Recreational Mathematics: An Avenue to Engaging in Mathematical Development

John Grant McLoughlin

This article features four examples of mathematical challenges or games that have been effectively drawn into my own teaching. The idea of playing games for fun is healthy, though it is the intentionality of drawing particular forms of challenges or games into play that makes the teaching and learning much more meaningful. That is, the examples here are integrated into situations that are designed to develop number sense, as with the first two examples, or bring forth aspects of mathematical structure and/or proof, as with the latter pair.

Developing Number Sense

Random trial and error is commonly employed to find numbers that meet particular requirements or constraints. Such trials are beneficial to the understanding of a problem, though examples here demonstrate how one can go beyond randomness to actually get at the mathematical underpinnings of a challenge.

Alphametics

This is an example of a puzzle that uses letters to represent distinct digits. It is typically assumed that the initial digit of any number represented in the problem is nonzero. Also, any letter that appears more than once must represent the same digit in all instances; further, a digit is to be represented by one letter consistently. For example, if K = 4, there is not another letter also representing 4.

Consider the following example, which I have used in a variety of contexts:

$ABCD \times 4 = DCBA$

What letters are represented by each of A, B, C, and D?

If students are given a few minutes to play with the challenge, it is common for some observations to be raised that, when put together, get us started. The first points may be that

- A is 1 or 2, since multiplication by 4 results in a product less than 10,000 [that is, with fewer than five digits]; and
- A must be even, because it is the last digit of the product and we multiplied by 4.

It follows that $A = 2$, and then the number sense may lead to the following points:

- D must be 8 or 9, since the product is greater than $2{,}000 \times 4$.
- $D \times 4$ ends in 2, so $D = 8$.

Some students are inclined to revert at this point to trial and error again, though number sense can be applied neatly here. Recall that $4 \times 8 = 32$ and hence, there is a carry of 3 into the tens, thus guaranteeing that the value of B must be odd. Why? We know that the value of $4C + 3$ is odd. Quickly it follows that $B = 1$ to keep the product to four digits, and finally, $C = 7$.

The above example is one illustration of how number sense can be developed—or unearthed, as it may already be there beneath the surface. My tendency is to use the example as a collective problem. Then the following example with a parallel structure and fewer restrictions can be offered for unsupported work. (It is important to note that the challenges are independent and all digits are available again.)

$EFGH \times 9 = HGFE$

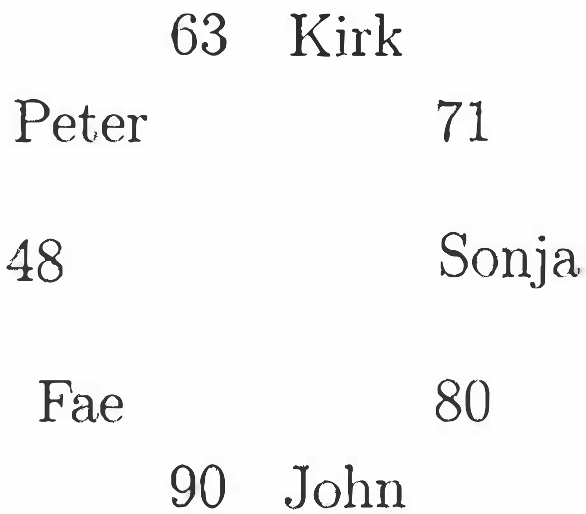
Interestingly, 1,089 and 2,178 are the only four-digit numbers that can be multiplied by a single digit greater than 1 to produce their reversals.

Pick and Find a Number

This problem can serve as an icebreaker. The example below was initially shared at the public lecture opening Sharing Mathematics: A Tribute to Jim Totten in May 2009.[1] The names have been maintained, as all were colleagues of Jim.

[1]Editor's note: In March 2008, the mathematical community lost a dear colleague, Dr James Totten, who had been a professor of mathematics at Thompson Rivers University, from which he retired as professor emeritus in 2007. Friends and colleagues celebrated his spirit by hosting a conference in his honour from May 13–15, 2009, at Thompson Rivers University in Kamloops. The conference included a public lecture, outreach activities for the whole community, and invited and contributed talks. Proceedings were published.

Kirk, John (Ciriani), Fae, Sonja and Peter are seated in a circular arrangement. Each person selected a number. The neighbours then added their numbers together and the results are shown. What was John's number?



Again, there is a tendency for students to use trial and error in many cases. It is helpful to try a possible number and see what happens as a means of understanding the problem. However, number sense or some form of mathematical organization allows for a swift and elegant solution. Observe that twice the sum of the five selected numbers equals 71 + 80 + 90 + 48 + 63, or 352.

Hence, the sum of the five numbers is 176.

One possible approach is to exclude John's number from a sum containing each of the other four numbers. Leaving John's number out, the total of the remaining four numbers is 71 + 48 (that is, Kirk + Sonja added to Fae + Peter). Hence, John's number must be 176 – (71 + 48) = 57. Other methods can also be employed. This usually surfaces in the discussion, thus providing an experience in which multiple approaches are used. The middle school student can do this as a reasoning problem, whereas the high school student can opt for an algebraic representation. In fact, a way to ensure that the problem is understood is to form groups of five to generate a set of sums to be exchanged with other groups for the purpose of solving.

Mathematical Structure and Proof

Two examples of games lending themselves to other aspects of mathematics are provided here.

The first of these, Fifteen Finesse, appeared in a Martin Gardner book, *aha! Insight* (Freeman, 1978). The rules of the game are described. Playing the game a few times helps one to understand the core principles. The second example, Sim, is spatial in nature. While ties are commonplace in the first of these two games, the proof in the second example depends upon the fact that a tie is impossible in Sim.

Fifteen Finesse

The numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 are used in this game—one of each being available for use. Players alternate turns selecting one of the available numbers, with the object being to have three numbers that sum to 15. (Note that 7 and 8, or 2, 3, 4, and 6 would not satisfy the requirement, as it must be exactly three numbers.) The game ends in a tie if no player gets three numbers that sum to 15.

What makes this game mathematically intriguing to play? My experience is that students generally play this in a way that that sharpens their thinking. For instance, consider the following sequence of opening number selections with the choices of Player A in bold: **6** 8 **2** 7. A misconception is that the 6 and 2 are no longer helpful when the choice of 7 blocks the possibility of using them to make 15. However, the 6 and 2 remain with Player A, and a subsequent selection of 4 here would actually guarantee a win, as both 5 and 9 remain available for either (6, 4, 5) or (2, 4, 9). Reasonable players will soon find that many games are ending without a winner. Why?

The answer lies in the structure of the game. This game has a structure that is isomorphic to tic-tac-toe with a playing board that consists of a magic square.

4	9	2
3	5	7
8	1	6

That is, the object of the game is to claim any of the triples that form a diagonal, row or column in the magic square. In order to appreciate the structural similarities, it is helpful to have the students record the sequencing of number selections in a few games prior to delving into the isomorphism itself. One can readily replay the moves on the magic square board to see how they obviously won or lost a game.

Sim

Six dots are drawn on a piece of paper to form the vertices of a (convex) hexagon. Two players are each assigned a colour. The players take turns joining any two of the dots with a line segment, using their assigned colours. The loser is the player who completes a triangle with three of the original six dots as its vertices and with all three edges the same colour.

Why is it that the game of Sim is mathematically enriching to play? Practically speaking it offers a curious "equalizer" quality, in that spatial perception and logic blend to bring forth strengths/weaknesses not so apparent day to day in a class. My experience is that some struggling math students have gained confidence by matching up with or perhaps bettering students and teachers known to be more successful in mathematics. Indeed the game helps to focus attention on detail—a valuable skill for mathematical development. Further, the game does not take long to play because there are only 15 possible segments that can be drawn.

Mathematically there is a lovely connection to proof. Unlike the preceding game, it is a fact that every game of Sim must have a winner. (In fact, in dire situations with an imminent losing position late in a game it may be worth checking if you won already and did not notice!) Why is a tie impossible?

The essence of the why lies in the idea of a proof by contradiction that is outlined here. Assuming that a tie is possible would require that all five possible segments be drawn from each vertex. So we know that at least three segments from any vertex must be the same colour. So suppose that three red segments are drawn from a given vertex to connect with three other vertices. This will create a situation in which two edges of three different triangles, each containing two of the original vertices, are the same colour. Hence, a tie is possible only if the third edge of each of these triangles is the other colour. However, this creates a triangle with three edges all having the other colour. The contradiction is evident.

Conclusion

Select games and other recreational mathematical ideas offer valuable teaching examples that can be intentionally drawn into the development of mathematical work at elementary or advanced levels. The beauty of many examples comes from the invitational space created for the student to engage directly in the mathematical process, thus enabling deeper appreciation of what is essentially being learned.

Follow-up Note

Sim and Fifteen Finesse are two of five ideas shared in "Playing Games with Mathematics (Part I)" and "Playing Games with Mathematics (Part II)" in *Crux Mathematicorum*, issues 32, no 5 and 32, no 6 respectively. Both articles are publicly accessible, at https://cms.math.ca/crux/v32/n5 and https://cms.math.ca/crux/v32/n6. The articles appear under the heading of "Polya's Paragon" in the "Mathematical Mayhem" section. Part I provides the games and challenges and Part II offers insights into the mathematics underlying them, as it is hoped that readers will have played with the ideas beforehand.

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