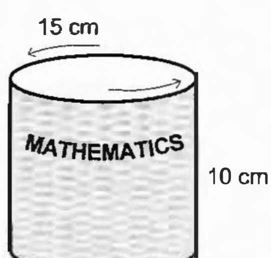
Calgary Junior High School Mathematics Contest 2014

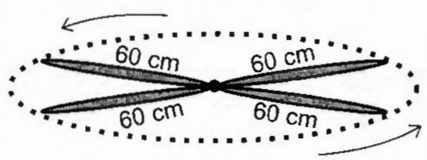
The Calgary Junior High School Mathematics Contest takes place every spring. The 90-minute exam is primarily for Grade 9 students; however, all junior high students in Calgary and surrounding districts are eligible. Participants write the exam in their own schools. School and individual prizes include trophies, medals, a cash award to the student achieving the highest mark, and the opportunity for the top students (and their teacher sponsors) to attend a banquet at the University of Calgary. The 38th annual contest took place on April 30, 2014.

Part A: Short Answer

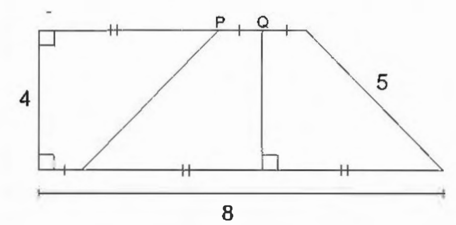
1. From the set {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} all prime numbers are removed. How many numbers are remaining?
2. Alex, Betty and Chi have a total of 87 candies. If Chi gives four candies to Betty and three candies to Alex, they each then have the same number of candies. How many candies does Chi start with?
3. Roll three dice so that each die shows one number from 1 to 6, and multiply these three numbers together. What is the smallest positive even number which cannot be obtained?
4. A glass in the shape of a cylinder is 10 cm high and 15 cm around, as shown. The glass has a logo on it occupying 2% of the curved side of the glass. What is the area (in square centimetres) of the logo?



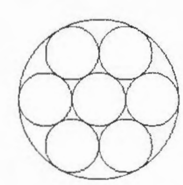
5. A book has 200 pages. How many times does the digit 5 appear in the page numbers?
6. A home has some fish, some birds and some cats. All together there are 15 heads and 14 legs. If the home has more than one of each animal, how many fish are there?
7. A ceiling fan has blades 60 cm long, and rotates at a rate of two revolutions per second. The speed of the end of a blade can be written in the form $N\pi$ cm/s, where N is a positive integer. What is N?



8. In the diagram below, similarly marked segments are equal in length. Find the length of the segment PQ.



9. Ruby cuts seven equal-sized round cookies from a big round piece of cookie dough, as in the diagram. What fraction of the original cookie dough is left?



Answers

1. 6
2. 36
3. 14
4. 3
5. 40
6. 10

Solution

Let there be F fish, B birds and C cats. From the legs, we get $2B + 4C = 14$, so $B + 2C = 7$, and from the heads $F + B + C = 15$. Now there are more than one of each type, so there are at least two birds. But then there can be at most two cats, so $C = 2$ and $B = 3$. From this, we get $F = 15 - 2 - 3 = 10$.

7. 240

Solution

The tip of a blade moves through the circumference of a circle with radius 60 cm twice in a second, so it travels $2 \times 2\pi \times 60$ cm in a second.

8. 2/3

Solution

Note that the three small rectangular trapezoids are congruent, so the length of both nonvertical lines is 5. Drop a line from P perpendicular to the base to meet the other side in R. This makes a 3–4–5 right triangle. So the length of the base of the trapezoids is 3 + PQ. This gives PQ + 2 × (3 + PQ) = 8. From this, 6 + 3PQ = 8 and PQ = 2/3.

9. 2/9

Solution

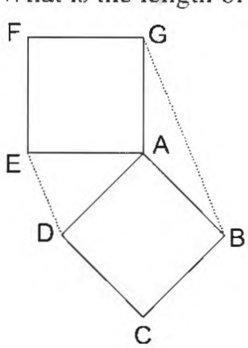
If R is the radius of the large circle and r that of the small circle, then $R = 3r$. The area of the big circle is $\pi R^2 = \pi(3r)^2$. Each little circle has area πr^2, so the amount of dough left is $\pi(3r)^2 - 7\pi r^2 = 2\pi r^2$. The fraction is then $2\pi r^2/(9\pi r^2) = 2/9$.

Part B: Long Answer

1. A truck is delivering heavy goods from city A to city B. When travelling from A to B, the truck has an average speed of 45 km/h. On the return trip, the empty truck has an average speed of 90 km/h. The total time spent travelling from A to B and returning from B to A is four hours. Find the distance in kilometres from A to B.

2. There are 2,014 digits in a row. Any two consecutive digits form a number that is divisible by 17 or 23.
   (a) If the last digit is 1, then what are the possibilities for the first digit?
   (b) If the first digit is 9, then what are the possibilities for the last digit?

3. Two squares, ABCD and AEFG, each with side length 25, are drawn so that the two squares only overlap at vertex A. Suppose DE has length 14. What is the length of BG?

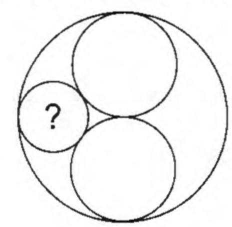


4. We will call a positive integer a "2-timer" if its digits can be arranged to make a number of shape $2 \times 2 \times 2 \times \cdots \times 2$. For example, 2,014 is a 2-timer because its digits can be arranged to make 1,024, which is

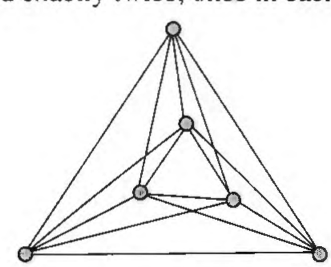
$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2.$$

Positive integers cannot start with the digit 0.
   (a) What is the smallest 2-timer larger than 2,014? Be sure to justify your answer.
   (b) What is the largest 2-timer less than 2,014? Be sure to justify your answer.
   (c) Show that 12,345,678,987,654,321 is not a 2-timer.

5. Three circles of radii 12 cm, 6 cm and 6 cm each touch the other two. What is the radius in centimetres of the smaller circle that touches all three?



6. Six friends live in six houses, with a separate path connecting each pair of houses, as shown. One day, each person leaves his or her house and visits each of the other five houses, one after the other, using paths and stopping at the last house visited. Nobody can change paths except at one of the houses. Show how all this can be done so that every path is travelled exactly twice, once in each direction.



Solutions

1. The answer is **120 km.**

Solution 1

Let D be the distance from A to B. The time for the truck to go from A to B is $D/45$, and the time to return from B to A is $D/90$, so the total time is

$$\begin{aligned}\text{total time} &= (\text{time from A to B}) \\ &\quad + (\text{time from B to A}) \\ &= D/45 + D/90 \\ &= 3D/90 \\ &= D/30.\end{aligned}$$

Since the total time spent travelling from A to B and returning from B to A is four hours, we have $D/30 = 4$. Hence, $D = 120$. Therefore, the distance from A to B is 120 km.

Solution 2

Since the truck goes twice as fast from B to A as from A to B, it takes only half as long to make the trip from B to A as it does to go from A to B. Thus, it should spend 1/3 of the time going from B to A, or 4/3 hours. It is travelling at 90 km/h at this time, so it will travel $90(4/3) = 120$ km. So 120 km must be the distance from A to B.

2a. The first digit is **6.**

The two-digit multiples of 17 are 17, 34, 51, 68 and 85. Those of 23 are 23, 46, 69 and 92. Thus, any two consecutive digits xy must have one of the forms from {17, 23, 34, 46, 51, 68, 69, 85, 92}.

We now can work backward from right to left, working two digits at a time. Since the last digit is 1, we start with $x1$. This gives 51, since no other two-digit multiple of 17 or 23 ends in a 1. Now, with $x5$ we get 851. Repeating this procedure, we get

6,851
46,851
346,851
2,346,851
92,346,851
692,346,851
⋮

It is now clear that the pattern 92,346 of length 5 will repeat. Now, $2{,}014 = 402 \times 5 + 4$, so the number must be 402 blocks of 69,234 followed by 6,851, so the first digit is 6.

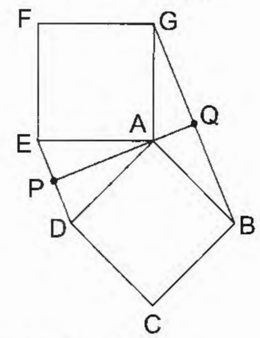
b. The last digit can be **4 or 7.**

With the same method as before but starting with $9y$, we get 92, then 923, then 9,234 and then 92,346. At this point, we have both a multiple of 17 (namely, 68) and a multiple of 23 (namely, 69) to consider.

CASE 1. Let us first pursue the path with 68. We then continue to 923,468, then 9,234,685, and 92,346,851, and next 923,468,517, where we are stuck since we cannot have $7y$ appearing.

CASE 2. What happens with the possibility from 69? Here we observe, as in the first example, that a cycle 92,346 of length 5 is formed. Now we see that two possible numbers could be formed. One is 402 blocks of 92,346 ended by 8,517, and the other is 402 blocks of 92,346 ended by 9,234. So the two possibilities for the last digit are 4 and 7.

3. The answer is **48.**



Solution 1

Let P be the midpoint of ED. Then, since triangle EAD is isosceles, PA is the bisector of $\angle$EAD. Extend PA to meet GB at Q. Since AEFG is a square, $\angle QAG = 90° - \angle EAP$. Similarly, $\angle QAB = 90° - \angle DAP$, and as PA is a bisector we get AQ as the perpendicular bisector of GB in isosceles triangle AGB. This shows that in fact ED and GB are parallel, and that triangle APE is similar to triangle GQA. Since the hypotenuse in each is of length 25, they are in fact congruent, so

$$GQ = AP = \sqrt{25^2 - 7^2} = 24.$$

Now, $GB = 2GQ = 2 \times 24 = 48$.

Solution 2

This solution uses the law of cosines, which is not likely to be familiar to many of the students, except for those who have done work beyond junior high. Notice that $\angle$DAE and $\angle$GAB are supplementary, since their sum plus the two right angles in the squares gives 360°. This means that $\cos(\angle GAB) = -\cos(\angle EAD)$. Using the law of cosines in the two triangles, we get

$$14^2 = 25^2 + 25^2 - 2 \times 25 \times 25 \times \cos(\angle EAD)$$

and

$$GB^2 = 25^2 + 25^2 + 2 \times 25 \times 25 \times \cos(\angle EAD).$$

From the first, we get

$$\cos(\angle EAD) = (2 \times 25^2 - 14^2)/(2 \times 25^2).$$

Using this in the second, we get

$$\begin{aligned} GB^2 &= 2 \times 25^2 + [(2 \times 25^2) \times (2 \times 25^2 - 14^2)/(2 \times 25^2)] \\ &= 4 \times 25^2 - 4 \times 7^2 \\ &= 4 \times (625 - 49) \\ &= (2 \times 24)^2. \end{aligned}$$

It follows that $GB = 48$.

4a. The answer is **2,041.**

The numbers of shape $2 \times 2 \times \cdots \times 2$ (which we call powers of 2) that have exactly four digits are 1,024, 2,048, 4,096 and 8,192.

Arranging the digits of each of these in all possible ways, the smallest number larger than 2,014 that we find is 2,041, found by rearranging 1,024. Since any other power of 2 would have either fewer than four digits or more than four digits, the same would be true for any arrangement of the digits of such a number. So all such arrangements would either be smaller than 2,014 or larger than 2,041. Therefore, the answer to the problem is 2,041.

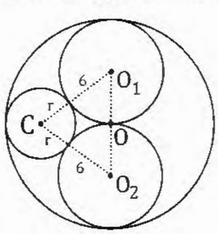
b. The answer is **1,982.**

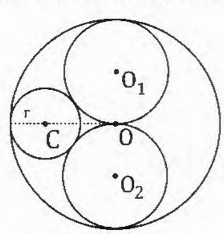
Notice that 1,982 is a 2-timer less than 2,014. This will be the answer if we can check that there are none between 1,982 and 2,014, and this is easily done, since the number must start with a 1 or a 2, making it a permutation of either 1,024, 2,048 or 8,192. None coming from 2,048 can work, and it is easy to see that you cannot rearrange 1,024 or 8,192 starting with a 2 and get something smaller than 2,014 and larger than 1,982.

c. Since any rearrangement of this number uses exactly the same digits, to check whether it is a 2-timer we can apply "casting out nines." We repeatedly sum digits, always recording only the remainder on division by 9. Applying this process to 12,345,678,987,654,321 leads to a final result of 0, meaning it is divisible by 9. Since no integral power of 2 can be divisible by 9, the number is not a 2-timer.

5. The answer is **4.**

Let O be the centre of the largest circle, O_1 and O_2 be the centres of the two circles of radius 6, and C be the centre of the smallest circle, with radius r.





As shown in the figure on the left, the lengths of CO_1 and CO_2 are both $6 + r$. Also, the length of O_1O_2 is 12, and O is its midpoint. As shown in the figure on the right, the length $r + CO$ must equal the radius of the large circle. This gives us $r + CO = 12$.

Applying the theorem of Pythagoras gives

$$(CO)^2 + (OO_1)^2 = (CO_1)^2.$$

This gives

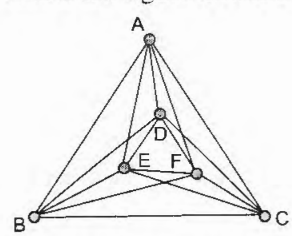
$$(12 - r)^2 + 6^2 = (6 + r)^2.$$

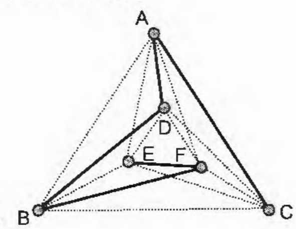
Now we have

$$6^2 = (6 + r)^2 - (12 - r)^2$$
$$= 18 \times (2r - 6).$$

Now we get $r - 3 = 6 \times 6/(2 \times 18) = 1$. So the radius is $3 + 1 = 4$.

6. Label the figure as shown on the left.





Consider the route followed by the person in house C as shown in the figure on the right: C–A–D–B–F–E.

Note that this route uses one of the three sides of the outer triangle (CA), one of the three sides of the inner triangle (FE), one of the three short edges connecting the two triangles (AD) and one of the three pairs of longer edges connecting the two triangles (DB and BF).

Thus, we may rotate this configuration to get two more routes: A–B–E–C–D–F and B–C–F–A–E–D, which are the routes followed by the people in houses A and B, respectively. These three routes together use each path exactly once. Reverse the three routes to get the routes followed by the people in houses D, E and F: D–E–A–F–C–B, E–F–B–D–A–C and F–D–C–E–B–A, respectively.

With these six routes, every path is used once in each direction.