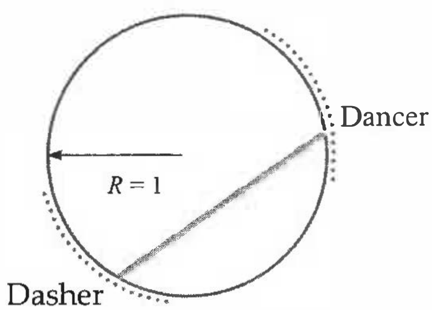
A Northern Lights Circle Problem

Gregory V Akulov

When December is incredibly frosty, the famous Dasher and Dancer preface their holiday flight to the Land of Living Skies with a spectacular warm-up. They canter gracefully at a constant rate along the Northern Lights Circle of a radius of 1 km and can be seen by everyone in the Land of the Midnight Sun. Some inhabitants even state that the curious bear Arctic, an awesome navigator from the Coffee Club Island, periodically records the exact straight-line distance between the reindeer. If the map drawn one morning is as shown below, find d. Give an exact answer.



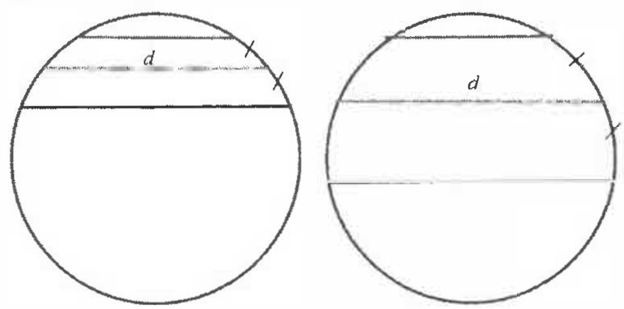
Time	Distance
AM	km
6:50	1.2
7:05	d = ?
7:20	1.6

Author's Solution

A_1, A_2 and A_3 and B_1, B_2 and B_3 are the positions of the reindeer at 6:50 AM, 7:05 AM and 7:20 AM, respectively. Since they are moving at a constant rate, it is easy to see that

$$A_1A_2 = A_2A_3 \text{ and } B_1B_2 = B_2B_3. \qquad (1)$$

Therefore, if the 1.2 km A_1B_1, d km A_2B_2 and 1.6 km A_3B_3 chords are redrawn so that $A_1B_1 \parallel A_2B_2 \parallel A_3B_3$, then (1) will remain true.



Both possible cases can be solved using an arc midpoint computation (see http://mathcentral.uregina.ca/RR/database/RR.09.10/akulov2.html):

$$d = \sqrt{(1 + 0.6)(1 + 0.8)} \pm \sqrt{(1 - 0.6)(1 - 0.8)},$$

which gives two answers $d = \sqrt{2}$ or $d = (7\sqrt{2})/5$ km.

For a student solution to this problem, see the Student Corner on page 8.

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A Solution to "A Northern Lights Circle Problem"

Dennis Situ

The following is a Grade 9 student's solution to Gregory Akulov's "A Northern Lights Circle Problem" on page 7.

Editor's note: When he considers one possible case below, Dennis rounds sin (36.87) = 0.6 and sin (53.13) = 0.8. Actually, sin (36.87) > 0.6 and sin (53.13) < 0.8.

First of all, we must note that the distances between the two reindeer are not directly correlated with time; rather, it is the angle formed by the radii attached to both reindeer. The reindeer can be thought of as hands on a clock, which move at a constant rate: each minute, the minute hand moves 6° and the hour hand moves 0.5°. However, the straight-line distance between them does not change at a constant pace. Therefore, we must find the angles before we move any further.

At 6:50, we have a diagram as shown (Figure 1), with OA drawn as the perpendicular bisector of the line connecting Dasher (D^I) and Dancer (D^{II}). Because of the RHS (right hypotenuse side) property, the two triangles shown are congruent. This means that OA is also the angle bisector of D^IOD^{II}. We also have a right-angled triangle with two sides known. Therefore, we can use trigonometry to find the angles. Label angle D^IOA as α.

We have $\sin \alpha = 0.6$, so $\alpha = 36.87°$, which means that angle $D^IOD^{II} = 73.74°$.

We now examine the case at 7:20, where we draw a similar diagram (Figure 2), but with OB as the new perpendicular bisector. Angle D^IOB is to be labelled β.

We have $\sin \beta = 0.8$, so $\beta = 53.13°$. Now, $D^IOD^{II} = 106.26°$.

In order to find the distance d at 7:05, we must find the average angle, as 7:05 is the average time between 6:50 and 7:30. The average angle is given by

$$\frac{73.74 + 106.26}{2} = 90°.$$

At 7:05, the radii of the reindeer form a right angle. The diagram is given below (Figure 3).

Although we could use trigonometry to find the length d, the easiest way is to simply use Pythagoras' theorem to find that $d = \sqrt{1^2 + 1^2} = \sqrt{2}$.

At 7:05, the distance d is $\sqrt{2}$ km.

Figure 1

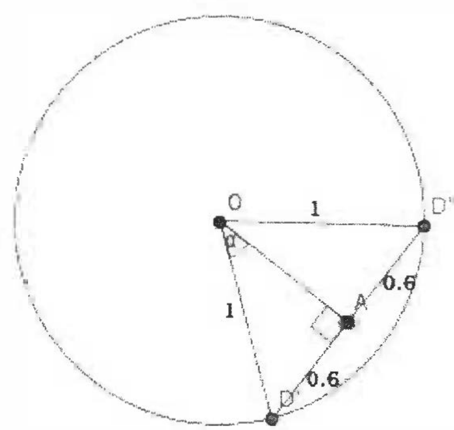


Figure 2

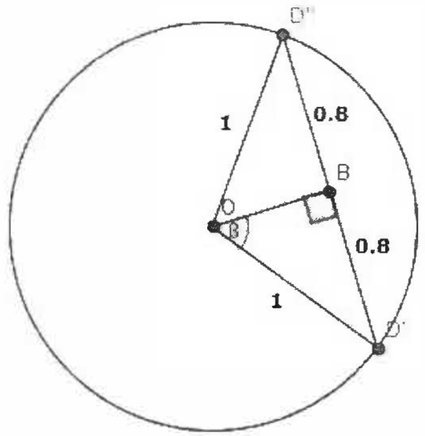
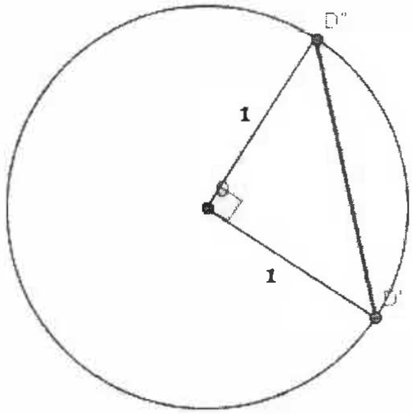


Figure 3



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