delta-K

Journal of the Mathematics Council of the Alberta Teachers' Association

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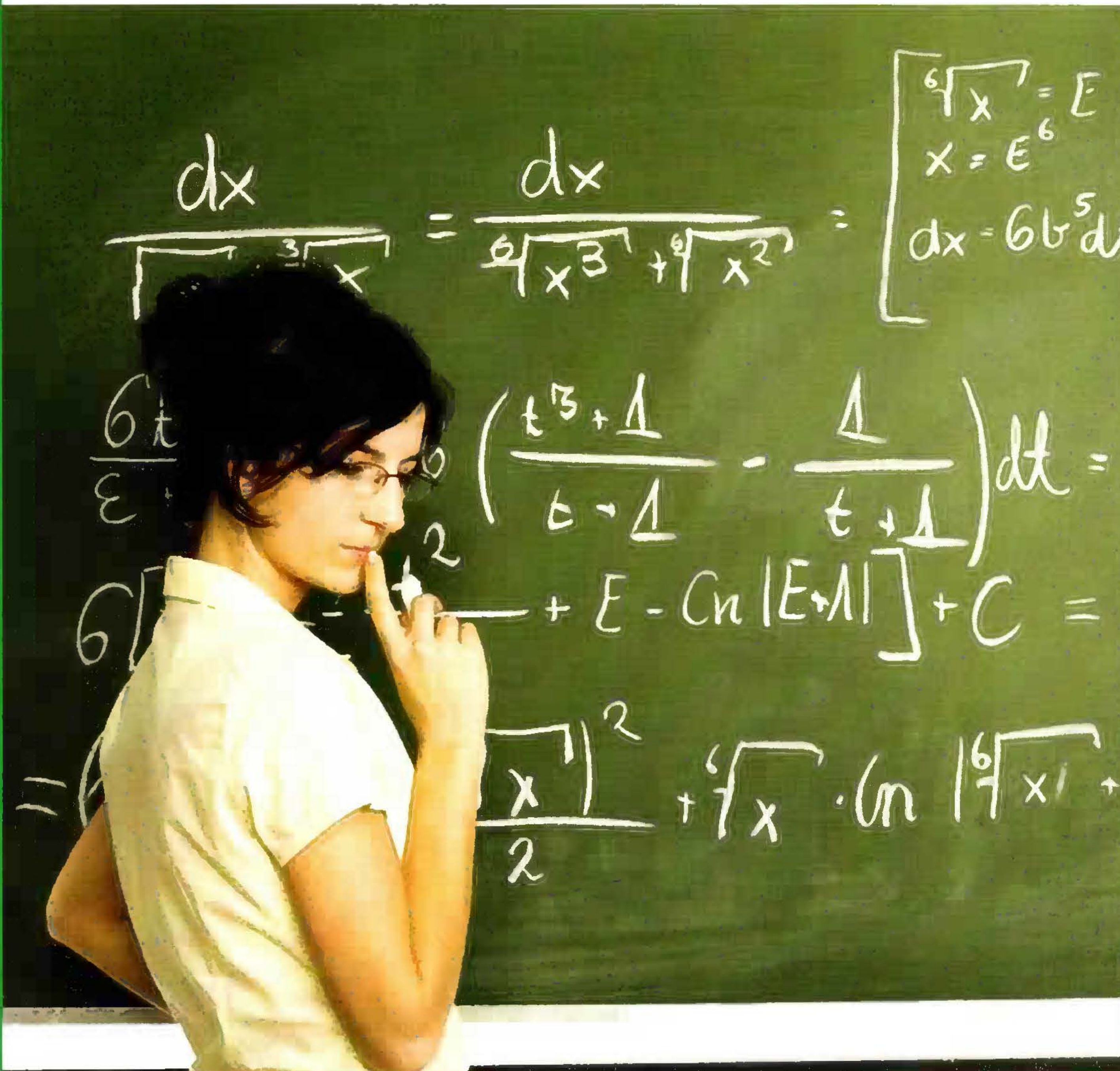

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. All manuscripts should be typewritten and properly referenced. All pages should be numbered.
4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author-date system or the American Psychological Association (APA) style manual.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB, T1S 2L4; e-mail gsterenberg@mtroyal.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 51, Number 2 June 2014

CONTENTS

 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB T1S 2L4; e-mail gsterenberg@mtroyal.ca. EDITORIAL AND PRODUCTION SERVICES: Document Production staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in CBCA Education. ISSN 0319-8367

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From the Editor's Desk

Gladys Sterenberg

Sometimes the theme of an issue develops as I work on it; sometimes it emerges through the articles submitted. Sometimes I cannot make connections until I sketch out my ideas for the editorial. Such is the case with this issue. As I write, I am distracted by my need to meet a variety of upcoming deadlines involving both my work and my holiday preparations with family and friends. I am trying to clear my inbox and am sorting through a variety of messages about the PISA results just released. A number of newspaper articles have been forwarded to me, and I am expected to make some response. I return to my work on *delta-K* and scan the articles I have compiled, and suddenly a theme jumps out at me: making sense of abstract ideas.

Carol Matsumoto, the Canadian director for the National Council of Supervisors of Mathematics (NCSM), sent me the NCSM position paper that is included in this issue. It focuses on the importance of using manipulatives to make sense of abstract ideas. This organization's membership includes supervisors, teachers, principals, department heads, department of education personnel and math coaches. The voices of its members have been significant in shaping math curricula across North America. Jerry Ameis considers the tensions inherent in making sense of numbers by presenting a one-act play. The importance of manipulatives and making sense of formulas is captured by Gerald Krabbe, who sent me a photo montage of one of his students, Michelle, creating a visual demonstration of the formula for finding the area of a circle using pizza.

I have included a reprint of an editorial by Elena Braverman as a feature article. Here she identifies the issue with a one-size-fits-all approach to math education and provides us with the arguments surrounding the importance of learning technical skills in mathematics. This serves as a catalyst for what I hope will be a flood of reader responses.

Making sense of how to assess student thinking is the focus of the lessons presented by Trisha Vadnais and Carole Kamieniecki. Both authors grapple with how to meaningfully engage their students in thinking mathematically.

John Grant McLoughlin invites us to reconsider the role of recreational mathematics in our attempts as teachers to engage our students in making sense of abstract ideas. His work resonates with me as I prepare to teach a mathematics education course to elementary preservice teachers.

In the current context of media responses that describe frustrated parents, an ineffective math curriculum, inadequate preparation of teachers, a lack of professional development opportunities and low PISA test results, I offer a challenge to you as we consider the concerns of members in our communities. It may be time to engage in a deliberate process of educating the public. It may be time to present ourselves as professionals who understand the implications of emerging brain research and research on learning. It may be time to actively participate in the design and implementation of math curricula. It may be time to make a significant difference to views of mathematics education. I invite you to submit responses for publication in upcoming issues of *delta-K*.

Enjoy the summer holidays!

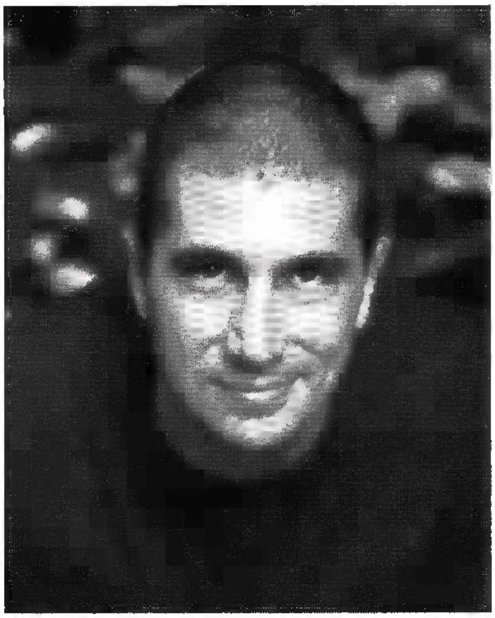
MCATA Conference Committee 2013

Back Row, L–R: Alicia Burdess, Mark Mercer, Rod Lowry, Daryl Chichak, John Scammell, Carmen Wasylynuik
Front Row, L–R: Marj Farris, Tancy Lazar (conference cochair), Debbie Duvall (conference cochair)

Conference Fun

Conference Cochairs

Tancy Lazar and Debbie Duvall



Keynote Speakers

David Moursund and Karim Ani

Awards

NCSM award for a one-year membership presented to Lillian Lundstrom by Carol Matsumoto

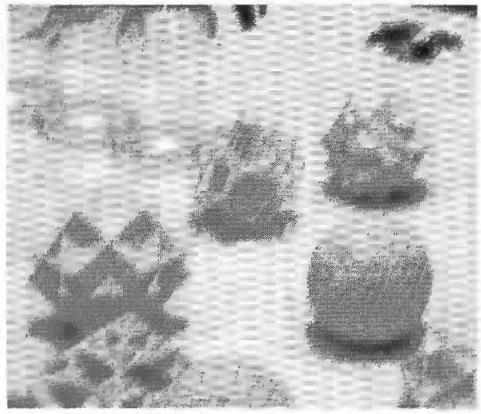
Friends of MCATA award presented to Cheryl O'Brien by Donna Chanasyk. Chris Smith and Maggie Shane also received Friends of MCATA awards (no photo available).

Alberta Mathematics Educator Award presented to Leann Miller by Marj Farris

Symposium Presenter Olive Chapman

Executive, registration, preservice teacher volunteers

Participants



The National Council of Supervisors of Mathematics
Improving Student Achievement Series No 11/Spring 2013

Improving Student Achievement in Mathematics by Using Manipulatives with Classroom Instruction

The NCSM Improving Student Achievement Series is a set of position papers designed to provide research-based practices for school and district mathematics education leaders.

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> Manipulative materials help students make sense of abstract ideas, provide students ways to test and verify ideas, are useful tools for solving problems, and make mathematics learning more engaging and interesting by lifting mathematics off textbook and workbook pages
>
> —Burns, 2007

Our Position

It is the position of the National Council of Supervisors of Mathematics (NCSM) that in order to develop every student's mathematical proficiency, leaders and teachers must systematically integrate the use of concrete and virtual manipulatives into classroom instruction at all grade levels. This position can be accomplished when leaders and teachers

- understand that manipulatives are not toys but are powerful learning tools that build conceptual understanding of mathematics;
- use research to guide instructional use of manipulatives;
- provide sustained professional learning opportunities in the use of manipulatives; and
- recognize that learners—both adults and students—progress through varying levels of proficiency as they use manipulatives before they can realize their full impact.

The *Common Core State Standards* emphasize that concrete models are an essential tool for learning mathematics across all grade levels, K–12. This assertion is articulated most clearly in the Standard for Mathematical Practice 5, "Use Appropriate Tools Strategically," where students choose from concrete models (including manipulatives) and technology. Beyond this, the standards regularly suggest using models in initial steps of learning mathematics before students move to other representations. Therefore, students should have a variety of manipulatives and tools available to them at all times.

Manipulatives used in classroom instruction are physical objects handled by individual students and small groups. Virtual manipulatives are important tools for teacher modelling and demonstration and, additionally, provide students access to manipulatives both inside and outside of the school day via computers. However, virtual manipulatives do not replace the power of physical objects in the hands of learners.

Research That Supports Our Position

John Van de Walle and his colleagues (2013) define a mathematical tool as, "any object, picture, or drawing that represents a concept or onto which the relationship for that concept can be imposed. Manipulatives are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (eg, connecting cubes) or for other purposes (eg, buttons)" (p 24). Virtual manipulatives are "digital objects that resemble physical objects and can be manipulated with a mouse of a computer" (Moyer, Bolyard and Spikell 2002, 372). For example, virtual versions of Cuisenaire Rods and Tangrams are readily available online for instructional purposes. Users should be careful that the virtual versions are accurate matches for the physical tools.

In the opening quote, Marilyn Burns (2007) provides four reasons manipulative materials are fundamental to mathematics instruction. These ideas appear repeatedly in research and thoughtful commentary on the teaching of mathematics. The National Research Council's *Adding It Up* (2001) concludes its review of research on the role of manipulatives with the following statement:

> The evidence indicates, in short, that manipulatives can provide valuable support for student learning when teachers interact over time with the students to help them build links between the object, the symbol, and the mathematical idea both represent. (p 354)

Numerous studies have examined the effectiveness of specific manipulatives to teach specific topics. For example, the Milken Family Foundation analysis of NAEP data suggests that the use of hands-on materials is highly effective. The findings note that "when students are exposed to hands-on learning on a weekly rather than a monthly basis, they prove to be 72 per cent of a grade level ahead in mathematics" (Wenglinsky 2000, 27). Additionally, Sowell (1989) conducted a meta-analysis of studies focused on teaching with manipulatives and found them to have a positive impact on mathematics learning. Cramer, Post and delMas (2002) compared the performance of 1,600 fourth- and fifth-grade students studying fractions using both manipulative-based curricula and nonmanipulative-based curricula. They found that students in the manipulative-based program had higher mean scores at the end of the unit as well as higher retention scores.

Manipulatives are also considered an important element of teacher preparation. For example, the Conference Board of Mathematical Sciences' 2012 report, the *Mathematical Education of Teachers II,* includes numerous references to the use of manipulatives in classroom instruction and the importance of teacher preparation for this use. The authors continue by pointing out that teachers must work to help students see the connections between the manipulatives or other tools and the mathematical concept being taught. A number of studies cited in Van de Walle, Karp and Bay-Williams (2012) suggest that manipulative instruction that follows a pattern of "do as I do" is one of the most widespread misuses of manipulatives. Stein and Bovalino (2001), for example, suggest three key features of successful manipulative lessons that avoid this pitfall. They conclude that 1) teachers have extensive training in the use of manipulatives; 2) teachers prepare by using manipulatives to complete the same instructional activities they would ask of their students; and 3) teachers prepare the classroom for activities by organizing students in groups, preparing materials and thinking through the logistics of the lesson.

Similar findings on the importance of effective instructional strategies when teaching with manipulatives appear in the 2009 Institute for Education Sciences report on response to intervention in mathematics (Gersten et al 2009). The report states that "research shows that the systematic use of visual representations and manipulatives may lead to statistically significant or substantively important positive gains in math achievement" (30–31). The report goes on to discuss the importance of transitioning from concrete objects to visual representations and then to abstract notation. It provides a comprehensive summary of the evidence supporting the use of manipulatives, including evidence supporting the concrete–representational–abstract (CRA) method of instruction. This method, grounded in Bruner's (1966) constructivist discussion of enactive/iconic/symbolic progression in learning, provides a basis for an effective framework for teaching with manipulatives. Under this framework, teachers begin with concrete manipulative experiences, transition students to using visual representations (drawings), and finally transition to using abstract mathematical notation.

Hattie (2012) states "when teachers see learning occurring or not occurring, they intervene in calculated and meaningful ways. In particular, they provide students with multiple opportunities and alternatives for developing learning strategies based on the surface and deep levels of learning some content or domain matter, leading to students building conceptual understanding of this learning, which the students and teachers then use in future learning" (p 15). Hattie later cites research on the power of balance in the classroom: "There is a balance between teachers talking, listening, and doing; there is a similar balance between students talking, listening, and doing" (p 76). Manipulatives provide a foundation around which teachers and students can talk, listen, and do. Other research from Hattie (2009) concludes that, more often than not, when students do not learn, they do not need "more;" rather, they need "different" (p 83). Again, to ensure that every student learns mathematics, a wide range of different strategies are needed for teaching, and both physical and virtual manipulatives are a critical part of this toolkit.

Witzel, Mercer and Miller (2003) describe an example of successful implementation of the CRA approach in teaching algebra to middle-grades students. The Association of Middle Level Education's research summary, "Manipulatives in Middle Grades Mathematics" (Goldsby 2009), provides further information about this and other studies.

How NCSM Members Can Implement Our Position

As leaders, NCSM members must work to ensure that research-based recommendations are implemented in their schools, districts, states and provinces. NCSM members must act to create and sustain the conditions and structures that will enable every mathematics teacher to use manipulatives successfully. Moreover, NCSM members must act to alert teachers, coaches and administrators that it is time to move away from incidental to systematic approaches to manipulative-based instruction. NCSM members must act to move communities away from the "Yes, but … I learned math without manipulatives" or "They're playing with toys instead of learning mathematics" toward the power of multimodal, conceptually based, hands-on instruction.

More specifically, NCSM members must

- ensure that curriculum documents K–12 support the use of manipulatives by their inclusion as an instructional tool on par with textbooks, technological tools, or other resources;
- ensure access to manipulatives for every teacher and every student;
- ensure ongoing professional development around the use of manipulatives;
- ensure that teachers work collaboratively on grade level or subject area teams to provide equity among all student opportunities in using manipulatives;
- ensure that the use of manipulatives is not viewed as optional by teachers, while recognizing that the nature and frequency of use will vary from course to course;
- ensure the support of manipulatives to scaffold learning and in the problem-solving process;
- ensure that teachers use manipulatives within the concrete–representational–abstract learning cycle;
- ensure that parents are educated about the place of manipulatives in the mathematics classroom;
- ensure that manipulative-based activities are used for formative assessment in classrooms;
- ensure that student background knowledge is considered in the variety of student choices; and
- ensure that students have manipulatives available to help provide evidence in visualizing their thinking.

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One of a series of position papers of the National Council of Supervisors of Mathematics
mathedleadership.org

National Council of Supervisors of Mathematics

Mission Statement

The National Council of Supervisors of Mathematics (NCSM) is a mathematics leadership organization for educational leaders that provides professional learning opportunities necessary to support and sustain improved student achievement.

Vision Statement

NCSM envisions a professional and diverse learning community of educational leaders that ensures every student in every classroom has access to effective mathematics teachers, relevant curricula, culturally responsive pedagogy, and current technology.

To achieve our NCSM vision, we will:

N: Network and collaborate with stakeholders in education, business, and government communities to ensure the growth and development of mathematics education leaders

C: Communicate to mathematics leaders current and relevant research; and provide up-to-date information on issues, trends, programs, policies, best practices and technology in mathematics education

S: Support and sustain improved student achievement through the development of leadership skills and relationships among current and future mathematics leaders

M: Motivate mathematics leaders to maintain a life-long commitment to provide equity and access for all learners

July, 2007

Sigmund and Joe: A Moment in Math

Jerry Ameis

A one-act play with three scenes, set in 1950. A male adult, Joe, is having a therapy session about his math woes with Sigmund, a psychologist.

Scene 1

J: Sigmund, I think I have QREUS and TINK syndromes.

S: Oh, dear! You may have what psychologists like to call DDT (double double tinder). Why do you think you have the syndromes?

J: First let me tell you about Grade 2. The teacher wanted us to know our addition facts by heart. He gave us sheets and sheets of 30 addition questions to do. We had to do them in one minute. I could do them correctly on time even though it was boring. It is a good thing I liked math. Many of my friends could not do the questions correctly fast enough. Some even cried. Doing the sheets made them hate math.

I think it is a good thing to know the facts by heart, but I also understood why 11 is the answer to a fact like 5 + 6. Good thing I understood what addition meant and when to use it to solve problems before I learned the addition facts.

S: Why did you know the addition facts already?

J: My buddy and I helped each other learn. We made special cards that we used five minutes a day. Took us three months to learn the facts, but we did it and had fun doing it. Doing the sheets over and over again didn't seem to help most of my friends learn the facts. They had to get help elsewhere.

Scene 2

S: Okay, but what about the QREUS and TINK syndromes? I only see hints so far.

J: The syndromes really started to rear their heads when I was learning to add big numbers. The teacher told us to line them up vertically and add each column in turn, beginning with the right-hand column. That might have been okay, but when I asked the teacher why to do that he told me because it was the most efficient thing to do. When I asked why the method gave the right answer, the teacher said, "Because." All I could do was say, "But, but, but ..." to myself. And you know very well, Sigmund, that repression is not healthy.

S: There must be more to tell.

J: There is. When I was older I thought about what *efficient* meant. I concluded that it meant getting the correct answer in the fastest way. I tested *efficient* on addition questions. Here is an example. For 768 + 999, lining up the numbers and adding the columns from right to left is not the fastest way to get the answer. It is much faster to add 1000 to 768 and then subtract 1.

S: Aren't you cheating by looking at a special kind of question?

J: Sort of, but the example does poke a hole in the argument that the line-up-vertically-and-begin-with-the-right-hand-column method is the most efficient. The way of thinking in my example can be used in bigger questions to get answers quickly. There also are other ways to add quickly. Sigmund, make up a big addition question.

S: Okay, add 278, 3,456, 991 and 1,425.

J: Let's have a competition. In real life, numbers are not lined up vertically for you. They come in the form of a problem where you have to find the numbers, determine what arithmetic operation to use and, finally, do the arithmetic. We'll leave out the find and determine parts of the problem-solving process and start with the four numbers. You get the answer by the vertical method. I'll get the answer another way. Ready, set, GO!

(Sigmund writes the numbers vertically and adds each column, starting at the right-hand column. Joe writes the four numbers in a row, four-digit numbers first, then the three-digit. He begins adding by looking at the four-digit numbers. Time passes.)

J: Done. The answer is 6,150.

S: How did you do that so fast? I am just beginning to add the last column. You didn't write much down, either.

J: I began by adding 3,400 and 1,400 mentally, getting 4,800 in my head. Then I added 1,000, getting 5,800. I wrote down –9 to help me remember that, by adding 1,000, I added 9 too much because 991 is 9 less than 1,000. I added 200 in my head, getting 6,000. I added 25 and 56 in my head and added the result to 6,000, getting 6,081. I added 80 to that, getting 6,161 and, to help me remember, wrote down –2 because 78 is 2 less than 80. Finally, I subtracted 11 (9 + 2) from 6,161, getting 6,150 for the answer to the addition.

S: I am convinced. The vertical addition method is not really the most efficient. Your method could even be faster when solving real problems. If the numbers were already there, you wouldn't have to write them down. You could just look at them and begin adding. For the vertical method, you usually would have to begin by writing down the numbers when working with real problems. Only with artificial school arithmetic questions are the numbers already written vertically.

Scene 3

S: Ah, I see it now. You have QREUS and TINK syndromes because you are curious and want to think. When you learned math, you had to repress those longings. This brought on inner turmoil, one that needs to be resolved if you want to live a healthy fulfilled life.

J: Yes, help me. I don't want to be a robot following orders. I want to wonder and feel joy at unravelling wonders. Is there a conspiracy at work? Doesn't the word *robot* come from the eastern European word *robota*, which means *labour*? And you know the political system that governs many of those eastern European countries.

S: Careful, careful. Don't get carried away. Stalin is not responsible for all of the world's ills. Let's just deal with your syndromes. We'll use the context of addition for that. It seems to be a struggle between training people to be robotic calculators and helping people to be thinkers. Are you aware of that? Which do you prefer?

J: Yes, I see that now. I don't want to be a robot. When I was young I felt that I was being held down, with my desire to explore and be creative being crushed.

S: Good. You are part way to resolving your inner turmoil. The nature of a human being is to explore and be creative. You were trying to exert your humanity, but it was being squashed by outside forces.

Let's think more deeply about the vertical adding method. If you know the addition facts by heart, you should have success using the method. But it does not encourage mental arithmetic (other than using the facts). It does not encourage using mathematical principles such as that adding can be done in any order. It does not encourage understanding, because you were told what to do but not why it works. Furthermore, you did not participate in thinking about ways to add. Engaging in the creative process is important to future learning and to working and living. Feel good that you saw flaws with the vertical method. You showed strength, not weakness, in doing so.

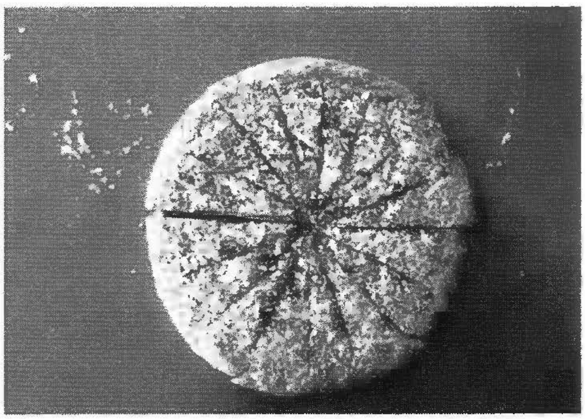
J: Yes, yes. I feel my turmoil slipping away. I also see more light through the swirl. There shouldn't be an emphasis on knowing how to do math without an accompanying emphasis on understanding why it works. Both should be part of a whole. A gestalt approach is the most sensible way to look at things.

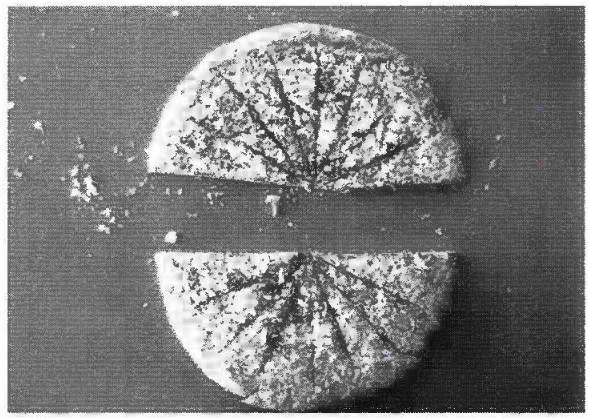
S: Marvellous! That will be $300, please. And, by the way, come and see me for a session on the goes-into division method. I can do it, but I have no clue why it works or why anyone should learn it. Seems like magic to me.

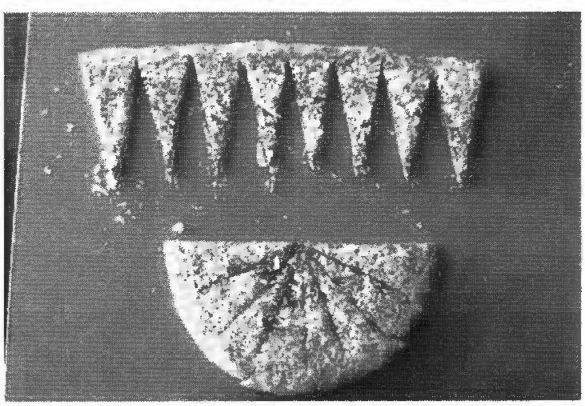
Jerry Ameis, PhD, is an associate professor at the Faculty of Education, University of Winnipeg. His PhD is in mathematics education and honours mathematics, and his work is primarily in the Access Program for K–8 teacher candidates, where he teaches a mathematics course specifically designed for them, as well as years 4 and 5 mathematics methods courses. His research interests centre on how K–8 students learn mathematics and what kind of mathematics they can learn. His research framework relies on complexity theory and cognitive science, with a strong hint of constructivism.

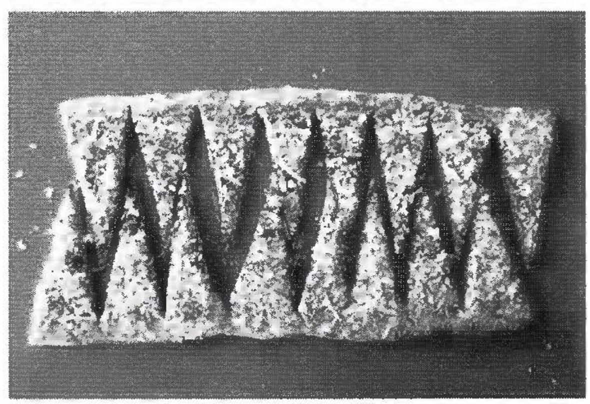
Shape Shifting

Michelle Peace









Feature Article

The Broken Link

Elena Braverman

Imagine a society where toddlers are discouraged from walking. It is strongly advised to keep them in strollers outdoors and in walkers inside the house; children younger than six are rarely seen walking in the street, and even then only when assisted by a parent. Parents who do not follow their kids' every step, even in their backyard, risk attracting the neighbours' attention. There are several reasons proclaimed for this: the risk that toddlers will fall and hurt themselves while they are too young to be aware of the dangers around them; even supervised.

young children can easily be involved in car accidents in the streets. There are handicapped kids who cannot walk, run and climb as successfully as the others, which creates a feeling of inequality at an age which can be handled only with great difficulty by psychologists. It is commonly agreed that at the age of six it is allowed to start learning how to walk. Unfortunately, the precious moment is lost, the beginner walkers are too heavy for their undeveloped muscles, and most children cannot really catch up, and can only walk slowly and clumsily for the rest of their lives. There is no public concern about this issue as it is not an obstacle to driving successfully ten years later. The system has many side effects: there are countless tutoring agencies, advertising a "smooth and easy way to walking," as well as private tutors, with many desperate parents of both young and older children relying on them. For rich people, there are elite clubs where walking skills are developed earlier by licensed instructors who do the training at the appropriate age. In addition, at the most advanced level, there are free outreach activities for active runners, especially for promising athletes. However, most students are neither interested nor qualified to participate. The whole idea does not stir society since most parents grew up in a similar environment, walk with difficulty and would definitely feel uncomfortable assisting their children once they move quickly; the same situation exists with teachers at school.

The situation described above seems absurd, but it is quite appropriate when applied to mathematics, which is sometimes seen as a scarecrow of our school education and even as an obstacle for socially underprivileged students to obtain a college or university education. This is especially surprising, since the development of mathematical skills requires neither expensive tools, when compared, for example, to many sports and experimental sciences, nor an "educated" environment to be cultivated, unlike language and humanities skills. The main reason why mathematics is blamed for causing depression in high school and first-year undergraduate students is probably that mathematics is not a mirror that is easily distorted. It does not allow anybody to believe what he or she desires, but tells the truth. We do not credit students for wrong solutions and answers, though there was once an article by a parent in the *Calgary Herald* (and in other sources, I believe) advocating that a partial score should be granted for totally incorrect mathematical solutions. I do not believe that first-year students are more successful in writing essays or analyzing natural science phenomena than in solving mathematical problems, but in the first case evaluation criteria may be not so strict. My colleagues in geology, for example, do not assume that students have any background, while in mathematics, we are very much dependent on the skills our students bring from high school. However, we should be careful when comparing disciplines. When I complained to my colleague who teaches history that our main problem is the poor background of students in math and that this poor background is probably not as much of an issue in the history department, she answered, "How would you teach history to students who have never heard of the first and second world wars?" (Oh, happy students! It would be interesting to know where they come from.)

One of the recommended books for junior high school is the series Math Makes Sense. There was a justified critique of this series by Dr Malgorzata Dubiel (see, for example, http://blogs.vancouversun.com/2012/04/13/math-makes-sense-not-with-these-textbooks-expert-says and https://sfu.ca/pamr/issues-experts/2012/assessing-math-teaching.html). I have just reviewed the textbooks for Grades 8 and 9 in this series; I recommend this activity to every colleague involved in teaching first-year calculus, as it is quite revealing. Your students have difficulties with adding fractions? There is no reason to be surprised: unit 3 of the Grade 8 book, titled "Operations with Fractions," deals with multiplication and division but not addition of fractions. Bringing fractions to the common denominator is not outlined; the phrase *least common multiple* is not even in the index. So, next time, in the topic of "integration with partial fractions," when students choose the product of all the denominators of algebraic fractions as a common denominator, just note that they do the best they can. By the way, concerning algebra, the only element introduced by the end of junior high school is addition of polynomials, and multiplication and division by a number and by a monomial (in all the exercises, a polynomial with integer coefficients is a result). My first impression is that the book was inspired by lazy schoolchildren who wanted material that could cause any difficulty removed from the curriculum. It does not matter that you will need these things in the future; if the addition of fractions, squaring a binomial and solving a quadratic equation can be postponed to senior high school, they probably can be avoided.

However, the lack of technical skills acquired by students in the first nine years of their school life, in elementary and junior high schools, is only a symptom. The main problem of the approach accepted in the books is that what is presented is not really mathematics (which is all about why), but some collection of recipes. In the books, I could not find any explanation for why the sum of measures of angles in a triangle is $180°$ (or for that matter, any other fact in geometry). The best justification that could be found in the book is to draw some triangle, measure, compute and check. Reviewing the books for Grades 8 and 9 was very educational for me: now I can understand what my students may mean by "I am good at math" and "I am bad at math" and why their image of their mathematical abilities may be so far from reality. Presumably, being good at math is viewed as being able to implement some numerical or algebraic operations according to some prescribed (and even not justified) algorithms quickly and accurately after lengthy training. It is fortunate that I was not informed earlier; personally, I struggled with long addition and subtraction in Grades 2 and 3.

There are several aspects of mathematical education: the first and most important is developing the mind in general and logical thought in particular. The second one is encouraging independent thinking and judgment: even beginners make minor discoveries, and students can challenge their instructors and parents, since reason, not authority, establishes what is true. The third part is acquiring some technical skills and learning certain methods. For some reason, only the third aspect seems to be involved in the present junior high school curriculum, and even then at a rather limited level. For example, the textbook includes exercises outlining problem-solving skills; however, practically all the problems require only one step for their solution.

The main cause of this limited representation of mathematics is the one-size-fits-all education; sometimes it may mean that the slowest students have no chance to catch up with the material, while the more advanced students have no chance to be exposed to much mathematics (which is all about reasoning, not a set of recipes). I believe while there is still a certain percentage of junior high school students in the former category who would benefit from less intensive curriculum or curriculum spread out over the years, an even larger proportion falls into the latter category. There are free outreach activities (math nights at the University of Calgary and similar clubs at practically every Canadian university), but until a certain age it is, in fact, the parents' initiative: students cannot get to the universities on their own. And when the students grow up and get to high school it is, unfortunately, too late.

The situation can be resolved in the following way, with some foreign school systems serving as an example. First, in elementary school, when the pressure of school reports is not yet high, students should be exposed to operations with integers (mastering oral counting to 100, for example) and fractions (including addition), as well as some ideas in geometry and basic word–problem-solving skills. In the beginning of Grade 7, as a result of qualification testing, students are divided into several groups (four in the program of reference), run at the same time for several classes at the same level. Starting with senior high school, there are several streams in mathematics (A, B, C); level A includes not only enriched calculus, which is at a level between Math 31 and calculus in IB programs, but also a high level of algebra, geometry, and trigonometry. Level B roughly corresponds to the well-mastered Math 30 program, while C is a weaker level. Students can switch to a higher stream by taking exams. It is true that some of them will employ tutors for this; however, this will not be comparable to the level of private assistance that our high school and first-year undergraduate students are seeking now. Their present situation can be compared to a joint school race for all grades: first-graders have no chance; graduates have no challenge.

For our high school students to transition smoothly to universities, a solid mathematical background plays an important role. However, this link for many of our first-year science and engineering students is broken, probably at an earlier stage than senior high school.

Elena Braverman is vice-president, western provinces and territories, of the Canadian Mathematical Society. This article is reprinted from CMS Notes *45, no 3, pages 1 and 4, with the author's permission. Minor changes have been made in accordance with ATA style.*

Exploring Angles by Making Paper Airplanes

Trisha Vadnais

Learning from Students

I introduced the task of making paper airplanes using angles in a Grade 6 math class with 16 students, 8 of whom had special needs. The students used various types of paper to compare how far the planes flew and noted if paper weight influenced accuracy of flight. They used rulers to design the airplanes and geometry sets to identify the angles when they were labelling them. A metre stick was used to measure the distance the airplanes flew. Students used templates to estimate, identify and categorize angles and to fold paper into airplanes and test flight. They used a distance chart to record, compare and analyze each plane's flight and features.

For this paper, I focused on the two students that stood out the most to me, both boys. I will refer to them as Paul and Jim. Paul appeared to be an advanced student; he knew his angles and was able to answer all of the questions. Jim was a special needs student who really struggled with the mathematical words and estimation of angles.

The lesson was 80 minutes long. I began by reviewing angles by writing on the board "What are angles?" I drew a picture of each of the angles—acute, right, obtuse, straight, reflex, and circle—and marked the angle I wanted them to solve. In a class discussion, I asked the students to tell me what each angle was called and why it fit into that angle classification by estimating the degree of the angle. I asked these questions:

- Can you tell me what each angle is called?
- How do you know it is called that?

Paul knew that the angle was called acute because it was smaller than the right angle and that the obtuse angle was bigger than the right angle. He knew that the other angle was 180 degrees because it was a straight line. He was able to articulate that right angles were always square and were 90 degrees and that 360 degrees made a circle, so the reflex angle had to be between 180 and 360 degrees.

Jim had a much more difficult time. He didn't know what that angle was called, and he couldn't remember the vocabulary, but he identified that one angle was bigger and the other smaller than the right angle. He didn't know that a straight line was an angle or that if an angle goes all the way around it makes a circle. Jim was able to recognize the right angle and the acute and obtuse angle when he used the right angle as a reference.

- Using estimation, what degree is each of these angles?
- How do you know you're right?

Paul was able to identify that the acute angle was 45 degrees because it looked like it was half the size of a right angle. He thought the obtuse angle was 135 degrees because it was half again as much as the right angle, and the reflex angle was half again as much as the straight angle. Paul already knew how many degrees a straight line and a circle are, so he was able to use that as a reference for the reflex angle. I drew the reflex angle at 270 degrees, so he was also able to use the right angle as a reference.

Jim knew that the smaller angle had to be less than a right angle and that the bigger angle was more than the right angle. I asked Jim how he could use the right angle to figure out the other two angles. Jim thought for a bit and then said he could draw a pretend line for a right angle and then add to make it bigger or subtract to make it smaller. He was still unsure about estimating a number. I asked Jim to use a piece of paper and fold the corners to make a right angle and then fold it in half again. Jim was able to see that the acute angle he just made was half of 90 degrees. He was then able to take half of 90 and make 45. Jim was able to connect angles to making paper airplanes.

After we were finished our discussion on angles, the students labelled the angles and degrees on a template. I threw a regular piece of paper into the air to demonstrate that a piece of paper cannot fly without angles. Then I had the students design their own paper airplanes. When they were done, they competed against each other to see whose design could go the greatest distance and whose design had the best accuracy. After we were finished flying, we discussed why some designs worked better than others and how angles make a difference.

Reflecting on the Work

I was really surprised by Jim. He struggled with estimating the angles; however, his design won first place in both distance and accuracy. His plane flew three times farther than the plane that came in in second place. He was also able to explain why his design won. This was really unexpected, and I was very impressed. Jim understood the concepts of angles and flight but needed help breaking down the angles to figure out the degrees.

Paul, on the other hand, was able to use strategies and reference points for estimating angles. However, it was more difficult for him to make the connection between angles and flight. He was not able to explain how he could improve the design of his paper airplane, although he could explain how angles affect flight.

Having a class discussion on angles before having the students label the template worked really well. The students would have been lost and confused without a class discussion. As we discussed angles, I could see the students light up when they were remembering how to label and estimate. It would have been an impossible task without the discussion.

In many ways I think my task would have been easier to conduct if I had been working with only two or three students instead of such a large group. I had to adjust my task as I went along so that it could accommodate so many students with special needs. One of the downfalls was that I needed to be able to help all of the students in the class instead of just focusing all of my time on Paul and Jim. In a smaller environment, I would have been able to ask more in-depth questions as I assessed the boys' thinking. I also would have been able to keep it closer to a 30-minute time frame.

The students could not remember very much about angles until I started asking questions. I think that the communication went well. The students were able to learn from each other and share ideas through the discussion. I was able to get all of the students to participate by asking them to show me using their arms what a particular angle looks like. Even if some of the students could not remember what each angle was called or what it looked like, they were able to copy the other students with the positioning of their arms. This helped trigger everyone's memory and was a huge icebreaker; it also encouraged more students to participate in the conversation.

The mathematical language was tricky for some because not everyone understood the meaning of *acute*, *obtuse* and *reflex*; some also forgot that a straight line was also a straight angle. This is where using the positioning of their arms came in handy. The students were quickly able to relate the angle names to the positioning of their arms. This helped them to recognize the angles when they saw a picture of them.

The students knew that the paper was not going to fly without angles. After they finished making and flying the paper airplanes, we discussed how the angles affected the flight. The students mentioned how folding smaller angles and making the plane narrower made it fly farther. The larger angles and bigger wingspan helped the plane stay in the air longer. They also discovered that some designs caused the plane to do loops or fly back to them. When we finished discussing paper airplanes, I asked them if they could think of any other ways that angles are used and who uses them. I got a lot of different responses: building houses and sheds, carpenters, electricians, hunters practising shooting clay pigeons, and playing pool. I think that the students understand the concept of angles and the importance of angles in real-life situations. By understanding the operations and relationships, students will be able to connect to other areas in mathematics. Knowing relationships is essential when learning addition, subtraction, multiplication and division.

One modification I could make would be to have the students label a premade airplane rather than having them label a template. They could take it apart, label it and put it back together. This way they could see the actual angles that are being used in that particular airplane. They could test the flight and see how the angles affect its flight. The students could do the same thing with a different premade airplane and then compare the angles between the two airplanes. This would give the students a visual reference to the differences that angles make in how the plane flies. They could continue working on improving the design of the plane to get the results they want.

Tricia Vadnais completed her BEd in elementary education at the University of Alberta in 2013. This article is based on an assignment in a mathematics education course she took during her last year of studies.

Geometry with *Three Pigs, One Wolf and Seven Magic Shapes*

Carole Kamieniecki

Learning Geometrically

According to van Hiele (1999), children go through five stages as they develop geometric concepts. At the first level, visualization, children identify shapes based on a prototypical shape; the shape is judged based on its overall appearance only, and specific attributes are not considered (Hannibal 1999; van Hiele 1999; Ryan and Williams 2007). One example of a problem that can occur at this stage is that when a square is rotated 1/8 turn, so that one vertex is pointed down, children may no longer identify it as a square; they may call it a diamond (Margerm 1999). Teachers need to question the child to help correct this type of misconception. In the previous example, questions such as

- What changed about the square when I turned it to make it a diamond? Why is it no longer a square? and
- What happens when I turn it some more (turn it another 1/8 turn so the vertex is on the bottom left)?

should allow the child to think about what really changed. One of the ways to help prevent children from developing this misconception is to expose them to a variety of geometric shapes in different forms and positions (Hannibal 1999).

The second stage of van Hiele's (1999) levels is the descriptive level, at which children judge a shape based on its properties. For example, it is a square because it has four straight sides that are the same length. At this stage, children may still base their judgements on a prototypical shape and may not recognize a shape as being of a specified type. For example, a triangle like Triangle 1 below may not be called a triangle because it does not resemble the child's prototypical triangle concept (Ryan and Williams 2007).

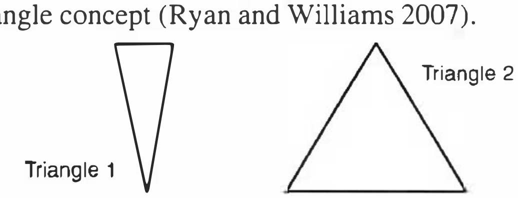


Figure 1

Triangle 1 may not match the prototypical triangle, Triangle 2, that a child pictures in his or her mind.

When students have a misconception, such as the one above, teachers once again need to provide some probing questions to help dispel the misconception. Here are some questions that would apply to the triangle problem above:

- What is different about this shape (Triangle 1)? (Children may respond that the pointy part is down and the flat part is up.)
- What is the same about the two shapes? (You want the child to indicate that both triangles have three sides and three points.)
- What happens if I turn this shape (Triangle 1) a half-turn? (Turn the triangle as you ask the question.)

These questions should help the children think about why they thought Triangle 1 was not a triangle and reconsider how they determine what attributes make a triangle.

Van Hiele (1999) indicates that mathematical "language is important for describing shapes" (p 311) at this level, so students need exposure to the language of geometry. Terms such as *face*, *edge* and *line* may hold different meanings for children, and teachers need to define and use these terms frequently so children can assimilate the new definition (Ryan and Williams 2007).

The third level, abstraction and informal deduction, is when the "properties [of shapes] are logically ordered" (van Hiele 1999, 311). Ryan and Williams (2007) indicate that "the concept [of the shape] becomes ...'concrete'... because its invariance ... under different transformations is recognized" (p 84). Teachers need to expose students to a wide variety of shapes and help students to understand what attributes of shapes are "relevant and irrelevant" (p 85) so students can proceed to this stage. Van Hiele (1999) feels that many students do not achieve this level of geometric thinking and have difficulty when encountering more advanced geometric thinking in higher grades. Ryan and Williams (2007) also indicate that many students 12 to 15 years old have not achieved this stage. It is important to provide probing questions to students in division I to dispel misconceptions so that, as they progress through division II and into secondary education, students can achieve this third level and continue on to van Hiele's (1999) fourth and fifth levels.

A Geometrical Task

1. Give each student a tangram. Explain that a tangram is a Chinese puzzle with seven geometric pieces (tans) that can be used to create different shapes. Demonstrate how the tans go together to form the square. Have students make the square (standard tangram shape) with their tans. Ask students to answer the first question on the worksheet.
2. Have students separate the tangram, examine each piece and fill in the chart on the worksheet. Ask students to demonstrate a flip and a turn with the tans.
3. Explain to students that we will read the book *Three Pigs, One Wolf, and Seven Magic Shapes* (Maccarone 1997) and students will construct the different shapes in the story with their tangram pieces. Read the story, allowing time for students to build each of the shapes in the story. Place an enlarged picture of the shape on the interactive whiteboard or overhead. Tell students to indicate if they are having difficulty making the shapes in the story. Check to see how the students approach the task of building the shape. If they are having difficulty, provide an outline of the shape for the students to fill in with their tans.
4. After the story is done, have students create their own shapes with their tans. Students should choose one of their shapes to draw on their worksheet and then colour it with pencil crayons.
5. As the students are filling out the chart, ask the following questions where applicable:
   - Why does flipping the triangle/square/parallelogram make it/not make it different? How is it different or why is it not different?
   - Why does turning the triangle/square/parallelogram make it/not make it different? How is it different or why is it not different?
6. Both as students are building the shapes in the story and afterward, ask them some of the following questions (pick a few that are appropriate for what the student is doing):
   - Why do you think that shape fits there?
   - Why doesn't that (pick the shape the child is using) fit in that spot?
   - Why did you make the square with those two triangles?
   - Why do you think the square fits in that spot?
   - What would happen if you flipped that piece? (National Council of Teachers of Mathematics [NCTM] 2012)
   - Can you use a different arrangement of pieces to make the shape? (NCTM 2012)
   - I see you rotated the square. Did rotating it change the shape? If the student says yes, ask why; if the student says no, ask why not.
7. To increase the students' mathematical vocabulary, make comments about what they are doing to the shapes as they build shapes in the story and after (NCTM, 2012).
   - I see that you are rotating that triangle to make it fit.
   - I like how you tried flipping the parallelogram to see if it fit.

Curriculum Connections

Grade 3 Mathematics

Specific Outcome 7—sort regular and irregular polygons, including

- triangles
- quadrilaterals
- pentagons
- hexagons
- octagons

according to the number of sides (Alberta Education 2007, 42)

Grade 3 Art

Purpose 2: Students will illustrate or tell a story.

- Concept A: A narrative can be retold or interpreted visually
- Concept C: Material from any subject discipline can be illustrated visually
  (Alberta Education 1985, C.7)

Grade 3 Language Arts

General Outcome 2—Students will listen, speak, read, write, view and represent to comprehend and respond personally and critically to oral, print and other media texts.

Specific Outcome 2.2—Respond to texts.

- Experience various texts—choose a variety of oral, print and other media texts for shared and independent listening, reading and viewing experiences, using texts from a variety of cultural traditions and genres, such as nonfiction, chapter books, illustrated storybooks, drum dances, fables, CDROM programs and plays.
  (Alberta Learning 2000, 31)

Engaging in the Task

I chose three children of different ages to work on my math task because I was interested to find out if there were any differences in their geometric ability; all three children are mine. The youngest child is a 7-year-old male in Grade 1; he enjoys working with numbers and is at a beginning reading level. The second child is a 9-year-old male in Grade 4; he is an average to above-average student who does well in math, science and reading when he takes his time and applies himself. The third child is a 12-year-old female in Grade 7; she is an above-average student who excels in all subjects. I worked with the two boys together for 35 to 40 minutes and I worked individually with the girl for about 30 minutes. Both interactions took place on the same day; the girl did not observe the interaction with the boys.

All three children filled in the worksheet, but I needed to assist the 7-year-old male in filling out his worksheet. The completed worksheets are attached at the end of this article. I have also included some of the interactions and pictures from the videos in the "Reflection" section.

Reflection

All three children were reluctant participants at first, because the activity entailed doing math and was expected to be like school. The thing that surprised me the most was that the older two children had not used tangrams previously in math. All three children were surprised at how hard it was to make the square out of the seven tans. I expected this, because I have played with tangrams and I still find it difficult to make the square. Once the children had become frustrated with trying to make the square, I gave them the tangram template sheet showing how the square is made with the tans. I was surprised that the 7-year-old found it easier to use the template than the 9-year-old did. If I did this activity again, I would start by giving just the square outline and have the children try to fill it in with the tans. Having the children fill in the outline first would allow them to do more exploration with flipping and rotating the tans.

By doing this expanded initial exploration with flipping and rotating the tans, the children should begin to understand how the shape changes when the tan is flipped or rotated. One of the things I could have improved during the task was getting the children to relate how they manipulated the tans to make the square to flipping and rotating th tans when filling in the chart. Relating these two activities should provide additional assistance for the children to see how a flip or rotation would change the tan.

When the children started filling in the chart, there were both things I expected and things that surprised me. I expected the young children to not be clear on what they needed to do to flip the tan. At first, the two boys were unsure about what flipping the shape involved, but when I asked a question about pancakes ("What do you do to a pancake when you cook it?") they immediately understood a flip. Some children may not have had the experience of making pancakes, so the teacher would need to find another analogy that may be relevant in the life of that child. Another word that caused problems was the word *different* in the chart. The boys told me that nothing would make the shape different because it was always the same; they stated, "All you do is move the shape around, you don't change it in any way" (N Kamieniecki and C Kamieniecki, personal communication). After I asked the boys to flip their tan I asked them what had changed; the answer I got from the youngest was "It's pointed in a different direction" (C Kamieniecki, personal communication). I indicated that this is what *different* means. Both boys worked individually for a few minutes at flipping the triangle to see if it changed anything. After they decided on how many flips made the triangle different, I asked them to show me how they came up with their answer.

Figure 2: 7-year-old flipping a triangle

The youngest child would only flip the triangle along the base, and he rationalized that the flip made the triangle different because it was pointed in a different direction. Both the 9-year-old and the 12-year-old were not limited to flipping the triangle at one point of the shape; they would perform a flip using a vertex, a base or one of the sides. One of the things that the 12-year-old girl mentioned is that flipping does not change the shape itself; it simply changes the position of the shape relative to where it was at the start. As a Grade 7 student, she has experienced working with the translocation of geometric shapes; for her, any translocation of a tan was considered as making the shape different even if the position of the vertices and faces were the same.

I was pleasantly surprised to find out that the 7-year-old did still call the square a square even when one vertex was pointed downward so that the square resembled a diamond shape. I asked him why it was still a square when it was in that position and his response was "It is still a square because the only thing that that is different is the corner is pointed down" (C Kamieniecki, personal communication). I had expected the 7-year-old to have the misconception that the square was a diamond when it was turned in this position. I believe that this child would be considered to be starting to understand the third stage of van Hiele's (1999) geometric thinking because he judges a shape based on its properties but does not limit himself to a prototypical shape.

The other thing that surprised me was that none of the three children knew what to call the parallelogram. The 12-year-old called it a quadrilateral, which it is, but did not remember the term *parallelogram*, although she did remember what a trapezoid was. When she completed her chart, the term was provided by her brother who had already completed the task. As an aside, another child who is 14 years old and in Grade 9 also did not remember the term *parallelogram*. I believe that the two older children had difficulty remembering the term because geometry is not given a lot of teaching time in elementary school, as evidenced by the number of 2-D and 3-D shape and space outcomes in relation to the total outcomes for each grade.

Having the children build the shapes along with the story was quite engaging for the two boys. The boys enjoyed the humorous relationship of the book to the original story of the Three Little Pigs. Both boys found building the different shapes in the book challenging but not frustrating. The 7-year-old boy had some slight difficulty with the sailboat shape, which used most of the tans in diagonal positions (relative to their prototypical positions). When he became frustrated with trying to build the sailboat, I asked him to stop and look at the shapes in the picture. While he was looking I asked him to pick one shape to start with; he started with the large triangle in the sail and was then able to complete the sailboat easily. The 9-year-old boy had a lot of difficulty with the candle shape, which required proper placement of the correctly sized triangle. While he was attempting to construct the triangle I asked him if rotating the triangle he had chosen would change the shape so that it would fit. After he tried rotating the triangle, he said it would not fit. I asked him to show me why he did not think it worked, and he pointed to the picture of the candle in the book and said that it stuck out and was too big.

For the last activity, the 9-year-old created his own shape, a house, and after he had constructed it and recorded it, we explored what would happen if he chose different shapes or changed the position of the shape by rotating or flipping a tan. He discovered that he could make different houses, but he could not make the same house. When the 7-year-old was doing the last activity, building his own shape with the tans, I asked him if the parallelogram would still fit if he flipped it. He answered no and when I asked him why, he said he would show me; he demonstrated what would happen if he flipped the parallelogram. The 12-year-old girl did not enjoy the story but did engage with building the shapes, as did the 14-year-old, who was just passing through.

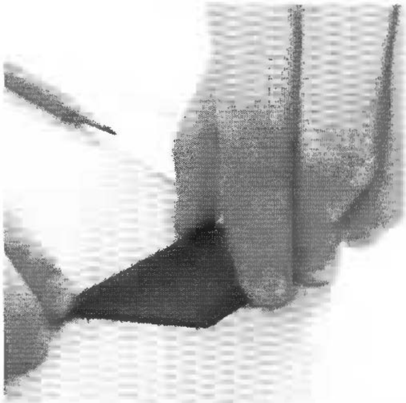


Figure 3: 9-year-old flipping the parallelogram

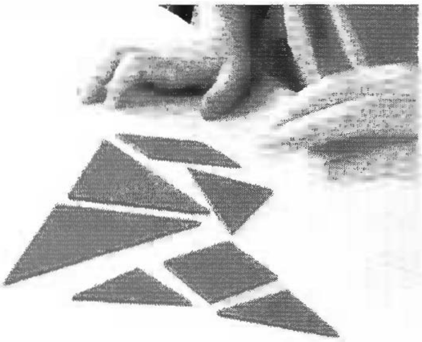


Figure 4: 7-year-old building the sailboat from the book

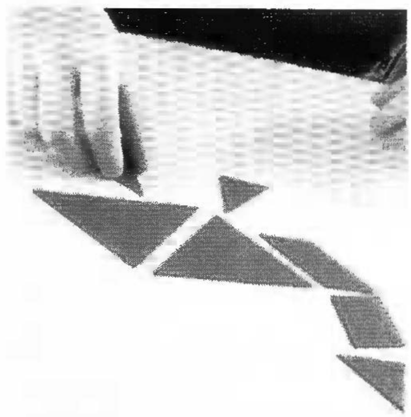


Figure 5: 12-year-old building the duck from the book

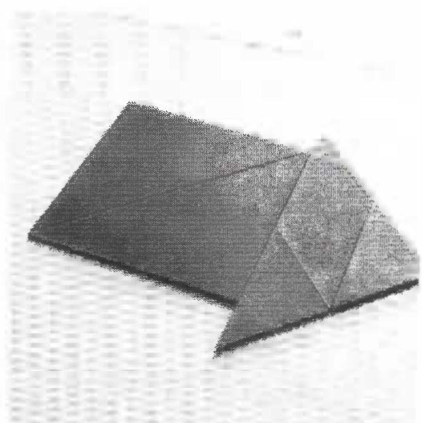


Figure 6: 9-year-old's completed shape

Assessment of Task

The two boys were very engaged with the task and even expressed their enjoyment of playing with the tangrams. The girl did not appear to be as engaged, but when I asked her what she did not like, she replied "It's math and I always have this attitude about math" (M Kamieniecki, personal communication). She was also my testing station for math last year and so has a jaded opinion of being a math guinea pig. The three children followed the task instructions and required few directions. The task connected math to language arts and art. I was able to note student thinking about the task using question prompts.

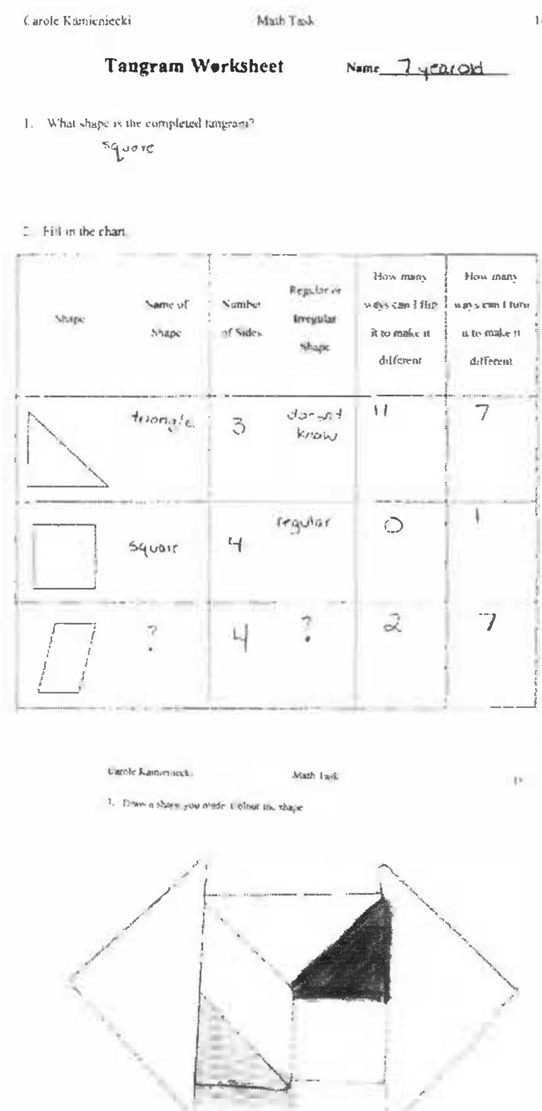
Modifications of the Task

The first modification I would make is to give students the square tangram outline to fill in if they have difficulty making the standard square tangram shape. I would also ask questions about how they are manipulating the individual tans to make them fit, such as

1. Why did you select that piece to go in that spot?
2. Why did you flip that piece to make it fit in that spot? What made it different?
3. Why did you rotate that piece to make it fit in that spot? What made it different?

I think these questions would allow the children to think about the word *different* and what it means to say that a shape is different. Asking why may make children feel uncomfortable; if it does, the questions could be rephrased:

1. How does this piece fit in that spot?
2. Show me how flipping that piece made it fit.
3. Show me how rotating that piece made it fit.



Carole Kamieniecki Math Task 14

Tangram Worksheet Name 7 year old

1. What shape is the completed tangram?

squore

2. Fill in the chart

Shape	Name of Shape	Number of Sides	Regular or Irregular Shape	How many ways can I flip it to make it different	How many ways can I turn it to make it different
	triangle	3	don't know	11	7
	squair	4	regular	0	1
	?	4	?	2	7

Carole Kamieniecki Math Task 15

1. Draw a shape you made. Colour the shape

Figure 7: 7-year-old's worksheet

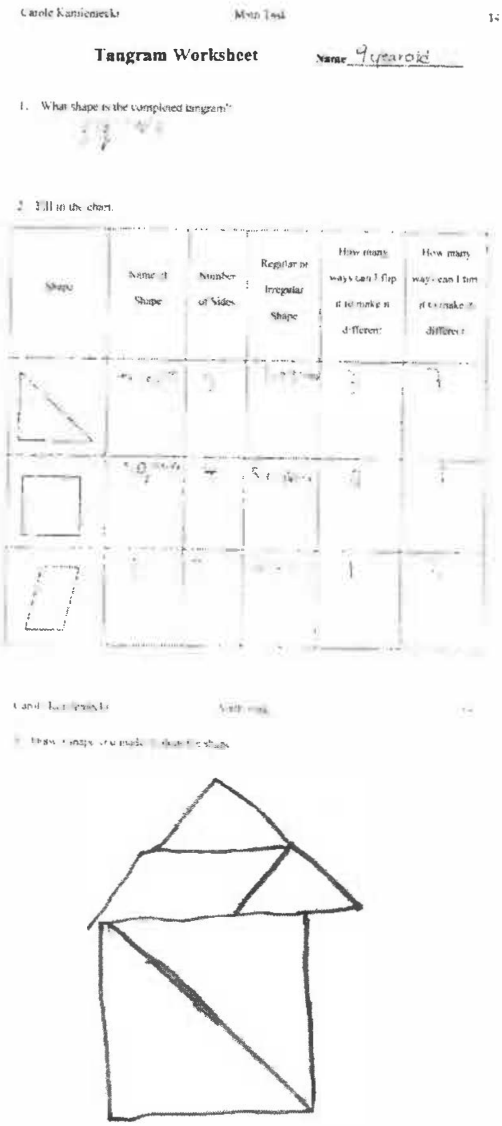


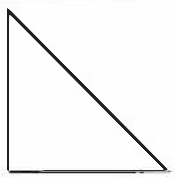
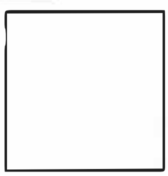
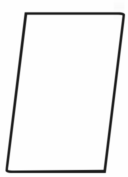
Figure 8: 9-year-old's worksheet

Figure 9: 12-year-old's worksheet

Tangram Worksheet Name

1. What shape is the completed tangram?

2. Fill in the chart.

Shape	Name of shape	Number of sides	Regular or irregular shape	How many ways can I flip it to make it different?	How many ways can I turn it to make it different?
					
					
					

3. Draw a shape you made. Colour the shape.

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Prior to earning her BEd, Carole Kamieniecki worked as a veterinarian and thought that she would like to focus on teaching science. However, during her math curriculum classes she developed an intense interest in math. She received her BEd in elementary education in November 2013; she is currently working as a substitute teacher and volunteer math and science tutor for junior and senior high school students. She continues to attend math-related professional development sessions to discover more ways that students can explore mathematical concepts, to better understand where students can become confused, and to develop a better understanding of how students can find the connections between mathematical concepts.

Recreational Mathematics: An Avenue to Engaging in Mathematical Development

John Grant McLoughlin

This article features four examples of mathematical challenges or games that have been effectively drawn into my own teaching. The idea of playing games for fun is healthy, though it is the intentionality of drawing particular forms of challenges or games into play that makes the teaching and learning much more meaningful. That is, the examples here are integrated into situations that are designed to develop number sense, as with the first two examples, or bring forth aspects of mathematical structure and/or proof, as with the latter pair.

Developing Number Sense

Random trial and error is commonly employed to find numbers that meet particular requirements or constraints. Such trials are beneficial to the understanding of a problem, though examples here demonstrate how one can go beyond randomness to actually get at the mathematical underpinnings of a challenge.

Alphametics

This is an example of a puzzle that uses letters to represent distinct digits. It is typically assumed that the initial digit of any number represented in the problem is nonzero. Also, any letter that appears more than once must represent the same digit in all instances; further, a digit is to be represented by one letter consistently. For example, if K = 4, there is not another letter also representing 4.

Consider the following example, which I have used in a variety of contexts:

$ABCD \times 4 = DCBA$

What letters are represented by each of A, B, C, and D?

If students are given a few minutes to play with the challenge, it is common for some observations to be raised that, when put together, get us started. The first points may be that

- A is 1 or 2, since multiplication by 4 results in a product less than 10,000 [that is, with fewer than five digits]; and
- A must be even, because it is the last digit of the product and we multiplied by 4.

It follows that $A = 2$, and then the number sense may lead to the following points:

- D must be 8 or 9, since the product is greater than $2{,}000 \times 4$.
- $D \times 4$ ends in 2, so $D = 8$.

Some students are inclined to revert at this point to trial and error again, though number sense can be applied neatly here. Recall that $4 \times 8 = 32$ and hence, there is a carry of 3 into the tens, thus guaranteeing that the value of B must be odd. Why? We know that the value of $4C + 3$ is odd. Quickly it follows that $B = 1$ to keep the product to four digits, and finally, $C = 7$.

The above example is one illustration of how number sense can be developed—or unearthed, as it may already be there beneath the surface. My tendency is to use the example as a collective problem. Then the following example with a parallel structure and fewer restrictions can be offered for unsupported work. (It is important to note that the challenges are independent and all digits are available again.)

$EFGH \times 9 = HGFE$

Interestingly, 1,089 and 2,178 are the only four-digit numbers that can be multiplied by a single digit greater than 1 to produce their reversals.

Pick and Find a Number

This problem can serve as an icebreaker. The example below was initially shared at the public lecture opening Sharing Mathematics: A Tribute to Jim Totten in May 2009.[1] The names have been maintained, as all were colleagues of Jim.

[1]Editor's note: In March 2008, the mathematical community lost a dear colleague, Dr James Totten, who had been a professor of mathematics at Thompson Rivers University, from which he retired as professor emeritus in 2007. Friends and colleagues celebrated his spirit by hosting a conference in his honour from May 13–15, 2009, at Thompson Rivers University in Kamloops. The conference included a public lecture, outreach activities for the whole community, and invited and contributed talks. Proceedings were published.

Kirk, John (Ciriani), Fae, Sonja and Peter are seated in a circular arrangement. Each person selected a number. The neighbours then added their numbers together and the results are shown. What was John's number?

63 Kirk

Peter 71

48 Sonja

Fae 80

90 John

Again, there is a tendency for students to use trial and error in many cases. It is helpful to try a possible number and see what happens as a means of understanding the problem. However, number sense or some form of mathematical organization allows for a swift and elegant solution. Observe that twice the sum of the five selected numbers equals 71 + 80 + 90 + 48 + 63, or 352.

Hence, the sum of the five numbers is 176.

One possible approach is to exclude John's number from a sum containing each of the other four numbers. Leaving John's number out, the total of the remaining four numbers is 71 + 48 (that is, Kirk + Sonja added to Fae + Peter). Hence, John's number must be 176 – (71 + 48) = 57. Other methods can also be employed. This usually surfaces in the discussion, thus providing an experience in which multiple approaches are used. The middle school student can do this as a reasoning problem, whereas the high school student can opt for an algebraic representation. In fact, a way to ensure that the problem is understood is to form groups of five to generate a set of sums to be exchanged with other groups for the purpose of solving.

Mathematical Structure and Proof

Two examples of games lending themselves to other aspects of mathematics are provided here.

The first of these, Fifteen Finesse, appeared in a Martin Gardner book, *aha! Insight* (Freeman, 1978). The rules of the game are described. Playing the game a few times helps one to understand the core principles. The second example, Sim, is spatial in nature. While ties are commonplace in the first of these two games, the proof in the second example depends upon the fact that a tie is impossible in Sim.

Fifteen Finesse

The numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 are used in this game—one of each being available for use. Players alternate turns selecting one of the available numbers, with the object being to have three numbers that sum to 15. (Note that 7 and 8, or 2, 3, 4, and 6 would not satisfy the requirement, as it must be exactly three numbers.) The game ends in a tie if no player gets three numbers that sum to 15.

What makes this game mathematically intriguing to play? My experience is that students generally play this in a way that that sharpens their thinking. For instance, consider the following sequence of opening number selections with the choices of Player A in bold: **6** 8 **2** 7. A misconception is that the 6 and 2 are no longer helpful when the choice of 7 blocks the possibility of using them to make 15. However, the 6 and 2 remain with Player A, and a subsequent selection of 4 here would actually guarantee a win, as both 5 and 9 remain available for either (6, 4, 5) or (2, 4, 9). Reasonable players will soon find that many games are ending without a winner. Why?

The answer lies in the structure of the game. This game has a structure that is isomorphic to tic-tac-toe with a playing board that consists of a magic square.

4	9	2
3	5	7
8	1	6

That is, the object of the game is to claim any of the triples that form a diagonal, row or column in the magic square. In order to appreciate the structural similarities, it is helpful to have the students record the sequencing of number selections in a few games prior to delving into the isomorphism itself. One can readily replay the moves on the magic square board to see how they obviously won or lost a game.

Sim

Six dots are drawn on a piece of paper to form the vertices of a (convex) hexagon. Two players are each assigned a colour. The players take turns joining any two of the dots with a line segment, using their assigned colours. The loser is the player who completes a triangle with three of the original six dots as its vertices and with all three edges the same colour.

Why is it that the game of Sim is mathematically enriching to play? Practically speaking it offers a curious "equalizer" quality, in that spatial perception and logic blend to bring forth strengths/weaknesses not so apparent day to day in a class. My experience is that some struggling math students have gained confidence by matching up with or perhaps bettering students and teachers known to be more successful in mathematics. Indeed the game helps to focus attention on detail—a valuable skill for mathematical development. Further, the game does not take long to play because there are only 15 possible segments that can be drawn.

Mathematically there is a lovely connection to proof. Unlike the preceding game, it is a fact that every game of Sim must have a winner. (In fact, in dire situations with an imminent losing position late in a game it may be worth checking if you won already and did not notice!) Why is a tie impossible?

The essence of the why lies in the idea of a proof by contradiction that is outlined here. Assuming that a tie is possible would require that all five possible segments be drawn from each vertex. So we know that at least three segments from any vertex must be the same colour. So suppose that three red segments are drawn from a given vertex to connect with three other vertices. This will create a situation in which two edges of three different triangles, each containing two of the original vertices, are the same colour. Hence, a tie is possible only if the third edge of each of these triangles is the other colour. However, this creates a triangle with three edges all having the other colour. The contradiction is evident.

Conclusion

Select games and other recreational mathematical ideas offer valuable teaching examples that can be intentionally drawn into the development of mathematical work at elementary or advanced levels. The beauty of many examples comes from the invitational space created for the student to engage directly in the mathematical process, thus enabling deeper appreciation of what is essentially being learned.

Follow-up Note

Sim and Fifteen Finesse are two of five ideas shared in "Playing Games with Mathematics (Part I)" and "Playing Games with Mathematics (Part II)" in *Crux Mathematicorum*, issues 32, no 5 and 32, no 6 respectively. Both articles are publicly accessible, at https://cms.math.ca/crux/v32/n5 and https://cms.math.ca/crux/v32/n6. The articles appear under the heading of "Polya's Paragon" in the "Mathematical Mayhem" section. Part I provides the games and challenges and Part II offers insights into the mathematics underlying them, as it is hoped that readers will have played with the ideas beforehand.

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