical ideas they will be teaching, by re-examining K–12 mathematics content from an advanced perspective (Conference Board of the Mathematical Sciences 2001; Fi 2003; Usiskin et al 2003).

Mathematical Knowledge in the Flatlands

Recognizing the research presented above, we will now critique the education that prospective math teachers are receiving, specific to the University of Saskatchewan and the Saskatoon Public School Division.

As but one example, the College of Education at the University of Saskatchewan is a professional development school that allows teacher candidates to work with nearby school divisions to get classroom experience. This, alongside educational foundations classes, fulfills many areas in which teacher candidates need education, such as curriculum studies, anti-racism and anti-oppression teaching, experiential learning, and differentiated learning. However, there seems to be a serious lack of content knowledge and, more specifically, pedagogical content knowledge.

Secondary teacher candidates take only one methods course devoted to each of their teaching areas (whereas, for example, all primary teachers must take a math methodology course). This methods course, which has a lot to cover, is not able to devote all of its allotted time to pedagogical content knowledge. This equates to under 40 hours of education on content knowledge for math (and other subjects). Yes, teacher candidates do have prerequisite postsecondary courses in their subject areas prior to gaining entrance to the College of Education. However, as discussed earlier, research demonstrates that these prerequisites do not necessarily translate into the content knowledge teachers require for effective teaching in the future. Thus, it would appear that teacher candidates potentially come up short in their content knowledge of the subjects they are going to teach. Further compounding the issue, once teacher candidates graduate from the college and apply for a teaching certificate, they are considered to be qualified to teach any subject at any grade level. Thus, a teacher candidate could obtain none of the necessary mathematical knowledge during his or her time in the College of Education, but could nevertheless be hired to teach math. Lacking the content knowledge necessary to be an effective math teacher, how well will this teacher teach mathematics?

The local school divisions, Saskatoon Public Schools in particular, seek teacher candidates who have qualifications for "accreditation." In other words, they want teacher candidates who have extra postsecondary courses in mathematics, which are essentially meaningless for teachers, as we have discussed. Teacher interviews are done by subject area and are based on the subject area in which prospective teachers have the most postsecondary courses, not the most pedagogical content knowledge training. The entire process in Saskatoon of teacher education, teacher qualification and even hiring practices appears misguided given the pedagogical content knowledge needs of teachers. Teacher candidates are not adequately supported in learning the required content knowledge, and teachers with inadequate mathematical knowledge are consistently hired to teach math. Even the hiring process that does try to target mathematical knowledge does so inadequately by focusing on postsecondary courses in math that do not provide the knowledge teachers need in order to teach math effectively. Based on (1) the research that shows the importance of proper mathematical content knowledge for teachers and (2) the lack of mathematical content knowledge teachers have in other parts of Canada and in the United States, we contend that teacher candidates in Saskatoon continue to be inadequately served.

Mathematical Knowledge and Student Equity

Those math teachers who do have proper pedagogical content knowledge and mathematical understanding are misplaced in current education systems. For example, Hill (2007) showed that qualified math teachers are unevenly distributed across the United States, resulting in affluent students consistently getting the best math teachers. Similar issues arise in the way qualified math teachers are viewed in both the United States and Canada. Currently, the common practice is to have the most qualified or most senior teachers teaching the higher-level content, such as calculus, while the less experienced and weaker teachers end up in front of students with a history of low achievement in mathematics (Brahier 2013). This discrepancy often results in a downward spiral in which low success, less experienced teachers and fewer supports prevent students from advancing in mathematics. Even with the assumption that all students have the right to equal access to all areas of the curriculum, as well as to high-quality instruction, these practices do not support equity for students (Brahier 2013). By contrast, increased mathematical content knowledge of teachers can facilitate equity for students (Ball, Hill and Bass 2005). In the end, it is the students who suffer the most from these misguided knowledge requirements for teacher candidates.

Conclusion

Mathematical knowledge, especially pedagogical content knowledge, is of vital importance to effective math teaching. However, teacher candidates tend to be inadequately educated in math while also being pressured into taking the wrong math courses in order to get hired. Accounting for how students understand a content domain is a key feature of the work of teaching that content (Ball, Thames and Phelps 2008). The research and theories examined here show the importance of mathematical knowledge in the teaching of mathematics. The distinction between pedagogical content knowledge and subject knowledge itself highlights the importance of teachers mastering the content they are to teach beyond simply passing a course. Perhaps aspiring math teachers should be encouraged (whether in colleges of education or elsewhere) to take their education into their own hands by seeking out opportunities to gain mathematical knowledge, the kind of knowledge that is needed to effectively teach mathematics.

Research shows the positive effects of professional development (Ball, Hill and Bass 2005; Hill and Ball 2004) and, further, that it is important for teachers to be active in their own professional development (Brahier 2013). Given the importance of mathematical knowledge, despite the fact that the current environment does not adequately support education in pedagogical content knowledge, aspiring teachers—in this age of access to information about modern techniques and theory in math teaching (for example, massive open online courses [MOOCs] by Jo Boaler, Keith Devlin and others)—should adhere to an age-old adage: where there is a will, there is a way. This may become a crucial attitude in future math teacher education.

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Aboriginal Hand Games and Probability

Jessica Shaw and Maxine Elter

Objective

The purpose of this lesson is to integrate the tradition and culture of Aboriginal hand games with the probability outcomes of Alberta's Math 8 curriculum. This is meant as a wrap-up project to summarize all understanding in this topic.

Specific Learner Outcome

The following outcome from Alberta's Math 8 program of studies is covered:

- Statistics and Probability: Solve problems involving the probability of independent events.

Time Required

- Approximately 80 minutes

Materials

- Two blankets
- Pebbles/stones for each student
- Sticks
- Aboriginal drumming

Lesson

Introduce this topic by providing students with information about the tradition and culture of Aboriginal hand games. It would be beneficial to bring in an Aboriginal elder from the community in which you reside to help you address the historical, cultural and community aspects of this cross-curricular activity. The knowledge and guidance of Aboriginal elders are vital to the success of any Aboriginal traditional teaching. In many communities, elders (or knowledge keepers) are identified by the community population in which they reside. Resources such as the First Nations, Métis and Inuit liaison worker in your school; a Native Friendship Centre; Aboriginal interagencies; and Child and Family Services offices can help locate an Aboriginal elder. If an Aboriginal elder is not available, the rules and play for Aboriginal hand games, as well as demonstrations, are available from the following online resources:

- Dene Games (http://denegames.ca/introduction/index.html)
- "Hand Games Prove Popular for Everyone" (www.ammsa.com/publications/alberta-sweetgrass/hand-games-prove-popular-everyone)
- "Dene Hand Games Tournament Takes Off in Whati, N.W.T" (www.cbc.ca/news/aboriginal/dene-hand-games-tournament-takes-off-in-whati-n-w-t-1.2547769)
- "Aboriginal Hand Games All About Mind Trickery" (www2.canada.com/edmontonjournal/news/story.html?id=39c4ea22-8a80-405f-8b82-1bdd86b37052)

An Aboriginal hand game is a community game played with traditional hand drums, sticks, stones and blankets. It is based on the simple concept of hiding objects and using elaborate hand signals and gestures to both hide the object and find the object. Two opposing teams attempt to deceive each other through chanting, body movements and hand movements. A long time ago Aboriginal peoples would gather several times during the year to celebrate seasonal changes and special events (such as births, passings and joinings). Aboriginal hand games were played at these events in celebration and as friendly competition between communities. This math lesson explores this traditional game's relevance to mathematical probability.

Give students an opportunity to play one round of the game so that they are familiar with strategy and game play. This could take a whole class period or more, and could include a short written reflection about game strategy or whatever students noticed about succeeding in the game.

Then, have a class discussion based on the following questions:

- What is the probability that you can choose the correct hand for one person? [Answer: 0.5, 1/2, 50%]

- What is the probability that you can choose the correct hand for two people? [Answer: $0.5 \times 0.5 = 0.25$, 1/4, 25%]
- What is the probability that you can get everyone out the first time (that is, choose the correct hand for all people)? [Answer: 0.5^n, where n is the number of people]
- Are these theoretical or experimental probabilities? How do you know? What is the difference? [Answer: These are theoretical probabilities. They are based on what should happen as opposed to what actually will happen.]
- How might the experimental probability differ from the theoretical probability? [Answer: Experimental probability is what actually happens. It will vary from game to game and will not be consistent.]
- What could you do to improve or reduce the probability of choosing the correct hand? How does this relate to the strategy of the game? [Answer: You could distract the members of the other team to reduce their likelihood of choosing correctly.]

Once students have discussed how probability is related to the hand game, as well as how they can affect the probability of the game by making conscious decisions, allow them to play the game once more to test their strategies.

Extending the Lesson

This lesson can be extended by exploring the difference between theoretical probability and experimental probability. Have students generate a list of various conditions under which the game could be played (for example, all the distractions possible, drumming but no taunting with words, drumming but no taunting of any kind, or no drumming but taunting). Keep track of the experimental probabilities under all these conditions.

Additional Note

If students enjoy this game, they may be interested in participating in an Aboriginal hand games tournament in your area.

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Success by Numbers: Math Competitions Help Prepare Students for Challenges Ahead

Johan Rudnick

Math is often misunderstood and maligned, especially by students. Math is dull and boring. Math is way too hard. Math is anything but cool. At the same time, math can be fun and rewarding. Math can be beautiful and creative. And, perhaps most important, math can nurture strong analytical and problem-solving skills—key ingredients for career success. One tool for bridging the gap between general perceptions and critical reality is the math competition.

Since 1969, the Canadian Mathematical Society (CMS) has been staging national math competitions to encourage students to explore, discover, and learn more about mathematics and problem solving. Along the way, thousands of students have become more comfortable with math and more confident about what they can achieve. The most popular of the CMS national competitions is the Sun Life Financial Canadian Open Mathematics Challenge, which is held in November each year.

Why Math Competitions?

Math competitions are a primary extracurricular activity that can both stimulate and cultivate student interests. Like sports, whether for recreation or competition, math competitions require basic knowledge and understanding. Coupled with discipline and practice, exercising math skills will lead to improved self-confidence, enjoyment and success. And math competitions are much more than just an intellectual pursuit for gifted students. A good math competition provides opportunities for "Hey, I can do that!" and nurtures the attitude of "Now maybe I can try that!"

Teachers play a critical role in math education, and competitions represent much more than another opportunity to organize. Most math competitions provide a plethora of supplementary resource materials, including sample and past problem sets, weekly or monthly problems, and other resources. In the case of the CMS, the resources are quite diverse. There is the book *The Alberta High School Math Competitions 1957–2006: A Canadian Problem Book*, edited by Andy Liu (Mathematical Association of America/Canadian Mathematical Society, 2009). For shorter treatments, there is the book series A Taste of Mathematics (ATOM) and the international problem-solving journal *Crux*. There is also the Problem of the Week (http://cms.math.ca/Competitions/COMC/2014/potw.html). And then there are the CMS math camps (for example, the Alberta/CMS Math Camp held in August for Grades 7–10 participants. These resources are valuable and rich tools in their own right, and they can be tailored to classroom or student interests.

Teachers can choose from many math competitions. Some are offered by a university, such as the Centre for Education in Mathematics and Computing (CEMC) contests at the University of Waterloo. Others are designed for regional or provincial participants, such as the Edmonton Junior High Math Contest. The CMS national math competitions are built on partnerships with universities across Canada; in Alberta, the CMS partners with the University of Calgary.

The Competitions

Sun Life Financial COMC

The Sun Life Financial Canadian Open Mathematics Challenge (COMC) is a flagship national competition open to any student in any location at any grade level. It attracts thousands of participants from across Canada and internationally each year. Although the competition is targeted at upper-level high school students, performance awards are available for multiple grade levels. Furthermore, every student in Canada who participates is equally eligible for prizes. Top-performing students receive plaques and certificates, and their school receives a plaque as well. Graduating students who perform well may be considered for scholarships, and other students are eligible for an invitation to a CMS regional, specialty or national math camp.

The two-and-a-half-hour competition is usually held at participants' schools in early November. Although the competition is nationally focused, performance is recognized at both the national and the provincial levels (including specific grade levels).

In addition to receiving awards, plaques, certificates and prizes, top-performing students are invited to participate in the advanced CMS competition. For students with advanced interests, this is the only competition that can lead to a student's being chosen by the CMS for Math Team Canada and participation at the International Mathematical Olympiad.

Sun Life Financial Repêchage

Students who come close to qualifying for an invitation to the advanced CMS competition are invited to participate in the take-home Repêchage competition in early February. About 75 students are given one week to complete eight questions. The best performing students will be invited to the advanced competition, and each student receives a book from the ATOM series.

Sun Life Financial CMO

The Sun Life Financial Canadian Mathematical Olympiad (CMO) is a three-hour advanced competition that is usually written in school in late March and typically consists of five challenging math problems. Approximately 80 students are invited to participate. The top-scoring students receive awards and prizes, and the student with the best score is awarded the Sun Life Financial Cup. All students receive a one-year online subscription to the journal *Crux*. Top-performing students may also be invited to a national math camp or to be members of Math Team Canada.

Math Team Canada and the IMO

Canadian students have consistently performed very well on the world stage at the International Mathematical Olympiad (IMO). The IMO is an intense world-class competition that takes place over three days. Each day students have four and a half hours to solve three questions. CMS selects six students to be on Canada's team and, as with any sport, CMS assembles training and coaching staff to provide an intense preparation program. Training takes place at the Banff International Research Station, and then the team flies off to the IMO venue. At the July 2014 IMO, in South Africa, the team took two gold medals, one silver medal and three bronze medals and ranked ninth out of 101 countries (with 560 participants).

Other Competitions

In addition to staging national competitions, the CMS is also the supporting organization for Canada's participation in the Asian Pacific Mathematics Olympiad (APMO). Domestically, the CMS also supports provincial and regional math competitions across Canada. For example, the top three students from the Alberta High School Mathematics Competition are invited to participate in the CMO. Students can explore these competitions as they work toward the CMS math competitions and Math Team Canada.

Teacher Appreciation

None of these math competitions could be staged without the support and efforts of the teachers who organize events at their schools. The CMS is especially grateful to those teachers who support the CMS competitions, and it has a teacher appreciation program, including the distribution of sponsored educational materials, as a small token of appreciation for making a big difference.

Success by Numbers

If web stats are to be believed, the websites associated with the CMS national math competitions attract an awful lot of students and teachers, who seem to enjoy the breadth of materials available. Meanwhile, direct participation in the CMS competitions grows each year as more and more students and teachers become aware of the opportunities.

While it is always reassuring to note success by numbers, perhaps the more rewarding aspect of the CMS math competitions is the knowledge that each student can participate and compete at his or her own level. The collateral benefit is students who are better prepared for career competitions ahead, where success with numbers can really mean success by numbers.

Johan Rudnick is the executive director of the Canadian Mathematical Society. Information on the CMS math competitions and other competitions can be found on the CMS website at http://cms.math.ca/Competitions/. Readers are welcome to contact Johan directly at director@cms.math.ca.

Alberta High School Mathematics Competition 2013/14

The Alberta High School Mathematics Competition is a two-part competition taking place in November and February of each school year. Book prizes are awarded for Part I, and cash prizes and scholarships for Part II. Presented here are the problems and solutions from the 2013/14 competition.

Part I

November 19, 2013

1. Of the first 100 positive integers 1, 2, . . . , 100, the number of those not divisible by 7 is
   (a) 14 (b) 15 (c) 85 (d) 86 (e) none of these

2. The total score of four students on a test is 2,013. Ace scores one point more than Bea, Bea scores three points more than Cec, and Cec scores two points more than Dee. None of their scores are divisible by
   (a) 3 (b) 4 (c) 5 (d) 11 (e) 23

3. Of the following five fractions, the largest one is
   (a) $\frac{1}{75}$ (b) $\frac{2}{149}$ (c) $\frac{3}{224}$ (d) $\frac{4}{299}$ (e) $\frac{6}{449}$

4. Two teams, A and B, played a soccer game on each of seven days. On each day, the first team to score seven goals won. There were no ties. Over the seven days, A won more games than B. but B scored more goals than A overall. The difference in the total number of goals scored by B and A is at most
   (a) 17 (b) 18 (c) 19 (d) 20 (e) none of these

5. ABCD is a quadrilateral with AB = 12 and CD = 18. Moreover, AB is parallel to CD, and both ∠ADC and ∠BCD are less than 90°. P and Q are points on side CD such that AD = AP and BC = BQ. The length of PQ is
   (a) 6 (b) 7 (c) 8 (d) 9 (e) 10

6. Each of four cows is either normal or mutant. A normal cow has four legs and always lies. A mutant cow has either three or five legs and always tells the truth. When asked how many legs they have among them, their respective responses are 13, 14, 15 and 16. The total number of legs among these four cows is
   (a) 13 (b) 14 (c) 15 (d) 16 (e) none of these

7. Let a and b be positive integers such that $ab < 100$ and $a/b > 2$. Denote the minimum possible value of a/b by m. Then we have
   (a) $m \leq 2.15$ (b) $2.15 < m < 2.2$ (c) $m = 2.2$
   (d) $2.2 < m < 2.25$ (e) $m \geq 2.25$

8. Let ABCD be a quadrilateral with ∠DAB = ∠CBA = 90°. Suppose there is a point P on side AB such that ∠ADP = ∠CDP and ∠BCP = ∠DCP. If AD = 8 and BC = 18, the perimeter of the quadrilateral ABCD is
   (a) 70 (b) 72 (c) 74 (d) 76 (e) 78

9. Two bus routes stop at a certain bus stop. The A bus comes at one-hour intervals, and the B bus comes at regular intervals of a different length. When Grandma rests on the bench by the bus stop, one A bus and two B buses come by. Later, Grandpa rests on the same bench, and eight A buses come by. The minimum number of B buses that must have come by during that time is
   (a) fewer than 4 (b) 4 or 5 (c) 6 or 7 (d) 8 or 9
   (e) more than 9

10. Suppose that $16^{2,013} = a^b$, where a and b are positive integers. The number of possible values of a is
    (a) 2 (b) 8 (c) 11 (d) 16 (e) 24

11. The following five statements are made about the integers a, b, c, d and e: (1) ab is even and c is odd, (2) bc is even and d is odd, (3) cd is even and e is odd, (4) de is even and a is odd, and (5) ea is even and b is odd. The maximum number of these statements that may be correct is
    (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

12. A very small cinema has only one row of five seats, numbered 1 to 5. Five moviegoers arrive one at a time. Each takes a seat not next to any occupied seat, if this is possible. If not, then any seat will do. The number of different orders in which the seats may be taken is
    (a) 24 (b) 32 (c) 48 (d) 64 (e) 72

13. Let $f(x) = x^2 + x + 1$. Let n be the positive integer such that $f(n) = f(20)f(21)$. Then the number of distinct prime divisors of n is
    (a) 1 (b) 2 (c) 3 (d) 4 (e) more than 4

14. The number of pairs (x, y) of integers satisfying the equation $x^2 + y^2 + xy - x + y = 2$ is

(a) 3 (b) 4 (c) 5 (d) 6 (e) none of these

15. A triangle ABC with AB = 7, BC = 8 and CA = 10 has an interior point P such that $\angle$APB = $\angle$BPC = $\angle$CPA = 120°. Let r_1, r_2 and r_3 be the radii of the circles passing through the vertices of triangles PAB, PBC and PCA, respectively. The value of $r_1^2 + r_2^2 + r_3^2$ is

(a) 71 (b) 72 (c) 73 (d) 74 (e) 75

16. The list 1, 3, 4, 9, 10, 12, 13, . . . contains in increasing order all positive integers that can be expressed as sums of one or more distinct integral powers of 3. The 100th number in this list is

(a) 981 (b) 982 (c) 984 (d) 985 (e) 999

Solutions

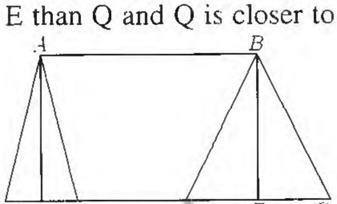
1. When 100 is divided by 7, the quotient is 14,with a remainder of 2. Thus, 14 of the first 100 positive integers are divisible by 7. It follows that the number of these integers not divisible by 7 is 100 – 14 = 86. The answer is **(d)**.

2. Since the average score is just over 500, we try 500 as a score for Dee. Then Cec's score is 502, Bea's is 505, and Ace's is 506. The total is indeed 2,013, so no adjustment is necessary. Now, 500 is divisible by 4; 500 and 505 are divisible by 5; and 506 is divisible by 11 and 23. The answer is **(a)**.

3. The lowest common numerator is 12. The fractions then become
$$\frac{12}{900}, \frac{12}{894}, \frac{12}{896}, \frac{12}{897} \text{ and } \frac{12}{898}$$
respectively. The answer is **(b)**.

4. B won at most three games, and for each of those games, B won by at most seven goals, with a 7–0 score. In the other four games, B lost by at least one goal, with a 7–6 score. The difference in the number of goals is at most (7 × 3) – (1 × 4) = 17. The answer is **(a)**.

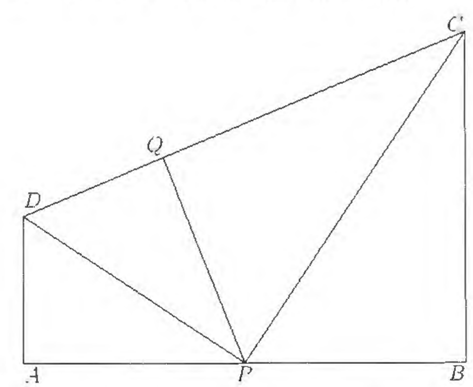
5. From A and B, drop perpendiculars onto CD at the points E and F, respectively. Since both $\angle$ADC and $\angle$BCD are less than 90°, E and F do lie on the segment CD. Note that
$$DE + FC = CD - EF = CD - AB = 6.$$
Since ED = EP and FC = FQ, EP + FQ = 6 < 12 = EF, P is closer to E than Q and Q is closer to F than P. Therefore, PQ = EF – (EP + FQ) = 12 – 6 = 6. The answer is **(a)**.



6. Since all four responses are different, at least three of them are wrong. If all four are wrong, then the cows are all normal, and they will have 16 legs among them. However, this makes one of the responses right. Hence, one of the responses is indeed right. The three normal cows have 12 legs among them. Hence, the mutant cow must have three legs in order to make one of the responses right. The answer is **(c)**.

7. We have $100 > ab > b(2b + 1)$ so that $b \leq 6$. Thus, the minimum value of m is 2 + 1/6 = 2.1666 . . . . The answer is **(b)**.

8. Since $\angle$DAB + $\angle$CBA = 180°, AD is parallel to BC. Therefore, $\angle$ADC + $\angle$BCD = 180°. Hence, $\angle$PDC + $\angle$PCD = 90°. Consequently, $\angle$DPC = 90°. Let Q be the foot of the perpendicular on CD from P. Note that triangle PDA is congruent with triangle PDQ, and triangle PCB is congruent with triangle PCQ. Hence, DQ = DA = 8, CQ = CB = 18 and PA = PQ = PB. Since $\angle$DPC = 90°, triangle DPQ is similar to triangle PCQ. Hence, DQ/QP = PQ/QC . Therefore, $PQ^2 = 8 \times 18 = 144$ and PQ = 12. The perimeter of ABCD is therefore AP + PB + BC + CQ + QD + DA = 12 + 12 + 18 + 18 + 8 + 8 = 76. The answer is **(d)**.



9. To minimize the number of B buses coming by, we stretch the length of their intervals as far as possible. Suppose Grandma sees the A bus that comes at 10:00. Then she does not see those that come at 9:00 and 11:00. Thus, the length of the interval between two B buses is strictly less than two hours. The two B buses Grandma sees may have come at 9:01 and 10:59, in which case the interval is 118 minutes. Grandpa is on the bench for at least seven hours. This is longer than three intervals for the B buses, so he must have seen at least three of them. Suppose he sees the A buses at 11:00, 12:00, 1:00, 2:00, 3:00, 4:00, 5:00 and 6:00. Then he will see only the B buses at 12:57, 2:55 and 4:53. The answer is **(a)**.

10. Since $16^{2,013} = 2^{4\times 2,013}$, a must be of the form 2^k, where k is a positive integer divisor of $4 \times 2,013 = 2^2 \times 3 \times 11 \times 61$. The prime factorization of k may contain up to two 2s, one 3, one 11 and one 61, so that there are $(2 + 1)(1 + 1)^3 = 24$ possible values of k and, therefore, of a. The answer is **(e).**

11. If a, c and d are odd, while b and e are even, then (1), (2) and (4) are all correct. Suppose at least four statements are correct. By symmetry, we may assume that they are (1), (2), (3) and (4). However, by (1) and (2), c and d are both odd, and yet by (3), cd is even. This is a contradiction. The answer is **(c).**

12. The first two moviegoers to arrive may take the pair (1, 5), (1, 4), (2, 5), (1, 3), (2, 4) or (3, 5) of seats. In each case, there are 2! = 2 ways for them to do so. If they take (1, 5), (1, 3) or (3, 5), then the third moviegoer has only one choice of seat. The remaining two seats may be occupied in 2! = 2 ways. Otherwise, the last three moviegoers may take any vacant seats, and this can be done in 3! = 6 ways. Hence, the total number of orders in which the seats may be taken is $2 \times 3 \times (2 + 6) = 48$. The answer is **(c).**

13. Note that $f(m-1)f(m) = (m^2 - m + 1)(m^2 + m + 1) = m^4 + m^2 + 1 = f(m^2)$. Substituting $m = 21$ yields $f(20)f(21) = f(441)$. Therefore, $n = 441 = 3^2 7^2$. The answer is **(b).**

14. The equation can be written as $x^2 + (y-1)x + y^2 + y - 2 = 0$. By the quadratic formula, the solutions are

$$x = \frac{-(y-1) \pm \sqrt{(y-1)^2 - 4(y^2 + y - 2)}}{2}$$
$$= \frac{-(y-1) \pm \sqrt{-3(y-1)(y+3)}}{2}.$$

These are real if and only if $(y + 3)(y - 1) \le 0$. Since y is an integer, it must be one of $-3, -2, -1$, 0 or 1. If $y = -3$, then $x = 2$. If $y = -2$, then $x = 0$ or 3. If $y = -1$, x is not an integer. If $y = 0$, then $x = -1$ or 2. Finally, if $y = 1$, then $x = 0$. The answer is **(d).**

15. Let O be the centre of the circle passing through the vertices of triangle PAB. Note that since $\angle APB > 90°$, O lies on the perpendicular bisector of AB outside of triangle PAB. Since OA = OP = PB,

$\angle AOB = \angle AOP + \angle POB = 180° - 2\angle APO + 180° - 2\angle BPO = 360° - 2\angle APB = 120°$.

Hence, $AB = \sqrt{3}OA$ so that $OA = AB/\sqrt{3}$. It follows that

$$r_1^2 = \frac{AB^2}{3}.$$

Similarly,

$$r_2^2 = \frac{BC^2}{3}$$

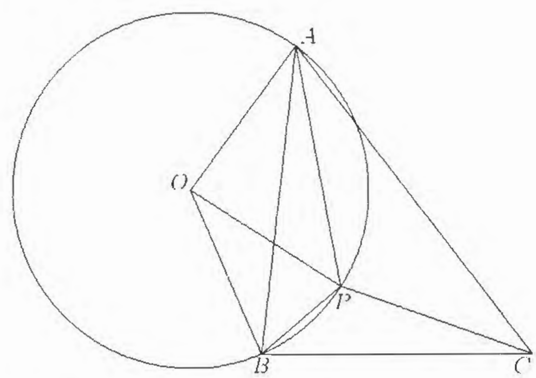
and

$$r_3^2 = \frac{CA^2}{3}.$$

Hence,

$$r_1^2 + r_2^2 + r_3^2 = \frac{7^2 + 8^2 + 10^2}{3} = 71.$$

The answer is **(a).**



16. If we switch the powers of 3 to the powers of 2, then we get all the positive integers. Hence, we convert 100 into base 2, obtaining $100 = 2^6 + 2^5 + 2^2$. It follows that the 100th number on the list is $3^6 + 3^5 + 3^2 = 981$. The answer is **(a).**

Part II

February 5, 2014

1. The side length of square ABCD is 2. The circle with diameter AB intersects the circle with centre C and radius BC again at E. Determine the length of the segment AE.

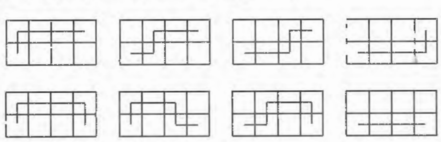
2. A family consists of two parents, of the same age, and a number of children, all of different ages. The average age of the children is 15. The average age of the whole family is 21. When each child was born, the parents were at least 25 years old and at most 35 years old. All ages are given in whole years. Find all possible values of the number of children in this family.

3. Two cars 100 m apart are travelling in the same direction along a highway at the speed limit of 60 km/h. At one point on the highway, the speed limit increases to 80 km/h. A little later, it increases

to 100 km/h, and still later it finally increases to 120 km/h. Whenever a car passes a point where the speed limit increases, it instantaneously increases its speed to the new speed limit. When both cars are travelling at 120 km/h, how far apart are they?

4. Let $p(x)$ be a polynomial with integer coefficients such that $p(1) = 5$ and $p(-1) = 11$.
   (a) Give an example of $p(x)$ that has an integral root.
   (b) Prove that if $p(0) = 8$, then $p(x)$ does not have an integral root.

5. On a $2 \times n$ board, you start from the square at the bottom left corner. You are allowed to move from square to adjacent square, with no diagonal moves, and each square may be visited at most once. Moreover, two squares visited on the path may not share a common edge unless you move directly from one of them to the other. We consider two types of paths: those ending on the square at the top right corner and those ending on the square at the bottom right corner. The diagram shows that there are four paths of each type when $n = 4$. Prove that the numbers of these two types of paths are the same when $n = 2{,}014$.



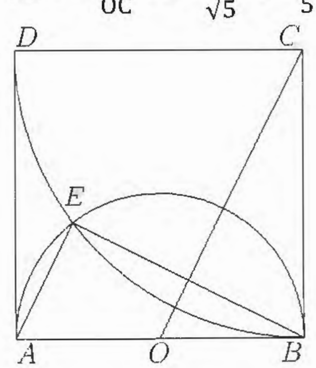
Solutions

1. Let O be the midpoint of AB. Then OB = 1 and

$$OC = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

The line OC is perpendicular to BE. AE is also perpendicular to BE since E lies on the circle with diameter AB. Now $\angle ABE = 90^\circ - \angle COB = \angle OCB$. Hence, triangles ABE and OCB are similar, so that

$$AE = \frac{AB \cdot OB}{OC} = \frac{2 \times 1}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$



2. Let the age of the parents be p, and let the number of children be n. Then the total age of the family is $15n + 2p = 21(n + 2)$, which simplifies to $p = 3n + 21$. Since the children are of different ages and their average age is 15, the age of the eldest one is at least

$$15 + \tfrac{n-1}{2} = \tfrac{n+29}{2}.$$

It follows that

$$3n + 21 = p \geq \frac{n + 29}{2} + 25 = \frac{n + 79}{2}.$$

This simplifies to $5n \geq 37$, which implies that $n \geq 8$. On the other hand, the age of the youngest child is at most

$$15 - \tfrac{n-1}{2} = \tfrac{31-n}{2}.$$

It follows that

$$3n + 21 = p \leq \frac{31 - n}{2} + 35 = \frac{101 - n}{2}.$$

This simplifies to $7n \leq 59$, which implies that $n \leq 8$. It follows that the only possible value is $n = 8$. This may be realized if the children are of ages 11, 12, 13, 14, 16, 17, 18 and 19, and both parents are of age 45.

3. Let the first car be at a point B while the second car is at a point A, both in the 60 km/h zone. Then AB = 100 m. Let the first car be at a point D while the second car is at a point C, both in the 120 km/h zone. Now, the amount of time the second car takes to go from A to C is the same as the amount of time the first car takes to go from B to D. Both cars take the same amount of time going from B to C. Hence, the amount of time the second car takes to go from A to B at 60 km/h is the same as the amount of time the first car takes to go from C to D at 120 km/h. It follows that CD = 2AB = 200 m.

4a. We are given two pieces of information. So we seek a polynomial with two undetermined coefficients. The first attempt is $p(x) = ax + b$. Then $5 = p(1) = a + b$ and $11 = p(-1) = -a + b$. Hence, $a = -3$ and $b = 8$, but the only root of $-3x + 8 = 0$ is $x = 8/3$, which is not integral. However, it is easy to modify our polynomial to $p(x) = 8x^2 - 3x$. We have $p(1) = 5$ and $p(-1) = 11$, but this time we have an integral root $x = 0$ in addition to $x = 3/8$.

b. Suppose $p(x)$ has an integral root $x = r$. Then $r - 1$ divides $p(r) - p(1) = -5$, so that r is one of -4, 0, 2 or 6. Also, $r + 1 = r - (-1)$ divides $p(r) - p(-1) = -11$, so that r is one of -12, -2, 0 or 10. The only common value between the two lists is $r = 0$, but $p(0) = 8$. This is a contradiction.

5. The path of the marker is uniquely determined by its vertical moves. The only condition is that

no two vertical moves can be made in adjacent columns. Whether the path ends in the upper or lower right corner is determined by the parity of the number of vertical moves. Let the columns be represented by elements in the set $\{1, 2, \ldots, n\}$. Consider all subsets that do not contain two consecutive numbers. Let a_n be the number of such subsets of even size, and b_n be the number of such subsets of odd size. Then $a_0 = a_1 = a_2 = 1$ because of the empty subset, $b_0 = 0$, $b_1 = 1$ and $b_2 = 2$.

For $n \geq 3$, classify the subsets of $\{1, 2, \ldots, n\}$ into two types: those containing $n - 1$ and those not containing $n - 1$. A subset of the first type cannot contain either $n - 2$ or n. Hence, the number of such subsets of even size is b_{n-3} and the number of such subsets of odd size is a_{n-3}. The subsets of the second type may be divided into pairs such that in each pair the two subsets are identical except that one contains n and the other does not. Hence, the number of such subsets of even size is equal to the number of such subsets of odd size. It follows that $a_n - b_n = b_{n-3} - a_{n-3}$. Hence,

$$a_{3k} - b_{3k} = (-1)^k(a_0 - b_0) = (-1)^k,$$
$$a_{3k+1} - b_{3k+1} = (-1)^k(a_1 - b_1) = 0 \text{ and}$$
$$a_{3k+2} - b_{3k+2} = (-1)^k(a_2 - b_2) = (-1)^{k+1}.$$

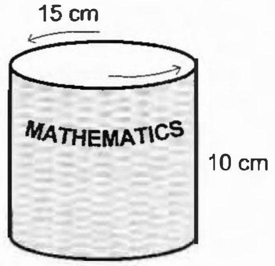
In particular, $a_{2,014} = b_{2,014}$.

Calgary Junior High School Mathematics Contest 2014

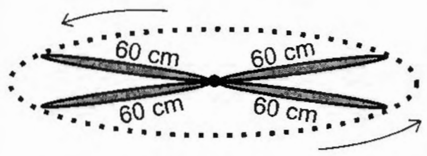
The Calgary Junior High School Mathematics Contest takes place every spring. The 90-minute exam is primarily for Grade 9 students; however, all junior high students in Calgary and surrounding districts are eligible. Participants write the exam in their own schools. School and individual prizes include trophies, medals, a cash award to the student achieving the highest mark, and the opportunity for the top students (and their teacher sponsors) to attend a banquet at the University of Calgary. The 38th annual contest took place on April 30, 2014.

Part A: Short Answer

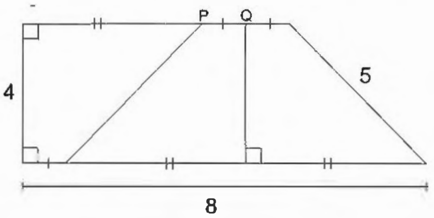
1. From the set {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} all prime numbers are removed. How many numbers are remaining?
2. Alex, Betty and Chi have a total of 87 candies. If Chi gives four candies to Betty and three candies to Alex, they each then have the same number of candies. How many candies does Chi start with?
3. Roll three dice so that each die shows one number from 1 to 6, and multiply these three numbers together. What is the smallest positive even number which cannot be obtained?
4. A glass in the shape of a cylinder is 10 cm high and 15 cm around, as shown. The glass has a logo on it occupying 2% of the curved side of the glass. What is the area (in square centimetres) of the logo?



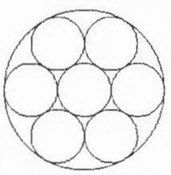
5. A book has 200 pages. How many times does the digit 5 appear in the page numbers?
6. A home has some fish, some birds and some cats. All together there are 15 heads and 14 legs. If the home has more than one of each animal, how many fish are there?
7. A ceiling fan has blades 60 cm long, and rotates at a rate of two revolutions per second. The speed of the end of a blade can be written in the form $N\pi$ cm/s, where N is a positive integer. What is N?



8. In the diagram below, similarly marked segments are equal in length. Find the length of the segment PQ.



9. Ruby cuts seven equal-sized round cookies from a big round piece of cookie dough, as in the diagram. What fraction of the original cookie dough is left?



Answers

1. 6
2. 36
3. 14
4. 3
5. 40
6. 10

Solution

Let there be F fish, B birds and C cats. From the legs, we get $2B + 4C = 14$, so $B + 2C = 7$, and from the heads $F + B + C = 15$. Now there are more than one of each type, so there are at least two birds. But then there can be at most two cats, so $C = 2$ and $B = 3$. From this, we get $F = 15 - 2 - 3 = 10$.

7. 240

Solution

The tip of a blade moves through the circumference of a circle with radius 60 cm twice in a second, so it travels $2 \times 2\pi \times 60$ cm in a second.

8. 2/3

Solution

Note that the three small rectangular trapezoids are congruent, so the length of both nonvertical lines is 5. Drop a line from P perpendicular to the base to meet the other side in R. This makes a 3–4–5 right triangle. So the length of the base of the trapezoids is 3 + PQ. This gives PQ + 2 × (3 + PQ) = 8. From this, 6 + 3PQ = 8 and PQ = 2/3.

9. 2/9

Solution

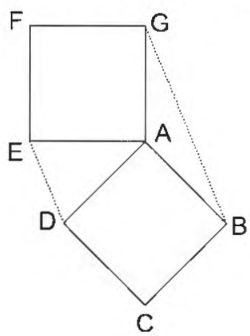
If R is the radius of the large circle and r that of the small circle, then $R = 3r$. The area of the big circle is $\pi R^2 = \pi(3r)^2$. Each little circle has area πr^2, so the amount of dough left is $\pi(3r)^2 - 7\pi r^2 = 2\pi r^2$. The fraction is then $2\pi r^2/(9\pi r^2) = 2/9$.

Part B: Long Answer

1. A truck is delivering heavy goods from city A to city B. When travelling from A to B, the truck has an average speed of 45 km/h. On the return trip, the empty truck has an average speed of 90 km/h. The total time spent travelling from A to B and returning from B to A is four hours. Find the distance in kilometres from A to B.

2. There are 2,014 digits in a row. Any two consecutive digits form a number that is divisible by 17 or 23.
   (a) If the last digit is 1, then what are the possibilities for the first digit?
   (b) If the first digit is 9, then what are the possibilities for the last digit?

3. Two squares, ABCD and AEFG, each with side length 25, are drawn so that the two squares only overlap at vertex A. Suppose DE has length 14. What is the length of BG?

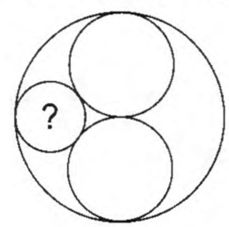


4. We will call a positive integer a "2-timer" if its digits can be arranged to make a number of shape $2 \times 2 \times 2 \times \cdots \times 2$. For example, 2,014 is a 2-timer because its digits can be arranged to make 1,024, which is

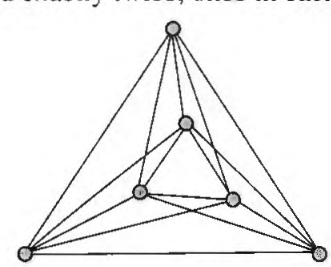
$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2.$$

Positive integers cannot start with the digit 0.
   (a) What is the smallest 2-timer larger than 2,014? Be sure to justify your answer.
   (b) What is the largest 2-timer less than 2,014? Be sure to justify your answer.
   (c) Show that 12,345,678,987,654,321 is not a 2-timer.

5. Three circles of radii 12 cm, 6 cm and 6 cm each touch the other two. What is the radius in centimetres of the smaller circle that touches all three?



6. Six friends live in six houses, with a separate path connecting each pair of houses, as shown. One day, each person leaves his or her house and visits each of the other five houses, one after the other, using paths and stopping at the last house visited. Nobody can change paths except at one of the houses. Show how all this can be done so that every path is travelled exactly twice, once in each direction.



Solutions

1. The answer is **120 km.**

Solution 1

Let D be the distance from A to B. The time for the truck to go from A to B is $D/45$, and the time to return from B to A is $D/90$, so the total time is

$$\begin{aligned}\text{total time} &= (\text{time from A to B}) \\ &\quad + (\text{time from B to A}) \\ &= D/45 + D/90 \\ &= 3D/90 \\ &= D/30.\end{aligned}$$

Since the total time spent travelling from A to B and returning from B to A is four hours, we have $D/30 = 4$. Hence, $D = 120$. Therefore, the distance from A to B is 120 km.

Solution 2

Since the truck goes twice as fast from B to A as from A to B, it takes only half as long to make the trip from B to A as it does to go from A to B. Thus, it should spend 1/3 of the time going from B to A, or 4/3 hours. It is travelling at 90 km/h at this time, so it will travel 90(4/3) = 120 km. So 120 km must be the distance from A to B.

2a. The first digit is **6.**

The two-digit multiples of 17 are 17, 34, 51, 68 and 85. Those of 23 are 23, 46, 69 and 92. Thus, any two consecutive digits xy must have one of the forms from {17, 23, 34, 46, 51, 68, 69, 85, 92}.

We now can work backward from right to left, working two digits at a time. Since the last digit is 1, we start with $x1$. This gives 51, since no other two-digit multiple of 17 or 23 ends in a 1. Now, with $x5$ we get 851. Repeating this procedure, we get

6,851
46,851
346,851
2,346,851
92,346,851
692,346,851
⋮

It is now clear that the pattern 92,346 of length 5 will repeat. Now, 2,014 = 402 × 5 + 4, so the number must be 402 blocks of 69,234 followed by 6,851, so the first digit is 6.

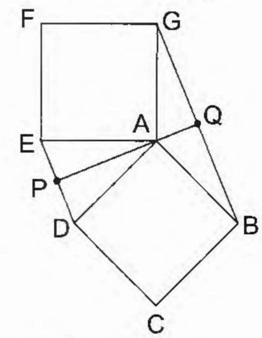
b. The last digit can be **4 or 7.**

With the same method as before but starting with $9y$, we get 92, then 923, then 9,234 and then 92,346. At this point, we have both a multiple of 17 (namely, 68) and a multiple of 23 (namely, 69) to consider.

CASE 1. Let us first pursue the path with 68. We then continue to 923,468, then 9,234,685, and 92,346,851, and next 923,468,517, where we are stuck since we cannot have $7y$ appearing.

CASE 2. What happens with the possibility from 69? Here we observe, as in the first example, that a cycle 92,346 of length 5 is formed. Now we see that two possible numbers could be formed. One is 402 blocks of 92,346 ended by 8,517, and the other is 402 blocks of 92,346 ended by 9,234. So the two possibilities for the last digit are 4 and 7.

3. The answer is **48.**



Solution 1

Let P be the midpoint of ED. Then, since triangle EAD is isosceles, PA is the bisector of ∠EAD. Extend PA to meet GB at Q. Since AEFG is a square, ∠QAG = 90° – ∠EAP. Similarly, ∠QAB = 90° – ∠DAP, and as PA is a bisector we get AQ as the perpendicular bisector of GB in isosceles triangle AGB. This shows that in fact ED and GB are parallel, and that triangle APE is similar to triangle GQA. Since the hypotenuse in each is of length 25, they are in fact congruent, so

$$GQ = AP = \sqrt{25^2 - 7^2} = 24.$$

Now, GB = 2GQ = 2 × 24 = 48.

Solution 2

This solution uses the law of cosines, which is not likely to be familiar to many of the students, except for those who have done work beyond junior high. Notice that ∠DAE and ∠GAB are supplementary, since their sum plus the two right angles in the squares gives 360°. This means that cos(∠GAB) = –cos(∠EAD). Using the law of cosines in the two triangles, we get

$$14^2 = 25^2 + 25^2 - 2 \times 25 \times 25 \times \cos(\angle EAD)$$

and

$$GB^2 = 25^2 + 25^2 + 2 \times 25 \times 25 \times \cos(\angle EAD).$$

From the first, we get

$$\cos(\angle EAD) = (2 \times 25^2 - 14^2)/(2 \times 25^2).$$

Using this in the second, we get

$$\begin{aligned} GB^2 &= 2 \times 25^2 + [(2 \times 25^2) \times (2 \times 25^2 - 14^2)/(2 \times 25^2)] \\ &= 4 \times 25^2 - 4 \times 7^2 \\ &= 4 \times (625 - 49) \\ &= (2 \times 24)^2. \end{aligned}$$

It follows that GB = 48.

4a. The answer is **2,041.**

The numbers of shape 2 × 2 × · · · × 2 (which we call powers of 2) that have exactly four digits are 1,024, 2,048, 4,096 and 8,192.

Arranging the digits of each of these in all possible ways, the smallest number larger than 2,014 that we find is 2,041, found by rearranging 1,024. Since any other power of 2 would have either fewer than four digits or more than four digits, the same would be true for any arrangement of the digits of such a number. So all such arrangements would either be smaller than 2,014 or larger than 2,041. Therefore, the answer to the problem is 2,041.

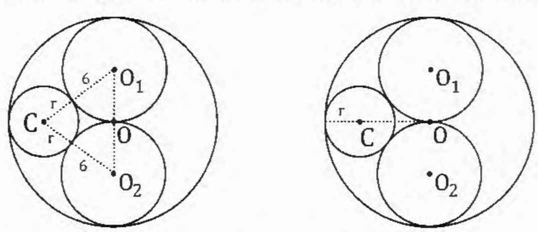
b. The answer is **1,982.**

Notice that 1,982 is a 2-timer less than 2,014. This will be the answer if we can check that there are none between 1,982 and 2,014, and this is easily done, since the number must start with a 1 or a 2, making it a permutation of either 1,024, 2,048 or 8,192. None coming from 2,048 can work, and it is easy to see that you cannot rearrange 1,024 or 8,192 starting with a 2 and get something smaller than 2,014 and larger than 1,982.

c. Since any rearrangement of this number uses exactly the same digits, to check whether it is a 2-timer we can apply "casting out nines." We repeatedly sum digits, always recording only the remainder on division by 9. Applying this process to 12,345,678,987,654,321 leads to a final result of 0, meaning it is divisible by 9. Since no integral power of 2 can be divisible by 9, the number is not a 2-timer.

5. The answer is **4.**

Let O be the centre of the largest circle, O_1 and O_2 be the centres of the two circles of radius 6, and C be the centre of the smallest circle, with radius r.



As shown in the figure on the left, the lengths of CO_1 and CO_2 are both $6 + r$. Also, the length of O_1O_2 is 12, and O is its midpoint. As shown in the figure on the right, the length $r + CO$ must equal the radius of the large circle. This gives us $r + CO = 12$.

Applying the theorem of Pythagoras gives

$$(CO)^2 + (OO_1)^2 = (CO_1)^2.$$

This gives

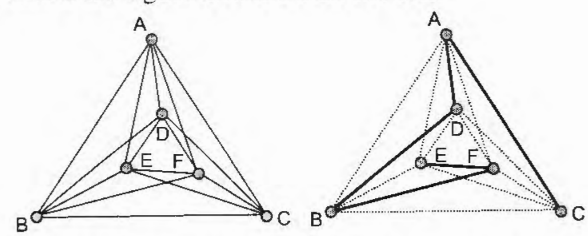
$$(12 - r)^2 + 6^2 = (6 + r)^2.$$

Now we have

$$\begin{aligned} 6^2 &= (6 + r)^2 - (12 - r)^2 \\ &= 18 \times (2r - 6). \end{aligned}$$

Now we get $r - 3 = 6 \times 6/(2 \times 18) = 1$. So the radius is $3 + 1 = 4$.

6. Label the figure as shown on the left.



Consider the route followed by the person in house C as shown in the figure on the right: C–A–D–B–F–E.

Note that this route uses one of the three sides of the outer triangle (CA), one of the three sides of the inner triangle (FE), one of the three short edges connecting the two triangles (AD) and one of the three pairs of longer edges connecting the two triangles (DB and BF).

Thus, we may rotate this configuration to get two more routes: A–B–E–C–D–F and B–C–F–A–E–D, which are the routes followed by the people in houses A and B, respectively. These three routes together use each path exactly once. Reverse the three routes to get the routes followed by the people in houses D, E and F: D–E–A–F–C–B, E–F–B–D–A–C and F–D–C–E–B–A, respectively.

With these six routes, every path is used once in each direction.

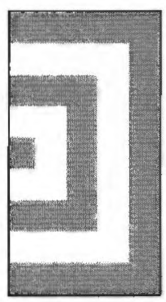
Edmonton Junior High Math Contest 2014

Part A: Multiple Choice

1. Which of these numbers is greater than its reciprocal?
   (a) $-1.\overline{5}$ (b) 0.995 (c) −99.9% (d) $0.\overline{3}$ (e) 2/5

2. What number is doubled when 3/4 of it is subtracted from 99?
   (a) 32 (b) 36 (c) 40 (d) 44 (e) 52

3. A target is made of dark and white strips of equal width, as shown. If a dart is thrown and lands randomly inside the target, what is the probability that it will land on white?



   (a) 2/5 (b) 3/8
   (c) 4/9 (d) 1/2
   (e) 1/3

4. How many two-digit whole numbers less than 40 are divisible by the product of their digits?
   (a) 5 (b) 4 (c) 3 (d) 2 (e) more than 5

5. A florist has 72 roses, 90 tulips and 60 daffodils, and uses all of them to make as many identical bouquets as possible. How many flowers does the florist put in each bouquet?
   (a) 6 (b) 18 (c) 24 (d) 29 (e) 37

Solutions

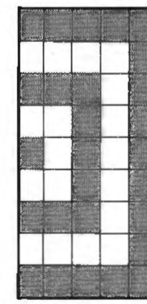
1. (a) $-\frac{14}{9} < -\frac{9}{14}$
   (b) $\frac{199}{200} < \frac{200}{199}$
   (c) $-\frac{999}{1,000} > -\frac{1,000}{999}$
   (d) $\frac{1}{3} < 3$
   (e) $\frac{2}{5} < \frac{5}{2}$

The answer is **(c)**.

2. Let n be the number.

$$2n = 99 - 3/4(n)$$
$$11n/4 = 99$$
$$n = 36$$

The answer is **(b)**.



3. The shape can be divided into 45 individual squares. The white squares are 18/45 = 2/5 of the entire target. The answer is **(a)**.

4. There are exactly five of them: 11, 12, 15, 24 and 36. The answer is **(a)**.

5. Find the greatest common factor: GCF (72, 90, 60) = 6. This gives six bouquets with 12 roses, 15 tulips and 10 daffodils—a total of 37 flowers in each bouquet. The answer is **(e)**.

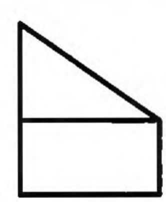
Part B: Short Answer

6. A rectangle has an area of 48 cm^2 and a perimeter of 28 cm. What is the length of the rectangle's diagonal, rounded to the nearest whole centimetre?

7. When a two-digit number is multiplied by the sum of its digits, the product is 952. What is the two-digit number?

8. Twenty-six people are seated in a circle and are lettered alphabetically from A to Z. Beginning with person A, and proceeding in a clockwise direction, each alternate person leaves the circle. What is the letter of the last person to leave?

9. In the rectangle BCDE, BC = 30 cm. A is on the extension of EB, and AC = 34 cm. The area of triangle ABC is 30 cm^2 less than half the area of BCDE. What is the perimeter of the quadrilateral ACDE?



10. The age of a tortoise is 52 years more than the combined age of two elephants. In 10 years, the tortoise will be twice as old as the two elephants combined. How old is the tortoise now?

11. The angle bisectors of the two acute angles of obtuse triangle XYZ intersect at point W. The measure of ∠Z is 98°. What is the measure, in degrees, of ∠XWY?

12. Maria purchased a number of peaches and apples. The mean mass of the peaches is 170 g. The mean mass of the apples is 140 g. The mean mass of all the fruit is 152 g. What is the ratio of the number of peaches to apples purchased?

13. Two sides of a scalene acute triangle measure 12 cm and 13 cm. If the length of the third side is also an integer, then how many different lengths are possible for the third side?

14. What is the largest n such that n^n is an n-digit number?

Solutions

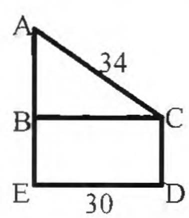
6. Let l be length and w be width. Knowing that $l(w) = 48$ and $l + w = 14$, we have $l = 8$ and $w = 6$. The diagonal is

$$\sqrt{8^2 + 6^2} = 10 \text{ cm}.$$

7. The prime factorization of 952 is $2 \times 2 \times 2 \times 7 \times 17$. Two-digit divisors are 14, 17, 28, 34, 56 and 68. Checking all cases, we have $952 = 68(6 + 8)$.

8. After the first round, BDFHJLNPRTVXZ are left, and the next to go is B. After the second round, DHLPTX are left, and the next to go is H. After the third round, DLT are left, and the next to go is L. After the fourth round, DT are left and the next to go is D. After the fifth round, only T is left.

9. By Pythagoras's theorem, AB = 16 cm and the area of triangle ABC is 240 cm^2. Hence, the area of BCDE is 2(240 + 30) or 540 cm^2, so that CD = 18 cm. The perimeter of the trapezoid ACDE is 34 + 18 + 30 + 18 + 16 = 116 cm.



10. Suppose the tortoise is x years old and the two elephants together are y years old. Then $x - y = 52$. In 10 years' time, $x + 10 = 2(y + 20)$. Hence, $y + 52 = 2y + 30$ so that $y = 22$ and $x = 74$.

11. In degrees, $\angle X + \angle Y = 82°$, $\angle WXY + \angle WYX = 41°$, and $\angle XWY = 139°$.

12. Let p = number of peaches and a = number of apples. This gives

$$\frac{170p+140a}{p+a} = 152$$

or $18p = 12a$. Thus, the ratio of $p{:}a = 2{:}3$.

13. Let ABC be the triangle, where AC = 13 cm and BC = 12 cm. When AB = 5 cm, we have a right-angled triangle at $\angle$ABC. When $\angle$ACB = 90°, AB = 17.69. We now have 5 < third side < 17.69, so that it is an acute triangle. This gives 12 possible lengths for the third side, from 6 cm to 17 cm. For the triangle to be scalene, we need to eliminate both 12 cm and 13 cm from the list; hence, there are a total of 10 possible lengths for the third side.

14. A quick check reveals that 10^{10} gives a total of 11 digits. In fact, when n is greater than 10, the resulting power will always have more than n digits. Thus, the greatest $n = 9$, giving $9^9 = 387{,}420{,}489$ (nine digits).

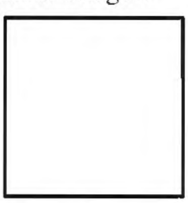
Part C: Short Answer

15. Consider a 2,014-digit number consisting of 2,013 9s followed by one 1.

$$\overbrace{99\ldots99}^{2013}1$$

The smallest factor is 1, and the largest factor is the number itself. Let M be the second smallest factor and N be the second largest factor. What is the sum of the digits of M and N?

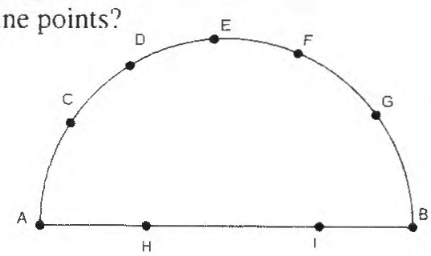
16. ABCD is a square with AC = 49.5 cm. P is a point inside ABCD such that PB = PC, and the area of triangle PCB is a third of the area of ABCD. What is the length, in centimetres, of PA? Round your answer to the nearest integer.



17. A three-digit number is equal to 17 times the product of its digits, and the hundreds digit is one more than the sum of the other two digits. Find all such three-digit numbers.

18. A magazine receives 32 articles, of length 1, 2, . . . , 32 pages, respectively. The first article starts on page 1, and all other articles start on the page after the preceding article. The articles may be arranged in any order. What is the maximum number of articles that can start on an odd-numbered page?

19. The diagram shows nine points. How many triangles are there whose vertices are chosen from the nine points?



Solutions

15. First, we know that the number is not divisible by 3, as it divides into all the 9s but not the last digit of 1.

As for 7, it will divide into six 9s evenly. The longest string of 9s would be 2,010 digits. This leaves 9,991, which 7 does not divide into evenly.

Next, 11 divides into pairs of 99 but won't divide evenly into 91.

The number 13 will go into six 9s evenly. Similar to 7, it does not divide evenly into 9,991.

The number 17 divides evenly into a string of sixteen 9s. Leaving thirteen 9s followed by a 1, which 17 does not divide into evenly.

The number 19 divides evenly into a string of eighteen 9s. Leaving fifteen 9s followed by a 1, which 19 also does not divide evenly.

Similarly, 23 divides evenly into a string of twenty-two 9s. Leaving eleven 9s followed by a 1, which 23 does not divide evenly.

The next prime number to try is 29. Like the previous quotients, 29 divides evenly into a number made up of a string of twenty-eight 9s. The quotient is 0,344,827,586,206,896,551,724,137,931. The sum of its digits is 126. This also means that there are 71 sets of this number, giving a total of 1,988 digits of 9s. We still have a number with twenty-five 9s followed by a 1—a total of 26 digits left. Fortunately, 29 multiplied by the quotient less the last two digits (31) results in exactly twenty-five 9s followed by a 1. This concludes that 29 divides evenly into the original number.

To recap, the second smallest factor is 29. The sum of the digits is 2 + 9 = 11.

The second largest factor is a number of the form

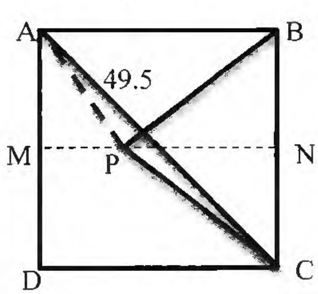
$$\overbrace{0344827586206896551724137931}^{71\text{ sets}}\,03448275862068965517241379$$

The sum of the digits is 126 × 72 – 4 = 9,068.

Therefore, the total sum of the digits *M* and *N* is 11 + 9,068 = 9,079.

For related problems, see the appendix.

16. Let l be the length of one side of the square. Using Pythagorean property, we have $2l^2 = 49.5^2$. This gives l^2 = 1,225.125 and l = 35 cm. As well, knowing that three times the area of triangle PCB is equal to l^2, we have



$$3\left(\frac{\overline{PN}\times 35}{2}\right) = 35^2,$$

or

$$\overline{PN} = 23\frac{1}{3}$$

and

$$\overline{PM} = 11\frac{2}{3}.$$

It follows that

$$\overline{PA} = \sqrt{17.5^2 + 11\frac{2}{3}^2} = 21 \text{ cm}.$$

17. Let *a*, *b* and *c* be the three digits not necessarily different. As well, we should consider only product that is less than 999 ÷ 17 = 59. Since we have the hundreds digit one more than the sum of the other two digits, we could use the following table to sort out the three digits.

Original Number	*a*	*b*	*c*	Product *abc*
	9	1	7	63
	9	2	6	108
	9	3	5	135
	9	4	4	144
17(48) = 816	8	1	6	48
	8	2	5	80
	8	3	4	96
17(35) = 595	7	1	5	35
17(56) = 952	7	2	4	56
	7	3	3	63
17(24) = 408	6	1	4	24
17(36) = 612	6	2	3	36
17(15) = 255	5	1	3	15
17(20) = 340	5	2	2	20
17(8) = 136	4	1	2	8
	3	1	1	3

Only one such number exists, and it is 816.

Alternative Solution

The number is divisible by any of its digits. Using its hundreds digit, the quotient is greater than 100 and less than 111. It is also a multiple of 17, so it has to be 102. Now, 102 = 17 × 6. So the last two digits are 1 and 6, or 2 and 3. It is easy to check that 861, 632 and 623 are not multiples of 17 but 816 is.

18. Put all 16 articles of even length first, so that they all start on odd-numbered pages. Of the other 16, half of them will start on odd-numbered pages, for a total of 24. This cannot be higher, because an article of odd length changes the parity of the starting page number. This parity must change at least 15 times, so that at least eight articles must start on even-numbered pages.

19. There are six ways to choose two points from the straight line, and each pair can form a triangle with each point on the curve. This gives $6 \times 5 = 30$ triangles.

    There are 10 ways to choose two points from the curve, and each pair can form a triangle with each point on the straight line. This gives $10 \times 4 = 40$ triangles.

    Last, all three vertices can be chosen from the curve alone. There are 10 ways to do so.

    In total, there are $30 + 40 + 10 = 80$ triangles.

Appendix

Below are several problems related to problem 15 (Part C: Short Answer).

1. Prove that the 2,014-digit number n is a composite number.
2. Prove that n is not a square.

   Problem 2 is needed to set up the next problem. A positive integer, which is not a square, has an even number of positive divisors because they form pairs whose product is n. If n is a square, which means that $\sqrt{n}$ is a positive integer, then it is paired with itself. It counts as only one divisor, making the total number of divisors odd. By problem 2, n has $2k$ positive divisors for some positive integer k (namely, $1 = d_1 < d_2 < \cdots < d_k < d_{k+1} < \cdots < d_{2k-1} < d_{2k} = n$.
3. Find the combined digit sum of d_k and d_{k+1}.
4. Find the combined digit sum of d_2 and d_{2k-1}.

Solutions

1. Note that $n = 10^{2,014} - 9 = (10^{1,007})^2 - 3^2 = (10^{1,007} + 3)(10^{1,007} - 3)$. Since each factor is clearly greater than 1, n is a composite number.
2. Note that $n = 999 \ldots 991 = 999 \ldots 9 \times 100 + 91 \equiv 0 + 3 = 3 \pmod 4$ since $100 \equiv 0 \pmod 4$. Since all squares are congruent to 0 or 1 (mod 4), n is not a square.
3. Note that $d_k = 10^{1,007} - 3 = 999 \ldots 997$ so that its digit sum is $1,006 \times 9 + 7 = 9,061$. On the other hand, $d_{k+1} = 1,000 \ldots 003$ so that its digit sum is $1 + 3 = 4$. Hence, the combined digit sum is $9,061 + 4 = 9,065$.
4. Clearly, $d_2 \neq 2$ or 5. By the tests of divisibility, it is neither 3 nor 11. If $d_2 = 7$, then we must have $10^{2,014} \equiv 9 \equiv 2 \pmod 7$. Now, $10 \equiv 3 \pmod 7$, $10^2 \equiv 3 \times 3 = 9 \pmod 7$, $10^3 \equiv 3 \times 9 = 27 \equiv 6 \pmod 7$, $10^4 \equiv 3 \times 6 = 18 \equiv 4 \pmod 7$, $10^5 \equiv 3 \times 4 = 12 \equiv 5 \pmod 7$ and $10^6 \equiv 3 \times 5 = 15 \equiv 1 \pmod 7$. It is not necessary to go on any further. This is because $2,014 = 335 \times 6 + 4$, so that $10^{2,014} = (10^6)^{335} \times 10^4 \equiv 1^{335} \times 4 = 4 \neq 2$. It follows that $d_2 \neq 7$.

   A key step in the above argument is that $10^k \equiv 1 \pmod 7$ for some positive integer k, which happens to be 6. How do we know that such a k always exists, if we replace 7 with another prime number? Let us understand why $k = 6$ for the prime number 7. Suppose we wish to convert the fraction 1/7 into a decimal. By long division, we find that

   $$\frac{1}{7} = 0.\overline{142857},$$

   a decimal expansion consisting of repeating blocks of the six digits 142857. The reason there are six digits is that when we divide by 7, the only possible remainders are 0, 1, 2, 3, 4, 5 and 6. Here, 0 will not appear since no power of 10 is divisible by 7. By the time we have seen each of the non-zero remainders once, repetition must start. Thus, the repeating block of decimal digits has a length of at most 6. In this case, it happens to be exactly 6. This means that

   $$\frac{1}{7} = \frac{142,857}{999,999}$$

   so that 999,999 is divisible by 7. It follows that $10^6 \equiv 1$.

   In a similar manner, we can prove that $d_2 \neq 13, 17, 19$ or 23. We know that $10^{12} \equiv 1 \pmod{13}$, $10^{16} \equiv 1 \pmod{17}$, $10^{18} \equiv 1 \pmod{19}$ and $10^{22} \equiv 1 \pmod{23}$. As it turns out, $10^6 \equiv 1 \pmod{13}$, but the other powers (namely, 16, 18 and 22) cannot be reduced. Since $10^4 \not\equiv 9 \pmod{13}$, $d_2 \neq 13$.

   Now $2,014 = 125 \times 16 + 14$ but $10^{14} \not\equiv 9 \pmod{17}$, $2,014 = 111 \times 18 + 16$ but $10^{16} \not\equiv 9 \pmod{19}$, and $2,014 = 91 \times 22 + 12$ but $10^{12} \not\equiv 9 \pmod{23}$. Hence, $d_2 \neq 17, 19$ or 23.

   The next candidate for d_2 is 29. We know that $10^{28} \equiv 1$, but perhaps one of 10^2, 10^4, 10^7 and 10^{14} may be too. In modulo 29, we have $10^2 = 100 \equiv 13$, $10^3 \equiv 10 \times 13 = 130 \equiv 14$, $10^4 \equiv 10 \times 14 = 140 \equiv 24$, $10^7 \equiv 14 \times 24 = 336 \equiv 17$ and $10^{14} \equiv 17^2 = 289 \equiv 28$. So this does not happen. Since $2,014 = 28 \times 71 + 26$, what we need is $10^{26} \equiv 9$.

Now $10^5 \equiv 24 \times 10 = 240 \equiv 8$, $10^{25} \equiv 8^5 = 32{,}768 \equiv 27$ and $10^{26} \equiv 10 \times 27 = 270 \equiv 9$. This is exactly what we want. We are lucky that $n = 10^{2{,}014} - 9 = (10^{1{,}007} + 3)(10^{1{,}007} - 3)$ has a prime factor as small as 29. Each of $10^{1{,}007} + 3$ and $10^{1{,}007} - 3$ has more than 1,000 digits. Even if they were not prime numbers, they could have been products of prime numbers with over 500 digits. It would be very diffcult to find d_2 then.

From $d_2 = 29$, we have $d_{2k-1} = n/29$. There remains only the trivial matter of determining their combined digit sums, via the following long division:

$$\frac{9999999\ 999999\ 9999999\ 9999999}{29}$$

$$= 344{,}827{,}586{,}206{,}896{,}551{,}724{,}137{,}931.$$

The sum of the digits of the quotient is 126, and there are 71 such blocks. In the last incomplete block, the quotient is without the last two digits 3 and 1. It follows that the digit sum of d_{2k-1} is $126 \times 71 + 122 = 9{,}068$. Since the digit sum of d_2 is 11, the combined digit sum is 9,079.

Further Problems

The solutions for the following two problems are left to readers.

5. Determine which of $10^{1{,}007} + 3$ and $10^{1{,}007} - 3$ is divisible by 29.
6. Determine for what year $y > 2{,}014$ the second smallest positive divisor of $10^y - 9$ would be
   (a) 7,
   (b) 13,
   (c) 17,
   (d) 19 and
   (e) 23.

Mathematical Models for Teaching: Reasoning Without Memorization, by Ann Kajander and Tom Boland

Canadian Scholars' Press, 2014

Reviewed by Gladys Sterenberg

In Alberta's current political climate, we are challenged to address the needs of our stakeholders (including parents) when teaching math. The apparent conflict between teaching for understanding and teaching basic facts through rote memorization has been the focus of much debate. The authors of *Mathematical Models for Teaching: Reasoning Without Memorization* are explicit about the importance of reasoning in learning math, and they offer compelling strategies for facilitating reasoning in our classrooms.

The book is written for the classroom teacher. Ann Kajander and Tom Boland outline math concepts that are required for teaching math, and they are motivated by the need to provide comfortable experiences for teachers in order to mitigate math anxiety. Through a focus on mathematical models and manipulatives, the authors explicitly promote reasoning and sense-making over memorization. This is not what is known as discovery math but, rather, an emphasis on constructivist, inquiry and problem-based teaching strategies. What makes this book of particular interest is the Canadian context and the framing of teacher professional knowledge that relies on input from over 700 teachers who participated in the authors' research study.

The book's 15 chapters are organized by math strands that match Alberta's mathematics program of studies. It is not grade specific, but it certainly explicates math concepts relevant to K–9 classrooms. The first two chapters provide a research-based overview of the book. The subsequent chapters each focus on a specific math concept. Within each chapter is an exposition of concepts (including key terms and fundamental ideas), ideas for student explorations, tasks, examples, ideas for follow-up and discussion, problems for teachers, and suggestions for further reading. The concepts were chosen according to what the authors consider most important for teaching mathematics.

What makes this book different from other professional resources is its focus on mathematics for teaching. This is consistent with current and emerging research on teachers' pedagogical content knowledge, which is linked to effective teaching. The authors' choice to promote models and reasoning reframes a focus on manipulatives and communication in a way that promotes deep pedagogical understanding of why models are necessary, how to use models with children, and how to engage learners in communication to prompt reasoning. The authors provide explicit direction for teachers who might not be familiar or comfortable with teaching mathematics. While the authors note that their focus is not on memorization, they do provide a strong rationale for the way skills and proficiencies result from a focus on models and reasoning. Thus, the development of basic skills is an outcome of teaching through models and reasoning, not the reverse.

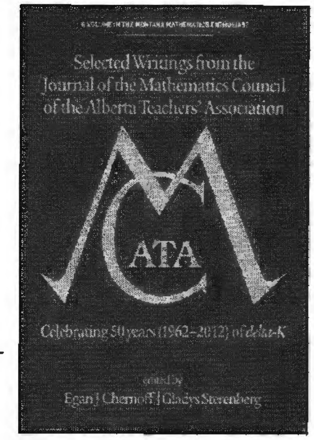
I recommend this book as a resource for teacher leaders who are working with inservice and preservice teachers, to support the work in the classroom. Kajander and Boland are deliberate about providing support for student learning of mathematics. However, within this context, teachers will also learn the mathematics they are encountering in Alberta's program of studies. I believe that this book provides a way to support teachers' own understanding of mathematics as they encounter strategies to enhance their teaching of mathematics.

New Book Information

Selected writings from the Journal of the Mathematics Council of the Alberta Teachers' Association: Celebrating 50 years (1962-2012) of *delta-K*

Edited by **Egan J Chernoff,** *University of Saskatchewan*
and **Gladys Sterenberg,** *Mount Royal University*

A volume in the series ***The Montana Mathematics Enthusiast: Monograph Series in Mathematics Education***. *Series Editor: Bharath Sriraman, The University of Montana*



The teaching and learning of mathematics in Alberta - one of three Canadian provinces sharing a border with Montana - has a long and storied history. An integral part of the past 50 years (1962-2012) of this history has been *delta-K: Journal of the Mathematics Council of the Alberta Teachers' Association.* This volume, which presents ten memorable articles from each of the past five decades, that is, 50 articles from the past 50 years of the journal, provides an opportunity to share this rich history with a wide range of individuals interested in the teaching and learning of mathematics and mathematics education. Each decade begins with an introduction, providing a historical context, and concludes with a commentary from a prominent member of the Alberta mathematics education community. As a result, this monograph provides a historical account as well as a contemporary view of many of the trends and issues in the teaching and learning of mathematics. This volume is meant to serve as a resource for a variety of individuals, including teachers of mathematics, mathematics teacher educators, mathematics education researchers, historians, and undergraduate and graduate students. Most importantly, this volume is a celebratory retrospective on the work of the Mathematics Council of the Alberta Teachers' Association.

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