Alberta High School Mathematics Competition 2013/14

The Alberta High School Mathematics Competition is a two-part competition taking place in November and February of each school year. Book prizes are awarded for Part I, and cash prizes and scholarships for Part II. Presented here are the problems and solutions from the 2013/14 competition.

Part I

November 19, 2013

1. Of the first 100 positive integers 1, 2, . . . , 100, the number of those not divisible by 7 is
   (a) 14 (b) 15 (c) 85 (d) 86 (e) none of these

2. The total score of four students on a test is 2,013. Ace scores one point more than Bea, Bea scores three points more than Cec, and Cec scores two points more than Dee. None of their scores are divisible by
   (a) 3 (b) 4 (c) 5 (d) 11 (e) 23

3. Of the following five fractions, the largest one is
   (a) $\frac{1}{75}$ (b) $\frac{2}{149}$ (c) $\frac{3}{224}$ (d) $\frac{4}{299}$ (e) $\frac{6}{449}$

4. Two teams, A and B, played a soccer game on each of seven days. On each day, the first team to score seven goals won. There were no ties. Over the seven days, A won more games than B, but B scored more goals than A overall. The difference in the total number of goals scored by B and A is at most
   (a) 17 (b) 18 (c) 19 (d) 20 (e) none of these

5. ABCD is a quadrilateral with AB = 12 and CD = 18. Moreover, AB is parallel to CD, and both ∠ADC and ∠BCD are less than 90°. P and Q are points on side CD such that AD = AP and BC = BQ. The length of PQ is
   (a) 6 (b) 7 (c) 8 (d) 9 (e) 10

6. Each of four cows is either normal or mutant. A normal cow has four legs and always lies. A mutant cow has either three or five legs and always tells the truth. When asked how many legs they have among them, their respective responses are 13, 14, 15 and 16. The total number of legs among these four cows is
   (a) 13 (b) 14 (c) 15 (d) 16 (e) none of these

7. Let a and b be positive integers such that $ab < 100$ and $a/b > 2$. Denote the minimum possible value of a/b by m. Then we have
   (a) $m \le 2.15$ (b) $2.15 < m < 2.2$ (c) $m = 2.2$ (d) $2.2 < m < 2.25$ (e) $m \ge 2.25$

8. Let ABCD be a quadrilateral with ∠DAB = ∠CBA = 90°. Suppose there is a point P on side AB such that ∠ADP = ∠CDP and ∠BCP = ∠DCP. If AD = 8 and BC = 18, the perimeter of the quadrilateral ABCD is
   (a) 70 (b) 72 (c) 74 (d) 76 (e) 78

9. Two bus routes stop at a certain bus stop. The A bus comes at one-hour intervals, and the B bus comes at regular intervals of a different length. When Grandma rests on the bench by the bus stop, one A bus and two B buses come by. Later, Grandpa rests on the same bench, and eight A buses come by. The minimum number of B buses that must have come by during that time is
   (a) fewer than 4 (b) 4 or 5 (c) 6 or 7 (d) 8 or 9 (e) more than 9

10. Suppose that $16^{2,013} = a^b$, where a and b are positive integers. The number of possible values of a is
    (a) 2 (b) 8 (c) 11 (d) 16 (e) 24

11. The following five statements are made about the integers a, b, c, d and e: (1) ab is even and c is odd, (2) bc is even and d is odd, (3) cd is even and e is odd, (4) de is even and a is odd, and (5) ea is even and b is odd. The maximum number of these statements that may be correct is
    (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

12. A very small cinema has only one row of five seats, numbered 1 to 5. Five moviegoers arrive one at a time. Each takes a seat not next to any occupied seat, if this is possible. If not, then any seat will do. The number of different orders in which the seats may be taken is
    (a) 24 (b) 32 (c) 48 (d) 64 (e) 72

13. Let $f(x) = x^2 + x + 1$. Let n be the positive integer such that $f(n) = f(20)f(21)$. Then the number of distinct prime divisors of n is
    (a) 1 (b) 2 (c) 3 (d) 4 (e) more than 4

14. The number of pairs (x, y) of integers satisfying the equation $x^2 + y^2 + xy - x + y = 2$ is

(a) 3 (b) 4 (c) 5 (d) 6 (e) none of these

15. A triangle ABC with AB = 7, BC = 8 and CA = 10 has an interior point P such that ∠APB = ∠BPC = ∠CPA = 120°. Let r_1, r_2 and r_3 be the radii of the circles passing through the vertices of triangles PAB, PBC and PCA, respectively. The value of $r_1^2 + r_2^2 + r_3^2$ is

(a) 71 (b) 72 (c) 73 (d) 74 (e) 75

16. The list 1, 3, 4, 9, 10, 12, 13, . . . contains in increasing order all positive integers that can be expressed as sums of one or more distinct integral powers of 3. The 100th number in this list is

(a) 981 (b) 982 (c) 984 (d) 985 (e) 999

Solutions

1. When 100 is divided by 7, the quotient is 14,with a remainder of 2. Thus, 14 of the first 100 positive integers are divisible by 7. It follows that the number of these integers not divisible by 7 is 100 – 14 = 86. The answer is **(d)**.

2. Since the average score is just over 500, we try 500 as a score for Dee. Then Cec's score is 502, Bea's is 505, and Ace's is 506. The total is indeed 2,013, so no adjustment is necessary. Now, 500 is divisible by 4; 500 and 505 are divisible by 5; and 506 is divisible by 11 and 23. The answer is **(a)**.

3. The lowest common numerator is 12. The fractions then become

$$\frac{12}{900}, \frac{12}{894}, \frac{12}{896}, \frac{12}{897} \text{ and } \frac{12}{898}$$

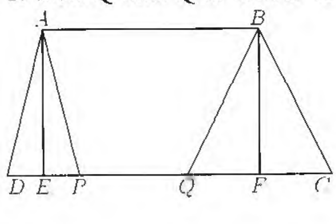
respectively. The answer is **(b)**.

4. B won at most three games, and for each of those games, B won by at most seven goals, with a 7–0 score. In the other four games, B lost by at least one goal, with a 7–6 score. The difference in the number of goals is at most (7 × 3) – (1 × 4) = 17. The answer is **(a)**.

5. From A and B, drop perpendiculars onto CD at the points E and F, respectively. Since both ∠ADC and ∠BCD are less than 90°, E and F do lie on the segment CD. Note that

DE + FC = CD – EF = CD – AB = 6.

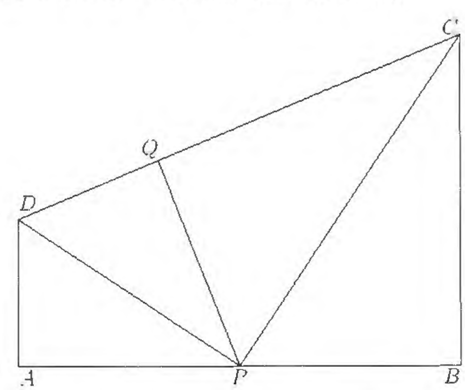
Since ED = EP and FC = FQ, EP + FQ = 6 < 12 = EF, P is closer to E than Q and Q is closer to F than P. Therefore, PQ = EF – (EP + FQ) = 12 – 6 = 6. The answer is **(a)**.



6. Since all four responses are different, at least three of them are wrong. If all four are wrong, then the cows are all normal, and they will have 16 legs among them. However, this makes one of the responses right. Hence, one of the responses is indeed right. The three normal cows have 12 legs among them. Hence, the mutant cow must have three legs in order to make one of the responses right. The answer is **(c)**.

7. We have 100 > ab > $b(2b + 1)$ so that $b \le 6$. Thus, the minimum value of m is 2 + 1/6 = 2.1666 . . . . The answer is **(b)**.

8. Since ∠DAB + ∠CBA = 180°, AD is parallel to BC. Therefore, ∠ADC + ∠BCD = 180°. Hence, ∠PDC + ∠PCD = 90°. Consequently, ∠DPC = 90°. Let Q be the foot of the perpendicular on CD from P. Note that triangle PDA is congruent with triangle PDQ, and triangle PCB is congruent with triangle PCQ. Hence, DQ = DA = 8, CQ = CB = 18 and PA = PQ = PB. Since ∠DPC = 90°, triangle DPQ is similar to triangle PCQ. Hence, DQ/QP = PQ/QC . Therefore, PQ^2 = 8 × 18 = 144 and PQ = 12. The perimeter of ABCD is therefore AP + PB + BC + CQ + QD + DA = 12 + 12 + 18 + 18 + 8 + 8 = 76. The answer is **(d)**.



9. To minimize the number of B buses coming by, we stretch the length of their intervals as far as possible. Suppose Grandma sees the A bus that comes at 10:00. Then she does not see those that come at 9:00 and 11:00. Thus, the length of the interval between two B buses is strictly less than two hours. The two B buses Grandma sees may have come at 9:01 and 10:59, in which case the interval is 118 minutes. Grandpa is on the bench for at least seven hours. This is longer than three intervals for the B buses, so he must have seen at least three of them. Suppose he sees the A buses at 11:00, 12:00, 1:00, 2:00, 3:00, 4:00, 5:00 and 6:00. Then he will see only the B buses at 12:57, 2:55 and 4:53. The answer is **(a)**.

10. Since $16^{2,013} = 2^{4\times 2,013}$, a must be of the form 2^k, where k is a positive integer divisor of $4 \times 2,013 = 2^2 \times 3 \times 11 \times 61$. The prime factorization of k may contain up to two 2s, one 3, one 11 and one 61, so that there are $(2 + 1)(1 + 1)^3 = 24$ possible values of k and, therefore, of a. The answer is **(e).**

11. If a, c and d are odd, while b and e are even, then (1), (2) and (4) are all correct. Suppose at least four statements are correct. By symmetry, we may assume that they are (1), (2), (3) and (4). However, by (1) and (2), c and d are both odd, and yet by (3), cd is even. This is a contradiction. The answer is **(c).**

12. The first two moviegoers to arrive may take the pair (1, 5), (1, 4), (2, 5), (1, 3), (2, 4) or (3, 5) of seats. In each case, there are 2! = 2 ways for them to do so. If they take (1, 5), (1, 3) or (3, 5), then the third moviegoer has only one choice of seat. The remaining two seats may be occupied in 2! = 2 ways. Otherwise, the last three moviegoers may take any vacant seats, and this can be done in 3! = 6 ways. Hence, the total number of orders in which the seats may be taken is $2 \times 3 \times (2 + 6) = 48$. The answer is **(c).**

13. Note that $f(m-1)f(m) = (m^2 - m + 1)(m^2 + m + 1) = m^4 + m^2 + 1 = f(m^2)$. Substituting $m = 21$ yields $f(20)f(21) = f(441)$. Therefore, $n = 441 = 3^2 7^2$. The answer is **(b).**

14. The equation can be written as $x^2 + (y - 1)x + y^2 + y - 2 = 0$. By the quadratic formula, the solutions are

$$x = \frac{-(y-1) \pm \sqrt{(y-1)^2 - 4(y^2+y-2)}}{2}$$
$$= \frac{-(y-1) \pm \sqrt{-3(y-1)(y+3)}}{2}.$$

These are real if and only if $(y + 3)(y - 1) \le 0$. Since y is an integer, it must be one of $-3, -2, -1$, 0 or 1. If $y = -3$, then $x = 2$. If $y = -2$, then $x = 0$ or 3. If $y = -1$, x is not an integer. If $y = 0$, then $x = -1$ or 2. Finally, if $y = 1$, then $x = 0$. The answer is **(d).**

15. Let O be the centre of the circle passing through the vertices of triangle PAB. Note that since $\angle APB > 90°$, O lies on the perpendicular bisector of AB outside of triangle PAB. Since OA = OP = PB,

$\angle AOB = \angle AOP + \angle POB = 180° - 2\angle APO + 180° - 2\angle BPO = 360° - 2\angle APB = 120°$.

Hence, $AB = \sqrt{3}OA$ so that $OA = AB/\sqrt{3}$. It follows that

$$r_1^2 = \frac{AB^2}{3}.$$

Similarly,

$$r_2^2 = \frac{BC^2}{3}$$

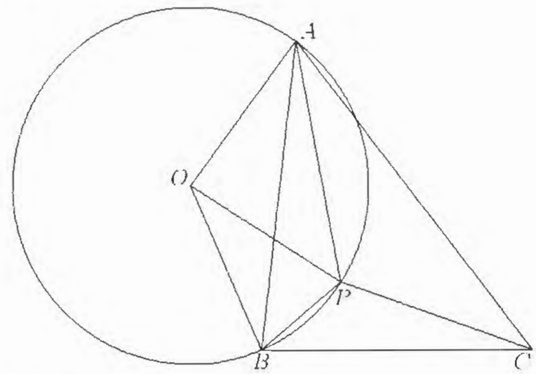
and

$$r_3^2 = \frac{CA^2}{3}.$$

Hence,

$$r_1^2 + r_2^2 + r_3^2 = \frac{7^2 + 8^2 + 10^2}{3} = 71.$$

The answer is **(a).**



16. If we switch the powers of 3 to the powers of 2, then we get all the positive integers. Hence, we convert 100 into base 2, obtaining $100 = 2^6 + 2^5 + 2^2$. It follows that the 100th number on the list is $3^6 + 3^5 + 3^2 = 981$. The answer is **(a).**

Part II

February 5, 2014

1. The side length of square ABCD is 2. The circle with diameter AB intersects the circle with centre C and radius BC again at E. Determine the length of the segment AE.

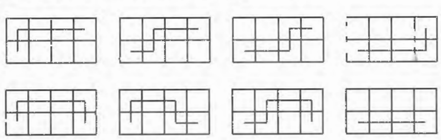
2. A family consists of two parents, of the same age, and a number of children, all of different ages. The average age of the children is 15. The average age of the whole family is 21. When each child was born, the parents were at least 25 years old and at most 35 years old. All ages are given in whole years. Find all possible values of the number of children in this family.

3. Two cars 100 m apart are travelling in the same direction along a highway at the speed limit of 60 km/h. At one point on the highway, the speed limit increases to 80 km/h. A little later, it increases

to 100 km/h, and still later it finally increases to 120 km/h. Whenever a car passes a point where the speed limit increases, it instantaneously increases its speed to the new speed limit. When both cars are travelling at 120 km/h, how far apart are they?

4. Let $p(x)$ be a polynomial with integer coefficients such that $p(1) = 5$ and $p(-1) = 11$.
   (a) Give an example of $p(x)$ that has an integral root.
   (b) Prove that if $p(0) = 8$, then $p(x)$ does not have an integral root.

5. On a $2 \times n$ board, you start from the square at the bottom left corner. You are allowed to move from square to adjacent square, with no diagonal moves, and each square may be visited at most once. Moreover, two squares visited on the path may not share a common edge unless you move directly from one of them to the other. We consider two types of paths: those ending on the square at the top right corner and those ending on the square at the bottom right corner. The diagram shows that there are four paths of each type when $n = 4$. Prove that the numbers of these two types of paths are the same when $n = 2,014$.



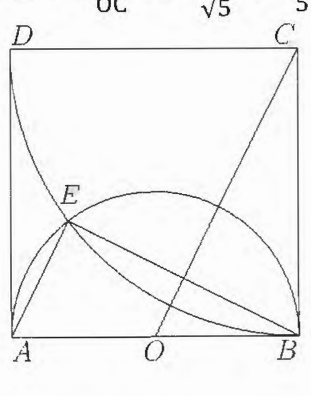
Solutions

1. Let O be the midpoint of AB. Then OB = 1 and

$$OC = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

The line OC is perpendicular to BE. AE is also perpendicular to BE since E lies on the circle with diameter AB. Now $\angle ABE = 90^\circ - \angle COB = \angle OCB$. Hence, triangles ABE and OCB are similar, so that

$$AE = \frac{AB \cdot OB}{OC} = \frac{2 \times 1}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$



2. Let the age of the parents be p, and let the number of children be n. Then the total age of the family is $15n + 2p = 21(n + 2)$, which simplifies to $p = 3n + 21$. Since the children are of different ages and their average age is 15, the age of the eldest one is at least

$$15 + \tfrac{n-1}{2} = \tfrac{n+29}{2}.$$

It follows that

$$3n + 21 = p \geq \frac{n + 29}{2} + 25 = \frac{n + 79}{2}.$$

This simplifies to $5n \geq 37$, which implies that $n \geq 8$. On the other hand, the age of the youngest child is at most

$$15 - \tfrac{n-1}{2} = \tfrac{31-n}{2}.$$

It follows that

$$3n + 21 = p \leq \frac{31 - n}{2} + 35 = \frac{101 - n}{2}.$$

This simplifies to $7n \leq 59$, which implies that $n \leq 8$. It follows that the only possible value is $n = 8$. This may be realized if the children are of ages 11, 12, 13, 14, 16, 17, 18 and 19, and both parents are of age 45.

3. Let the first car be at a point B while the second car is at a point A, both in the 60 km/h zone. Then AB = 100 m. Let the first car be at a point D while the second car is at a point C, both in the 120 km/h zone. Now, the amount of time the second car takes to go from A to C is the same as the amount of time the first car takes to go from B to D. Both cars take the same amount of time going from B to C. Hence, the amount of time the second car takes to go from A to B at 60 km/h is the same as the amount of time the first car takes to go from C to D at 120 km/h. It follows that CD = 2AB = 200 m.

4a. We are given two pieces of information. So we seek a polynomial with two undetermined coefficients. The first attempt is $p(x) = ax + b$. Then $5 = p(1) = a + b$ and $11 = p(-1) = -a + b$. Hence, $a = -3$ and $b = 8$, but the only root of $-3x + 8 = 0$ is $x = 8/3$, which is not integral. However, it is easy to modify our polynomial to $p(x) = 8x^2 - 3x$. We have $p(1) = 5$ and $p(-1) = 11$, but this time we have an integral root $x = 0$ in addition to $x = 3/8$.

b. Suppose $p(x)$ has an integral root $x = r$. Then $r - 1$ divides $p(r) - p(1) = -5$, so that r is one of -4, 0, 2 or 6. Also, $r + 1 = r - (-1)$ divides $p(r) - p(-1) = -11$, so that r is one of -12, -2, 0 or 10. The only common value between the two lists is $r = 0$, but $p(0) = 8$. This is a contradiction.

5. The path of the marker is uniquely determined by its vertical moves. The only condition is that

no two vertical moves can be made in adjacent columns. Whether the path ends in the upper or lower right corner is determined by the parity of the number of vertical moves. Let the columns be represented by elements in the set $\{1, 2, \ldots, n\}$. Consider all subsets that do not contain two consecutive numbers. Let a_n be the number of such subsets of even size, and b_n be the number of such subsets of odd size. Then $a_0 = a_1 = a_2 = 1$ because of the empty subset, $b_0 = 0$, $b_1 = 1$ and $b_2 = 2$.

For $n \geq 3$, classify the subsets of $\{1, 2, \ldots, n\}$ into two types: those containing $n - 1$ and those not containing $n - 1$. A subset of the first type cannot contain either $n - 2$ or n. Hence, the number of such subsets of even size is b_{n-3} and the number of such subsets of odd size is a_{n-3}. The subsets of the second type may be divided into pairs such that in each pair the two subsets are identical except that one contains n and the other does not. Hence, the number of such subsets of even size is equal to the number of such subsets of odd size. It follows that $a_n - b_n = b_{n-3} - a_{n-3}$. Hence,

$$a_{3k} - b_{3k} = (-1)^k(a_0 - b_0) = (-1)^k,$$
$$a_{3k+1} - b_{3k+1} = (-1)^k(a_1 - b_1) = 0 \text{ and}$$
$$a_{3k+2} - b_{3k+2} = (-1)^k(a_2 - b_2) = (-1)^{k+1}.$$

In particular, $a_{2,014} = b_{2,014}$.