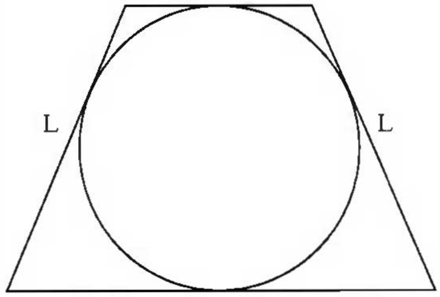
Calgary Junior High School Mathematics Contest 2013

The Calgary Junior High School Mathematics Contest takes place every spring. The 90-minute exam is primarily for Grade 9 students; however, all junior high students in Calgary and surrounding districts are eligible. Participants write the exam in their own schools. School and individual prizes include trophies, medals, a cash award to the student achieving the highest mark, and the opportunity for the top students (and their teacher sponsors) to attend a banquet at the University of Calgary. The 37th annual contest took place on April 24, 2013.

Part A: Short Answer

1. From the set {1, 2, 3, 4, 5, 6, 7, 8, 9} all odd numbers are removed. How many numbers are remaining?
2. A bag contains red, blue and green marbles. Two-thirds of the marbles are *not* red, and three-quarters of the marbles are *not* blue. What fraction of the marbles are *not* green? Express your fraction in lowest terms.
3. Ajooni walked 9 km at 4 km/h, and then biked for 4 hours at 9 km/h. What was her average speed (in km/h) for the entire trip?
4. Notice that the digits of 2013 are four consecutive integers (because 0, 1, 2 and 3 are consecutive integers). What was the last year (before 2013) whose digits were four consecutive integers?
5. A circle is inscribed in an isosceles trapezoid, as shown, with parallel edges of lengths 8 cm and 18 cm and sloping edges of length L cm each. What is L?



6. Mary has a large box of candies. If she gives a third of her candies to her mom, then a third of the remaining candies to her dad, and finally a third of what's left to her little sister, there will only be 16 candies in the box. How many candies are in the box at the beginning?
7. I have half a litre of solution, which is 40% acid and the rest water. If I mix it with 2 L of solution that is only 10% acid, what is the percentage of acid in the mixture?
8. A two-digit positive integer is said to be *doubly divisible* if its two digits are different and non-zero, and it is exactly divisible by each of its two digits. For example, 12 is doubly divisible since it is divisible by 1 and 2, whereas 99 is not doubly divisible since its digits are equal, and 90 is not doubly divisible because it contains a zero. What is the largest doubly divisible positive integer?
9. What is the remainder when 2^{2013} is divided by 7?

Answers

1. 4
2. 7/12
3. 36/5 = 7.2
4. 1432
5. 13
6. 54
7. 16
8. 48
9. 1

Part B: Long Answer

1. You currently have \$100 and two magic wands, A and B. Wand A increases the amount of money you have by 30%, and wand B adds \$50 to the amount of money you have. You may use each wand exactly once, one after the other. In which order should you use the wands to maximize the amount of money you have? How much money would you have?

2. Put one of the integers 1, 2, . . ., 13 into each of the boxes, so that 12 of these numbers are used once (and one number is not used at all), and so that all four equations are true. Be sure to explain how you found your answers.

$$\square + \square = \square$$
$$\square - \square = \square$$
$$\square \times \square = \square$$
$$\square \div \square = \square$$

3. On planet X, an X-monkey has two legs and one head, while an X-hypercow has three legs and four heads. Robert has a herd of X-monkeys and X-hypercows on his farm, with a total of 87 legs and 86 heads in his herd. How many animals of each kind does Robert have?

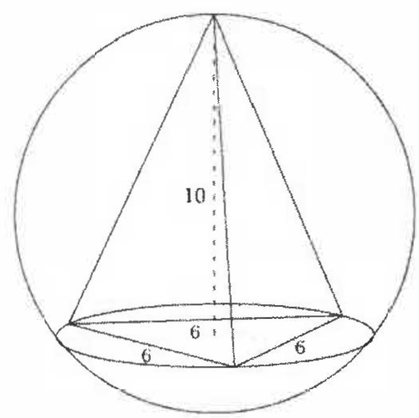
4. A pie is cut into a equal parts. Then one of these parts is cut into b smaller equal parts. Finally, one of the smaller parts is cut into c smallest equal parts. One of the original parts, together with a smaller part and a smallest part, makes up exactly three-fifths of the pie. What are a, b and c (assuming a, b and c are integers greater than 1)?

5. In a hockey tournament, five teams participated and each team played against each other team exactly once. A team received 2 points for a win, 1 point for a tie and 0 points for a loss. At the end of the tournament, the results showed that no two teams received the same total points, and the order of the teams (from highest point total to lowest point total) was A, B, C, D, E. Team B was the only team that did not lose any games, and team E was the only team that did not win any games. How many points did each team receive, and what was the result of each game?

	Total Points
A	
B	
C	
D	
E	

	Winner (or Tie)
A vs B	
A vs C	
A vs D	
A vs E	
B vs C	
B vs D	
B vs E	
C vs D	
C vs E	
D vs E	

6. The three edges of the base of a triangular pyramid (tetrahedron) each have length 6 units, and the height of the pyramid is 10. The other three (sloping) edges are equal in length. A sphere passes through all four corners of the pyramid. What is the radius of the sphere?



Solutions

1. If you first use wand A, the \$100 becomes \$130, and then applying wand B produces \$130 + \$50 = \$180. However, if you first use wand B, you obtain \$150, which (after using wand A) becomes \$150 × 1.3 = \$195. So the maximum amount is \$195, obtained by using wand B first, then wand A.

2. One answer is shown here. The fourth equation (A ÷ B = C) is the same as B × C = A, which is the same form as the third equation. Neither of these equations can use the number 1 (or

$$\boxed{6} + \boxed{7} = \boxed{13}$$
$$\boxed{9} - \boxed{8} = \boxed{1}$$
$$\boxed{3} \times \boxed{4} = \boxed{12}$$
$$\boxed{10} \div \boxed{5} = \boxed{2}$$

else there would be a repeated number), so in one of these two equations the smallest number must be 2, and in the other the smallest number must be 3. If the smallest number is 3, the only possibility is $3 \times 4 = 12$. This leaves $2 \times 5 = 10$ as the only possibility for the other equation (since we cannot repeat the numbers 3 and 4). So the last two equations must use the six numbers 3, 4, 12, 2, 5 and 10, and there are various ways this can happen. For example, we could have used $5 \times 2 = 10$ and $12 \div 4 = 3$ instead of what we wrote above.

Now, the second equation (X − Y = Z) can be written as Z + Y = X, which is the same form as the first equation. So we need to find two equations of the form Z + Y = X using only numbers from 1, 6, 7, 8, 9, 11 and 13. One of these equations cannot use the number 1, so it must be $6 + 7 = 13$. Then the only possibility for the other equation is $1 + 8 = 9$. So the first two equations must use the numbers 6, 7, 13, 1, 8 and 9 in some order. The missing number must be 11.

3. Let a be the number of X-monkeys and b the number of X-hypercows. Since X-monkeys have two legs, they contribute $2a$ legs to the total. There are $3b$ legs from the X-hypercows, for a total of 87 legs. This gives the equation

$$2a + 3b = 87.$$

Counting heads, we get the equation

$$a + 4b = 86.$$

So we can write $a = 86 - 4b$ and substitute this into the first equation to get

$$2(86 - 4b) + 3b = 87.$$

Then $5b = 85$, so $b = 17$ and then $a = 86 - 4 \times 17 = 18$. So the herd has 18 X-monkeys and 17 X-hypercows.

Another way to proceed is to note that taking one X-monkey and one X-hypercow gives five heads and five legs. Now, $85 = 5 \times 17$, so 17 X-monkeys and 17 X-hypercows would give 85 heads and 85 legs. So you need one more head and two more legs, which is another X-monkey. So there are 18 X-monkeys and 17 X-hypercows.

4. From the information, the size of each slice from the first cut is $1/a$ of the whole pie, the size of each slice from the second cut is $1/ab$ of the whole pie, and the size of each slice from the third cut is $1/abc$ of the whole pie. So we have

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} = \frac{3}{5}.$$

Multiplying both sides by $5abc$, we get the equation

$$5bc + 5c + 5 = 3abc.$$

Now, c is a factor of all the summands except 5, so c must divide into 5. Since 5 is prime and c is bigger than 1, $c = 5$. Using this in the equation, we obtain

$$25b + 25 + 5 = 15ab,$$

so $5b + 6 = 3ab$. This tells us that b must divide into 6. Let us look at the possibilities. Trying $b = 2$ gives $10 + 6 = 6a$, which isn't possible since 6 doesn't divide into 16. Next, try $b = 3$. This gives $15 + 6 = 9a$, which doesn't work since 9 doesn't divide into 21. So $b = 6$. Then $a = 2$. The answer is $a = 2$, $b = 6$ and $c = 5$.

5.

	Total Points
A	6
B	5
C	4
D	3
E	2

	Winner (or Tie)
A vs B	B
A vs C	A
A vs D	A
A vs E	A
B vs C	T
B vs D	T
B vs E	T
C vs D	C
C vs E	T
D vs E	D

Note that if we total the points for each match, we obtain 2, so the point total recorded for the 10 games is 20. Since B was the only team that did not lose a game, A lost at least one game, making its maximum possible score 6. Its score could not be 5 since $5 + 4 + 3 + 2 + 1 = 15 < 20$. Thus, A has three wins and one loss. Since B has no losses, the game A lost must have been to B. Then B must have a tie in all three of its other games (otherwise, it has at least 6 points, at least as many as A). All the teams but E won some games, so both C and D won some game. Neither of them could win both games (excluding those with A and B, about which we already know) because that would give a score of 5, which B got. Thus, C won one of the other games and had one tie, and D won one game and had one loss, which means that E had one loss and one tie in the remaining games. Thus, E lost the game with D and had a tie with C.

6. Note that triangle ABC is equilateral, so the medians of the three sides intersect at the centre O of the triangle. Let D be the midpoint of side AB, so that CD is a median of the triangle. Then triangle ADO is a right triangle. This triangle is similar to triangle CDA, so OD : AD = AD : AC = 1 : 2, and AO = 2OD. So, since AO = CO, you get OD = (1/3)CD. Also,

$$CD = \sqrt{6^2 - 3^2} = 3\sqrt{3}.$$

Then,

$$OA = \frac{2}{3} \times 3\sqrt{3} = 2\sqrt{3}.$$

Let V be the other vertex of the pyramid, and S the centre of the sphere. Then triangle SOA is a right-angled triangle, and if h (= 10) is the height of the pyramid and r the radius of the sphere, we get from Pythagoras' theorem that

$$OA^2 = SA^2 - SO^2.$$

This becomes

$$\left(2\sqrt{3}\right)^2 = r^2 - (h - r)^2$$

and

$$12 = 2rh - h^2 = 20r - 100.$$

Thus, $112 = 20r$, and the radius of the sphere is 5.6 cm.