Alberta High School Mathematics Competition 2012/13

The Alberta High School Mathematics Competition is a two-part competition taking place in November and February of each school year. Book prizes are awarded for Part I, and cash prizes and scholarships for Part II. Presented here are the problems and solutions from the 2012/13 competition.

Part I

November 21, 2012

1. Each day Mr Sod visited pubs A, B, C and D, in that order, always spending \$35, \$12, \$40 and \$27 at the respective places. His total expenditure at the pubs, from the beginning of the month up to a certain moment that month, was \$1,061. Which pub would he be visiting next?

   (a) A (b) B (c) C (d) D (e) impossible total

2. Meeny, Miny and Moe were playing tennis. From the second game on, the one who sat out the preceding game would replace the loser of that game. At the end, Meeny played 17 games and Miny played 35 games. How many games did Moe play?

   (a) 18 (b) 26 (c) 36 (d) 52 (e) not uniquely determined

3. A circle of diameter r is drawn inside a circle of diameter R. For which of the following pairs (r, R) is the area of the smaller circle closest to half the area of the larger circle?

   (a) (1, 3) (b) (2, 4) (c) (3, 5) (d) (4, 6) (e) (5, 7)

4. A quadratic polynomial $f(x)$ satisfies $f(0) = 1$, $f(1) = 0$ and $f(2) = 3$. What is the value of $f(3)$?

   (a) −3 (b) 1 (c) 2 (d) 10 (e) none of these

5. ABCD is a square. E and F are points on the segment BC such that BE = EF = FC = 4 cm. The segments AF and DE intersect at G. What, in cm², is the area of triangle EFG?

   (a) 6 (b) $4\sqrt{3}$ (c) 8 (d) 12 (e) none of these

6. For how many integers $n \geq 2$ is the sum of the first n positive integers a prime number?

   (a) 0 (b) 1 (c) 2 d) 3 (e) more than 3

7. In a test, Karla solved four-fifths of the problems and Klaus solved 35 problems. Half of the problems were solved by both of them. The number of problems solved by neither was a positive one-digit number. What was this number?

   (a) 1 or 2 (b) 3 or 4 (c) 5 or 6 (d) 7 or 8 (e) 9

8. What is the largest possible integer a such that exactly three of the following statements are true: $a < 1$, $a > 2$, $a < 3$, $a > 4$ and $a < 5$?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

9. A rectangle with integer length and width in cm has area 70 cm². Which of the following, in cm, cannot be the length of the perimeter of the rectangle?

   (a) 34 (b) 38 (c) 74 (d) 98 (e) 142

10. The positive integer n is such that between $n^2 + 1$ and $2n^2$ there are exactly five different perfect squares. How many such n can we find?

    (a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

11. ABCD is a rectangle such that AD − AB = 15 cm. PQRS is a square inside ABCD whose sides are parallel to those of the rectangle, with P closest to A and Q closest to B. The total area of APSD and BQRC is 363 cm², while the total area of APQB and CRSD is 1,113 cm². What, in cm², is the area of PQRS?

    (a) 900 (b) 1,600 (c) 2,500 (d) 3,600 (e) not uniquely determined

12. Weifeng writes down 28 consecutive numbers. If both the smallest and the largest numbers are perfect squares, what is the smallest number she writes down?

    (a) 9 (b) 36 (c) 100 (d) 169 (e) not uniquely determined

13. If the positive numbers a and b satisfy

$$\frac{1}{a^2+4b+4}+\frac{1}{b^2+4a+4}=\frac{1}{8},$$

    what is the maximum value of $a + b$?

    (a) 3/2 (b) 2 (c) 5/2 (d) 4 (e) none of these

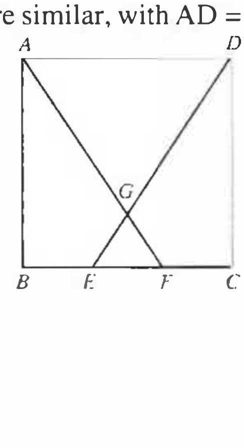
14. The incircle of triangle ABC is tangent to AB and AC at F and E, respectively. If BC = 1, $\angle A = 90^\circ$ and $\angle B \neq \angle C$, what is the distance from the midpoint of BC to EF?
(a) $\sqrt{2}/4$ (b) $\sqrt{2}/2$ (c) $(3\sqrt{2})/4$ (d) $\sqrt{2}$
(e) not uniquely determined

15. At the beginning of the year, there were more robots than androids. On the first day of each month, each robot made seven androids and each android made seven robots. The next day, each old android would pick a fight with a new android, and they would destroy each other. At the end of the year, there were 46,875 million robots and 15,625 million androids. What was the difference between the numbers of robots and androids at the beginning of the year?
(a) less than 10 (b) at least 10 but less than 100 (c) at least 100 but less than 1,000 (d) at least 1,000 but less than 10,000 (e) at least 10,000

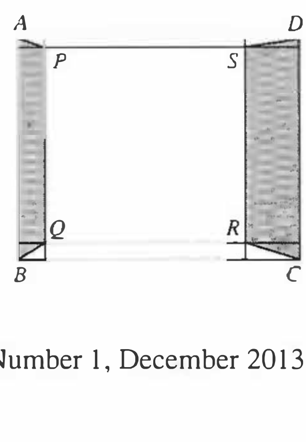
16. Let m and n be positive integers such that 11 divides $m + 13n$ and 13 divides $m + 11n$. What is the minimum value of $m + n$?
(a) 24 (b) 26 (c) 28 (d) 30 (e) 34

Solutions

1. Note that 35 + 12 + 40 + 27 = 114 and 1,061 = 9 × 114 + 35. Thus, Mr Sod had spent \$35 on the 10th day of that month at pub A. The answer is **(b)**.
2. Since Meeny played 17 games, Miny and Moe played each other at most 17 + 1 = 18 times, and each could play at most 18 + 17 = 35 games. As Miny played 35 games, Moe did not play Meeny but played Miny 18 times. The answer is **(a)**.
3. We want $(r/R)^2$ to be close to 1/2. We have $(1/3)^2 < (2/4)^2 < (3/5)^2 < (4/6)^2 = 4/9$ and $(5/7)^2 = 25/49$. Since $25/49 - 1/2 = 1/98 < 1/18 = 1/2 - 4/9$, the answer is **(e)**.
4. Let $f(x) = ax^2 + bx + c$. Then $1 = f(0) = c$, $0 = f(1) = a + b + c$ and $3 = f(2) = 4a + 2b + c$. We have $c = 1$, $a + b = -1$ and $2a + b = 1$. Hence, $a = 2$ and $b = -3$, so that $f(3) = 10$. The answer is **(d)**.
5. Triangles GAD and GFE are similar, with AD = 3EF. Hence, the vertical height of triangle EFG is 1/3 of the vertical height of triangle ADG. Hence, it is equal to (1/4)AB = 3 cm so that the area of triangle EFG is 1/2 × 3 × 4 = 6 cm². The answer is **(a)**.



6. The sum of the first n positive integers is $[n(n + 1)]/2$. Suppose n is even. Then we must have either $n/2 = 1$ or $n + 1 = 1$. Both lead to $n = 2$. Suppose n is odd. Then we must have either $(n + 1)/2 = 1$ or $n = 1$. However, $n = 1$ is not allowed by the hypothesis. The answer is **(b)**.
7. The fraction of problems solved only by Karla was 4/5 − 1/2 = 3/10 so that the total number of problems was a multiple of 10. The fraction of problems solved by Klaus was at most 1 − 3/10 = 7/10. Thus, the total number of problems was at least 50. If it was 50, then 10 problems were solved by Klaus alone, and as Karla solved 4/5 × 50 = 40 problems, the number of problems solved by neither was 0. The total number of problems could only be as large as 70, since 35 problems would be solved by both. In this case, the number of problems solved by neither was 1/5 × 70 = 14. It follows that the total number of problems must be 60, of which 30 were solved by both, 5 by Klaus alone, 3/10 × 60 = 18 problems by Karla alone, and 60 − 30 − 5 − 18 = 7 by neither of them. The answer is **(d)**.
8. Note that $a > 2$ and $a < 3$ cannot both be true as there are no integers between 2 and 3. Similarly, $a > 4$ and $a < 5$ cannot both be true. Since exactly three of the statements are true, $a < 1$ must be true. Hence, the largest possible value is $a = 0$, and for this value, the three statements $a < 1$, $a < 3$ and $a < 5$ are true and the two statements $a > 2$ and $a > 4$ are false. The answer is **(a)**.
9. We have 70 = 1 × 70 = 2 × 35 = 5 × 14 = 7 × 10. Thus, there are four possible shapes of the rectangle, with respective perimeters 142 cm, 74 cm, 38 cm and 34 cm. The answer is **(d)**.
10. Solving $(n + 5)^2 < 2n^2 < (n + 6)^2$ yields $50 < (n - 5)^2$ and $(n - 6)^2 < 72$. Thus, $8 \leq n - 5$ and $n - 6 < 9$ or $13 \leq n \leq 14$. The answer is **(c)**.
11. We shade the regions APQB and CRSD, while leaving the regions APSD and BQRC unshaded. Extend the sides of PQRS to the perimeter of ABCD, creating four rectangles at the corners, each of which consists of two congruent triangles, one shaded and one unshaded. The difference between the total area of the unshaded regions (not counting PQRS) and the total area of the shaded regions is 1,113 − 363 = 750 cm². The difference in the lengths of AD and AB is 15 cm.



Hence, the side length of PQRS is 750 ÷ 15 = 50 cm, and the area of PQRS is 2,500 cm^2. The answer is (**c**).

12. Let the smallest and the largest numbers Weifeng writes down be n^2 and m^2 respectively. Since they are the ends of a block of 28 consecutive numbers, $(m + n)(m - n) = m^2 - n^2 = 27$. We may have $m + n = 27$ and $m - n = 1$, whereby $m = 14$ and $n = 13$. We may have $m + n = 9$ and $m - n = 3$, whereby $m = 6$ and $n = 3$. Thus, the smallest number Weifeng writes down may be $3^2 = 9$ or $13^2 = 169$. The answer is (**e**).

13. We have

$$\frac{1}{8} = \frac{1}{a^2 + 4b + 4} + \frac{1}{b^2 + 4a + 4}$$

$$= \frac{1}{(a-2)^2 + 4a + 4b} + \frac{1}{(b-2) + 4b + 4a}$$

$$\leq \frac{1}{4(a+b)} + \frac{1}{4(a+b)}$$

$$= \frac{1}{2(a+b)}.$$

Hence, $a + b \leq 4$. This maximum value is attained if and only if $a = b = 2$. The answer is (**d**).

14. Let M be the midpoint of BC and D the point where the circle is tangent to BC. Let B', C' and M' be the respective projections of B, C and M on EF. Now AEF is a right isosceles triangle. Hence, so are BB'F and CC'E. Hence, BB' = BF/$\sqrt{2}$ and CC' = CE/$\sqrt{2}$ so that

$$\text{MM}' = \frac{1}{2}(\text{BB}' + \text{CC}')$$

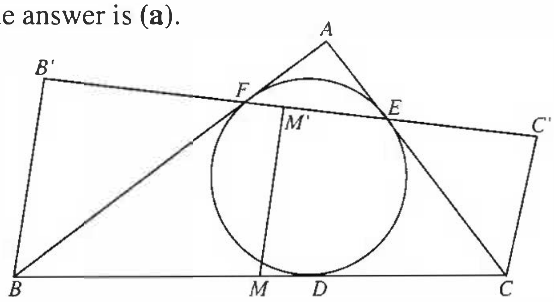
$$= \frac{1}{2\sqrt{2}}(\text{BF} + \text{CE})$$

$$= \frac{1}{2\sqrt{2}}(\text{BD} + \text{CD})$$

$$= \frac{\text{BC}}{2\sqrt{2}}$$

$$= \frac{\sqrt{2}}{4}.$$

The answer is (**a**).



15. Let the numbers of robots and androids be r and a respectively. After one month, these numbers became $r + 7a$ and $7r - a$. After another month, they became $(r + 7a) + 7(7r - a) = 50r$ and $7(r + 7a) - (7r - a) = 50a$. Hence, after a two-month period, the number of robots became 50 times the original number, and the same goes for the number of androids. There being six two-month periods in a year, the initial number of robots was 46,875,000,000 ÷ 50^6 = 3, and the initial number of androids was 15,625,000,000 ÷ 50^6 = 1. The answer is (**a**).

16. Since 13 divides $6(m + 11n) = (6m + n) + 13(5n)$, 13 divides $6m + n$. Since 11 divides $6(m + 13n) = (6m + n) + 11(7n)$, 11 also divides $6m + n$. Hence, $11 \times 13 = 143$ divides $6m + n$, so that $6m + n = 143k$ for some integer k. Since $6(m + n) = 143k + 5n = 6(24k + n) - (k + n)$, 6 divides $k + n$ so that $k + n \geq 6$. Now $6(m + n) = 143k + 5n = 138k + 5(k + n) \geq 138 + 30 = 168$. Consequently, $m + n \geq 28$, and this is attained if $m = 23$ and $n = 5$. The answer is (**c**).

Part II

February 6, 2013

1. Determine all pairs of positive integers (a, b) with $a \leq b$ such that

$$\left(a + \frac{6}{b}\right)\left(b + \frac{6}{a}\right) = 25.$$

2. A set S of positive integers is called *perfect* if any two integers in S have no common divisors greater than 1. Candy wants to build a perfect set of numbers between 1 and 20 inclusive, in such a way that her set contains as many numbers as possible.
(a) How many elements will her set have?
(b) How many different such sets can she build?

3. Randy plots a point A. Then he starts drawing some rays starting at A, so that all the angles he gets are integral multiples of 10°. What is the largest number of rays he can draw so that all the angles at A between the rays are unequal, including all angles between nonadjacent rays?

4. In a convex pentagon of perimeter 10, each diagonal is parallel to one of the sides. Find the sum of the lengths of its diagonals.

5. Find all integers $r > s > t$ and all quadratic polynomials of the form $f(x) = x^2 + bx + c$ such that b and c are integers, $r + t = 2s$, $f(r) = 1$, $f(s) = b$ and $f(t) = c$.

Solutions

1. The given equation may be rewritten as $ab + 36/ab + 12 = 25$. Therefore,

$$(ab)^2 - 13ab + 36 = (ab - 4)(ab - 9) = 0.$$

Hence, $ab = 4$ or $ab = 9$. Note that a and b are positive integers with $a \le b$. If $ab = 4$, we have $(a, b) = (1, 4)$ or $(2, 2)$. If $ab = 9$, we have $(a, b) = (1, 9)$ or $(3,3)$. It is easy to verify that all four are indeed solutions.

2. (a) Candy's perfect set may be {1, 2, 3, 5, 7, 11, 13, 17, 19}. We claim that this number is the highest possible. Now a maximal perfect set must contain the element 1, as otherwise we can add 1 and obtain a larger perfect set. Also, a maximal perfect set cannot contain an element that is divisible by two distinct primes, as otherwise we can replace that element by the two primes and obtain a larger perfect set. Hence, each element other than 1 is a positive power of a prime. Moreover, distinct elements are powers of distinct primes. Since there are only eight primes less than 20—namely, 2, 3, 5, 7, 11, 13, 17 and 19—our claim is justified.

   (b) Every maximal perfect set Candy can build must have the form

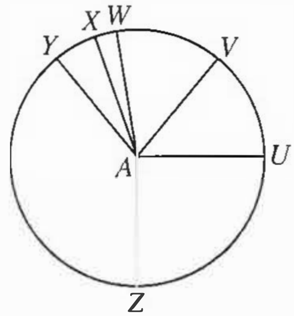
$$S = \{1, 2^{i_2}, 3^{i_3}, 5^{i_5}, 7^{i_7}, 11^{i_{11}}, 13^{i_{13}}, 17^{i_{17}}, 19^{i_{19}}\},$$

   where each exponent is a positive integer. Since $5^2 > 20$, the exponent for all primes greater than or equal to 5 must be 1. Since $2^4 \le 20 \le 2^5$ and $3^2 \le 20 \le 3^3$, the exponent for 2 must be 1, 2, 3 or 4, and the exponent for 3 must be 1 or 2. This yields eight different maximal perfect sets.

3. Let $n \ge 2$ be the number of rays drawn by Randy. Then there are

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

pairs of rays. Each pair determines two angles adding up to 360°. Hence, the total number of angles between two of the n rays is exactly $n(n - 1)$. The measure of such an angle is clearly less than 360°. Since it is supposed to be an integral multiple of 10°, there are at most 35 values for the measures of these angles. Since they are distinct, $n(n - 1) \le 35$. Now, $6 \times 5 = 30 < 35 < 42 = 7 \times 6$. Hence, $n \le 6$. It is possible for Randy to draw six rays, determining 30 distinct angles. In the diagram below, ∠UAV = 60°, ∠VAW = 40°, ∠WAX = 10°, ∠XAY = 20°, ∠YAZ = 140° and ∠ZAU = 90°.



We now verify that the 30 angles between two rays are distinct. We have ∠WAY = 30°, ∠VAX = 50°, ∠VAY = 70°, ∠UAW = 100°, ∠UAX = 110°, ∠UAY = 130°, ∠ZAV = 150°, ∠XAZ = 160° and ∠WAZ = 170°. These are nine different angles distinct from the six between adjacent rays. All have measures less than 180°. Corresponding to these 15 angles, we have 15 other angles greater than 180°, yielding a total of 30 distinct angles.

4. Let L be the point of intersection of EC and DB. Let M be the point on the extension of AB such that MC is parallel to AE. Then ABLE and AMCE are parallelograms. Note that triangles DLC and EAB are similar, as are triangles AMC and ELD. It follows that

$$\frac{EC}{AB} = \frac{EL+LC}{AB} = 1 + \frac{LC}{AB} = 1 + \frac{DL}{EA} = 1 + \frac{DL}{CM} = 1 + \frac{AB}{EC}.$$

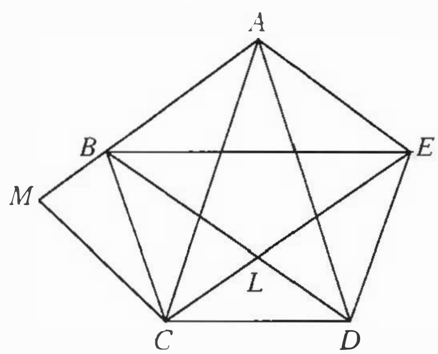
Let $x = EC/AB$. Then, $x = 1 + 1/x$ so that $x^2 - x - 1 = 0$. Hence,

$$x = \frac{1+\sqrt{5}}{2}.$$

Similarly, we have

$$\frac{DB}{AE} = \frac{AC}{ED} = \frac{AD}{BC} = \frac{EB}{DC} = \frac{1+\sqrt{5}}{2},$$

so that $EC + DB + AC + AD + EB = 5(1 + \sqrt{5})$.



Remark: The regular pentagon is used in the illustrative diagram. Many students may get the correct answer by treating only this special case, essentially proving that $\cos 36° = (1 + \sqrt{5})/4$.

5. The conditions are

$$r^2 + br + c = 1, \quad (1)$$
$$s^2 + bs + c = b, \quad (2)$$
$$t^2 + bt + c = c, \quad (3)$$
$$r + t = 2s. \quad (4)$$

From (3), $t(t + b) = 0$ so that either $t = 0$ or $t = -b$. We consider these two cases separately.

CASE 1: $t = 0$. From (4), we have $r = 2s$. Substituting into (1), we have $4s^2 + 2bs + c = 1$. Subtracting (2) from this, we have $3s^2 + bs = 1 - b$, which may be rewritten as $(s + 1)(3s - 3 + b) = -2$. Hence, 2 is divisible by $s + 1$, so that $s = -3, -2, 0$ or 1. However, since $s > t = 0$, we may only have $s = 1$. It follows that $b = -1$. Hence, $f(x) = x^2 - x - 1$, with $r = 2$, $s = 1$ and $t = 0$.

CASE 2: $t = -b$. From (4), we have $r = 2s + b$. Substituting into (1), we have $4s^2 + 6sb + 2b^2 + c = 1$. Subtracting (2) from this, we have $3s^2 + 5sb + 2b^2 = 1 - b$, which may be rewritten as $(3s + 2b + 3)(s + b - 1) = -2$. Hence, -2 is divisible by $s + b - 1$. From $r > s > t = -b$, we have $s + b > 0$. Hence, $s + b - 1 > -1$ so that $s + b - 1 = 1$ or 2. If $s + b - 1 = 1$, we have $3s + 2b + 3 = -2$ so that $s = -9$ and $b = 11$. Hence, $f(x) = x^2 + 11x + 30$ with $r = -7$, $s = -9$ and $t = -11$. If $s + b - 1 = 2$, we have $3s + 2b + 3 = -1$ so that $s = -10$ and $b = 13$. Hence, $f(x) = x^2 + 13x + 43$, with $r = -7$, $s = -10$ and $t = -13$.