# delta-k

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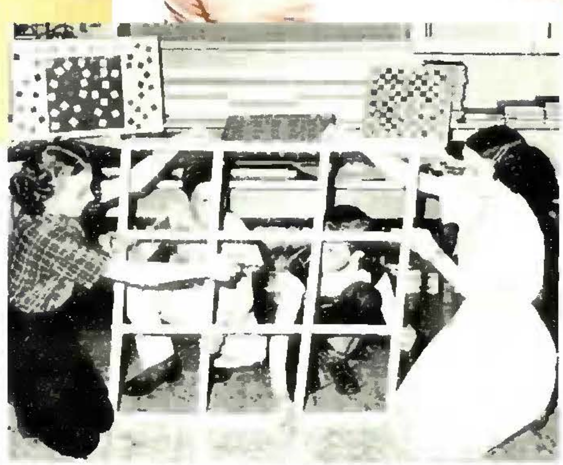

## Special Issue: Celebrating 50 Years of *delta-K*

# Guidelines for Manuscripts

*delta-K* is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

## Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. All manuscripts should be typewritten and properly referenced. All pages should be numbered.
4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author-date system or the American Psychological Association (APA) style manual.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB, T1S 2L4; e-mail gsterenberg@mtroyal.ca.

### MCATA Mission Statement

***Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.***

**Cover photo courtesy of the Edmonton Public School Board archives.**

delta-K

Volume 50, Number 2 March 2013

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**CONCLUSION**

# Down Memory Lane

***Gladys Sterenberg***

The project of working on this special edition, which celebrates 50 years of *delta-K*, has been fascinating and enjoyable. For this issue, I have assembled a collection of articles that have appeared in *delta-K* throughout its history. This issue coincides with the publication of a celebratory book, so many articles included here overlap with those I considered for the book. I used the following criteria in choosing the articles. First, I collected and examined the tables of contents of all issues and looked for repeated themes. I noted curricular changes and related teaching and research ideas. Second, I looked for articles that represented a range of grades. Third, I focused primarily on authors from Alberta.

As a brief introduction to these articles, it is important to note the themes that were evident to me. Much of the 1960s focused on modern math. Problem solving was introduced and geometry found a renewed place in the curriculum. Research, especially linked to the University of Alberta, formed much of the content. Technology use focused on the overhead projector, and computer programming was introduced. In the 1970s, we saw a great deal of experimentation in both curriculum and pedagogical approaches. Lab approaches, math options, models for teaching with manipulatives and the use of calculators emerged alongside the introduction of the metric system. A focus on problem solving defined the 1980s. Math applications, student projects and geometrical investigations through Logo programming were prominent. An attention to gender issues and a shift to constructivist learning theories took place. And … this is the decade that brought us achievement testing! Articles published in *delta-K* in the 1990s had a huge scope and variety. Topics included the use of manipulatives, math anxiety, communication, estimation, the use of calculators, gifted education, fractals, assessment, professional development, conics, data management, multiple intelligences and multiculturalism. In this decade, we became preoccupied with math processes, the Western Canadian Protocol and the National Council of Teachers of Mathematics Curriculum Standards. This variety of topics continued into the 2000s as we became interested in graphing calculators, listening to children's reasoning and formative assessment.

Throughout my study of the articles published in the past 50 years, I have been impressed with the dedication of teachers, math educators and mathematicians involved in the Math Council of the Alberta Teachers' Association. Articles written for *delta-K* have reflected the viewpoints of many people interested in mathematics education in Alberta.

As we mark this milestone, I wish to convey my deep gratitude to previous editors of *delta-K* who made strong contributions to our history. In his editorial "Thirty-Four Years and Counting," the late Art Jorgensen (1995) wrote

> As we look forward to the next 30 years, what will MCATA's future be? Will it grow and prosper and continue to be a voice to be heard, or will it wither and die? It is really up to the members. Today's executive members, like those of the past years, will do their best. Then they will move on and pass the torch to you. Personally, I have a great deal of confidence in our members, and believe that MCATA has a bright future. Let's make it so. (p 4)

I believe he would be proud of the vibrancy of our community.

The last word in this issue of *delta-K* is given to Werner Liedtke, a teacher and math educator who has supported *delta-K* since its early days. Both Art and Werner have been inspirations to me as I continue the editorial work of this journal. I hope you enjoy this journey down memory lane.

# Overview of Change—Or a Look at the Forest Before We Can't See It for the Trees

*E A Krider*

*Note: This article first appeared in the* Mathematics Council Newsletter *volume 2, number 2, pages 1–5 (1960). This publication was renamed* delta-K *in 1971. Minor changes have been made in accordance with current ATA style.*

The development of the mathematics curriculum in North America has been closely associated with the changing views of transfer of learning. In the half century before 1900, the theory of mental discipline held sway and it was accepted that transfer took place more or less automatically. Mathematics was of the sequential type and, generally, all high school students were required to take it without regard to what practical use it might be put.[2] At this time it was usual for subject-matter specialists to determine the content of the mathematics curriculum.

In the first decades of the twentieth century, we see a reaction against overemphasis on factual knowledge and also the theory of mental discipline being discredited. The emphasis on specific transfer as opposed to general transfer, the ascendency of pragmatic philosophy, the stimulus–response psychology, and the increased proportion of the population in our secondary schools all led to more emphasis being put on skills and specific information in mathematics. In the twenties and thirties, the stress was on social adjustment and training for democracy—"preparing the well-informed citizen."[3]

As the first half of the century comes to a close, we see the gap between the subject-matter specialist and the educationalist at its widest and the scholars at the forefront of knowledge starting to demand a voice in designing school curricula. Another facet of the development of mathematics that deserves mentioning is the emphasis in the forties on classes for the less gifted and in the fifties on classes for the gifted.[3]

Finally we come to the big turning point in the development of the mathematics curriculum in the mid-fifties. Here we see the results of the reaction to the extremes of progressive education, the stimulus–response psychology, the overemphasis on skills and specific information, the overemphasis on the social and the utilitarian aspect of education, the extreme negative views on transfer. The following quotation illustrates the changing view on transfer:

> Virtually all the evidence of the last two decades on the nature of learning and transfer has indicated that, while the original theory of formal discipline was poorly stated in terms of the training of the faculties, it is a fact that massive general transfer can be achieved by appropriate learning.[1, p 6]

The changes in ideas of transfer and the Gestalt psychology gave the reformers the psychological grounds for their movement. Bruner says

> What may be emerging as a mark of our generation is a widespread renewal of concern for the quality and intellectual aims of education—but without the abandonment of the ideal that education should serve as a means of training well-balanced citizens for democracy.[1, p 1]

With this movement we see the subject-matter specialist moving back into the picture.

> Curriculum programs such as SMSG and UICSM sprang from the dissatisfaction of the subject specialists with the preparation being given for their discipline in the schools.[4, pp 187–92]

Although this turning point seems to have taken place suddenly about 1954, the proponents of the need for radical change in emphasis were actively campaigning long before this. Professor Cecil B Read, of Wichita University, lists quotations all taken from articles written between 1917 and 1932, registering the same complaints as voiced by the "revolutionists" of the fifties.[7, pp 181–86] Why did these people suddenly become the authorities in the field of curriculum building? First, the gap between what was taught in schools and what was known in the field became acute because of the explosion of knowledge. Second, the shortage of scientists and mathematicians came to the public's attention with the first Sputnik. With the millions of dollars poured into the cause by the American government, the reformers were away. A number of professional groups attacking the problem of producing a new mathematics curriculum were set up. The three most influential groups are

- the Commission on Mathematics of the College Entrance Examination Board (usually referred to as the Commission on Mathematics),
- the University of Illinois Committee of School Mathematics, headed by Professor Max Berberman (abbreviated UICSM) and
- the School Mathematics Study Group headed by Professor Edward G Gegle at Stanford (abbreviated SMSG).

These groups are made up of professional mathematicians, professional educators, psychologists and, usually, practising teachers. It is hard to overemphasize the impact that these three groups have made not only on mathematics curriculum but in the whole spectra of the school curriculum building.[1, p 70] One cannot discuss recent mathematics curriculum change without referring to these groups.

In conclusion, and at the risk of oversimplification, one might infer from this brief survey that the development of the mathematics curriculum in North America since the turn of the century has been a series of actions and reactions. If one is to extrapolate from this, we would expect a reaction to the modern approach to curriculum building as exemplified by Bruner, and the workers in the specific subject matter fields, to be discernible. In the case of mathematics this reaction is not only discernible, but is well established with a substantial following. C Stanley Ogilvy, Hamilton College, Clinton, New York, writes

> After 20 years of propaganda in favor of the introduction of new mathematics, we can now discuss the beginning of a swing in the other direction. In almost every new issue of the *Mathematics Teacher* and the *American Mathematics Monthly* we find one or two articles cautioning us to move ahead slowly, to guard against discarding good and valuable material merely, to make room for something new for the take.[6]

And, from a statement signed by 64 mathematicians in the United States and Canada,

> Mathematicians, reacting to the dominance of education by professional educators who may have stressed pedagogy at the expense of content, may now stress content at the expense of pedagogy and be equally ineffective. Mathematicians may unconsciously assume that all young people should like what present day mathematicians like or that the only students worth cultivating are those who might become professional mathematicians.[5]

Could there be a little bit of truth in the statement that, in education, if you're old-fashioned long enough, you'll be modern!

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(3) Harris, Chester W. "Mathematics." *Encyclopedia of Educational Research*, Brett-MacMillan, Ltd: 1960.

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(5) *Ontario Mathematics Gazette*, Bulletin 1, No 2 (October 1962) p S, "On the Mathematics Curriculum of the High School."

(6) Ogilvy, Stanley C, "Second Thoughts on Modernizing the Curriculum." *The Mathematics Teacher*: November 1960.

(7) Read, Cecil B, "What's Wrong with Mathematics." *School Science and Mathematics* Vol LVIII, No 509: March 1958.


*Editor's note [original publication date]: Mr Krider is a former principal at Oyen, Alberta. During the past year he has been a teaching assistant in mathematics education while working toward his master of education degree.*

# Discovery or Programming

*William F Coulson*

*Note: This article first appeared in the* Mathematics Council Newsletter *volume 4, number 1, pages 1–3 (1960). This publication was renamed* delta-K *in 1971. Minor changes have been made in accordance with current ATA style.*

Is it possible that one sees two different trends in mathematics education as he surveys current publications from afar? On the one hand we have the experimental developments by such groups as the University of Illinois Committee on School Mathematics, the School Mathematics Study Group, the Ball State Project and the Madison Project. Upon closer inspection, one finds the discovery approach built into the material. Many present-day authors are trying to imitate their style and their approach to the material.

Each of the groups mentioned previously and most of the authors currently producing material for mathematics courses recognize and make use of pupil discovery of relationships and structure. Mathematics is recognized as a subject area that has a structure that is meaningful to the student. Thus, the students are able to discover relationships and to build one concept upon another. Mathematics does not become a series of isolated bits of fact and computations to be mastered until they become habit.

Taking the teacher and the teaching method into consideration, this approach is designed to give the student an opportunity to think for himself. The teacher must know a great deal about the material. He must know where he has been and where he is going and must know how to guide the pupils to develop the structure for themselves. Questioning techniques take on greater importance. Rules must not be presented to the student to be applied blindly to a multitude of simple examples until the mechanics become habit and these rules can be recalled upon the receipt of the proper stimulus.

It has been argued that the statement of the generalization by the student is not necessary. Some even feel that this is a hindrance. The spoken or written language gets in the way of the mathematical ideas. When a student is able to apply the generalization in an unfamiliar situation, then he knows what it tells him.

Others argue that it is necessary for the student to express the generalization. Only then can one be certain that the concept is known and understood.

On the other hand we have the type of curriculum exemplified by programmed materials. This development is not new but in recent years it has gained impetus, especially in mathematics.

One of the most significant points involved in programming is that the student is led down a very definite path. The material to be mastered is presented in very small steps to ensure understanding and correctness. The student is able to proceed very slowly and along a path determined by the author of the program. At no point is he permitted to meditate upon a related topic. His attention is always directed toward the mastery of one specific concept.

If one sticks to the traditional definition of mathematics, one thinks of it more as a tool subject. Is this all that mathematics is in this day and age? Many very prominent people do not accept this view alone. Mathematics has become more than mere subject matter to be mastered because it is useful in some other field of endeavour. It is thought of more as a way of thinking, as an academic discipline to be studied for itself. A great many mathematicians study mathematics just for the sake of the mathematics involved and not because of its utilitarian value.

Looking for a moment at the mathematics curriculum or at mathematics education, can we note any relationship in the trends? How do they appear to be affecting the curricula in mathematics? What is the effect of each of them on the teaching of mathematics?

It would appear that the two ideas are not very closely related. They would seem to be worlds apart. The discovery approach adopted by the UICSM and the SMSG would seem to give the pupil credit for being able to think for himself, for being able to recognize meanings, for being able to direct his attention toward a series of related learning tasks.

The approach adopted by those who advocate programmed instruction would appear to deny the ability of the pupil to do independent thinking. A stimulus is presented to the student to which he must make one and only one response. Since this response is right 90 per cent or more of the time, he has little or no opportunity to analyze. His attention is directed toward a rather narrow, limited topic.

The effect on the curriculum in the one instance seems to be a freeing one. Pupils are given an opportunity to act as mature mathematicians. Observations are made. These are accepted or rejected by proof. New observations or relationships are introduced, not necessarily by the teacher or text. These, too, are accepted or rejected by proof from what has gone before. The pupil is an active participant in the development of mathematical concepts.

Programmed instruction tends to do the opposite, as far as the pupil is concerned. Pupils are not given an opportunity to make independent observations. They have little chance to analyze so that they might accept or reject a relationship. The opportunity to act as a mathematician is absent.

As far as the teaching of mathematics is concerned, one of these trends would permit the individual to develop as a skilled craftsman. The teacher would have a vast storehouse of knowledge which he would need to rely upon to keep the class moving in a correct fashion. For example, if a student wanted to solve quadratic equations, the teacher would know immediately whether or not this could be done with the knowledge possessed by the student. The teacher could then guide the student through the discovery of the various processes of solving this particular type of problem. Knowledge of his subject, then, is very important to the teacher who wants to follow the discovery approach used by the SMSG or the UICSM.

Programmed instruction would seem to leave very little for the teacher to do. When a student is unable to understand a specific point, the teacher could assist the student in mastering this concept. One other aspect of programming comes into play when a teacher builds a program of his own. During his labours, he becomes more intimately acquainted with the particular topic, with some of the problems involved in learning this topic, and with some of the problems involved in teaching this topic.

In this paper a brief look at two apparently divergent trends in mathematics education was attempted. Each teacher of mathematics must look more closely at each of these trends to see how they will or will not affect his teaching. It seems obvious that no teacher will remain untouched by these trends. Many people are advocating one or the other of these two approaches, people who are recognized as authorities in mathematics education. Perhaps it will be best for each teacher to conduct a little action research in his own classroom to help him decide. There can be no fence straddlers.

# Euclid Must Go!

*Marshall P Bye*

*Note: This article first appeared in the* Mathematics Council Newsletter *volume 9, number 2, pages 2–12 (1970). This publication was renamed* delta-K *in 1971. Minor changes have been made in accordance with current ATA style.*

Euclid must go! Surely anyone who utters such sentiments must be sacrilegious. Yet these are the words of the outstanding mathematician Professor J Dieudonné in his address to the Organization of European Economic Council in France, in 1959.[1] Why did he make the statement? Perhaps I can bring some light to this.

We read so much today about what should and should not be included in the school curriculum that, I am sure, we all wonder just what mathematics will become in another decade. One such indicator is the *Report of the Commission on Mathematics* of the College Entrance Examination Board, published in 1959.[2] The very bold programs set forth in that document (very bold for its day) are being realized in varying degrees around the world today—just 10 years later. A number of topics and concepts listed have yet to be included in the Alberta curriculum, but we are surprisingly close to the programs outlined.

The report of the Cambridge Conference on School Mathematics—*Goals for School Mathematics*,[3] published in 1963—listed a still more startling set of objectives for mathematics from K through 12. This document might well be the preview of the next decade.

Why do I mention these two reports? I do because in both reports strong reference is made to transformations. In the Commission report,[4] some time is spent on the three primary weaknesses of the so-called Euclidean geometry as it has been presented for so many years. I shall be concerned primarily with only one of the weaknesses. Professor Dieudonné had these weaknesses in mind when he made the statement "Euclid must go!"

There has been a pronounced trend away from "traditional" geometry in countries outside of North America. A number of the British programs—to name three, the School Mathematics Project, the Nuffield Project and the Scottish Mathematics group—emphasize transformation.

Belgium, perhaps as a result of Papy's work, leans heavily on transformations from elementary school up. Here in Canada, Professor Dienes of Sherbrooke, Quebec, has made transformations an integral part of his program. Del Grande and Egsgard of Toronto have come out with high school texts integrating transformations into the program.[5] The Secondary School Mathematics Curriculum Improvement Study (SSMCIS),[6] produced by Teachers College, Columbia University—two of the authors are Dr Julius Hlavaty and Professor Ray Cleveland—has utilized transformations in algebra and geometry. The NCTM publication *Geometry in the Secondary School* (1967)[7] devotes nearly half of its space to discussions about transformations of one type or another and hardly mentions traditional Euclidean geometry as it has been taught for years.

## Transformations

My objective will be to show quickly and easily how transformations may be used in high school geometry and, at the same time, not get involved with "motion" of a geometric figure or set of points. (At times, I shall call upon your intuition as to the motion of a figure.) I shall not be rigorous in such a brief presentation. I shall also make statements that, in a more formal presentation, would need more firm and rigorous attention.

### Definition of Transformation

A transformation is a one-to-one mapping. Since we will be talking about plane geometry, I will say that a transformation is a one-to-one mapping in which the domain and range are the set of points of a plane.

Let us now look at a particular set of transformations—the set known as *isometries*.

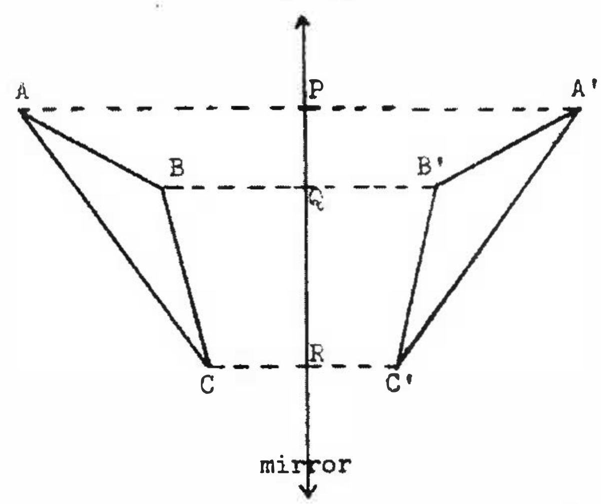
### Definition of Isometry

An isometry is a distance-preserving function. Any figure transformed under an isometry is said to be invariant; that is, a figure is its own image under an isometry. Another way to say this, and perhaps crucial to this discussion, is that a figure transformed under an isometry is congruent to its image.

## Reflection (in a Line)

Consider a triangle reflected in a mirror.

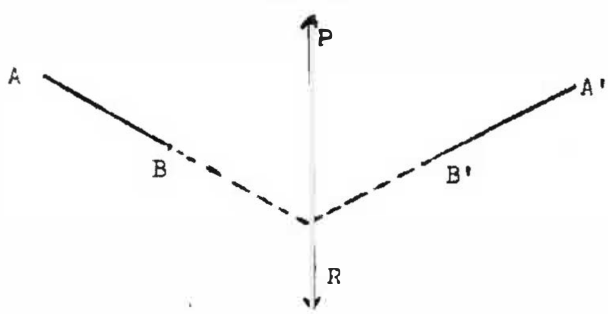
Figure 1




This is the intuitive concept of a reflection. Now let me draw to your attention some of the pertinent details.

1. Every point of the figure ABC is associated with one—and only one—point in its image figure A'B'C'.
2. Points: A→A'; B→B'; C→C'; P→P; Q→Q; R→R.
3. Notice that the points in the mirror line are invariant: each maps on to itself.
4. Segments: $\overline{AB}\rightarrow\overline{A'B'}$; $\overline{AC}\rightarrow\overline{A'C'}$; $\overline{BC}\rightarrow\overline{B'C'}$; $\overline{AP}\rightarrow\overline{PA'}$; $\overline{BQ}\rightarrow\overline{QB'}$; $\overline{CR}\rightarrow\overline{RC'}$
5. Notice that the mirror line is invariant. $\overleftrightarrow{PR}\rightarrow\overleftrightarrow{PR}$
6. Consider the angles formed by $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ with $\overleftrightarrow{PR}$. The angles are congruent.

Figure 2




7. The perpendicular distance between a point and the mirror is congruent to the perpendicular distance between the image point and the mirror or, stated differently, the axis of reflection is the perpendicular bisector of the segment joining a point and its image.
8. $\Delta$ ABC $\cong$ $\Delta$ A'B'C'
9. The sense of $\Delta$ ABC is opposite to that of $\Delta$ A'B'C'. The order of vertices of the object triangle listed clockwise is A-B-C, whereas the order of vertices of the image triangle, clockwise, is A'-C'-B'.

Let us now look at a double reflection—a reflection of a reflection. In the first, the two axes of reflection are parallel (Figure 3), whereas in the second (Figure 4), the two axes are not parallel. Notice that we have one transformation followed by another. This is called *composition of transformations*.

Figure 3

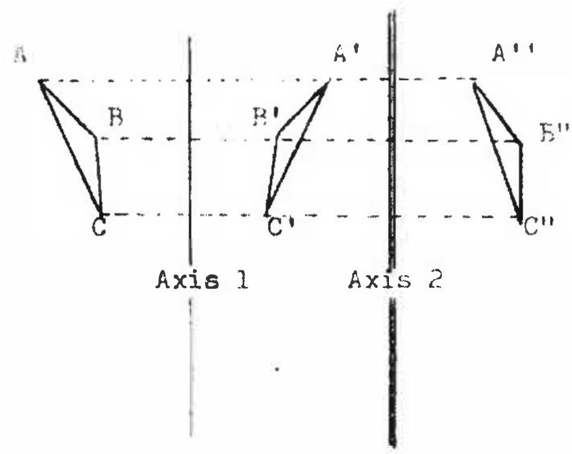
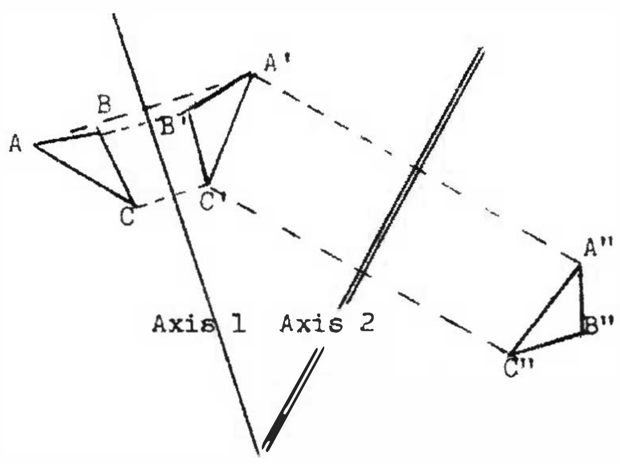



Figure 4

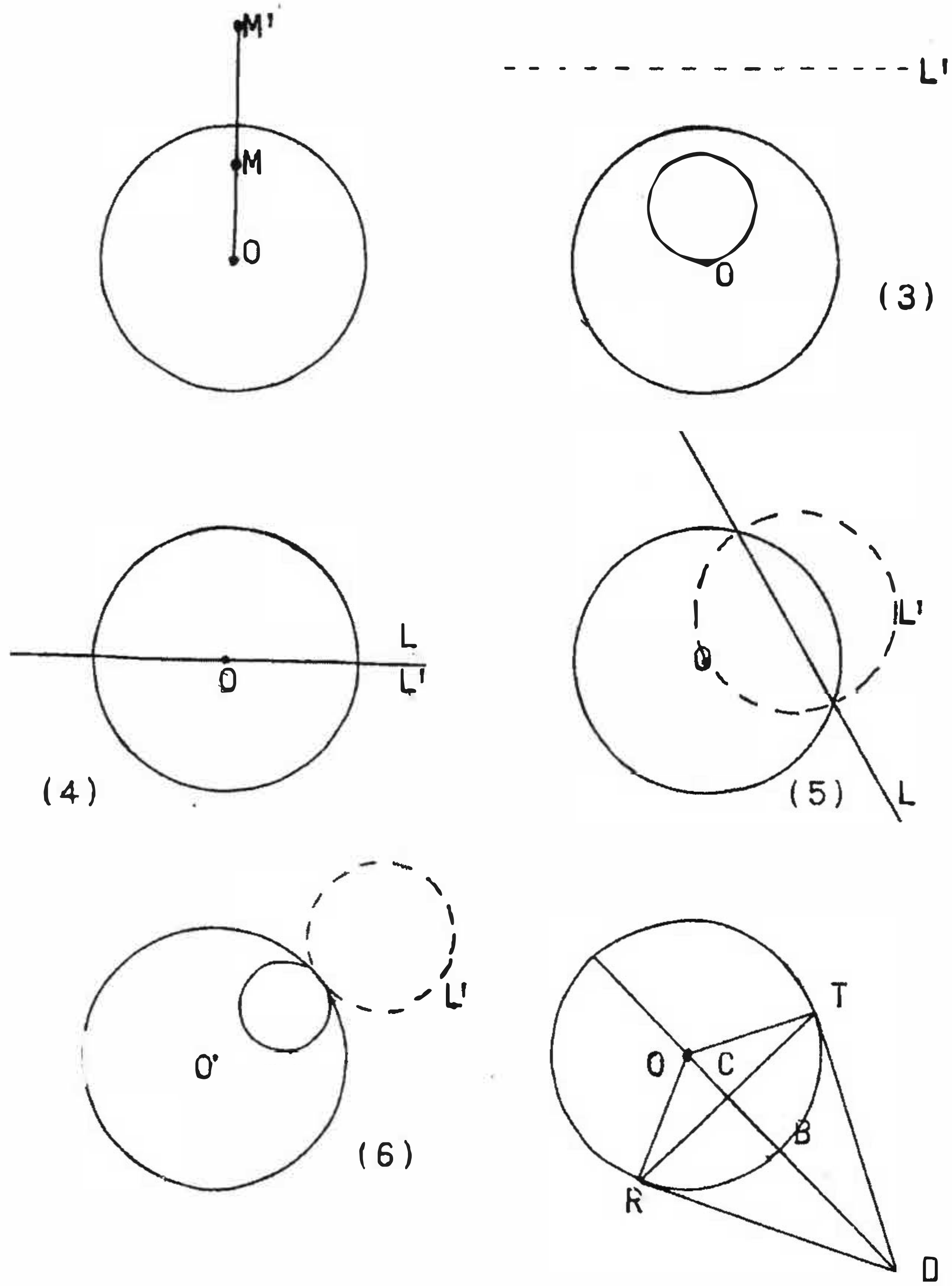



These two illustrations may lead us intuitively to accept the statement that given any two congruent triangles in a plane, there is a series of reflections *such that one triangle is mapped onto the other*. It is an interesting exercise to determine the maximum number of axes of reflection necessary to transform any given triangle into a specific congruent triangle and where those axes of reflection are.

At this time I wish to emphasize that we are not moving triangles or lines or points. When you look in a mirror and see your eyes, you do not, for a moment, have the notion that your eyes have moved behind the mirror. As one author states, in terms of a bowling lane, "We are setting up pins in another alley." As for motion in a plane to explain congruency, there is no motion that would permit you to move Δ ABC to coincide with Δ A'B'C' (Figure 1). The motion would have to come out of the plane.

To make my point clear, let me digress for a moment to a transformation that is not an isometry. Consider the inversive transformation. For this transformation, consider a circle in a plane with centre O and fixed radius r. Any point M is mapped into M' such that $m(\overline{OM}) \cdot m(\overline{OM'}) = r^2$. Refer to Figure 5.

**Figure 5**



This transformation results in some strange things:

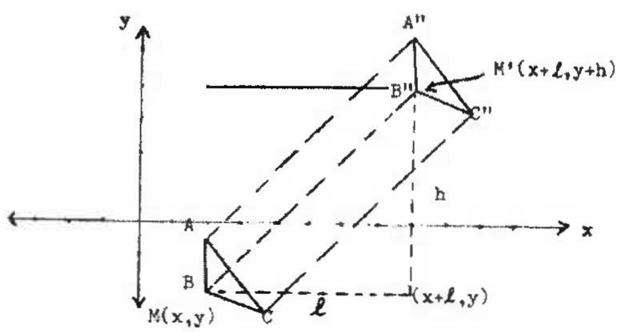
1. Every point in the interior of the circle is mapped into a point in the exterior of the circle and, conversely, every point in the exterior is mapped into the interior.
2. Every point on the circle remains fixed (mapped into itself).
3. Every circle in the interior of the circle that passes through the centre is mapped into a straight line.
4. Every line that contains the centre O of the circle is mapped into itself.
5. Every line that does not contain the centre O of the circle is mapped into a circle.
6. Every circle not containing the centre O of the circle is mapped into another circle.

Clearly, we have not "moved" figures—we have not preserved shape or size.

However, let us return to isometries. While we can use reflections to establish our mappings of one figure into congruent figures, other transformations may be used as well. We shall only spend time with two others in this paper.

Refer to Figure 6. Notice that we can think of A→A", B→B" and C→C". If we place this figure on the coordinate plane, it is easy to think of this transformation as mapping any point M(x,y) into M'(x+l, y+h).

**Figure 6**




Intuitively, a translation can be thought of as the transformation of the set of points, taken in order, through a certain fixed distance in some direction.

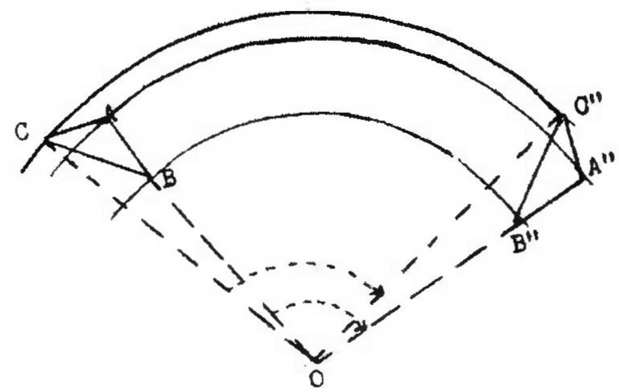
Review some of the properties of this invariant transformation:

1. Corresponding sides are parallel and congruent.
2. Corresponding angles are congruent.
3. The sense of the figure is preserved.

## Rotation

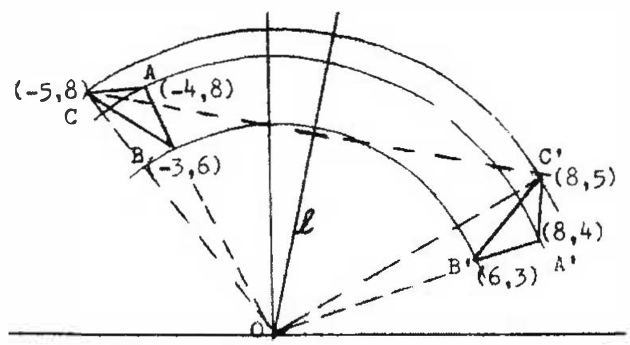
The third and last transformation discussed by me in this paper is illustrated in Figure 7. I have reproduced it here to show the mapping of Δ ABC onto Δ A"B"C". You can visualize the mapping: A→A", B→B" and C→C". As you know, this is a rotation.

**Figure 7**




In the next figure, we clearly see the mapping on the Cartesian plane.

**Figure 8**




The point of rotation is the origin. The angle of rotation is the measure of ∠AOA'.

Points to observe in this isometry:

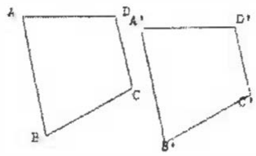
- ∠AOA' ≅ ∠BOB' ≅ ∠COC'
- The perpendicular bisector of the segment determined by two corresponding points contains the point of rotation O ($\ell$ bisects $\overline{CC'}$).
- The said perpendicular bisector of the segment $\overline{CC'}$ bisects ∠COC'.
- Sense is preserved.
- The point of rotation is the only point in the plane that is invariant.
- The image is congruent to the object.

Another isometry is the glide-reflection. It is a combination of the translation followed by a reflection. Some books use the glide-rotation. These are simply compositions of other isometries.

## Properties of an Isometry

At this point we will sum up briefly and state our understandings in the form in which we will be using them.

1. If there is an isometry or isometries which transform one geometric figure into another, the two figures are congruent.
2. Suppose in polygonsABCD and A'B'C'D' the mapping is an isometry and suppose A→A', B→B', C→C', D→D'.
   a. Distance is preserved:
   m(AB) = m(A'B')
   m(BC) = m(B'C')
   m(AC) = m (A'C')
   etc.



   b. Measure of each angle is preserved: m ∠ABC = m ∠A'B'C', etc.
   c. Straightness is preserved—that is, lines map into lines.
   d. Parallel lines map into parallel lines. If AB ∥ CD, then A'B' ∥ C'D'. (Hence, perpendicularity is preserved.)

Now we arrive at the main point of the discussion. I have gone neither into any detail on the method of presentation nor into interesting side trips. I have only laid the foundation for that which I want to present at this time.

## Geometric Proofs Using Isometries

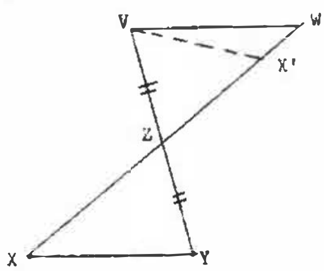
Definition: two figures are said to be *congruent* if there is an isometry (or a composition of isometries) that maps one of the figures onto the other. Let us look at specific instances.

**Example 1**

Figure XYZVW

Segments $\overline{XW}$ and $\overline{VY}$ intersect such that XY is parallel to $\overline{VW}$ and $\overline{YZ} \cong \overline{VZ}$.

Prove: $\overline{XZ} \cong \overline{WZ}$




Proof: Consider the 180° rotation of Δ XYZ about Z.

Thus Z→Z
Since $\overline{YZ} \cong \overline{VZ}$, Y→V
Let X→X'
Since X lies on $\overleftrightarrow{ZW}$, X' lies on $\overleftrightarrow{ZW}$

Now in a rotation of 180° a line not through Z maps onto a parallel line (property of rotations).

∵ $\overline{XY} \to \overline{VX'}$
∴ XY ∥ VX' and $\overline{XY} \cong \overline{VX'}$

But ∴ $\overline{XY} \parallel \overline{VW}$
$\overline{VX'} \parallel \overline{VW}$

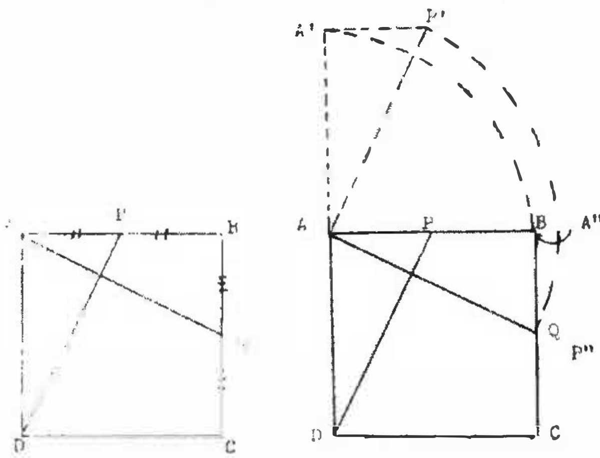
Two parallel segments with one common point must lie in the same line (Euclid—we have not banished him completely. Saccheri does not dethrone Euclid here!).

X' lies in $\overleftrightarrow{VW}$
But X' lies on line $\overleftrightarrow{ZW}$
W is the only point common to the two lines $\overleftrightarrow{ZW}$ and $\overleftrightarrow{VW}$
∴ X' = W
∴ $\overline{XZ} \to \overline{WZ}$
∴ $\overline{XZ} \cong \overline{WZ}$

Let us look at another example.

**Example 2**

Consider the square ABCD. P and Q are midpoints of $\overline{AB}$ and $\overline{BC}$ respectively. Prove $\overline{PD} \perp \overline{AQ}$



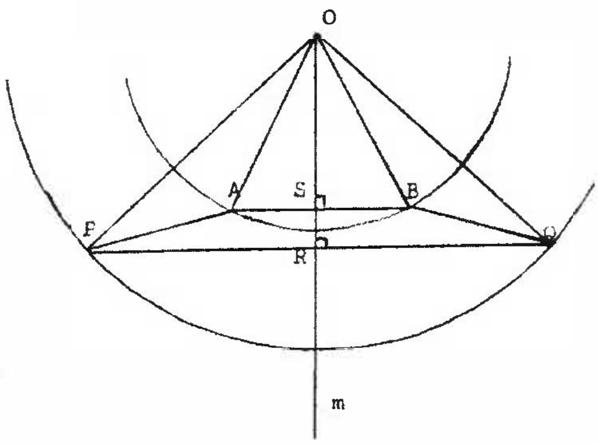
We will not set up the detailed proof, but I will work through the general approach. First, by translation, we transform Δ DAP along the $\overleftrightarrow{DA}$ the distance equal to the measure of $\overline{DA}$. Thus we get the Δ AA'P'. Now we rotate Δ AA'P' about point A, through an angle of rotation of $\pi/2$. We can then show that Δ AA'P' has been mapped onto Δ ABQ. Hence, AD' (which is parallel to $\overline{DP}$) maps onto $\overline{AQ}$ by a rotation of $\pi/2$. Hence $\overline{AQ} \perp$ DP.

Here is a final example to illustrate the use of a reflection.

**Example 3**

Given: In the figure, $\overline{AP}$ and $\overline{PQ}$ are parallel chords of two circles with a common centre O.

Prove: (a) ∠ AOP ≅ ∠ BOQ
(b) $\overline{AP} \cong \overline{BQ}$




Proof: Consider a reflection of Δ PAO in a line M through O perpendicular to $\overline{PQ}$. From previous work we know line M bisects $\overline{PQ}$. (Can prove, but will accept now.)

O→O
S→S
R→R
$\because \overline{PR} \to \overline{QR}$ and $\overline{PR} \perp \overline{OR}$ (previously proven)
$\therefore$ P→Q

Similarly, we can show A→B.

$\therefore \angle AOP \to \angle BOQ \quad \therefore \angle AOP \cong \angle BOQ$
$\therefore \overline{PA} \to \overline{QB}: \quad \therefore \overline{PA} \cong \overline{QB}$.

I have illustrated the use of transformations in proofs from plane geometry. Transformations also may be used to let students discover construction techniques and in turn make the work on constructions far more meaningful and a far more powerful unit in geometry. Time will not permit a discussion of this area.

## A Few Concluding Remarks

I have restricted my discussion to plane geometry. There is no need for this restriction; transformations allow an easy transition into 3-dimension or even n-dimension. This is an advantage of transformations.

Transformations can also be used to solve quadratic equations of the form $ax^2 + by^2 + 2yx + 2fy + c = 0$ by simply transforming them to the form $ax^2 + by^2 = c$. This is another advantage of transformations.

Finally, transformations provide ample opportunity to show that Euclidean geometry is one particular element of the set of geometries in which a certain set of properties are invariant. Whenever we change the set of invariant properties we have a new geometry. Within the scope of transformations lies a host of geometries of such a simple nature that students at an early age can develop, if given an opportunity, an intuitive understanding of them.

Professor Dieudonné viewed the broader field of mathematics that is possible through the new freedom provided by a break from Euclid. He does not advocate throwing out all of Euclid, but rather stresses that for young students there is a richness in geometry possible when parts of Euclid are set aside.

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# A Constructivist Approach to Teaching Mathematics

*Sol E Sigurdson*

*Note: This article first appeared in* delta-K *volume 26, number 1, pages 8–13 (1980). Minor changes have been made in accordance with current ATA style.*

Over the last few years, psychologists and educators have been interested in going beyond behavioristic and Piagetian views to new conceptualizations of learning, especially in using the computer as a model of how we think and learn. One of the new conceptualizations has been *information processing*. Proponents of this view claim that when we think, we basically process information; it's as simple as that. This, however, leads to a further question: How do we manage this processing? To account for the management of processing, it is suggested that the learner engages other processes called *metacognitive processes*. But, still we might ask, What manages the metacognitive processes? Although this is not a trivial question, most proponents presently do not differentiate between levels of management, simply naming all those processes above the cognitive level metacognitive processes. In fact, the difference between cognitive and metacognitive is not always clear. For the time being, let us say that strictly mathematics propositions, procedures and processes are called *cognitive*, while management decisions about such matters as when to use them, in what order and with what degree of confidence are called *metacognitive* processes.

Another related view of learning has been called a theory of *personal constructs*. The main tenet of this view is that all learners actively construct theories, no matter how minor, about what is appropriate action for responding to any particular situation. If a particular theory leads to inappropriate action, we revise the theory. This view, like information processing, also utilizes the notion of metacognitive processes managing our theory development. According to the personal-constructs view, learners of differing capabilities exist because both our cognitive capacities and our metacognitive (management) capabilities differ. Another explanation, which goes beyond differences in cognitive or metacognitive components, is that some learners' perceptions are blinded (by emotion, say), so that they are unable to differentiate between appropriate and inappropriate action and, consequently, construct poor theories.

What relevance do these new conceptions have to the mathematics classroom? The one outstanding impression that the personal-constructs view leaves is that our classrooms consist of 25 or so finely tuned, sensitive, self-initiating, theory-generating, learning "beings." The metacognitive aspect, on the other hand, leads us to question how much of a commitment we teachers have in attending to the development of metacognitive processes. The information processing aspect begs the question of how to present information for efficient storage and easy access. Psychologists and educators are still exploring answers to these questions and will be for many years. In the meantime, what aspect of these theories can be useful to teachers in dealing with the complex world of classroom instruction?

In order to make these ideas more available for teacher use, I will combine the three notions—information processing, personal constructs, and metacognitive processes—into one *constructivist* view of learning. In this article, I will describe constructivist principles of learning and further derive from them constructivist guidelines for classroom teaching of mathematics. Mathematics teachers are encouraged to think about, and use, these ideas to improve their classroom instruction. Psychologists and educators, who are continually striving for new insight into the learning process, would surely appreciate feedback from the most significant learning laboratory of all—the classroom. Curriculum examples will not be used to describe this view because these new conceptions of learning are equally relevant to all grade levels. The word *constructivist* has been around for many years. I am not concerned that my usage may be slightly different than that of others.

## Constructivist Principles of Learning

### 1. Purposeful Constructions

Students construct their own theories for responding to a given situation, and as they see their knowledge leading them to inappropriate action, they revise their theories. Learning proceeds from the current conceptions or theories of knowledge that the learner possesses. *Tuning*, that is, modifying or adjusting, is an important learning process. Appropriate theories are best constructed in the light of some acknowledged purpose.

### 2. Learning How to Learn

Learners' awareness of their knowledge (mathematical content and processes, and metacognitive processes) at any time aids learning. Metacognitive processes (management of cognitive knowledge) are especially important, and these may be a major source of individual differences between slow learners and others.

### 3. Confidence

Because learning means taking risks and experimenting with new cognitive constructions, the atmosphere for learning must be familiar and full of trust. Inaccurate perceptions can be caused by either strong positive or negative emotions.

### 4. Framework for Information

Learning occurs in a context that provides a framework for the organization of information. The most appropriate context is one which is most applicable to the future situation in which the knowledge will be used. A framework for mathematical knowledge can consist of mathematical, everyday and scientific elements.

### 5. Structure of Knowledge

All mathematical knowledge consists of propositional (conceptual and relational) structures and procedural (algorithmic and methods) structures. The process through which we understand and manipulate mathematical situations is grounded in specific content structures.

### 6. Complexity of Concepts

Propositional structures and procedural structures are complex content structures, a fact that is often disguised through rote learning and teaching. Although traditionally we teach through analyzing and breaking down knowledge, the constructivist sees "building up" as an equally valid learning process. Procedural structures (algorithms) are linked in important ways to propositional structures (concepts).

### 7. Transfer of Knowledge

As we learn, we learn context, as well as content and process. Transfer of knowledge must not be assumed; it occurs only as a new context is seen as the learned one.

Although a deeper understanding would require considerable elaboration on all of these principles, perhaps we can employ a constructivist teaching tactic, and let the reader come to understand the principles as they are used to develop the guidelines for classroom teaching. Classifying something as complex as human learning in seven principles seems to be an utterly futile undertaking. However, I would like to elaborate slightly on the structure and complexity principles. Recognized in the structure principle, first of all, is the importance of relationships among all mathematical concepts and that any understanding of mathematics is a matter of recognizing all these relationships. Also implied in the structure principle is that all mathematical activity, such as problem solving, is highly dependent on these structures. The complexity principle, while acknowledging the many-faceted aspect of even apparently simple concepts such as multiplication, stresses that understanding and use of knowledge must take into account all, or most, of these facets.

Of course, these learning principles can be applied to the teaching of any subject, but our concern here is what this might mean for the teaching of mathematics. In deriving these guidelines for classroom teaching, it became apparent that several possible interpretations would be valid. Once again, I have opted for seven, knowing that these can only serve as general suggestions.

## Constructivist Guidelines for Classroom Teaching

### 1. Unit Context

Mathematics should be taught in the context of a three- to four-week unit constructed around a mathematical, everyday or scientific application of the content. Students should feel comfortable and familiar with this application context.

**Rationale**: The purposeful-constructions and framework principles are satisfied by this. The actual application context would be a function not only of the content, but also of the grade level of the class, the characteristics of the students and the school environment.

## 2. Curriculum Tasks

The tasks that comprise the unit should be conducted with a view to the students engaging their current conceptions, mastering the task and learning from it. The focus of the task should be central to the unit application.

**Rationale**: The learning-how-to-learn and confidence principles suggest that the task be a manageable part of the unit. The structure principle suggests that relevant mathematics knowledge be an integrated part of the task.

## 3. Managing the Task

All students should be given assistance in dealing with the task—determining task difficulty, monitoring their understanding of it, apportioning time for it and predicting how well they can perform it. The teacher should pay special attention to the students' perception of the task. Individual differences should be noted and provided for in this aspect.

**Rationale**: The purposeful-constructions and learning-how-to-learn principles are important here, especially in helping students become aware of their knowledge and knowledge processes. This guideline is the core of the instructional process.

## 4. Task Variety

Tasks should include a range of learning activities, such as direct examples, reviewing, textbook use, note taking, concrete materials, understanding, amplification of basic concepts, problem solving, self-inquiry, practice exercises, group activities, discussion and questioning.

**Rationale**: The purposeful-constructions principle does not imply that student learning should be of a discovery nature, but only that learning should have some purpose. The complexity principle not only suggests that a considerable amount of guidance, even direct examples, is appropriate, but also that a variety of approaches is necessary to achieve an understanding of a mathematical topic.

## 5. Assessment Tasks

Assessment should be carried out primarily within the context of the unit.

**Rationale**: The transfer principle suggests that we should first apply learning to the context of the unit. If we do testing beyond the context of the unit, we should be conscious of how the new context relates to the learned one. In actual (real-life) use of mathematics, contexts that are important to the student are most often familiar ones.

## 6. Mathematical Learning

### *(a) Readiness*

Readiness for content learning must be noted, but only in the context of the learning task. What does the learner bring to the situation? Students' awareness of their own readiness is also important.

**Rationale**: Purposeful constructions are derived from previous "theories" that the student has. This is the central premise of the constructivist view. The learning-how-to-learn principle suggests a self-awareness of these previous theories.

### *(b) Concepts*

Concepts, the pivotal ingredients of mathematics learning, must be constructed from the student's prior knowledge. Learning of complex subject matter is achieved through many different propositional structures. Specific instructional devices, such as concept maps and structured apparatus, should be employed.

**Rationale**: The framework, structure, and complexity principles all indicate the necessity of a thorough conceptual basis for mathematics learning.

### *(c) Skills*

Skill development, as it relates to the curriculum unit, is important. Care should be taken in selecting the application context for curriculum units. Skills and algorithms (procedural structures) are founded upon certain propositional structures. Skills should be learned as broader "method" approaches.

**Rationale**: Although our principles do not address the matter of skills directly, the structure principle advocates a solid basis for all procedures, while purposeful constructions implies that all skill learning be in context.

### *(d) Applications*

All applications occur in the context of the unit. They should be dealt with as an indication of the use and usefulness of mathematics, and also as a way of relating the real world to the development of mathematics.

**Rationale**: The framework principle means that applications can be an important contribution to the framework for learning mathematics. The purposeful-constructions principle suggests applications as a primary reason for studying mathematics. Lastly, the teacher must be constantly aware of transfer and the problem of the context of learning.

### *(e) Problem Solving*

Problem solving should be approached through a study of the particular kinds of problems in each unit. Problem solving is a particular way of knowing content.

**Rationale**: The structure principle suggests that all mathematics is dependent on specific knowledge. The metacognitive processes of the learning-how-to-learn principle manage only cognitive knowledge. A constructivist view does not support broad, generalizable problem-solving strategies.

## 7. Goals of Mathematics Learning

The major goals of mathematics teaching are that students gain understanding of complex areas of mathematical knowledge, use this knowledge in relevant situations, and understand their own processes and capabilities for functioning in a mathematical environment.

**Rationale**: The constructivist view not only provides new insight into how mathematics should be taught, but also implies a somewhat revised goal for mathematics teaching; practice, feedback and coaching are not enough. Although the [constructivist] view expands upon what understanding means, one of the more interesting issues it raises is how teachers should regard their efforts toward improving students' capabilities for learning how to learn.

The strongest message of a constructivist approach is the desirability that teachers make clear to themselves and to students the purpose of learning mathematics. Making clear the purpose, without trivializing it, will be of great benefit in improving mathematics teaching. At this writing, I believe the weakest part of these guidelines is the matter of context and, therefore, the matter of what a sensible unit context might be. It seems essential that the context include, but go beyond the bounds of, mathematics itself. It certainly need not be confined to students' interests. Plausibility to the student might be a better guideline. Clearly, the broader the context, the more mathematics it will subsume. However, the greater breadth might tend to lose focus. Also, the notion of curriculum task and its position between the unit context and mathematics to be learned is somewhat problematic. An appropriate resolution of these weaknesses will need to be worked out in light of both the proposed principles of learning and the other guidelines.

Obviously, this interpretation of the constructivist perspective leaves many gaps. If a teacher were to conduct lessons solely on the basis of this statement (even assuming the availability of a textbook), I would predict chaos. The statement can only be seen as an attempt to modify already competent practice. Certainly, these are not prescriptions for teaching. Rather, I see them as interesting guidelines that can be tried, discussed, revised and reinterpreted. A constructivist would see a teacher interpreting these guidelines on the basis of the teacher's existing theories, and then, perhaps, rejecting them as invalid or tuning existing theories, using them, and then revising or discarding them.

At the very least, these guidelines should provide the basis for an interesting curriculum unit that would go far in explicating the guidelines. This would provide an opportunity for psychologists to say that their views have been misread or misinterpreted, which would be very useful. It might even serve to have them rethink their ideas in the light of feedback given by teachers. Whatever happens, teachers of mathematics are obligated to begin investigating ways that these new conceptualizations of learning can benefit them. Teachers certainly owe it to themselves and, in some sense, they owe it to psychologists and educators who are searching for new insight into the very important but, too often, frustrating process of learning mathematics.

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# Mathematics and the Alberta High School Curriculum

*John G Heuver*

*Note: This article first appeared in* delta-K *volume 26, number 1, pages 18–22 (1986). Minor changes have been made in accordance with current ATA style.*

From time to time, the teaching of mathematics changes. Since about 1980, the Alberta high school syllabus has undergone a certain reform, and while some of the reasons for such change seem sound, others are more obscure and questionable. The adoption of the metric system created a necessity for an update. The easy access to hand-held calculators required a different emphasis in the area of logarithms. Such traditional topics as geometry were to be treated from a different perspective because of developments in mathematics that had filtered down to the secondary school level. The inclusion of nontraditional areas, such as statistics and the minor topic of exponential growth and decay, have raised eyebrows. In this article, an attempt will be made to identify, by subject area, a few of the anomalies and difficulties that occur in our curriculum and textbooks.

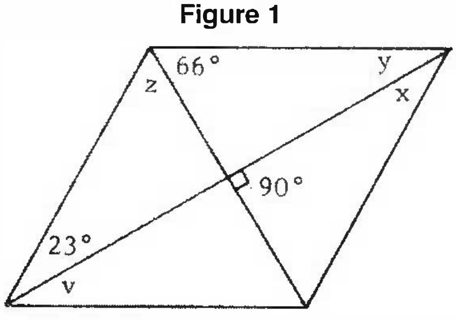
## Geometry as a High School Subject

For the high school curriculum the question of what part of geometry we present is a rather existential problem. Our present Grade 10 texts treat it very casually and with little sense of purpose, made worse by the fact that, in many places, the textbooks contain grave errors.

In the *Holt Mathematics 4* text (Hanwell, Bye and Griffiths 1980, 230), the following exercise occurs:

> Consider a parallelogram with three angles given and calculate the angles x, y, z, and v.

**Figure 1**




The unaware reader obtains results that correspond with the answers given in the back of the textbook. However, the exercise is completely ludicrous. Pictures and numbers have collided in a strange way. (A proper answer would be, This is a rhombus in which the diagonals are perpendicular and the diagonals also bisect the angles of the rhombus. Hence, the answer in the textbook is incorrect, and so is the "given" part.)

In order to see some of the difficulties encountered when deciding on which part of geometry should be presented in the school curriculum, we have to consider the development of geometry from a historical perspective and the more recent outlook on mathematics itself.

Once a proposition in mathematics has been settled, it becomes generally accepted. The acceptance is based on what we call proof. Over time, the significance of the proposition may change as it becomes part of a larger body of knowledge, but its quality stays the same. Since the time of Euclid, the validity of propositions in elementary geometry has been based on an axiomatic system, a collection of statements accepted as true. From these initial statements, a large collection of propositions is deduced by agreeing upon certain rules of inference. In 1931, Kurt Goedel proved that there exist axiomatic systems from which certain propositions belonging to the system can neither be proved nor disproved.

An illustration of Goedel's contention, which is even presentable in the classroom, is Goldbach's conjecture. The conjecture states that every even natural number greater than two is expressable as the sum of two primes, where primes are natural numbers divisible by one and themselves only. Up to now, no even number has been found that is not the sum of two primes. The conjecture may be true, but may not be derivable from the axioms of arithmetic. The same may apply to what is known as Fermat's last theorem. This theorem states that there are no natural numbers a, b and c such that $a^n + b^n = c^n$ for $n$ greater or equal to three and $n$ a natural number. These conjectures have the charm that they can serve as illustrations in the relatively simple setting of elementary mathematics, and that, even today, they draw considerable interest from mathematicians.

The closer scrutiny of the axiomatic system was largely caused by the development of different types of geometry. In Riemannian geometry, for example, Euclid's axiom that through a point P in the plane not on line *l* a line can be drawn parallel to *l* is denied. Of course, philosophical questions arise regarding the plausibility of these geometries.

The classical belief that the properties of Euclidean geometry are valid for the world in which we live has been undermined, as it becomes evident that other geometries are equally valid. In an article entitled "Elementary Geometry, Then and Now," I M Yaglom (Davis, Grünbaum and Scherk 1981, 165) speaks about geometries that draw considerable attention in this half of the twentieth century and makes a comparison to developments in the previous century. He says

> In contrast to discrete geometry, combinatorial geometry so far has no serious practical applications; in this respect, it resembles "classical" elementary geometry, which considered properties of triangles and circles, which beautiful though they were, were scientifically blind alleys—leading nowhere, giving nothing to science at large. Still "nineteenth-century elementary geometry" was closely bound up with what might be called the "scientific atmosphere" of those years ...

There are two pedagogical consequences to be drawn from Yaglom's argument. Certain aspects of geometry are culturally bound and do not necessarily lend themselves to so-called practical applications. The present curriculum seems to be preoccupied with these applications. Second, since Euclidean geometry is not the only valid system, we have to conclude that one of the significant objectives is to teach our students the method of a deductive system. The deductive character of a system is more easily established in Euclidean geometry than in any other part of high school mathematics. (For the 13–23–33 sequence of mathematics courses, a different perspective should prevail.)

## Exponential Growth and Decay

Euclidean geometry has been, traditionally, part of the secondary school curriculum. This cannot be said of the particular minor topic presented in both approved texts for Grade 12. In order to see what is going on, we will have to go through a more or less technical explanation with omission of mathematical techniques. In the *FMT Senior* text (Dottori, Knill and Stewart 1979, 153), the exponential growth rate is explained on an intuitive basis. Since bacteria multiply by splitting, the population increases by a power of two. Without much ado, the growth function is declared to be an exponential function with base two for any increasing biological population whatsoever. It could include mice. The model in the textbook is quite reasonable as long as the bacteria are declared immortal. Such a representation violates the laws of nature.

A correct way to derive the appropriate formula for the growth rate would be by means of a simple differential equation, which is beyond the scope of high school mathematics. The proper formulation of the problem lies in the assumption that a biological population has a growth rate that is proportional to its size. In this formulation of the problem the mortality rate is included in the hypothesis. A simple technique of elementary calculus yields the correct result. In this derivation the base two of the textbook can be shown not to be unique. Thus, a mice population increase no longer creates a hazard for the formula.

For decay of radioactive materials, the rate of decay is again assumed to be proportional to the original mass of the material. Again, the proper formula is derived by the same differential equation. However, the textbook explanation requires the observer to watch the material for 25 years to again halve the mass. After some mysterious reasoning, an exponential function emerges with the not unique base two. In *Calculus*, volume I, Tom Apostol (1964, 229) says

> Actually, the physical laws we use here are only approximations to reality, and their motivation properly belongs to the sciences from which the various problems emanate.

The opinion has been voiced that high school courses should contain practical applications. However, some sobering thoughts come to mind if one considers the examples cited here.

1. The problem of exponential growth and decay requires mathematical techniques that are not available to the high school student.
2. If a student were to try out the methods from the textbook on a science project, it would be doomed to failure. It would also require estimation of the constants in the formula that demands the method of least squares, which is also beyond the secondary school level.
3. It seems that so-called applications borrowed from mathematical literature past the high school level lead to disastrous results.

The final conclusion has to be that this topic should be abandoned unless somebody can come up with a proof that is presentable at the high school level.

## Statistics in High School

The field of statistics has grown enormously in this century and the results are being felt in almost every aspect of life. Who can imagine a political election without a poll? By its overwhelming presence, statistics has also found its way into the high school curriculum. In Grade 12, we study something about the normal distribution which, in two dimensions, is graphically represented by a bell-shaped curve. Assumptions about this distribution are, as a rule, verified by hypothesis testing. However, in high school, the experiment is absent, and so we are told that all necessary assumptions hold in order to simplify the case. Suddenly, the conclusion is drawn that we have obtained a "standardized normal distribution."

About 15 per cent of the questions on the departmental exams are based on this topic. The value of this type of mental exercise is highly questionable. At present, the student has been taught to manipulate some formulae that appear out of the blue yonder.

It may be necessary to look at the historical development of statistics in order to come up with a suitable secondary program. At the moment, we only deal with the normal distribution. The danger is that we give students the impression that this is the only distribution there is, which is not true. It is also very hard to explain that mean and standard deviation have the same meaning as the first two moments of a mass in physics. Interrelationships are not established. In *Mathematics and Logic – Retrospects and Prospects*, Mark Kac and Stanislaw Ulam (1968, 50) say "The theory (or calculus) of probability has its logical and historical beginnings in the simple problems of counting."

Indeed, it is simpler to present, in the classroom, the phenomena of tossing coins and dice than to give sound reasons for the continuous normal distribution. Since there is no long tradition in the teaching of statistics at the secondary level in any country, we are treading on very thin ice. It seems safer to go back to its original beginning and show something about the essence of its method than to show off with impressive-looking results. The normal distribution is a powerful tool in statistics, but the ability to see the full scope of its impact belongs to the professional statistician.

## Conclusion

There is a great need for rethinking parts of the mathematics program. I M Yaglom, in his article "Elementary Geometry, Then and Now" (Davis, Grünbaum and Scherk 1981), speaks about leading mathematicians who have written texts for secondary students. One of these is A N Kolmogorov, the Russian mathematician, who has written a text that is used by all secondary students in Russia. He speaks also about the French mathematician Jean Dieudonné, who wants to see geometry reduced to linear algebra and who has written a text for this purpose. Our school system cannot directly take over these ideas, but they can form a subject for study and comparison. If we want proper programs for our secondary schools, then we cannot leave the writing of textbooks to the book publishers and the forces of the marketplace.

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# Enhancing Mathematics Teaching in the Context of the Curriculum and Professional Standards of the National Council of Teachers of Mathematics

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This article briefly reviews the curriculum and professional standards of the National Council of Teachers of Mathematics (NCTM). These standards have driven and directed curriculum development, teaching, evaluation and professional development in mathematics since their publication in 1989. This article provides readers with an overview of the curriculum and professional standards for mathematics.

Inherent in the standards is a consensus that all students need to learn more, and often different, mathematics and that instruction in mathematics must be significantly revised. The need for standards for school mathematics is clearly evident in that they ensure quality, indicate goals and promote change. The NCTM considers all three reasons equally important. Schools, and in particular school mathematics, must reflect the important consequences of the current reform movement in mathematics if students are to be adequately prepared to live in the 21st century. Today's society expects schools to ensure all students have an opportunity to become mathematically literate, are capable of extending their learning, have an equal opportunity to learn, and become informed citizens capable of understanding issues in a technological society.

Educational goals for students must reflect the importance of mathematical literacy. Toward this end, the K–12 standards articulate five general goals for all students:

- That they learn to value mathematics
- That they become confident in their ability to do mathematics
- That they become mathematical problem solvers
- That they learn to communicate mathematically
- That they learn to reason mathematically

These goals imply that students should be exposed to numerous and varied related experiences that encourage them to value the mathematical enterprise, to develop mathematical habits of mind and to understand and appreciate the role of mathematics in human affairs; that they should be encouraged to explore, to guess and even to make and correct errors so that they gain confidence in their ability to solve complex problems; that they should read, write and discuss mathematics; and that they should conjecture, test and build arguments about a conjecture's validity.

The mathematics classroom must be permeated with these goals and experiences so that they become commonplace in students' lives. Exposing students to the experiences outlined in the standards will ensure that students gain mathematical power. This term denotes an individual's abilities to explore, conjecture and reason logically, as well as his or her ability to use various mathematical methods effectively to solve nonroutine problems. This notion is based on the recognition of mathematics as more than a collection of concepts and skills to be mastered; it includes methods of investigating and reasoning, means of communication and notions of context. In addition, for each individual, mathematical power involves development of self-confidence.

The NCTM (1989, 1991) has presented 78 standards divided among eight categories: Grades K—4 curriculum, Grades 5–8 curriculum, Grades 9–12 curriculum, evaluation of students, teaching mathematics, evaluation of the teaching of mathematics, professional development of teachers, and support and development of mathematics teachers and teaching.

## Curriculum Standards

Curriculum standards for school mathematics are value judgements based on a broad, coherent vision of schooling derived from several factors: societal goals, student goals, research on teaching and learning, and professional experience. Each standard starts with a statement of what mathematics the curriculum should include, followed by a description of the student activities associated with that mathematics and a discussion that includes instructional examples. Three features of mathematics are embedded in the standards. First, "knowing" mathematics is "doing" mathematics. Doing mathematics is different from mastering concepts and procedures. That is not to say that informational knowledge has no value, only that its value lies in the extent to which it is useful in the course of some purposeful activity. Students clearly must know the fundamental concepts and procedures from some branches of mathematics; established concepts and procedures must be relied on as fixed variables in a setting in which other variables may be unknown. However, instruction should persistently emphasize "doing" rather than "knowing that."

Second, some aspects of doing mathematics have changed in the last decade. The computer's ability to process large sets of information has made quantification and the logical analysis of information possible in such areas as business, economics, linguistics, biology, medicine and sociology. Because mathematics is a foundation discipline for other disciplines and grows in direct proportion to its utility, the mathematics community believes that the curriculum for all students must provide opportunities to develop an understanding of mathematical models, structures and simulations applicable to many disciplines.

Third, changes in technology and the broadening of the areas in which mathematics is applied have resulted in growth and changes in the discipline of mathematics. More than half of all mathematics has been invented since World War II. The new technology not only has made calculations and graphing easier but also has changed the very nature of the problems important to mathematics and the methods mathematicians use to investigate them.

Because technology is changing mathematics and its uses, the NCTM believes

- appropriate calculators should be available to all students at all times,
- a computer should be available in every classroom for demonstration purposes,
- every student should have access to a computer for individual and group work and
- students should learn to use the computer as a tool for processing information and performing calculations to investigate and solve problems.

Access to this technology is no guarantee that any student will become mathematically literate. Calculators and computers for users of mathematics are tools that simplify, but do not accomplish, the work at hand. Similarly, the availability of calculators does not eliminate the need for students to learn algorithms. Students should be aware of the choices of methods when calculating an answer to a problem. When an approximate answer is adequate, students should estimate. If a precise answer is required, students must be capable of choosing an appropriate procedure. Many problems should require students to conduct mental calculations or use paper and pencil. For more complex calculations (for example, long division or column addition), students should be able to use calculators. Finally, if many iterative calculations are needed, a computer program should be written or used to find answers (for example, finding a sum of squares).

With respect to mathematical content, the standards represent the minimum that all students will need to be productive citizens. The standards do not specify alternative instructional patterns prior to Grade 9. For Grades 9–12, the standards have been prepared in light of a core program for all students, with explicit differentiation in terms of depth and breadth of treatment and the nature of applications for college-bound students. There is an implied expectation that all students have an opportunity to encounter typical problem situations related to important mathematical topics.

Student activities are the second aspect of each standard. Two general principles have guided the description of these activities: first, activities should grow out of problem situations; and second, learning occurs through active as well as passive involvement with mathematics. Traditional teaching emphases on practice in manipulating expressions and practising algorithms as a precursor to solving problems ignore the fact that knowledge often emerges from the problems. Thus, present strategies for teaching may need to be reversed: knowledge often should emerge from experience with problems. Furthermore, students need to experience genuine problems regularly. A genuine problem is a situation in which, for the individual or group concerned, one or more appropriate solutions have yet to be developed. However, instruction should vary and include opportunities for

- appropriate project work,
- group and individual assignments,
- discussion between teacher and students and among students,
- practice on mathematical methods and
- exposition by the teacher.

Another premise of the standards is that problem situations must keep pace with the mathematical and cultural maturity and experience of the students. For example, the primary grades should emphasize the empirical language of the mathematics and whole numbers, common fractions and descriptive geometry. In the middle grades, empirical mathematics should be extended to other numbers and the emphasis should shift to building the abstract language needed for algebra and other aspects of mathematics. High school mathematics should emphasize functions, their representations and uses, modelling and deductive proofs.

The standards specify that instruction should be developed from problem situations. Situations should be sufficiently simple to be manageable but sufficiently complex as to provide for diversity in approach. They should be amenable to individual, small-group or large-group instruction; involve a variety of mathematical domains; and be open and flexible as to the methods to be used.

The first three standards for each grade level are problem solving, communication and reasoning, although these vary between the levels on what is expected of students and of instruction. The fourth curriculum standard at each level is mathematical connections. This label emphasizes the belief that, although it is often necessary to teach specific concepts and procedures, mathematics must be approached as a whole. Concept, procedures and intellectual processes are connected. Thus the curriculum should include deliberate attempts, through specific instructional activities, to connect ideas and procedures among different mathematical topics and with other content areas. Following the connections standard, nine or ten specific content standards are stated and discussed. Some have similar titles, which reflects that a content area needs emphasis across the curriculum; however, others emphasize specific content that needs to be developed at that level.

## Student Evaluation Standards

These standards are viewed and presented in three categories. The first set consists of three evaluation standards and discusses general assessment strategies related to the curriculum standards. These standards present principles for judging assessment instruments, including alignment, multiple sources of information, and appropriate assessment methods and uses.

The second set contains seven standards under student assessment and focuses on providing information to teachers for instructional purposes. They closely parallel the curriculum standards of problem solving, communication, reasoning, mathematical concepts and mathematical procedures, in addition to two separate standards on mathematical disposition and mathematical power. These seven standards are to be used by teachers to judge students and their mathematical progress.

The final set of four standards falls under program evaluation and addresses the gathering of evidence with respect to the quality of the mathematical program. The standards are indicators of program evaluation, curriculum and instructional resources, instruction, and evaluation team. These standards are to be used by teachers, administrators and policymakers to judge the quality of the mathematics program and the effectiveness of instruction.

## Standards for Teaching Mathematics

Central to the curriculum and evaluation standards is the development of mathematical power for all students. Mathematical power includes the ability to explore, conjecture, reason logically, solve nonroutine problems, communicate about and through mathematics, and connect ideas within mathematics and between mathematics and other intellectual activities. Mathematical power also involves the development of personal self-confidence and a disposition to seek, evaluate and use quantitative and spatial information in solving problems and in making decisions. Students' flexibility, perseverance, interest, curiosity and inventiveness also affect the realization of mathematical power.

To reach the goal of developing mathematical power for all students requires the creation of a curriculum and an environment in which teaching and learning are to occur that are very different from much of our current practice. The image of mathematics teaching needed includes elementary and secondary teachers who are more proficient in

- selecting mathematical tasks to engage students' interest and intellect (that is, worthwhile mathematical tasks);
- providing opportunities to deepen their understanding of the mathematics being studied and its applications;
- orchestrating classroom discourse in ways that promote the investigation and growth of mathematical ideas;
- using, and helping students use, technology and other tools to pursue mathematical investigations;
- seeking, and helping students seek, connections to previous and developing knowledge; and
- guiding individual, small-group and whole-class work.

The professional standards for teaching mathematics are a major shift away from the current practice to mathematics teaching for student empowerment. Current practice is characterized by a predictable sequence of activities. First, answers are given for the previous day's assignment. Difficult problems are worked on by the teacher or students at the chalkboard. This is followed by the introduction of a new concept and the assignment of homework. Students work on the homework for the remainder of the class period, with the teacher moving around the room to answer questions.

The professional standards for teaching mathematics call for a change in the environment of mathematics classrooms. We need to shift toward

- classrooms as mathematical communities and away from classrooms as simply a collection of individuals;
- logic and mathematical evidence as verification and away from the teacher as the sole authority for right answers;
- mathematical reasoning and away from merely memorizing procedures;
- conjecturing, inventing and problem solving and away from an emphasis on mechanistic answer finding; and
- connecting mathematics, its ideas and its applications and away from teaching mathematics as a body of isolated concepts and procedures.

The phrase *all students* is used throughout the standards. This means that schools and communities must accept the goal of mathematical education for every child from kindergarten to Grade 12. This does not mean that every child will have the same interests or capabilities in mathematics. It does mean that we will have to examine our fundamental expectations about what children can learn and do and that we will have to strive to create learning environments in which raised expectations for children can be met.

## Standards for the Evaluation of the Teaching of Mathematics

This section presents eight standards for evaluating the teaching of mathematics organized under two categories. The first category describes the process of evaluation and includes standards dealing with the evaluation cycle, teachers as participants in evaluation and sources of information.

The evaluation process should reflect that the overall intent is to improve instruction, that it should be a dynamic and continual process, that teachers should be an integral part of that process, and that because of the complexity of teaching, it should involve a variety of sources of information gathered in various ways. The standards emphasize that teachers should be encouraged and supported to engage in self-analysis and to work with colleagues in improving their teaching. When evaluation involves supervisors or administrators, their relationship with teachers should be collegial with the intent to improve instruction.

The second category of standards in this section describes the foci of evaluation and includes five standards dealing with mathematical concepts, procedures and connections; mathematics as problem solving, reasoning and communication; mathematical disposition; assessing students' mathematical understanding; and learning environment.

These standards, particularly the standard dealing with mathematical disposition, emphasize the importance of significant mathematics when re-evaluating mathematics teaching. Through encounters with significant mathematics, students develop mathematical power. But attaining mathematical power requires more. It requires a disposition to do mathematics and an environment in which the processes of doing mathematics are continually emphasized.

This can occur only when teachers present stimulating tasks and create an environment in which problem solving, reasoning and communication are valued and promoted. Further, the message teachers send students should not be limited to instruction alone; it must also include what and how mathematical learning is assessed. Through assessment, we communicate to our students what mathematical outcomes are valued.

A consistent message throughout the standards for the evaluation of teaching is the importance of teachers reflecting on their teaching and working with colleagues and supervisors to improve their teaching. While the standards provide a focus for improvement, such improvements will occur only when teachers consciously decide to engage in ongoing professional development. This, in turn, requires support and encouragement at all levels.

## Standards for the Professional Development of Teachers of Mathematics

Five standards are presented in this section: experiencing good mathematics teaching, knowing students as learners of mathematics, knowing mathematical pedagogy, developing as a teacher of mathematics and the teacher's role in professional development.

Mathematics teachers must have good role models during their preservice and continuing inservice training. Teachers often teach the way they have been taught. Therefore, preservice instructors need to address the major components of teaching: tasks, discourse, environment and analysis of teaching.

The education of mathematics teachers should develop their knowledge of the content and discourse of mathematics. Teachers' comfort with, and confidence in, their own knowledge of mathematics affects what they teach and how they teach it. Their conceptions of mathematics shape their choice of worthwhile mathematical tasks, the learning environments they create and the discourse in their classrooms. Knowing mathematics includes understanding specific concepts and procedures as well as the process of doing mathematics. Mathematics involves the study of concepts and properties of numbers, geometric objects, functions and their uses—identifying, counting, measuring, comparing, locating, describing, constructing, transforming and modelling. The relationships and recurring patterns among these objects and the operations on these objects lead to building such mathematical structures as number systems, groups or vector spaces and to studying the similarities and differences among these structures.

Such knowledge ought not to be developed in isolation. The ability to identify, define and discuss concepts and procedures; to develop an understanding of the connections among them; and to appreciate the relationship of mathematics to other fields is critically important.

The standards clearly state that, to sufficiently understand the mathematical topics specified for each level, teachers teaching mathematics should have not less than nine semester hours of coursework in content mathematics at the K–4 level, fifteen semester hours of coursework at the 5–8 level, and the equivalent of a major in mathematics at the 9–12 level.

The preservice and continuing inservice training of teachers of mathematics should provide multiple perspectives on students as learners of mathematics by developing teachers' knowledge of research on how students learn mathematics; the effects of students' age, abilities, interests and experience on learning mathematics; the influence of students' linguistic, ethnic, racial and socioeconomic backgrounds and gender on learning mathematics; and ways to affirm and support full participation and continual study of mathematics by all students.

Knowing mathematical pedagogy is an important standard that can be achieved through preservice and continuing inservice training of teachers of mathematics by developing

- teachers' knowledge of and ability to use and evaluate instructional materials and resources, including technology;
- ways to represent mathematics concepts and procedures;
- instructional strategies and classroom organizational models;
- ways to promote discourse and foster a sense of mathematical community; and
- means for assessing student understanding of mathematics.

Developing as a teacher of mathematics is really at the heart of teaching. Teachers of mathematics must have ongoing opportunities to examine and revise their assumptions about the nature of mathematics, how it should be taught, and how students learn mathematics; observe and analyze a range of approaches to mathematics teaching and learning, focusing on the tasks, discourse, environment and assessment; work with a diverse range of students individually, in small groups and in large-class settings with guidance from and in collaboration with mathematics education professionals; analyze and evaluate the appropriateness and effectiveness of their teaching; and develop dispositions toward teaching mathematics.

Essentially, being a teacher of mathematics means developing a sense of self as such a teacher. Such an identity grows over time and is built from many different experiences with teaching and learning. Further, it is reinforced by feedback from students that indicates they are learning mathematics, from colleagues who demonstrate professional respect and acceptance, and from a variety of external sources that demonstrate recognition of teaching as a valued profession.

Teachers develop as professionals on an ongoing basis. The standard regarding the teacher's role in professional development suggests that teachers of mathematics should take an active role in their own professional development by accepting responsibility for experimenting thoughtfully with alternative approaches and strategies in the classroom; reflecting on learning and teaching individually and with colleagues; participating in workshops, courses and other educational opportunities specific to mathematics; participating actively in the professional community of mathematics educators; reading and discussing ideas presented in professional publications; discussing with colleagues issues in mathematics and mathematics teaching and learning; participating in proposing, designing and evaluating programs for professional development specific to mathematics; and participating in school, community and political efforts to effect positive change in mathematics education. Schools and school systems must support and encourage teachers in accepting these responsibilities. What is essential is that teachers of mathematics view themselves as agents of change, responsible for improving mathematics education at many levels: the classroom, the school, the district, the region and even the nation.

## Standards for Supporting and Developing Mathematics Teachers and Teaching

*Professional Standards for Teaching Mathematics* (NCTM 1991) presents a vision of teaching that calls for a teacher who is educated, supported and evaluated in ways quite different from current practice. To create a teaching environment as described in the standards, teachers must have access to educational opportunities over their entire professional lives that focus on developing a deep knowledge of subject matter, pedagogy and students.

Teachers can, and do, implement successful mathematics programs with little help or encouragement. However, sustaining them or expecting that they flourish without adequate support is not reasonable. The changes called for by the curriculum and evaluation and the professional teaching standards need the support of policymakers in government, business and industry; school administrators, school board members and parents; college and university faculty and administrators; and leaders of professional organizations.

Policymakers in government, business and industry should take an active role in supporting mathematics education by accepting responsibility for

- participating in partnerships at the national, provincial and local levels to improve the teaching and learning of mathematics;
- supporting decisions made by the mathematics education professional community that set directions for mathematics curriculum, instruction, evaluation and school practices; and
- providing resources and funding for, and assistance in, developing and implementing high quality school mathematics programs that reach all students, as envisioned in *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) and *Professional Standards for Teaching Mathematics* (NCTM 1991).

School administrators and school board members should take an active role in supporting teachers of mathematics by accepting responsibility for

- understanding the goals for the mathematics education of all students set forth in the NCTM standards and the needs of teachers of mathematics in realizing these goals in their classrooms;
- recruiting qualified teachers of mathematics, with particular focus on the need for the teaching staff to be diverse;
- providing a support system for beginning and experienced teachers of mathematics to ensure that they grow professionally and are encouraged to remain in teaching;
- making teaching assignments based on the qualifications of teachers;
- involving teachers centrally in designing and evaluating programs for professional development specific to mathematics;
- supporting teachers in self-evaluation and in analyzing, evaluating and improving their teaching with colleagues and supervisors;
- providing adequate resources, equipment, time and funding to support the teaching and learning of mathematics as envisioned in the standards
- establishing outreach activities with parents, guardians, leaders in business and industry, and others in the community to build support for quality mathematics programs; and
- promoting excellence in teaching mathematics by establishing an adequate reward system including salary, promotion and conditions of work.

College and university administrators need to actively support mathematics and mathematics education faculty by accepting responsibility for establishing an adequate reward system, including salary, promotion and tenure, and conditions of work, so that faculty can and are encouraged to

- spend time in schools working with teachers and students;
- collaborate with schools and teachers in the design of preservice and continuing education programs;
- offer appropriate graduate courses and programs for experienced mathematics teachers;
- provide leadership in conducting and interpreting mathematics education research, particularly school-based research;
- cooperate with pre-college educators to articulate the K–16 mathematics program; and
- make concerted efforts to recruit and retain teacher candidates of quality and diversity.

The leaders of professional organizations need to take an active role in supporting teachers of mathematics by accepting responsibility for

- promoting and providing professional growth opportunities for those involved in mathematics education,
- focusing attention of the membership and the broader community on contemporary issues dealing with the teaching and learning of mathematics,
- promoting activities that recognize the achievements and contributions of exemplary mathematics teachers and programs, and
- initiating political efforts that effect positive change in mathematics education.

## Summary

In *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) and *Professional Standards for Teaching Mathematics* (NCTM 1991), the argument is made that what is needed is a design change strategy. This means that new ways of doing things within the system—new roles for teachers and students, new goals, new structures—must be explored to find solutions to persistent problems that result in students failing to become mathematically powerful.

While the standards have directed us to some degree to the issues that need to be addressed, they are not a prescription for what must be done at each grade level. However, if we make a long-term commitment to the standards set forth, if we endeavour to persevere and if we continue to modify our course as new knowledge comes to the fore, we will make progress toward the goal of developing mathematical power for all students. Such massive change as is proposed in the standards will take time and much work and dedication from teachers and many others.

First, we must challenge all those charged with responsibility to teach mathematics in our schools to work collaboratively in using the curriculum and evaluation and professional standards as the basis for change so that the teaching and learning of mathematics in our schools is improved.

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# Explanation and Discourse in 9th-Grade Mathematics Classes[1]

*Elaine Simmt, Florence Glanfield and Shannon Sookochoff*

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The NCTM's *Professional Standards for Teaching Mathematics* suggests that "the nature of classroom discourse is a major influence on what students learn about mathematics" (1991, 45); but what does discourse look like in a mathematics classroom, what are its features and how does it come about? Of particular interest to us is the development of mathematical understanding through student interaction in discourse (Cobb and Bauersfeld 1995). At minimum, classroom discourse implies talk among teachers and students in mathematics classes. A more elaborate understanding might suggest that it is social interaction in language that supports the construction of mathematics in the microculture of the classroom. This article offers our interpretations of classroom discourse that specifically involves mathematical explanations.

## Explaining and Explanations

It is not surprising that the NCTM (1989) asserts that communication and mathematical discourse are primary in mathematics teaching and learning. Various studies have demonstrated the importance of mathematical explanations; for example, clear explanations (Westbury et al 1994), conceptually oriented explanations (Fuchs et al 1996) and demanding explanations (Fennema et al 1996) have been found to be positively correlated with student achievement. While an explanation may be the "heart of any teaching episode" (Leinhardt 1988), some research suggests that discourse is at the heart of learning (see, for example, Cobb and Bauersfeld 1995). The connection between explanation and discourse is described in the NCTM's (1991) *Professional Standards for Teaching Mathematics*. In that document it is suggested that teachers should "consistently expect students to explain their ideas ... [and should] help students learn to expect and ask for justifications and explanations from one another" (p 58). It further suggests that teachers' own explanations must focus on underlying conceptual meanings.

What is it about an explanation that makes it a key element in the processes of teaching and learning? From our work we learned that students think good teachers explain things well, offer good explanations and explain things until everyone understands. Although the students' perceptions appear to imply that an explanation is a gift or something that is given to them by the teacher, we think an explanation is more than that. Might it be that explanations are important to students and key to their mathematical understanding because, as Reid (1995) suggests, "explaining provides connections between what is known in a way that clarifies why a statement is true" (p 24)? Could it be that asking a student to explain himself or herself encourages reflection on one's own understanding? Does the demand for an explanation serve the student by requiring that they talk about (Mason 1996) their own explanations and the explanations of others? In this article we investigate the source of the mathematics explanations and how classroom discourse is used to facilitate the learning of those explanations.

## The Teaching Practices Project

The Teaching Practices Project (Simmt et al 1998) was a study to identify features of the teaching practices in schools whose students have a history of performing well on the Grade 9 achievement test in mathematics. The schools selected to participate in the study had a three-year history of meeting provincial expectations at the acceptable standard and exceeding expectations at the standard of excellence. Eight case researchers observed and interviewed 15 Grade 9 mathematics teachers in 13 schools across the province. In this study, we were interested not only in learning what teachers did in their classrooms but also about the nature of the discourse in their classrooms.

## Explanations in Discourse and a Discourse of Explaining

One of the common observations that case researchers made was that the teachers' classes observed were highly interactive. This was evident from the quantity and quality of the talk between the participants in the class. Specifically, the researchers reported the extensive use of explanations in the classes they observed. Two things stand out for us: the first is that explanations and classroom discourse were characteristics that co-emerged in the mathematics lessons we observed; the second is that the source of the explanations differed from teacher to teacher. In some classes the teacher was the source of the explanation, while in others the students were the source of explanations.

We would like to offer vignettes[2] taken from two classes in which each teacher happened to be teaching the same topic. These two vignettes illustrate the ways in which these teachers promoted classroom discourse and prompted explanations. In each class it was not good enough to be able to do something; the students also had to be able to explain how it was done. In the first vignette we see the interaction as the teacher helps students to make sense of his [teacher] explanation of how to find the area of a regular polygon. The teacher encourages his students to talk to help them come to an understanding of a statement he made. In the second vignette, the interaction among students and between the students and their teacher is the source of a number of student explanations for finding the area of a regular polygon.

## Vignette 1: Bellcroft School[3]

Ron Flynn, the teacher featured in the case, has been teaching mathematics for more than 30 years. Most of those years have been at Bellcroft, a large urban junior high school. Mr Flynn is well prepared to teach his Grade 9 mathematics classes; his course overview reads like a textbook page and lessons unfold like clockwork. His students and their parents talk about Mr Flynn's availability for extra help outside of class, his clear explanations using practical and relevant examples and his use of manipulatives to help students understand mathematics.

The illustration begins with Mr Flynn asking the students to consider a common formula for the area of a regular polygon. Through a series of questions he helps the students make sense of the formula he has suggested. In lines 7 through 38, notice how Mr Flynn "funnels" (Bauersfeld 1988) the discussion toward his "standard" formula. There are both advantages and disadvantages to funnelling the discussion. On one hand, the students are kept focused on the ideas that Mr Flynn believes are needed for the explanation. On the other hand, such funnelling may prevent the students from making decisions for themselves about what is important about the explanation. It is interesting to note that when Kevin (in line 36) asks a question connected to the area of a circle, Mr Flynn does not attempt to explain relationship between the two ideas; rather, he keeps the class discussion moving toward the target—the teacher's formula for area of a regular polygon.

## Vignette 2: Gilhooly Junior/Senior High School

This case features the teaching and learning in Bill Wilchuk's classroom, where mathematical conversation is both commonplace and awe-inspiring. Students in Mr Wilchuk's class appreciate the way he begins class by talking. One student said, "We talk about something different every day. He asks the class to give answers. It makes me listen more." Mr Wilchuk credits this kind of relaxed conversation to the extra time that he gets to teach math; the school schedules more time for core subjects than is mandated by Alberta Learning. He says that, before he had the extra time, he used to take up homework at the beginning of class. But now that he has more time, he chooses to begin each class with something new.

In this vignette we see the interaction between students and their teacher as he tries to help them make sense of the area of a regular polygon. In this case we note that Mr Wilchuk encourages his students to offer their explanations of how to find the area of a regular polygon. Rather than have the students work toward understanding the teacher's explanation, the teacher solicits many explanations, each of which is treated as a valid possibility. These explanations are illustrated in lines 2 to 5, 6 to 41, 42 to 51, 52 to 86, 87 to 94, and 95 to 105. In this discourse pattern, the teacher does not funnel the conversation but instead allows the conversation to remain open, thus providing the possibility that students might make sense of multiple explanations of a single concept. The vignette begins with Mr Wilchuk asking the students how they are going to find the area of a regular hexagon.

## Vignette 1

"Does anybody know what the formula is for the area of a regular polygon?" asked Mr Flynn.

"Susan?"

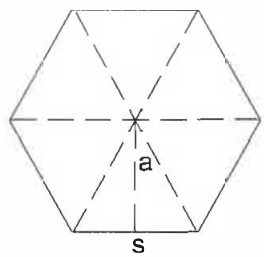
"No," she shook her head.

"Okay, this is probably one that you haven't seen before. So let's just take a moment to look at that."

Mr Flynn drew a hexagon on the board. "If we take our regular hexagon and divide it up. What do we have to divide it into?"

Susan answered, "Six triangles."

**Figure 1. Board Diagram for Area of a Regular Polygon**




"Six congruent triangles. That's right. This distance here …" he points to the apothem, "is the altitude of the triangle. And a side of the hexagon would be the base of each triangle. How do we find the area of triangle?"

"Base times height divided by two," Kevin suggested.

"Right." Mr Flynn gestured to the parts of the diagram that corresponded to the base and the height of the triangle. "This times that, divided by two. I'm just going to call this *a* for 'altitude' right now. There is another word that I'll tell you in a minute …. So we would have *a* times *s* divided by two. And how many of those triangles would we have?"

Many students called out, "Six."

"How many would we have if we had a regular pentagon?"

"Five."

"An octagon?"

"Eight."

"Right, so it's the number of sides." Mr Flynn concluded.

Mr Flynn returned to his board formula *as*/2 and added, "So, I'm just going to multiply that by *n*, standing for the number of sides that we have. Now, I'll write it in this form: 1/2 *ans* … or … *ans* over 2."

He continued, "*a* is the altitude of the triangle, but when you get a question like this, you're not going to have it divided up for you." He draws a diagram of the way a question might appear and puts a dotted line for the height of the apothem. "And the *a* will stand for, and write this word down so that you remember it, 'apothem.'"

Then he printed the word on the board and continued, "So for any regular polygon, that's how we find the area. The apothem times the number of sides times the length of each side divided by two."

Kevin asked, "Is that the same for a circle? Is that thing in a circle, the radius, called an apothem too?"

"No. This only works for regular polygons. The distance from the centre of the polygon to the midpoint of the side—that's called the apothem."

**Vignette 2**

"I'm curious as to what you're going to do for area."

Trevor blurted out, "Radius. The diameter."

"Okay, careful Trevor. Radius refers to basically circles as a subject."

Many students interjected with comments about Trevor's idea and tried to offer other possibilities. "Only one at a time," Mr Wilchuk reminded his class. He then called on Mike who suggested splitting the polygon into smaller shapes.

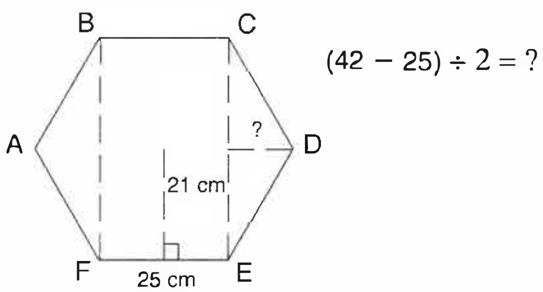
"Can you be more specific?"

Mike answered, "Two triangles and a rectangle."

A student sitting near Mike grasped the idea and said, "Oh, I see what he's saying."

Mr Wilchuk began to sketch Mike's idea on the overhead. "I'm going to try to copy what he's indicating. I don't know if Mike wanted me to split it this way, but at least this is one way I could do it." He sketched Mike's idea on the overhead (Figure 2). Then before making an attempt to solve using Mike's method, Mr Wilchuk asked the class, "Can we check to see whether or not his method will work?"

**Figure 2. Mike's Method: Two Triangles and a Rectangle**




Andrew responded, "I think it will."

And there was a chorus of, "It will."

Mr Wilchuk began to calculate, "I can take 25 and I can multiply it by ..."

He overheard a student say, "42."

"Excellent," Mr Wilchuk commended, "42. That will give me the area of the rectangle. How about these triangles?"

A student volunteered, "25 and 25." She was talking about the two short sides of the triangles in the sketch (Figure 2).

"This is 25, that's 25 ..." Mr Wilchuk confirmed. "What's the distance from here to there? In other words, the height of the triangles in Mike's method."

Mike called out, "42 take away 25 divided by 2."

"Now you're going to have to slow down for us. Say that again?"

"42 ... take away 25 ... divided by 2."

Mr Wilchuk thought out loud, "42 ... take away 25 ..." With further explanation from Mike, Mr Wilchuk discovered that Mike thought that the distance from vertex A to vertex D was the same as twice the apothem. In this case, 42 centimeters. The clarifying took about 45 seconds. Several students got confused and began to murmur.

Mr Wilchuk did not correct Mike. Instead he deferred Mike's idea for a moment and said, "It may or may not work, but I'm with one of your colleagues here—it's sort of confusing. 'It hurts the head.' I'm not going to say right now that your method will not work, but I think that some of your fellow students are saying that it's a tough one."

Mr Wilchuk encouraged more discussion. "Other suggestions, one at a time. Okay, Allen?"

Allen suggested dividing the hexagon into six triangles.

Mr Wilchuk wondered aloud, "Six triangles..." Without being prompted students turned to their neighbors and chatted about whether Allen's idea would work. Mr Wilchuk began to divide his hexagon into six triangles and listened to the student discussion.

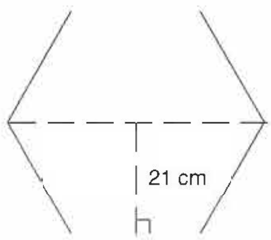
The conversation continued for about 10 seconds until Dean said, "Stop right here!"

So Mr Wilchuk interrupted the class discussion, "Hold it. Allen said 'six triangles' and in my attempt to start to divide it, Dean said, 'Stop right here!' So, Allen, I'm just going to put you on hold for a second—"

Some students giggled at the idea of Allen being put on hold.

"Now, Dean is suggesting that this will work." Based on Dean's suggestion, Mr Wilchuk divided the hexagon in half making two trapezoids (Figure 3).

**Figure 3. Dean's Method: Two Trapezoids**




"Do you know how to find the area of a trapezoid?"

Several students answered, "Yeah."

The teacher continued, "So, I could find the area of one, double it, and I've got the area of this thing. Now, do I have all my numbers? Do I know this distance?" He pointed to the base of the trapezoid and the distance that Mike had earlier thought was 42 cm.

"42," offered someone.

"It's not 42," said Mr Wilchuk.

"45," suggested Cynthia.

And Jeff echoed, "45."

"46?"

Then, "Uh-oh."

And another, "Uh-oh."

And, "Uh-oh."

Mr Wilchuk interrupted the confusion. "Actually, Dean the neat thing about a hexagon is that it is made up of six equilateral triangles. Or regular triangles. And, if this is 25," as he pointed to the side of the hexagon, "this side must be 25, from here to here." He gestured toward the distance from a vertex to the centre of the hexagon. "Believe it or not, I could find the base of this trapezoid."

"50," someone shouted.

"That is 50," Mr Wilchuk confirmed. Then he went back to the original question. "Do I have all my numbers?" And discovering that indeed he did, he proclaimed, "Hey, I can do this!"

A student wrinkling her nose caught Mr Wilchuk's attention.

"Okay, you may not like this method," Mr Wilchuk said to the student, "but, nonetheless, I can do what Dean suggested. Dean, can I get you to focus on *this* shape?" Mr Wilchuk indicated the pentagon on the overhead. "Does your method still work when I change the shape?"

"No," several students concluded.

"Okay, so even though his procedure will work for one, it's not universal in that it will work for all the regular polygons? Now, Mike's method. He's maybe saying, 'You didn't give my method a chance.' His method may have worked had I gone through it completely with him. Would his method work down here with the pentagon?"

"Yes," said Mike.

Mr Wilchuk questioned, "You can break it up into a rectangle and two triangles?" With that, Mike agreed that his procedure would not work for the pentagon and played at looking deflated.

Allen laughed. So did Mr Wilchuk.

"Okay, Allen, going back to yours." Then he teased, "We'll criticize *yours* in a minute. So, we'll see."

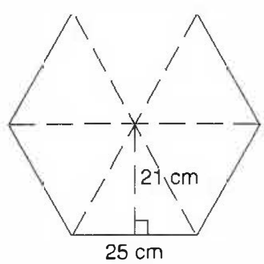
Others laughed, anticipating that Allen might be proven wrong.

"There's nothing wrong with it," Allen asserted, smiling.

"He says 'nothing's wrong with it.' Well, we'll see. You'll have to excuse my rough drawings here but I'm supposed to have six regular triangles. Before *you* draw, we better make sure that everybody's comfortable with this material. Okay, Allen, would you explain your procedure."

Allen explained, "You divide the hexagon into six triangles and use the area of a triangle formula to calculate the area of each triangle (Figure 4). Then, you multiply by 6."

**Figure 4. Allen's Method**




Mr Wilchuk asked Allen, "Dean's method worked for the first question. Is your method going to work for this guy, the pentagon?"

"Nope." Allen quickly accepted defeat for comic effect.

The students laughed.

"Why don't you show us the right method?" Jason asked.

"Slow down," said Mr Wilchuk, responding to Jason. "Okay, Allen, why didn't you say your method worked for the second one? I'm confused. Your method worked so nicely here."

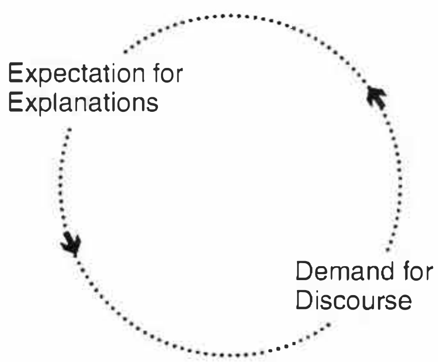
"Yes, but the second one ..." And so the conversation continued.

## Prompting Explanations and Promoting Discourse

In both of the classes observed, students were required to participate actively in the lessons, and teachers were very good at making sure students did so. Also common was that the discourse both between the students themselves and between students and the teacher was not simply around the answer to a mathematics question or problem (for example, calling out the answer to homework questions) but also consisted of explanations for the answers or explications of how a problem could be solved.

From our observations and analyses, we have come to understand the relationship between discourse and explanations as reflexive (see Figure 5). On one hand, the classroom discourse was fostered by the teacher's requests for explanations from the students; in complying, the students participated in the discourse. On the other hand, the expectation that the class be highly interactive, especially in terms of discourse, meant that as part of their teaching of mathematics teachers requested students to participate in the explaining.

**Figure 5.**
**Reflexive Nature of Explanation and Discourse**




Of particular interest to us was our observation that the explanations had two distinct sources depending on the teacher observed (and sometimes depending on what mathematics was being taught). Further, the discourse was intended to serve one of two purposes depending on the source of the explanation. In the first discourse pattern, we observed the teacher Mr Flynn to be the source of the explanation. That is, the teacher offered a mathematical explanation to the students and the discourse around this explanation was directed toward students making sense of the teacher's explanation. We will call this form of discourse teacher source/student matching discourse. Notice that this form of discourse assumes a preferred explanation and the need for students to construct or acquire the teacher's explanation. This was a common form of discourse in the classes we studied in the Teaching Practices Project.

The second discourse pattern is different from the first in that the students are the source of the explanations. This discourse pattern involves the teacher facilitating a discussion about a particular topic and soliciting student explanations. In the second vignette, we note how Mr Wilchuk encourages the students to offer their explanations in the discussion and for discussion. Here the students' explanations are treated as legitimate possibilities, and there is an acceptance of multiple explanations. Students are not expected to come to an understanding of a single explanation offered by the teacher; rather, they are expected to actively listen and participate in the explanations of others, formulate their own explanations and offer those explanations as contributions to the class. In this discourse pattern, which we will call student source/student fitting discourse, there is construction of mathematics knowledge in community with others. We witnessed this discourse pattern in a few of the classes we observed.

Based on our observations, the teacher's demand for talk and interaction around an explanation did not seem to depend on who was the initial source of the explanation. In both classes students were expected to actively participate in the large group discussion. In Mr Flynn's class, the students' role in discourse was to match the teacher's explanation, and in Mr Wilchuk's class, the students were expected to talk out their explanations to see how they fit with the explanations of others in the class. Any distinctions between the discourse patterns with respect to the meaning students made of the explanations was beyond the scope of this study. However, we think this is an important question and we would like to explore (in another study perhaps) the qualities and growth of student mathematical understanding in the two situations: when the students are themselves the source of the explanations and when the teacher is the source of the explanation.

From observing these teachers in action, we have been prompted to think about the importance of highly interactive classes where explanations are the focus of discussions. In classes where students actively participate in discussions, there is plenty of opportunity to learn *why* and *how* mathematics works—not just that it works. We invite teachers to reflect on their own teaching practices and ask themselves about the role explanation plays in their mathematics classes and to what extent they foster discourse around mathematical explanations. Based on our observations these appear to be very important questions for mathematics teachers to consider.

## Notes

1. This research was supported by a grant from Alberta Learning, Curriculum Standards Branch.

2. The cases are presented in full in *The Teaching Practices Project: Research into Teaching Practices in Alberta Schools that Have a History of Students Exceeding Expectations on Grade 9 Provincial Achievement Tests in Mathematics*. A copy of this report has been sent to all junior high schools in Alberta.

3. The names of the schools and teachers have been changed to protect anonymity.

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# Perceptions of Problem Solving in Elementary Curriculum

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## Introduction: A Look at the Meanings of Problem Solving

Recent research with intermediate teachers[1] indicates that the phrase *problem solving* often evokes multiple meanings in mathematics teaching and learning (Kajander and Mason 2007). While some teachers support the vision of problem solving espoused by the National Council of Teachers of Mathematics (NCTM) (2000) as "engaging in a task for which the solution method is not known in advance" (p 51), others view it as something to be done after students are taught and only if there is time (Kajander and Mason 2007). The goal of this article is to examine the usefulness and intent of the various meanings of this "problematic" phrase, while shedding light on the best way to engage in effective problem solving with students.

The inclusion of problem solving in mathematical learning is not new. Polya's (1957) famous model is even included in some provincial curricula (for example, Ontario's Ministry of Education 2005), as shown in Figure 1, and is one of the best known outlines of the possible processes involved in problem solving. What is perhaps new in many classrooms is that effective problem solving should be more than having students solve a problem using formulas or methods that *the teacher has previously shown.*

As a Grade 2 teacher in the United States, I was able to experience first-hand the effects of new legislation that required us to drill mathematics facts and algorithms into the minds of our students so that they could survive testing. Each year the same concepts had to be reviewed, because retention was minimal if at all. We met each year as a school staff to discuss ways to better meet the needs of the students with information handed to us from our school board. We discussed at length using problem solving to improve our mathematics test scores. Yet what this actually entailed, according to what we were told, was handing students a sheet of word problems to solve using the algorithm the teachers had already given them to use. It was suggested that we do a similar word problem with the students so that they would follow a similar process. Similar understandings of the implementation of problem solving have also been found in some Canadian classrooms (eg, Kajander and Mason 2007; Kajander and Zuke 2008).

Elementary mathematics curricular goals may refer to problem solving without exploring what the phrase really means. In Alberta, for example, the curriculum endorses the importance of using a problem-solving approach noting that "students need to explore problem-solving strategies in order to develop personal strategies and become mathematically literate" (Alberta Education 2007, 1). Ontario, as well, mentions that "problem solving forms the basis of effective mathematics programs and should be the mainstay of mathematical instruction" (Ontario Ministry of Education 2005, 11). These statements could be interpreted in multiple ways. From these statements in the mathematics curriculum guides, teachers could assume that problem solving means having students read a problem and find a correct solution using a given, previously taught method. Research has shown that this is not the most effective way to use problem solving in mathematics classrooms (Bay-Williams and Meyer 2005; Boaler and Humphreys 2005; Buschman 2004), nor is it the most effective method for teaching mathematics (Askey 1999; NCTM 2000; Van de Walle and Lovin 2006). This article will further examine some alternatives to this common interpretation.

## Problem Solving as Learning

I take the stance that true problem solving involves students really learning something new and not just applying a previously taught strategy to a new example or task. This position underscores the importance of problem solving as learning. As Bay-Williams and Meyer (2005) note, "teacher-directed instruction may help a teacher feel that more topics have been covered, but it reduces the chances that students are (1) making connections with other mathematical ideas and (2) understanding the concepts related to the skill" (p 340). In fact, students should engage in rich problem-solving tasks *in their daily mathematics classroom experience* in order to construct new knowledge and understanding by connecting it to their previous knowledge.

This interpretation of effective problem solving differs from the belief that students must be taught the concepts before they can engage in problem solving (Kajander and Mason 2007; Kajander and Zuke 2007, 2008). One view might be that by assigning the problem-solving questions in the textbook for homework, exercises and tests, students have the opportunity to problem solve. Typically, such problems are really applications of known formulas or methods to new examples. However, the goal with problem-solving tasks should be to allow students to figure out *how* they will solve the problem. The importance is placed on the *method for determining the solution*, as opposed to the solution itself. As McGatha and Sheffield (2006) point out, in problem-solving classrooms "students are pushed beyond simply *finding a right answer to questioning the answer*" (p 79; emphasis in original). The one single right answer is no longer the singular goal of the mathematics classroom; rather, the process taken to find an answer is where the real learning lies. Students should subsequently be given opportunities to discuss how they solved the problem so that they can learn from each other and see different ways of arriving at a possible solution. This is very different from classrooms in which the teacher tells the students how to go about solving problems so that they can arrive at the single right answer in this so-called correct way. True problem-based learning involves students constructing new ideas based on their experiences with appropriate problems, *not* applying known methods to new contexts.

## How Effective Problem Solving Is Accomplished

Effective problem-solving tasks can be implemented as part of a three-part lesson plan (Van de Walle and Lovin 2006). It is important to consider that the actual lesson may take more than a single mathematics class period to finish, depending on the students. In the first part of the lesson, the teacher sets up the current problem to be worked on. The teacher acquaints the students with any previously unknown vocabulary at this time. This portion of the lesson does not include the teacher showing the students a similar problem and how to solve it. After the teacher sets the stage for learning, the students begin to explore the given problem.

The second stage of a problem-solving lesson requires teachers to set up an environment and procedures that are conducive to exploratory learning. While exploring the problem, students may work individually, in pairs or in groups. Students need to be arranged in a way that allows them to share their ideas with each other. During this phase of the lesson, the students work with the problem to figure out a solution method that makes sense to them. As students work with the mathematics concepts embedded in the problem, they should record their thoughts to share during the final portion of an effective problem-solving lesson—the discussion.

Discussion is an absolutely essential phase of the problem-solving method because it allows students to come together and share while explaining their thinking. As Boaler and Humphreys (2005) note,

> students are not asked to present their answers; they are asked to show representations of their ideas and to justify why they make sense. None of the audience members will have the exact same answer, and all the students have a role. (p 50)

Not only are students more engaged while discussing ideas with their peers, they also learn more from each other and discover new ways of thinking about a problem. Students need to be able to put their solutions into words and discuss how they solved the problem so that they can explain their methods to others. This forces students to get at *how* their solution was found, not just what they decided was the correct answer. It is important that students learn "to *question the answers* by posing additional questions when solving the original problem [because this] is one way that teachers and students can develop mathematical power" (McGatha and Sheffield 2006, 79; emphasis in original). It is this power that helps further students' understanding of and learning in mathematics. Boaler and Humphreys (2005) suggest using the method of "convincing a skeptic" when trying to explain the solution the students came up with (originally from Mason, Burton and Stacey 1982). Their belief is that "this strategy … helps place responsibility on the person who is explaining to make his (sic) explanations understandable and gives permission for anyone who doesn't understand *yet* to play the role of being unconvinced rather than being just slow to catch on" (p 67; emphasis in original). Students are given the opportunity to question each other and refine their thought processes until everyone sees why the solution method works. Seeing alternative solutions is important because "if their knowledge is limited to the computational procedure without any idea why the procedure works, this is also not enough to build on. Students need both" (Askey 1999, 3). Through exploring a problem and discussing the solution, students learn how and why their method and procedures work and gain deeper mathematical understanding. At this point, teachers can help students see the generalizations or the procedures that are being developed through examining the students' solutions. Teachers play an important role in fostering this development of ideas. Since students are sharing their knowledge and understandings, or even misunderstandings, during this portion of the lesson, the teacher must create an environment where all contributions are valued and allowed to be expressed.

In order to use the problem-solving method effectively, students must be given opportunities to share their solution methods so that the teacher can see where any misunderstandings or confusions lie. These essential discussions also allow students to learn from each other. This very important aspect can be the deal breaker for the success of problem-based lessons if the teachers do not allow time for sufficient sharing of ideas. Sometimes issues or difficulties that arise during the discussion can prompt the teacher to suggest a new problem for the next class.

In a problem-solving lesson as just described, problem solving is the vehicle for knowledge and learning instead of simply the way that students showcase what they have learned. One issue with doing problem solving after the teacher has taught a concept is that students have trouble switching from a teacher-directed lesson one day to a lesson in which they control the learning path the next (Van de Walle and Folk 2007). Also, if students are to truly engage in problem solving, they need to know that the teacher is not about to step in and tell them the strategy or the answer eventually. If they feel that this will happen, my experience is that some students will simply wait for the instruction or answer to come from the teacher and therefore will not deeply engage in the task. In other words, they feel that their work will be devalued eventually when the teacher provides the right answer or method, and they have simply learned to wait for this to happen. By engaging in problem-solving lessons as the main curricular vehicle, students learn that their thoughts and ideas are important and are correct ways to solve a problem.

Teachers should choose problems that allow students to explore and construct knowledge for the big ideas or the overall expectations of the grade level. This allows the teacher to address multiple curricular expectations in one problem while, at the same time, addressing the needs of different learners. The benefit of well-chosen problems is that they "can be solved at different levels of sophistication, enabling all children to access the powerful mathematical ideas embedded in the problem" (English, Fox and Watters 2005, 156). For example, a problem like the handshake problem could be used with a class: "There are 20 students in a class. On the first day, the teacher asks each student to shake hands with each other student. How many handshakes were there?" (Small 2008, 567; similar problem in Kajander 2007). In order to make the problem accessible to all students, the teacher could make several different size classes, starting with 5 students and ranging up to 20. Students who are more advanced could begin looking for patterns in the different numbers in order to make the problem more challenging, while students who are struggling can simply tackle what would happen if five students shook hands. By having students solve the problems from their ability level, all students are engaged and learning from each mathematics lesson.

The problem-solving approach also allows all students to be included in the discussions. Students choose methods to solve the problem that make sense to them, which is more meaningful than just repeating what the teacher has said. Using problem solving in the classroom allows all students to reach mathematical understanding at a level that they are comfortable with. Since the goal is to have students use their prior knowledge, all students will be able to work with the problems using what they already know to build their own new ideas. As the Alberta curriculum asserts, "students learn by attaching meaning to what they do, and they need to construct their own meaning of mathematics" (Alberta Education 2007, 1). Teachers can also use this baseline knowledge to help students to come up with new ideas and more effective solution methods instead of teaching formulas that students apply without really understanding. For example, the solution to the handshake problem could be arrived at in many different ways, including drawing a picture, acting out the problem with children, looking for patterns or even solving algebraically. Students would be able to solve the problem with their own solution methods, but during the discussion would be exposed to all the different methods and thereby learn from the other students. The problem itself can be used to teach or review addition, multiplication, division, geometric patterning, numeric patterning, pattern rules and iterative patterns (Kajander 2007), depending on how the teacher guides the students through the discussions and what areas are highlighted as students present their solution methods.

One caution does need to be made in choosing effective problems to solve in order for students to gain the most benefits. Teachers should avoid forcing too much content into a single lesson; therefore, each lesson "focuses on investigating one rich problem, probing deeply into a different mathematical content strand each day" (McGatha and Sheffield 2006, 79). By narrowing the focus to one main concept each day, teachers can allow students to look further into the problem in order to reach a deeper understanding. For example, simply introducing a simple problem like the handshake problem with different-sized classes would allow the students to explore the solution methods; the teacher could then guide discussions to accomplish the necessary curriculum goals. By focusing on one problem, students are not overwhelmed by a worksheet full of problems and could be challenged to come up with multiple solution methods. Another important consideration is that, as one teacher said, "too much choice could be overwhelming for the children and difficult for me to manage" (Whitin 2004, 181). Putting too much into one lesson is not only hard for a teacher to organize and observe, but it can confuse students and prevent them from delving deeply into the topic being explored.

Another benefit of using problem solving extends beyond the mathematics classroom. Since the goal is not for teachers to show students a formula and the exact method to solve the problem, students use their own problem-solving skills to solve the problem. This can affect students' lives—not only do students learn mathematical concepts with deep understanding, they also gain skills that enable them to solve problems in their daily lives. The benefits of using problem solving and allowing students to learn how to solve a problem in their own daily lives are great. I turn now to showing how this can fit within the mathematics curriculum.

## Examples of Using Problem Solving with Ontario and Alberta Curriculum

It is my experience that curriculum guides mention problem solving while not explicitly laying out how to use problem solving in a classroom. For example, in the curriculum I am most familiar with, the Ontario Ministry of Education sets out several mathematical processes that should be included in the elementary curriculum: "problem solving; reasoning and proving; reflecting; selecting tools and computational strategies; connecting; representing; communicating" (Ontario Ministry of Education 2005, 11). These processes are listed as separate entities; yet an effective problem-solving approach to teaching would encompass all of these processes and would, therefore, be the only method necessary to accomplish these curricular goals. After being given a problem, students would have to *select tools* and the *computational strategies* needed to solve the problem. Since the students would be using prior knowledge to pursue a solution, they are *connecting* the new concept to previous knowledge and skills. Students would then be required to *reason* through their solution and *prove* that it works to the class and teacher. Through the discussion of the solution, students would have to *reflect* on whether or not their method makes sense in order to solve the problem. By sharing their solution with others, students would be required to *communicate* their thought processes and *represent* the solution so that others can see how they solved the problem. Using problems with a similar focus on different days would allow students to practise their skills and create more in-depth conceptual knowledge. By using a problem-solving approach to teaching, teachers are able to simplify their planning while meeting all the goals of the curriculum.

In Alberta, the curricular goals identified are that students will "use mathematics confidently to solve problems; communicate and reason mathematically; appreciate and value mathematics; make connections between mathematics and its applications; commit themselves to lifelong learning; [and] become mathematically literate adults, using mathematics to contribute to society" (Alberta Education 2007, 2–3). As with the Ontario curriculum, the curriculum guide provides the goal of using problem solving in the classroom as part of the routine. Effective problem solving would also help to accomplish the other goals by giving students ample opportunities to use mathematics in meaningful ways that will benefit students throughout their lives. Where Alberta's curriculum differs from Ontario's is that it explicitly states that students "must realize that it is acceptable to solve problems in a variety of ways and that a variety of solutions may be acceptable" (p 1). This statement lends itself more to the problem-solving methods described in this article, but teachers should keep in mind that this should not mean giving students different solution methods but allowing them to discover multiple solution methods. The two curriculums mention the importance of problem solving, but it is my experience that teachers are often left to their own devices to locate the problems that would address these goals. Some resources that I have used and found successful with students include *50 Problem-Solving Lessons: Grades 1–6*, by Marilyn Burns, and

*Big Ideas for Small Mathematicians*, by Ann Kajander. The problems in these books mention different curriculum strands addressed by the problems, and a single problem could be used in a lesson to give students a chance to delve deeply into the topics. In order to encourage more problem solving, the Ontario Ministry of Education has also created its own lessons that relate to the curriculum for Grades 7 to 10, called *Targeted Implementation and Planning Supports* (TIPS). Textbooks might also be a helpful tool—teachers could choose or create a single problem from the lesson that students could explore on their own in order to determine their own solution methods.

In Ontario, the Ministry of Education (2005) does provide a valuable framework that can be used with students while exploring a problem. During the exploratory phase of problem solving, the Ministry of Education suggests using Polya's problem-solving model (see Figure 1) to guide the students in thinking about how to solve a problem. The belief is that teachers should guide students in Grades 1 and 2 through the model without directly teaching the steps, whereas students in Grade 3 and above should be taught the terminology of each step of the model directly. For Grades 1 and 2, a simpler way of remembering the steps can be beneficial.

**Figure 1.**
**Polya's problem-solving model (Ontario Ministry of Education 2005, 13)**

**Understand the Problem**
**(the exploratory stage)**

- reread and restate the problem
- identify the information given and the information that needs to be determined

*Communication*: talk about the problem to understand it better

**Make a Plan**

- relate the problem to similar problems solved in the past
- consider possible strategies
- select a strategy or a combination of strategies

*Communication*: discuss ideas with others to clarify which strategy or strategies would work best

**Carry Out the Plan**

- execute the chosen strategy
- do the necessary calculations
- monitor success
- revise or apply different strategies as necessary

*Communication*:

- draw pictures; use manipulatives to represent interim results
- use words and symbols to represent the steps in carrying out the plan or doing the calculations
- share results of computer or calculator operations

**Look Back at the Solution**

- check the reasonableness of the answer
- review the method used: Did it make sense? Is there a better way to approach the problem?
- consider extensions or variations

*Communication*: describe how the solution was reached, using the most suitable format, and explain the solution

Thomas (2006) has suggested the use of the THINK strategy to get students organized in their thinking, which could be a useful mnemonic for Grades 1 and 2. First, students talk about the problem. Second, students look at how the problem could be solved. Third, students identify a strategy for solving the problem. Fourth, students notice how the strategy helped solve the problem. Finally, they keep thinking about the problem. As students continue working on the problem, they may need to cycle through this framework several times until they arrive at a solution that makes sense to them. According to the research study, Thomas notes that "students who used THINK demonstrated greater growth in problem solving than students who did not use the framework" (p 86). The use of a model is beneficial because a teacher who is aware of the model and who uses it to guide his or her questioning and prompting during the problem-solving process will help students internalize a valuable approach that can be generalized to other problem-solving situations, not only in mathematics but in other subjects as well. (Ontario Ministry of Education 2005, 13)

Students could be coached using this model while they are exploring the problem. The Polya model should be directly taught to higher grade levels (Ontario Ministry of Education 2005). While older students are working with the problem, the first step is for them to understand the problem. According to Outhred and Sardelich (2005), "understanding the problem requires children: to be able to read the problem; to comprehend the quantities and relationships in the problem; to translate this information into mathematical form; and to check whether their answer is reasonable" (p 146). Students begin by rereading the problem and deciding what the problem is asking them to figure out. Next, students make a plan for deciding how to solve the problem through examining different strategies to solve it. As Askey (1999) discovered when working with teachers, "the teachers argued that not only should students know various ways of calculating a problem [solution] but they should also be able to evaluate these ways to determine which would be the most reasonable to use" (p 6). Third, students enact the plan that they decide to use to solve the problem. Finally, students assess whether or not the solution is reasonable through re-examining the problem. If the solution is determined to be unreasonable, students would then go back through the model. This model is important because when students are used to traditional instruction they typically do not have the skills and strategies developed to effectively problemsolve (Van de Walle and Folk 2007; Kajander and Zuke 2007). Once students are given a problem, they need to be given a way of organizing their thinking in order to solve the problem.

## Summary

Problem solving may have varying definitions for different teachers, but effective problem solving should allow students to explore a problem for themselves to find a solution. I have argued that problem solving does not involve giving students a method or formula for how to get the answer; rather, it involves giving them a framework to think through the problem and work to develop their own method. Students need a structure to develop problem-solving skills, and this must be supported by peer and teacher-facilitated discussion at certain points in the learning. Neither of these can take place when problem solving is attempted in isolation as homework or on tests. True problem solving cannot happen for most students (or even most mathematicians!) in a time-limited situation such as a test. Students need time to reflect, discuss and try possibilities. Tests are simply not good places to attempt problem solving. While tests might play a role in efficiently assessing procedural skills, learning and assessment tasks are much better vehicles for learning through problem solving. Teachers in an effective problem-solving environment are no longer disseminators of knowledge but facilitators and coaches who help students create their own knowledge. In my experience, one of the most rewarding experiences can be watching students grapple with a problem and come to a solution after they have worked on the concepts within the problem. The excitement and feelings of accomplishment that accompany the final product can be empowering to their mathematical abilities, as well as foster the idea that they, too, can do mathematics! While using problem solving and discussion may be uncomfortable at first, the long-term benefits for both student learning and engagement are phenomenal (NCTM 2000). The goal of mathematics classrooms is to have students learn and understand mathematics, and engaging in effective problem-solving tasks is the best way to accomplish these goals.

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I would like to acknowledge the contributions of Dr Ann Kajander, at Lakehead University, for her invaluable input and guidance in writing this article. This work was funded by the NSERC University of Manitoba CRYSTAL grant for the project "Understanding the Dynamics of Risk and Protective Factors in Promoting Success in Science and Mathematics Education."

## Note

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# Assessing Young Children's Attention to Pattern and Structure

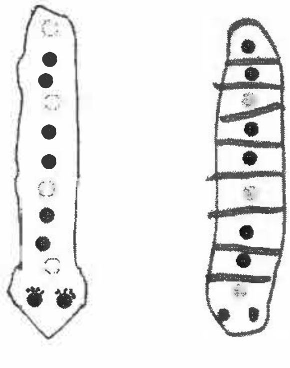
*Lynn McGarvey*

*Note: This article first appeared in* delta-K *volume 48, number 2 (2010).*

> A mathematician, like a painter or a poet, is a maker of patterns.
> —G H Hardy

Mathematics is sometimes described as "the science of patterns" (Devlin 1994; Steen 1988, 611). As Steen (1990) wrote, "Mathematics is an exploratory science that seeks to understand every kind of pattern—patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns" (p 8). Across North America, mathematics curricula in the early years emphasize the importance of pattern as a way for children to make connections to the world around them and as the foundation for algebraic thinking (NCTM 2000). From pre-K to Grade 2, children are expected to recognize, identify, duplicate, extend and translate simple sequential patterns using a variety of attributes including sounds, actions, colours, shapes, objects and numbers. Early childhood classroom walls are often adorned with a variety of colour- and shape-patterning products. However, these products often don't reveal the range of mathematical reasoning that takes place when the patterns are made. For example, examine the patterns in Figure 1 created by Jun and Mason, both age 6. Both children have created a similar repeating pattern successfully and independently, but their reasoning about patterning is very different.

**Figure 1.**
**Jun's (left) and Mason's (right) repeating patterns**



Jun described her pattern as "yellow-blue-blue-yellow-blue-blue-yellow ..." and pointed to each dot on her snake. When asked to describe her pattern, she said, "There are two blues between the yellows." And when asked how many dots made up her snake, she pointed and counted, "1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and then 11, 12 for eyes." Jun has met many of the expectations for repeating patterns, and we might assume that she knows repeating patterns well, but the curriculum expectations do not provide a clear indication of what teachers should be looking for in children's descriptions and do not help teachers recognize the link to algebraic thinking that underlies patterning activities. Jun's interpretation of the pattern as "two blues between the yellows" makes it difficult for her to see the structure of the pattern as a whole.

Mason's response provides a contrast in experience and reasoning about patterns. As Mason was making his pattern (before the lines were drawn), he was asked to describe his pattern he said, "It's a red-red-green pattern. That's the core. Do you want me to circle the core?"

"No, that's okay. Just keep making the pattern for your snake."

"I could change it by putting a green dot at the beginning [tail] and make it a green-red-red-green pattern ... No, wait. It would just be a green-red-red pattern, but I'm just going to keep it [as red-red-green]." He finished putting down his dots and I asked, "You used the word *core*. How many times does the core repeat?"

"Three."

"Do you know how many dots you used for your pattern?"

"Uh ... nine."

"Oh [expecting him to count]! How did you get that?"

"Well, I know that six and three is nine, so it was easy."

"Where did the six come from?"

"Two of these [two units of the core] are six and one more is another three. So nine."

Mason's description of his pattern, his identification of the pattern core, his flexibility in counting the core units as a group of three dots and then using that information to determine the number of dots altogether provide a solid basis for later understanding of multiplication, algebraic expressions and functional relationships.

This paper provides an example of a repeating-patterns assessment task that can be used with children aged 4 to 8. The task and variations of it reveal children's reasoning about patterns. Four types of reasoning are shown to orient teachers' attention during the patterning process and also provide guidance for instruction. Although the content of the task uses repeating colour patterns—which are the simplest form of pattern and attribute—the task may easily be adapted for other repeating patterns (for example, border, hopscotch) with a variety of visual attributes (for example, shapes, objects).

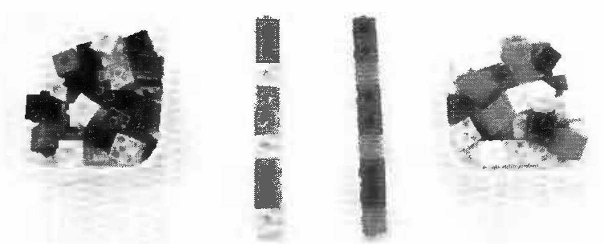
## Repeating Patterns Assessment Task

(adapted from Papic and Mulligan 2007)

### Materials and Preparation

- Connecting cubes in six colours: Create a two-colour ABB tower (for example, yellow-green-green) (see Figure 2) and a collection of individual cubes in the same two colours, plus a third colour used as a distracter (for example, black). Create a second two-colour ABB tower in different colours (for example, orange-blue-blue) and a collection of individual cubes in the same two colours, plus a third colour (for example, white).

**Figure 2.**
**ABB Towers**



- Strips of legal size paper cut in half (that is, 5.5" × 14")
- Coloured dot stickers in three or four colours
- Markers

### Set-Up

Working with pairs of students, give each child an ABB tower and coloured blocks (see Figure 2).

### Assessment Task

The following questions represent many of the outcomes for patterns in the early grades including identify, describe, copy, extend, compare and create patterns.

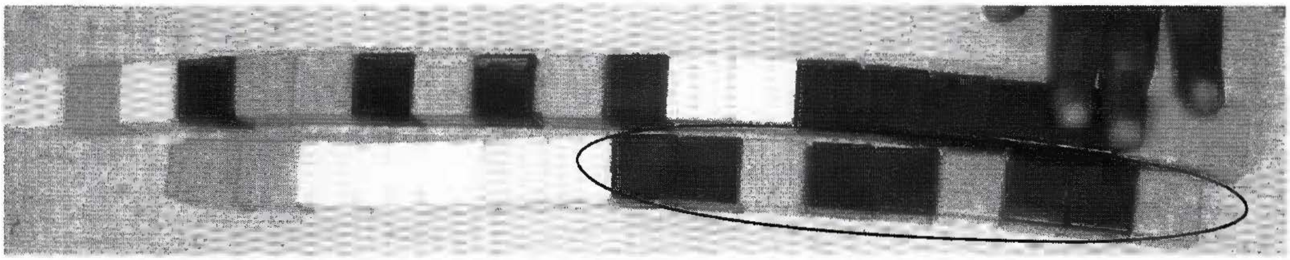
1. **Identify/describe:** Give each child a premade ABB tower and ask, "Is it a pattern? Why do you think it is (or isn't)? Describe the pattern." For kindergarten, ask, "What is the part that repeats?" For Grades 1 and 2, ask, "What is the pattern core?"
2. **Reproduce:** Give each child a set of individual cubes (two of the correct colour and a third colour) and say, "Make a tower exactly the same as this one." Depending on the children's previous experiences, either leave the tower on the table for them to make comparisons (preschool to Grade 1) or show the tower for five seconds and then hide it (Grades 1 to 3). If they have difficulty, show it again for a few seconds. If they still have trouble, leave the tower out to be copied.
3. **Extend:** "Can you add more blocks to your tower? What would come next on the tower? How do you know that block comes next?"
4. **Compare**: Have the pairs of children compare their towers with each other and ask, "How are the two towers the same? How are they different?"
5. **Create**: Remove the blocks and give each child coloured circle stickers and a strip of paper. "Make your own pattern with coloured stickers."
6. **Identify/describe**: "Did you make a pattern? How do you know? Describe your pattern. What is the pattern core? How many times does the core unit repeat?" It may be helpful to have the child circle the core units with a marker.

This assessment task may be modified for a variety of materials and for the experiences of the children being assessed. The general goal of the assessment task is to understand the children's reasoning about patterns. Not every question needs to be asked, and modifications may be made depending on the child's responses. The next section provides a range of children's patterning strategies, from preschool to Grade 2, in response to aspects of the assessment task.

## Children's Attention to Pattern and Structure

The assessment task is not a measure of understanding, but an indicator of how children perceive patterns and what strategies they use when working with patterns. The information gathered is intended to inform instruction. In this section, four types of

**Figure 3.**
**Inattention to pattern and structure**



responses are provided based on working with children from age 4 to 7. The range of responses is not intended to be developmental—that is, children will not necessarily go through each phase. In fact, children will attend to patterns differently, depending on the attribute. For example, children are often very successful with patterning tasks that focus on colour patterns, but they might have difficulty when patterns focus on shape, sound or other attributes. Differences in children's responses, such as those seen with Jun and Mason at the beginning of the paper, are due primarily to previous experiences and instructional orientation.

### 1. Inattention to pattern and structure

When asked, "What is a pattern?" Abed (age 4) did not have a definition or description. Not being able to define a pattern is not necessarily an indicator of understanding, so the assessment continued, and Abed was asked to make a copy of the orange-blue-blue tower he was given. Although I tried to encourage him to build the same tower, he either did not understand or was not interested. He was eager to build another tower, but he did so by randomly putting the blocks together (see Figure 3). When it got too long and started breaking apart, he began adding blocks to the original tower. The circle around the blocks in Figure 3 shows the original tower that remained intact. Abed appeared very motivated to build with the blocks, but he did not attend to the pattern as he did so.

### 2. Direct comparison strategy

Sophie (age 5) was given the yellow-green-green tower and was asked, "Is it a pattern?" She responded, "Yes," and described it as "yellow-green-green-yellow-green-green-yellow-green-green" as she pointed to each block in the tower.

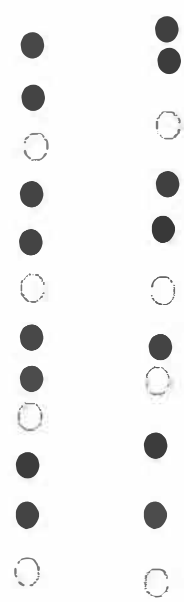
"How do you know it is a pattern?" She responded, "Because it has yellow and green and they keep going."

"Do you know what the core is?" She shrugged her shoulders.

"Here are some more blocks. I want you to make a tower exactly like this one, okay?" Sophie got a yellow block and then a green one and put them together. After this initial building she lined up her tower with the premade one to determine which colour would go on next. Her completed tower was identical to the original, but to examine her process more closely I created a revised task.

On a strip of paper I used yellow and green stickers to make a yellow-green-green pattern and asked her to make a copy. This time she also tried to use a direct comparison strategy by placing a finger on the original pattern at the left and putting a matching sticker on her pattern on the right. Once she had placed a sticker, she looked back to the original tower and found a dot just above the height of the sticker she had just placed to find the next sticker in line. Since there were longer gaps in her sticker tower, she missed one of the green stickers in the middle of the pattern.

**Figure 4:**
**Original pattern (left) and Sophie's direct comparison process (right)**



Children who use a direct comparison strategy will often be able to successfully copy patterns when objects fit together; however, they have more difficulty when they are asked to copy a pattern with stickers or stamps or by drawing, because the spaces between elements can vary, and it is more difficult to line patterns up to make a direct comparison. Children using this strategy may say that the original and copy (like that in Figure 4) are the same by looking at it. It is only when they read the pattern and hear the verbal pattern breaking down that they are able to correct the pattern. For example, when Sophie read her pattern, "Yellow-green-green-yellow-green-yellow," she heard the error and said, "Oh! I made a mistake."

### 3. Recursion strategy

Hua (age 6) described his tower as orange-blue-blue-orange-blue-blue-orange-blue-blue and said that it would keep going. He was able to copy and extend the pattern with blocks fairly easily. As he was building and extending the tower I asked, "How do you know which colour comes next?" He had just put an orange on and confidently said, "Blue comes next."

I asked, "And then what?"

"It's another blue."

"Then what?"

"Orange."

"So how do you know what comes next?"

"I look at this one [pointing to the last block put on] and then put the next one on."

Hua's response suggests that he is using a recursive strategy to build the tower. He knows what comes next by looking at the last block that was put on.

Children are often able to produce the expected pattern using a recursion strategy, but it is in the making and extending of patterns that the recursion process appears. A recursive strategy is used frequently by children (and adults), but it becomes less effective with more challenging patterns when the number of elements in the core unit gets longer, when the materials used are less familiar and when the shift is made to number patterns. For example, in the number pattern 4, 7, 10, 13, a child might use a recursive strategy of *plus 3* to determine that 16 comes next. However, the only way the child can determine, for example, the tenth number in the pattern using a recursive strategy is by adding 3 until the tenth number is reached.[1]

### 4. Core unit strategy

I showed Reagan (age 7) the yellow-green-green tower for three or four seconds and then put it behind my back and asked her to make the same pattern with stickers. She quickly and easily placed the stickers on the page.

**Figure 5:**
**Reagan's copy of tower using pattern unit strategy**

"Wow! That was fast. How did you know how to build the tower?"

Reagan said, "I remembered yellow-green-green [core unit] and there were three of them [units]."

If a child can see the pattern core, she doesn't have to remember every single block. Reagan showed that she needed to remember only the core unit and how many units there were. Looking for a core unit allowed Reagan to look for a relationship between the number of times the core unit is repeated and the number of elements in the core. Reagan demonstrated flexibility in being able to count with units other than one. A core unit strategy is also directly related to identifying a relationship in a function. A functional approach allows a person to determine any number of elements in a pattern without having to know all of the numbers in the sequence.

## Conclusion

Human beings are naturally inclined to make sense of their environment by searching for patterns in images, objects and events. While early patterning activities might produce pretty pictures for classroom walls, supporting young children's understanding of patterns provides an excellent starting place for mathematical thinking. This paper provides an example of an assessment task, but the questions asked during the task are also important for daily instruction in patterns:

- Is it a pattern? Why do you think so?
- How are the two patterns the same? How are they different?
- What is the pattern core? How many times does it repeat?

Instruction needs to draw children's attention to what is and what is not a pattern, finding similarities and differences in patterns and the structure of patterns by attending to the pattern core. Our assessment of children also needs to shift, from the patterning products that children produce to the reasoning and strategies they use in the process of copying, extending, comparing and creating patterns. Without a shift in our instruction and assessment, many children will continue to be successful in the outcomes related to patterns by focusing primarily on the repeating elements in a pattern (for example, red-green-red-green), but an understanding of patterns requires attention to the core unit that repeats (for example, red-green repeated three times). Understanding units and flexibly counting and comparing units are essential in many topics in mathematics, including place value, measurement, fractions, multiplication and unit circles in geometry. Patterns are at the heart of mathematics and mathematical thinking. Early childhood educators have the opportunity to help children see mathematics as the science of patterns, rather than just as exercises in counting and computation.

## Note

1. The more efficient alternative is to determine a functional relationship. In the example of 4, 7, 10, 13, the function rule is 'times 3 plus 1.' The tenth number would be $10 \times 3 + 1 = 31$.

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Teaching Ideas

# Open Search in Elementary Mathematics Experiences

*T P Atkinson*

*Note: This article first appeared in the* Mathematics Council Newsletter *volume 8, number 4, pages 4–6 (1969). This publication was renamed* delta-K *in 1971. Minor changes have been made in accordance with current ATA style.*

> The aim of elementary school mathematics is to foster continuous and maximum development of each child's potentialities in terms of the affective domain, the cognitive domain and the psychomotor domain.
> —Department of Education 1969

The teacher is challenged to be aware of the behavioural domains and the child's need for growth in each of them. My purpose is to examine Bloom's plan for classifying educational objectives within the cognitive domain (Bloom 1956), to examine a modification of Bloom's taxonomy suggested by Avital and Shettleworth (1968) and to make some suggestions that the teacher can follow. In a sense my purpose, on a small scale, is the same as that stated by Sanders (1966), who used the social studies as the vehicle for the presentation of his ideas.

> The objective of this book is to describe a practical plan to insure a varied intellectual atmosphere in a classroom. The approach is through a systematic consideration of questions that require students to use ideas, rather than simply to remember them.
> —Sanders 1966, 2

Bloom's taxonomy postulates six major levels of educational objectives, the categories being named Knowledge, Comprehension, Application, Analysis, Synthesis, Evaluation. The categories are assumed to be arranged sequentially, in that performance in any given category depends upon mastery of related materials in the preceding categories. It is also understood that by virtue of different degrees of experience, two persons may place an objective into different categories.

A set of multiple-choice questions, each applicable at some stage in the elementary school, is presented to illustrate the first five categories. The level Evaluation is omitted because it is difficult to illustrate with mathematics as the vehicle. [See Figure 1.]

Avital and Shettleworth identify three levels of mathematical thinking which they associate with objectives that fit the five levels in the taxonomy. In tabular form (Sanders 1966, 6, 7) they are

**Thinking Process**
1. Recognition, recall
2. Algorithmic thinking, generalization
3. Open search

**Taxonomic Level**
1. Knowledge
2. Comprehension, application
3. Analysis, synthesis

It is the third thinking process that gives me the title to my presentation.

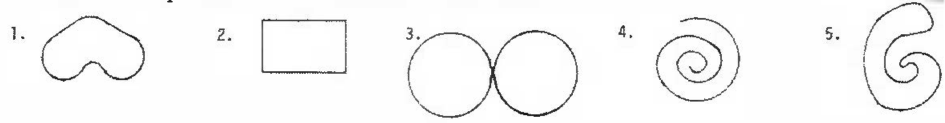
Among the mathematical experiences provided for elementary school children, all three thinking processes must be developed and practised. It is essential that children recognize and recall mathematical concepts, facts, terms and symbols. They must be able to develop, understand and use algorithms; they should be asked to generalize from data. However, we teachers fail to provide enough opportunities for open search. We create the impression that in mathematics there is one correct answer for a question and the pupil's sole responsibility is to learn techniques for producing that answer. We attach considerable importance to the one technique we think is the best and tend to downgrade all others.

In open search the pupil should be permitted, encouraged, forced, to use his ingenuity, and praised when he does so. In the synthesis example quoted earlier, the child who recognizes the pattern and simply writes the terms of the series, keeping a running total

**Figure 1.**
**Educational Objectives**

**Knowledge**

Which of the pictures does not show a closed curve?




**Comprehension**

If 16 × 16 = 256, which product is equal to 25,600?
1. 160 × 160  2. 16 × 160  3. 256 × 10
4. 160 × 16  5. 530 × 520

**Application**

A merchant sold canned corn at 2 cans for 35 cents on Friday and at 3 cans for 50 cents on Saturday. How much would a shopper save if she bought 12 cans at the cheaper rate?
1. 20 cents  2. 10 cents  3. nothing
4. 15 cents  5. 5/6 of 1 cent

**Analysis**

A gardener has two plots of land of equal area. One is square and the other is rectangular with its length four times its width. What is the ratio of the two perimeters?
1. 5/4  2. 5/1  3. 1/1
4. 3 /4  5. 3/1

**Synthesis**

Examine the pattern in these number sentences.
1 + 3 = 4
1 + 3 + 5 = 9
1 + 3 + 5 + 7 = 16
1 + 3 + 5 + 7 + 9 = 25
What is the last number in a set of this kind whose sum is 400?
1. 20  2. 39  3. 41
4. 72  5. 100

until the sum reaches 400, is using all three thinking processes. Do not worry about more sophisticated techniques; they will be developed in due course.

In open search there may be assumptions to be made, differences of opinion to be discussed, incomplete information to be supplemented. Consider the problem:

> A box contains steel bars, some weighing 4 pounds each and others weighing 6 pounds each. The total weight of the box and its contents is 50 pounds. If you removed all of the 6-pound bars, what would the box and its contents weigh?

Let two or three alert children at any grade level from 2 up wrestle with the problem. If they have not been too severely conditioned by the mathematical training they have encountered to date, their solutions should be interesting.

It is difficult to provide a group of teachers with problems that meet the dual criteria of being applicable to the elementary school and demanding the open search. I have prepared three activities for you to consider. You will find these in a future issue of the *Math Bulletin*.

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# Introducing Contemporary Concepts in Traditional Arithmetic

*James M Grasley*

*Note: This article first appeared in the* Mathematics Council Newsletter *volume 3, number 1, pages 15–18 (1963). This publication was renamed* delta-K *in 1971. Minor changes have been made in accordance with current ATA style.*

Although most elementary schools are still using traditional arithmetic texts, it is possible for teachers to introduce phases of the "new mathematics" without special fanfare or radical measures. Many teachers hesitate to stray too far from the text and its traditional approaches. If you were to try some of the ideas that follow, you might introduce the flavour of new mathematics into the present course. The content material is clearly traditional but its vocabulary and presentation are that found in contemporary approaches. Once you get used to the unfamiliar vocabulary of the contemporary material, you will see that most ideas presented in new mathematics are quite familiar. I am sure that as the new terms become familiar you will find yourself introducing them where they seem appropriate to your traditional teaching. Maybe there is some comfort in the fact that your pupils have less trouble with the new vocabulary than you will.

As you study the contemporary ideas being advanced, perhaps you can suggest procedures which will allow children to work with familiar operations and interpret them in the light of the new mathematics. By so doing we need not wait years to have the best of the new approach incorporated into our school program. It can be done now.

## Number Line

Some people find that the number line is a good way to introduce new material. Most traditional texts do very little with the number line, and its use to supplement the arithmetic program is excellent.

Careful study of the use of the number line by teachers, from primary to high school, will reveal much material and many approaches for classroom use. Many teachers find use of the number line helps pupils clarify their thinking. Why not have a number line painted on the floor in the room? Children could step off addition and subtraction facts on it. They could use the line to discover the commutative property of addition—from 2 you step off 3 additional spaces and you are at 5: $2 + 3 = 5$. Also from 3, by adding 2, you are at 5: $3 + 2 = 5$.

The associative property of addition could also be discovered by the pupils themselves. After doing the operations physically by stepping them off on the line, they could generalize that $(2 + 3) + 4$ is exactly the same as $2 + (3 + 4)$ and might even relate it to the previous idea of equality with $4 + (3 + 2)$. Whether or not the terminology, associative property and commutative property are used depends on the teacher and class. However, what is important is the idea.

Perhaps a bright pupil might ask why just one side of zero is used. What about numbers to the left of zero? I am sure that such an opportunity would arise long before high school, and negative numbers could be introduced. Using the number line on the floor, children could soon be doing examples like $3 + (-4) = -1$ and $(-4) + (-5) = -9$. With a little imagination, some reading of current articles on new mathematics and a few carefully planned lessons, most teachers could supplement their traditional programs with ideas from the new, modern mathematics.

## More Than One Name for a Number

Have a set of objects ready for your class to see. Suppose you use five objects in a set—the four fingers and thumb on the left hand, the five pussy willows on a twig, or the five girls in the row of desks next to the windows. Have different pupils write on the chalkboard ways of representing the number in the set. You could get 11111, 5, V, *five* and so on. You could encourage discussion of different ways of naming 5 such as $(4 + 1)$, $(3 + 2)$, $(5 \times 1)$, $(7 - 2)$. You could set up a table.

| | |
|---|---|
| $5 = 4 + 1$ | $1 + 4 = 5$ |
| $5 = 3 + 2$ | $2 + 3 = 5$ |
| $5 = 5 \times 1$ | $1 \times 5 = 5$ |
| $5 = 7 - 2$ | $6 - 1 = 5$ |

Emphasize that the numerals on either side of the equal sign are ways of naming the same number, 5. That these are *numerals*, not numbers, should be stressed, for this is basic to the new approach.

This idea of renaming numbers can be used profitably in practice and review exercises. For example, pupils may express 14 in different ways as the sum of two whole numbers:

| | |
|---|---|
| 14 = 7 + 7 | 14 = 13 + 1 |
| 14 = 2 + 12 | 14 = 0 + 14 |
| 14 = 3 + 11 | 14 = 5 + 9 |
| 14 = 4 + 10 | 14 = 8 + 6 |
| 14 = 6 + 8 | 14 = 11 + 3 |

Or express 14 as the difference of whole numbers three, five or a specific number of ways.

The exercises suggested above can be used by pupils to discover properties of numbers. For example, if they have written 17 as the sum of two whole numbers, they may be asked, "Did you use odd numbers to name 17 as the sum of two whole numbers? Did you use an odd and an even number to name 17 as the sum of two whole numbers?"

Some examples of how the idea that numbers have more than one name may be used in traditional arithmetic:

1. Write 6 as the sum of two addends where
   a) One addend is a proper fraction and the other is a mixed number, both fractions having the denominator 6.
   b) Both addends are mixed numbers, the fractions, having the denominator 3.
   c) One addend is the whole number 4 and the other is an improper fraction having the denominator 2.
2. Name each number as tens and ones in two ways.
   a) 27 (2 tens and 7 ones, 1 ten and 17 ones)
   b) 109 (10 tens and 9 ones, 9 tens and 19 ones)
3. Complete the following sentences.
   a) 286 = 2 hundreds + ____ tens + 6 ones
   b) 286 = 2 hundreds + 5 tens + ____ ones
   c) 286 = 1 hundred + ____ tens + 6 ones
4. Name each of the following in three different ways using hundreds, tens and ones.
   a) 405
   b) 312, etc
5. Name the following using thousands, hundreds, tens and ones.
   a) 9,125
   b) 3,047, etc

In addition to the ideas presented above on the use of the number line and on number–numeral distinctions, operations and their opposites and the properties of operations can also be used to blend some contemporary mathematics into the traditional approach.

*Editor's note [from original publication date]: Max Grasley of Saskatchewan has been a tireless worker in that province, in training elementary teachers in modern concepts. He has played a large part in preliminary work for the new course for their new course for Grade IX and is very active in the development of training courses at the University of Saskatchewan for junior high school teachers. This article deals with methods any elementary teacher may use in his interpretation of the "new mathematics."*

# The Laboratory Approach in Mathematics—Calgary Junior High Schools

*Bernice Andersen*

*Note: This article first appeared in the* Mathematics Council Newsletter *volume 9, number 3, pages 1–5 (1970). This publication was renamed* delta-K *in 1971. Minor changes have been made in accordance with current ATA style.*

Mrs Bernice Andersen, mathematics consultant with the Calgary Public School Board, assembled the following reports from several Calgary schools. Our thanks to Mrs Anderson and each of the contributors. The photographs were taken by Sharat C Sharma of the Calgary School Board Instructional Aids Department.

What is happening at *your* school? Please take time to send in a report to the *Newsletter*.

## The "Discovery Method" at Colonel Irvine Junior High School

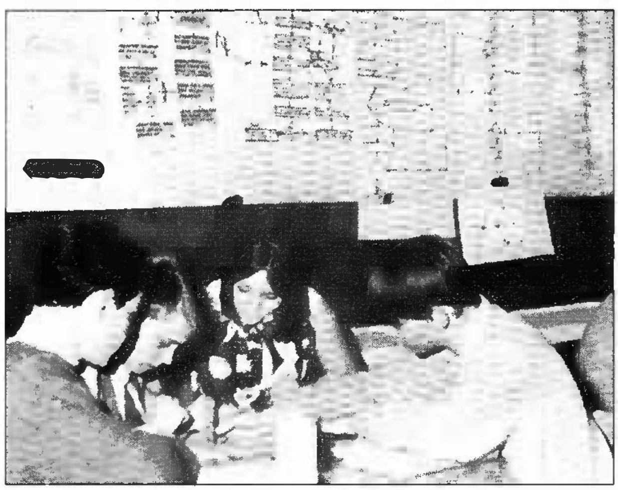
***Rose Makway***

*Mrs Makway teaches junior high mathematics at Colonel Irvine Junior High School, in Calgary, Alberta.*

At Colonel Irvine Junior High, in Calgary, some of the students were introduced to the "discovery method" of studying mathematics for the first time this year. Although we have no actual lab facilities, the regular classroom has been adapted to serve the purpose. Students work in groups at tables with complete freedom to discuss their discoveries and display their results in any way they wish, within the limits of our facilities. Usually no previous instruction has been given, and students discover new concepts with the guidance of activity cards and their own imagination. Although questions are always answered by the teacher, the students are encouraged to discover as much as possible on their own. In a follow-up lesson, each group presents its discoveries leading to the formation of general standard rules such as V = l w h.

In the following pictures a Grade 9 class is being introduced to algebra for the first time. No previous instruction has been given. With the guidance of activity cards and some help from the teacher, the class is discovering the rules for operations with powers and exponents. In a follow-up lesson the groups present their results. The class, together with the teacher's guidance, forms general rules such as $x^m \cdot x^m = x^{m+n}$, using the results of the groups' discoveries.

*In the background of this picture are charts made by a Grade 8 class studying geometric construction. Flow charts were introduced for the first time by using everyday situations such as "How to Comb Your Hair."*

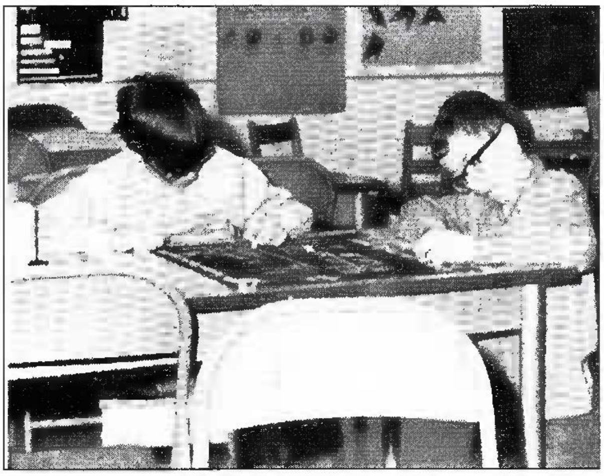


*In the next lesson, flow charts for geometric constructions were developed, using patterns discovered in the previous lesson. These were followed by working out problems that involved using the constructions in new situations.*

# The Workshop Approach at F E Osborne Junior High School

***Chuck Swaney and Wes Larson***

*Messrs Swaney and Larson, working cooperatively in F E Osborne Junior High School, Calgary, have developed a workshop approach in mathematics.*



*Grade 7 students are completing displays for percentage. They took a sampling of student opinion on their choice of an important question and represented the results in percentages. Some questions sampled were "Do you think you want to smoke?" "Should the Americans leave VietNam?" "Should girls be allowed to wear blue jeans in school?" and "Do you like our principal?"*

*The displays on the walls resulted from a workshop approach to mathematics. Some of the problems represented are keeping the area of a rectangle constant while varying the perimeter (Grade 8); prisms and pyramids; finding solutions to matching pairs of congruent sides of 30-60-90 triangles (Grade 9) by Grade 8 students from nets.*

# Studying Ratio by the Lab Approach at Sunalta Junior High School

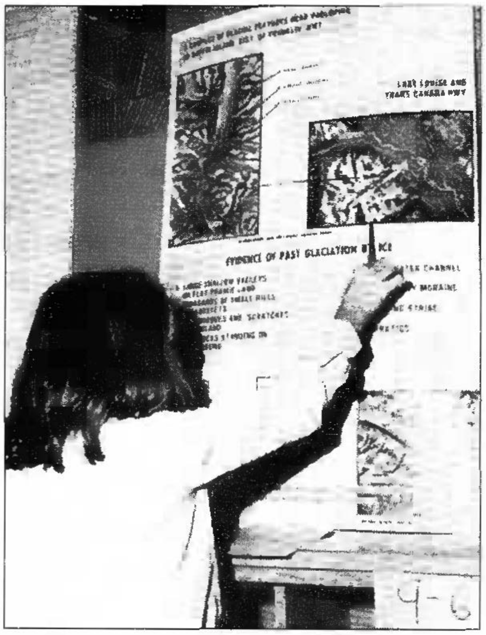
***Ron Cammaert***

*Mr Cammaert is a mathematics teacher at Sunalta Junior High School, in Calgary.*

The pictures refer to a Grade 7 laboratory mathematics class, dealing with ratio. Each student group was given a different problem. The problem usually required the students to gather some information from a situation and then to find a ratio or use a ratio to find a distance.

*The two boys with the globe decided to make a question on their own, which involved finding a distance in miles between two particular points. They then gave the problem to someone else to solve.*

*The four boys receiving help were attempting to change a particular ratio on a city map to another map that did not have a scale.*



*The girl with the aerial photograph was finding the ratio to which the photograph was made after being told the actual length of Lake Louise.*

*The girls and boy were working to find to what scale the area map of Calgary had been made. They had been told the elevation was exaggerated in a ratio of 3:1 and a particular elevation was pointed out to them.*

# Junior High School Math Option

*Harry Topolnitsky*

*Note: This article first appeared in the* Mathematics Council Newsletter *volume 9, number 4, pages 8–13 (1970). This publication was renamed* delta-K *in 1971. Minor changes have been made in accordance with current ATA style.*

The introduction of an option in junior high mathematics is a matter of some concern to teachers involved in setting up programs in their schools. To aid the teachers, a panel was formed for the annual fall mathematics conference. This panel, which has also been active at two conventions, includes Ted Rempel, of Londonderry Junior High, Edmonton, as chairman; Dennis Annesley, Avalon Junior High, Edmonton; Jim Barnes, Montgomery Junior High, Calgary; Dick Daly, D S MacKenzie Junior High, Edmonton; and Harry Topolnitsky, Ellerslie Junior High, County of Strathcona. The whole topic of junior high mathematics option was divided into the following subtopics, to which the members of the panel reacted.

## Personnel, Resources and Facilities

It is necessary to keep in mind that each school is individual in nature. Personnel, resources and physical facilities available or applicable in one situation may be totally unrealistic in another.

It seems logical to assume that the option should be handled by teachers of mathematics. This may be true for a large number of the possible topics, but when you consider the variety of available topics, you begin to realize that some of them may receive better guidance from teachers other than mathematicians. The section dealing with mathematical instructions could, perhaps, be best handled by a science teacher who is familiar with the calculator, sextant, micrometer, calipers, etc, and art teachers might prove valuable with mosaics, designs, symmetry and optical illusions. The history of mathematics or biographies of mathematicians may be in the domain of the language arts teacher; careers in mathematics may best be handled by the guidance counsellor. Thus we see that the realm of mathematics may be extended to include other teachers because the topics are so varied that no one teacher can be expected to be conversant with all of them. However, the teacher should have initiative and act as a resource person or consultant.

Resources and facilities vary with the size of the school and with the school boards in their allotment of funds for this purpose. It is difficult to say what is ideal in the way of resources and facilities. Some teachers may prefer to have a room assigned as the math room in which all the projects are done and the displays are stationary. Others may wish to have a mobile class. The science laboratory seems like the logical place to work with mathematical instruments. The social studies room, if such exists, might be used for map work, map colouring, topology, etc, which might be left on display for the social studies students. The library should be available for research work and display of art work done in relation to mathematics and mathematicians.

Many devices can be made for a fraction of the cost of commercial ones. A demonstration slide rule may be made by the industrial arts students instead of purchasing one for $47.50. Individual slide rules may be purchased for as low as 87 cents each. Once the students learn to use a slide rule, they prefer to have one for their personal use.

I fully realize that some of the suggestions offered are impractical in view of the resources and facilities of the individual schools. We may be content with or forced into using one corner of the home room for the option.

## Classroom Organization

The enrolment in the option may vary from 15 or fewer to 35 per class. Although the election of an option should be on the basis of strength and interest of the student, this is not always the case nor always practical. As a consequence the class may be composed of students who have interest but not much strength, strength but little interest, both strength and interest, and those who had no alternative.

In some situations it may be advisable to have the whole class work on one project. In this case the teacher would present a list of topics from which the students would democratically choose one. If, for example, the topic of paper folding is selected, it might involve all the students in making airplanes as a basis to paper folding. This might be followed by a lesson on drawing regular polygons. To complete the project, the students might work in groups or independently to construct polyhedrons.

Some teachers prefer to have the class divided into groups of students who have the same interest. Although an ideal group size may be four or five students, it will vary depending on the interest shown.

The interests of the students may be so varied that they would perform best independently. It is also possible to have a combination of groups and individuals. The teacher is in a position to assess which of the above would be suitable to his class.

## Time Allotment

The *Junior High School Handbook* suggests that the time allotted to the options be between 75 and 175 minutes, or the equivalent of two to five periods per week. Where feasible, blocking of the options on the timetable is perhaps the most desirable, but with the variety of situations in our schools, this is not always possible. A six-day timetable seems to be the most suitable. This allows for the options to be offered every second day. Schools on the semester or trimester system would offer the option to one group of students for only a part of the year and use a rotation. Another possibility is to offer the core in mathematics for four periods and to use the fifth period for exploring the mathematics option where no regular time can be timetabled for the option.

Setting a time limit for a project or topic is advisable. This should prevent the students' interests from waning. The anticipating of a new project should keep the students interested in completing the project at hand in a minimal time.

## Evaluation of Student Progress

The evaluation of student progress in the options varies with the requirements of the different school systems. Some of the methods for evaluation are (a) a five-point scale, (b) a three-point scale, (c) written comment, (d) interview and (e) no indication of progress.

However, where an evaluation is necessary or desirable, some criteria must be used. Following are several points that may be considered.

1. Student participation
2. Quality of records or finished project
3. Growth
4. Use of time
5. Independent study
6. Ability to organize
7. Ability to follow instructions
8. Effort
9. Enthusiasm
10. Completeness of project

A log may be kept for each student in which would be recorded the title the project, the date started, the date completed and any of the above criteria that are applicable.

## Evaluation of Program

This may be the most difficult task that we face. However, through participation we can learn about good points or ideas to propagate and pitfalls to avoid. Perhaps an indication of the program might be determined by the students themselves—their desire to re-enrol in the option for another year. Ordinarily, the proper evaluation of a program requires more time than we have had this year. The success of the program depends to a large extent on the teacher in charge—his interest, enthusiasm and suitability will be reflected in the final results.

The program has a lot of merit and gives the teacher an opportunity to use his initiative. It provides increased flexibility in the area of content and methodology. The option should cultivate interest, develop special abilities and provide for research.

To assist teachers and administrators in planning the mathematics option, the mathematics coordinators of the Edmonton Public school system have spent considerable time and effort in the preparation of the topics suggested below.

### Business Mathematics

1. Stock market—stocks, bonds, corporations, trading and quotations
2. Banking—savings accounts, chequing accounts
3. Instalment buying—kinds, interest rates
4. Consumer buying—discounts, comparative shopping
5. Mortgages—house, second mortgage
6. Insurance—home, car, life
7. Taxes—kinds, assessment, rates
8. Car operating expenses—fees, operating costs, credit cards, depreciation costs, instalment paying, insurance
9. Bookkeeping
10. Operation of a school business (eg, a store)—organization, shares, operation of the business, reports

### Recreational Mathematics

Magic squares, cross-number puzzles, paper folding, number sequences, numerology, math challenges or puzzles. codes, mathematics cartoons, math games (Krypto, Radix, Equations, Wff' n Proof, etc), aestheometry (three-dimensional string constructions)

## Number Theory

1. The natural numbers—sums of consecutive numbers, consecutive odd and consecutive even numbers; triangular numbers; Fibonacci numbers
2. Primes and composites—distribution of primes, relative primes, Sieve of Eratosthenes
3. Divisors of a number—prime numbers or factors, GCF, divisors, perfect numbers
4. Facts of number theory—Fermat's two-square theorem, Lagrange's theorem
5. Conjectures in number theory—Goldbach conjecture, Fermat's last theorem

## Topology—"Rubber Space Geometry"

1. Two-dimensional topology—simple closed, transformations
2. Networks
3. The Möbius strip
4. Map colouring problems
5. Mazes
6. Three-dimensional topology—Kline bottle

## Set Theory and Logic

1. Truth tables
2. Implication
3. Simple proof
4. Relating truth tables to set theory
5. Using intersection and union to teach GCF and LCM

## Probability and Statistics

1. Probability
   a) Mutually exclusive events
   b) Independent events
   c) Experiments (coins, dice)
   d) Odds and simple games
   e) Fair games (math expectations)
2. Statistics
   a) Histograms
   b) Measures of central tendency (mean, mode, medium)
   c) Surveys to illustrate the above concepts

## Math Instruments

Slide rule, transit, sextant, abacus, calculator, micrometer, calipers

## Measurement

1. Direct measure
   a) Metric and British
   b) Greatest possible and relative error
   c) Scientific and standard notation
   d) Significant digits
2. Indirect measure
   a) Scale drawing
   b) Similar triangles
   c) Numerical trigonometry
   d) Vectors

## Finite Math Systems

1. Group theory
   a) Modular arithmetic (clock)
   b) Permutations
   c) Plane transformations
   d) $2 \times 2$ matrices under addition
2. Fields
   a) Real numbers
   b) Prime modular systems

## Geometry

1. Plane constructions
2. Geo-board applications
3. Optical illusions
4. Mosaics and designs
5. Finite geometrics
6. Tangrams
7. Paper folding
8. Symmetry
9. 3-D constructions—prisms, pyramids

## Computer Science

1. Numeration systems—base 10, 5, 2
2. Functional relationships between parts of a computer
3. Flow charts
4. Programming in APL
5. Concentrated area in mathematics using flow charts
6. Field trip to a computing centre

## Graphs

1. Pictographs
2. Bar graphs—horizontal, vertical
3. Circle
4. Rectangular
5. Line graphs

## History

1. Ideas
   a) Need for counting—early methods, importance of zero, calendars
   b) Need for calculating—trade
   c) Need for measuring—surveying, building, navigation
2. People—Thales, Hipparchus, Pascal, Pythagoras, Ptolemy, Newton, Plato, Copernicus, Leibniz, Euclid, Galileo, Boole, Archimedes, Napier, Einstein, Eratosthenes, Kepler, Euler, Apollonius, Descartes, Gauss and others

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*Mr Topolnitsky was principal of Ellerslie Junior High, County of Strathcona, and a teacher of mathematics.*

# 1.4 Kilograms of Hamburger and a Litre of Milk, Please

*Dr S A Lindstedt*

*Note: This article first appeared in* delta-K *volume 14, number 3, pages 13–16 (1975). Minor changes have been made in accordance with current ATA style.*

Canada is adopting the metric system of measurement. Mr S M Gossage, chairman of the Metric Commission, has stated that he thinks we will be a "predominantly" metric country by 1980. In so doing we will join over 95 per cent of the countries of the world; this will help our international trade and facilitate worldwide understanding in the fields of commerce, industry and communications.

But, of course, international trade and communication is not the only reason for "going metric." There are other fundamental reasons for the adoption of this system of measurement.

**A. The metric system is easier.** Yes, it is. The units and subunits are all based on a decimal system and this means that conversion from one unit to another is just a matter of shifting the decimal point. For example, the length of a Canadian football field is 100.584 metres. The following chart shows how easy it is to change this measurement using other units of length.

Length of Football Field
100,584 millimetres (mm)
10,058.4 centimetres (cm)
1,005.84 decimetres (dm)
100.584 metres (m)
10.0584 decametres (dam)
1.00584 hectometres (hm)
0.100584 kilometres (km)

It is also easier to complete calculations. For example, if your granary is 20 ft 6 in long, 15 ft 4 in wide and 10 ft 2 in high, you need to perform the following calculations to find out the number of bushels it will hold: $20\frac{1}{2} \times 15\frac{1}{3} \times 10\frac{1}{6} \times 6\frac{1}{4} \times \frac{1}{8}$ bushels. That's a pretty awkward computation—even an electric calculator would have some difficulty with it.

The corresponding metric units to the same degree of precision would be length 6.25 metres, width 4.67 metres, height 3.10 metres. To find the capacity of the granary you would complete this calculation: $6.25 \times 4.67 \times 3.10 \times 1$ kilolitres. Not difficult at all.

**B. The metric system will simplify package sizes and make price comparisons much easier.** For example, washing detergent is sold in a great variety of sizes at various prices. In a recent survey I counted 28 different sizes on one shelf in a supermarket. Here are the sizes and prices of eight that I selected:

| | |
|---|---|
| 5 lb | $2.43 |
| 75 oz | $2.39 |
| 42 oz | $1.35 |
| 40 oz | $1.91 |
| 32 oz | $0.89 |
| 28 oz | $1.21 |
| 23 oz | $1.09 |
| 16 oz | $0.75 |

Quick now, which is the best buy?

Toothpaste, on the other hand, is now sold only in metric sizes. I noted the following on display:

| | |
|---|---|
| 150 ml | $1.43 |
| 100 ml | $1.03 |
| 50 ml | $0.66 |

(ml is the symbol for millilitre)

You see, you have a much better chance to compare prices.

Well, what is this marvellous, elegant system of measurement? What are the basics? Because we have been taught and have used Imperial units such as the inch, quart and pound, we may think that metric units are very numerous and very disorganized. Not so. There are three new basic units to learn for most of the everyday uses of measurement. They are

1. the metre (symbol m), a unit of length. It is about half the height of an ordinary door;
2. the litre (symbol l), a unit of capacity. It is just a bit smaller than the Canadian quart—and, as it happens, just a bit larger than the American quart (at least the use of the litre will eliminate that confusion); and
3. the gram (symbol g), a unit of mass (or weight, as it is commonly called). It is a very small unit—less than the weight of a paper clip. For that reason the

kilogram (symbol kg), which is 1,000 grams, will be in common use.

Now for each of the above three units we derive larger and smaller units indicated by the following six prefixes:

- For the bigger units
  - kilo-, meaning 1,000 times
  - hecto-, meaning 100 times
  - deca-, meaning 10 times
- For the smaller units
  - deci-, meaning 1/10 of
  - centi-, meaning 1/100 of
  - milli-, meaning 1/1,000 of

For different units of length we combine the above prefixes with the metre:

- A kilometre (symbol km) is 1,000 metres.
- A hectometre (symbol hm) is 100 metres.
- A decametre (symbol dam) is 10 metres.
- A metre (symbol m) is 1 metre.
- A decimetre (symbol dm) is 1/10 of a metre.
- A centimetre (symbol cm) is 1/100 of a metre.
- A millimetre (symbol mm) is 1/1,000 of a metre.

(Go back and review the example of the length of a football field.)

For different units of capacity we have a similar arrangement—we combine the same prefixes (and they keep their own meanings) with the [unit] to get a kilolitre (kl), a hectolitre (hl), a decalitre (dal), for the bigger units, and a decilitre (dl), a centilitre (cl), and a millilitre (ml), for smaller units. Although these units do exist, we will probably not use all of them in everyday practice. We will use the big one—the kilolitre—for measuring the capacity of storage tanks, granaries, oil tankers, reservoirs, etc. We will use the very small one—the millilitre—for measuring the capacity of toothpaste tubes, medicine drops, shampoo bottles, etc, and we will use the litre itself for milk, paint, gasoline, oil, antifreeze, etc.

Similarly, we combine the same prefixes with *gram* to get units of mass. The kilogram will be used in buying meat, vegetables, fruit, sugar, flour, fertilizer, lawn seed, cement. First-class passengers on airlines will be allowed 30 kg of luggage; economy class must get along with 20 kg.

Even the kilogram (1,000 grams) is a fairly small unit. Therefore a larger metric unit—the tonne (symbol t), sometimes called the *metric ton*, will be used for larger quantities. The tonne is equal to 1,000 kilograms; it is about 10 per cent bigger than the ordinary ton of 2,000 pounds. It will be used to measure loads of wheat, gravel, sand, bricks. The milligram (mg) is a tiny, tiny unit of mass. It will be used to measure pharmaceutical quantities.

We will not become metric overnight, nor by a certain date. We will move into the system at various places at different times. Because the students in our schools of today will undoubtedly graduate into a metric world of tomorrow, we should [do so] in school programs. All weather forecasts will be using metric units of measurement during the year 1975—snowfall will be measured in centimetres, rainfall in millimetres, wind velocity in kilometres per hour and temperature in degrees Celsius. During the year 1976, we can expect the metrication of highway signs—distances in kilometres, speeds in kilometres per hour, the heights of mountains in metres. In 1977, all grain will be measured, for local sales, in metric tons. Even at the present time, we sell our wheat overseas in metric tons. Many household products will start to appear in "metric" packages. As already mentioned, toothpaste tubes have been metricated. Heavy industries will take the first opportunity to replace worn-out or obsolete machines and tools with metric-calibrated equipment. Many have already made the change—the Ford Pinto is a metric car manufactured in the States; International Harvester, IBM and Stelco Steel are going metric. General Motors has announced similar intentions. In sports we are already accustomed to the 100-metre dash, the 50-metre swim, the high dive from the 10-metre board; the new racetrack at Stampede Park in Calgary is one kilometre in length.

Some things will not change. We often use units of measure just as a manner of speaking rather than as an application of serious measurement. We sing the song "I love you a bushel and a peck" without really thinking of measuring out the love. But I hope we won't destroy the charm of these little expressions by insisting on the metric translation, "I love you 36.369 litres and an additional 90.922 decilitres."

# Of Students, Computers and Learning

*Barry McGuire*

*Note: This article first appeared in* delta-K *volume 27, number 1, pages 38–41 (1988). Minor changes have been made in accordance with current ATA style.*

In the mid-1970s, like many other teachers, I became excited about the educational possibilities of the then-new microcomputer technology. This interest led, in 1979, to an innovative learning project designed to explore the uses of the microcomputer in a high school physics class. In 1984, I designed a curriculum for a locally approved course called Scientific Studies and Computing. The groundwork for the curriculum was taken from those aspects of the innovative learning project that offered the most interesting and constructive learning experiences.

Scientific Studies and Computing is, in all senses of the words, a science course. All of the objectives defined in the curriculum emphasize the nature, the knowledge and the processes of science. However, the curriculum displays one major difference from the regular science course: students in this course become the teachers; their pupils are the computers.

To fulfill the requirements of the course, students complete two science projects. The student chooses a topic from science and develops a computer application for science within that topic. Since the student is placed in one long problem-solving situation, the meta-lessons become the most important learning experiences.

After selecting a topic, the student prepares a project proposal. The proposal outlines the science content that the student expects to learn. The proposal also outlines the nature of the computer involvement in the project. Upon approval, students select the type and sequence of activities necessary to complete the project. My role is mainly that of a resource person who ensures that the students are on task while in the classroom. Neither task is particularly easy.

First, students often choose topics outside my range of expertise. In such cases, if the problem is complicated, I may do quite a bit of background reading. Second, what appears to be an unproductive approach on the surface may have several extremely beneficial long-term effects. If students give the impression that there is some design behind their activity, then I usually let them pursue it. On the other hand, students often perceive that the six to ten weeks allowed for a project is a long time and that to catch up on a little homework during class would not be a serious misuse of time. I do not permit this.

Initially, I was quite uncomfortable with the structure, or lack thereof, and to some extent I remain uneasy. Subconsciously, I want the students to be involved in activities that produce tangible results of their efforts every day. Consciously, I realize that problem-solving techniques evolve from a wide variety of behaviours. Therefore, I refrain from contributing when it is not essential. Indeed, students often request help, not because they are unable to solve a problem, but because they want reassurance that they are on the right track. In such cases, it would be easy to steer them in the direction I think their project should go. I try to give them the courage to proceed, although there is some doubt as to the outcome.

However, not all activities are unstructured. Students must have taken a 20-level science course as a prerequisite to ensure that they have some science background. During the first weeks of class, I establish the groundwork for science. Most students are rather naïve when it comes to this activity. I first establish the nature of science by discussing such activities as hypothesizing, interpreting, classifying, analyzing and problem solving.

Second, I introduce students to the Apple IIe and computer programming. While experience with computers and programming is recommended, it is not compulsory. To ensure that students have the fundamentals of BASIC programming, I give short programming assignments at the time that I discuss several of the science processes. For example, when discussing data and data analysis, I give students a parallel programming problem. This program requests the user to input several pieces of parametric data, find the mean of this data, find the standard error and display the results in a prespecified fashion on the video display terminal.

Obviously, students will have a wide range of programming experiences. Problems, such as the one discussed earlier, will challenge some students and be extremely simple for others. Peer tutoring is encouraged and students regularly consult each other on programming techniques Mini-lessons on programming (which most students require) are presented throughout the course. Books on programming are available in class, and students quickly become familiar with the resources and devise solutions to most of the programming tasks specific to their projects.

Selecting a project topic is very difficult (anyone who ever selected a thesis topic can relate to this task). Moreover, students inevitably perceive their project as a program that will teach the user all the neat information they have accumulated in preparing their project, a sort of computer assisted instructional program (CAI). On the contrary, their task is to create an application or utility program—a program, in other words, that makes the computer a useful tool to a scientist in the area of science from which the project originates.

The application of computers as scientific tools is a difficult concept for students to grasp. The most effective way to teach this concept is to use past projects that demonstrate several successes and failures. Even so, continued reinforcement is required to remind students that the nature of the computer application decides the ultimate validity of their projects. A computer application that attempts to do interesting things in science may have several programming flaws and yet be viewed much more favourably than a slickly programmed application that has a less scientifically valid application.

Here are some examples. The first time the course was offered, a student with considerable programming experience who was quite fluent in BASIC wanted to carry out a titration simulation. He wrote a good proposal describing a simulation of titration using graphics and a considerable amount of user interaction. Several discussions throughout the development of his project suggested that he was going not in the direction of his original proposal but more in the direction of a tutorial. When the project was presented, most of the graphics were found in a beautifully prepared title screen. The main routines of the program were merely a titration calculation sequence and a problem generation sequence. User interaction was limited to entering data to complete the calculations for the pH value of the unknown acid or base.

What of this student's learning objectives? Since he never studied titration, he certainly had to extend the base of his scientific knowledge to carry out the project. The output to the computer screen was extremely well designed and the data handling routines were excellent, but the scientific application of his program was weak. The project is typical of programs in which students misperceive the nature of computer applications in science.

Another student in the same class, however, completed the best project to date. The student undertaking the project had virtually no computer experience but possessed a great love of astronomy. His previous scientific experience was limited to project work and astronomy projects in the science fair. All of the programming skills used in his first project were picked up in developing his project. What made this project succeed? First, the student formulated a hypothesis that he wanted to test. Thus, he had a clear image of the science involved in his project. Second, there was a definite role for the computer to search the data for relationships verifying his hypothesis. Third, he had a clear image of what the final product should do.

The essence of his hypothesis was that the periodicity of the fluctuations in size of red giant carbon stars was related to the periodicity of the fluctuations in the intensity of the light given off in certain areas of their spectrum. Data for the periods of fluctuation in the size and light intensity in each area of the star's spectrum for about 100 red giant carbon stars were entered into a data file. The program grew as the student's programming experience increased.

First, he programmed the search sequence to look through the data to find stars that gave light in the regions of the spectrum specified by the user. Then the program plotted the period of the size of the start against the period of the light. If the graphed data resulted in an approximately straight line, a relating constant was calculated. In this way, he could explore relationships not only between the periods of the star's fluctuations in size versus light intensity but also between the periods of the intensities of various areas of the spectrum. Many of the programming techniques were quite sloppy, mostly due to the inexperience of the programmer, but the science of it was quite exquisite.

Another excellent program was carried out by a student who bred dogs as a hobby. The student was quite interested in the genetics of sex-linked characteristics. She proposed to develop a program that could track sex-linked characteristics throughout a breeding sequence of five generations. She had absolutely no previous computer experience and several times got stuck trying to debug quite convoluted programming sequences. In the end, she learned about programming as well as about preplanning problem-solving approaches.

The program enabled the user to specify the characteristic that was sex linked to specify the genetic structure on which the characteristic was found. Then the user could choose the genotype of the male and female. The computer generated the genotypes of the offspring and allowed the user to specify which offspring was to be bred for the next generation and to specify the genotype of the new breeding partner. The computer repeated the process until all five generations were traced and then presented a summary of the breeding sequence.

What students found most interesting during the in-class presentation was when the program traced the incidence of hemophilia in the royal families of Britain, Germany and Russia in the late 19th and early 20th centuries.

In some cases, previous programming seemed to be a drawback. Many students with programming experience viewed the class as a computer class rather than a science class. As a result of this confusion, several students dropped the class.

The best programmer in the class almost dropped the course. The student could program more proficiently in assembly language than most students could in a higher-level language. What kept this student from dropping the class was a discussion about the nature of gravity. We discussed the book *Flatland*[1] and the possibility that gravity could be a distortion of our three-dimensional world into the fourth dimension. The following day, the student presented a proposal for his project. He wanted to design a program that would allow the user to enter the data points (ordered triplets) for a "wire frame" diagram of an object in the third dimension. The program would then permit the user to rotate the object to any view in that space and then translate that view into one from the second dimension or extrapolate it to a higher-order space. For example, if a cube was entered in the third dimension, it could be rotated about any of the three orthogonal axes. Then the cube could be viewed as it would appear to a person living in the fourth dimension.

Two factors almost stopped me from approving his proposal. First, it lay on the fringes of science and really was a project in pure mathematics. Second, and more critical, the project was very complex. However, since the proposal was so clearly presented, I decided to allow it.

The student immediately plunged into researching the mathematics of drawing three-dimensional projections on a two-dimensional space—the display screen. (The techniques for programming this are found in BASIC.) The key is the matrix; a cube, for example, is a three-by-eight matrix containing the ordered triplets for its corners. To make the cube undergo a realistic rotation, a matrix multiplication with another matrix (when the trigonometric functions occupy the cells rather than numerical data) was required. Since this program ran so slowly in BASIC, the student translated the entire program to machine language. For weeks on end, he was immersed either in books on matrix mathematics or in books on spatial projections.

The next problem was that of dimensional translation; it proved much more difficult than he first imagined. In the case of matrix mathematics and spatial projections, all previous analysis was by mathematicians. All the student had to do was figure out what they were talking about and translate it into computer language. Because he had to create the mathematics before he could begin programming he had limited success. Nonetheless, during the four months he spent on the project he did learn an immense amount not only about mathematics but also about problem solving.

His presentation included the rotation of a wire-frame diagram of a simple car. However, the extrapolation of the car into the fourth dimension did not succeed entirely: portions disappeared in the translation because the matrix generated by the mathematical operations to translate even as simple solid from the third to the fourth dimension was so large that it required more memory than the computer had available. Therefore, much of the data generated by the program was lost. As a footnote, about a year and a half later I received a disk and the documentation for the completed version of this programming that he was preparing to market.

Students with sufficient programming background can undertake projects that involve interfacing the computer with laboratory equipment. The gameport on the Apple is fairly simple to access from any type of program. Several sensing devices are available and quite easy to use. Phototransistors, photoresistors, thermistors and transistors can be connected to the gameport, thus providing a relatively safe (for the computer) and easy (for the student) connection between the computer and the outside world.

One student's initial proposal was to develop an interface so that the computer, in conjunction with a spectroscope, could do spectral analysis. Unfortunately, the difficulty was that the light levels from the spectroscope were too low to activate the phototransistor. After several unsuccessful attempts to develop an amplifier to make the system more sensitive, the student changed the direction of his project. As a result, the new project tried to use the phototransistor as a light meter.

Although this project didn't have quite the romance of the original, it still had considerable merit. The student researched light intensity and luminance. He researched how the computer interacted with the phototransistor in order to tell his program how to read the phototransistor. Then, to translate the value the computer read from the phototransistor into an intelligible number, the student had to understand the nature and importance of instrument calibration. As a final step, the computer collected and stored the data in a format that made it intelligible to a commercially purchased graph analysis program. Using his own program, the student then collected data for luminance versus the separation of light emitted from both a point source (a bulb) and a rod source (a fluorescent tube) of light. Subsequent graph analysis of the data showed the inverse square law for the point source of light and the inverse first power law for the rod source.

At some point in developing any project, the original excitement of the project wears off. When that happens, persistence in problem solving becomes extremely important. Some students have confidence in their ability and do not need much encouragement; others need regular shots of enthusiasm. Once the first project is successfully complete, the difference in the students' approach to their second project is quite remarkable. Although the second project promises to be longer and more difficult than the first, students are much more confident of their ability to handle the challenge. They are more independent and flexible in their approaches to problem solving. They are less threatened by peer criticism than they were initially.

How students react to peer criticism is especially noticeable during the evaluation stage of the project. Projects are evaluated on the basis of three criteria. The science aspects of the project comprise 50 per cent of the final mark. The nature of the computer involvement and the final program contribute 30 per cent to the grade. Finally, an in-class presentation of the project, adjudicated by the students, earns 20 per cent. Not only do the students respond more positively to peer evaluation on their second project, but, having all been through a peer evaluation, they are much more perceptive and constructive in their criticisms of others the second time around.

I would eventually like to have several students cooperate in a major project. Each student would develop a separate segment of the computer program, which would then be merged with [other] students' sections. Considerable group planning in both the scientific and the computer aspects of project development would be realized. Perhaps the potential for disaster in this approach looms too large.

In the meantime, students benefit from their experiences in several ways: they increase their problem-solving ability, develop persistence, come to understand the nature and process of science, and learn to appreciate the symbiotic relationship between science and technology.

## Note

1. Abbott, E A. 1880. *Flatland: A Romance of Many Dimensions*. New York: Penguin. 1984.

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# Implementing Manipulatives in Mathematics Teaching

*A Craig Loewen*

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Manipulative materials have emerged in mathematics instruction as more than just a means to add variety to lessons; they are an essential element for effective mathematics instruction. However, for manipulatives to be used successfully in the classroom, a great deal of thought must precede their implementation. What role do manipulatives play in mathematics instruction? What factors influence the effective implementation of manipulatives in the instructional process? This paper presents alternative answers to these questions.

## A Definition of Manipulatives

The purpose of manipulatives is to make mathematics more concrete. Manipulatives enable students to play with, experience and develop for themselves mathematical principles, relationships and ideas. For manipulatives to have any place in the mathematics classroom, they must embody or physically represent specific mathematical concepts (Wiebe 1983). Consider two examples and one counterexample.

A concept that many elementary mathematics students struggle with is why the remainder after division can never exceed the divisor. This concept may be illustrated when teaching division using a balance beam (Knifong and Burton 1985). To model the equation $7 \div 2$, the student would place 1 weight on the balance a distance of 7 units to the left of the fulcrum (see Diagram 1). Because the divisor is 2, weights are hung 2 units from the fulcrum on the right side until the beam is balanced. The situation quickly arises that when 3 weights are hung on the right side, the beam tips to the left, but when another weight is added, the beam tips to the right. Where then should the final weight be hung in order to balance the beam? Through experimentation, it is obvious that hanging the weight any further to the right (eg, a value greater than the divisor) is counterproductive, and thus a position closer to the fulcrum (eg, a value less than the divisor) must be selected. In this case, the weight must be hung 1 unit to the right of the fulcrum to completely balance the beam. The remainder must always be less than the divisor; this mathematical concept is actually embodied within the manipulative materials.

Poker chips may be manipulated to model the subtraction of negative integers. Assume that a blue poker chip represents +1 and a red poker chip represents -1. Thus the subtraction of -4 from 3 in the equation $3 - -4 = ?$ may be modelled as follows. Set out 3 blue poker chips (+3) and then remove 4 red ones (-4). It is obvious that no red chips may be removed because there are only blue chips available. However, note that a blue and a red chip together total zero (ie, $-1 + 1 = 0$). Thus, any number of pairs may be added without changing the value of the expression. Pairs are added until there are enough red chips such that 4 may be removed (add 4 pairs). Now, when 4 red chips are removed, 7 blue chips remain. This process illustrates that $3 - -4 = 7$. This model makes it clear why the difference is greater than the minuend when subtracting a negative subtrahend.

As a counterexample, consider the common exercise in which students pair numbered cards with corresponding word cards (see Diagram 2). This activity, and others like it, may be called manipulative only in that students are given some object (cards) that they may touch and move. The cards do not embody any mathematical concept, however, and this exercise only serves to help students develop correspondence between names and symbols (a vocabulary exercise). For the student, no greater understanding of "oneness," "twoness," or "fiveness" is developed simply by matching symbols to words; memory skills are drilled.

## Diagram 1

Using a balance beam to illustrate division. Solve $7 \div 2 = ?$

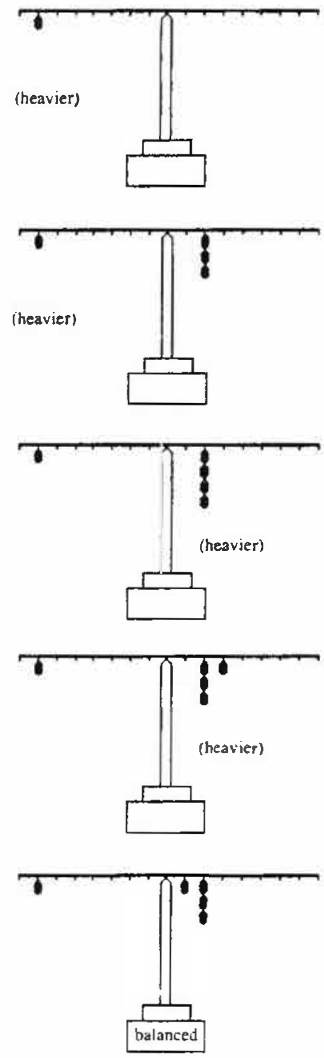
A. Begin with one weight seven units to the left of the fulcrum.

B. Add weights 2 units to the right of the fulcrum to balance the beam. The left side is still too heavy.

C. The right side is now too heavy, so the whole number quotient must be 3. Experiment with one weight to find the remainder.

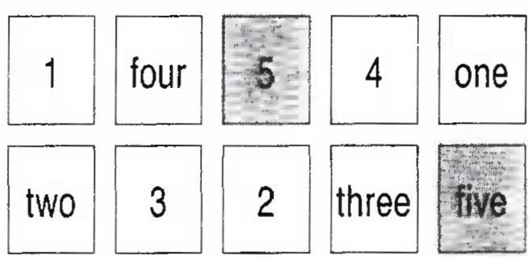
D. The right side is still heavier, so the remainder must be less than 2, which is the divisor.

E. The beam is now balanced. The remainder must always be less than the divisor.




## Diagram 2

Matching numbered cards with name cards.




The shaded pair is a match and may be removed. This activity is sometimes played as a game which begins with all cards face down. Two cards, one at a time, are turned over by a player. If a match is found then those cards are removed from the game. If no match is found, the cards are turned back face down and the other player takes a turn. The winning player is the one having made the greatest number of matches once all the cards have been used.

## Three Implementation Models

What role do manipulatives play in mathematics instruction? Where do manipulatives fit into the typical instructional sequence: introduce, develop, review and evaluate? The following three general models for implementing manipulatives offer some alternative answers to these questions. These models may be applied to either individual lessons or to complete units.

The first implementation model is called the *introductory model* (see Diagram 3) because the manipulatives are used only in the beginning stages of instruction. The purpose of the manipulative in this model is to introduce the mathematics concept to be learned and to provide a body of concrete experiences that can be drawn upon or synthesized during later formal instruction. The manipulative also fulfills the purpose of increasing student interest and motivation. In some cases the manipulative also provides a sense of relevance to the later formal instruction delivered by the teacher. The learning sequence flows from the concrete to the abstract. In this model, knowledge is organized from the general to the specific; general concrete experiences are provided first and followed by highly structured formal sessions in which specific concepts are revealed through an oral exposition delivered by the teacher. The major assumptions of this model are that students require a context for effective formal instruction, and that general concrete experiences facilitate the learning of specific abstract concepts.

The second implementation model is called the *tertiary model* (see Diagram 3) because the manipulatives are not introduced until the latter stages of instruction. Early instruction is teacher controlled, but later experimentation is less closely monitored.

In the initial stages of this model, the teacher provides formal focused instruction on specific abstract concepts; the focus is not on understanding but on the awareness of principles. These specific principles are later linked to create more general knowledge through informal experimentation with manipulatives; knowledge is organized from the specific to the general, while experiences are organized from the abstract to the concrete. The manipulative serves a synthesis role and functions as a context in which learned concepts may be applied. In this model, the manipulative may also serve as a means for the teacher to evaluate student progress and understanding, as well as a means to undertake review of specified concepts. The second model is built upon the assumption that students require basic skills and knowledge before they can fully benefit (eg, draw conclusions and formalize mental structures) from the experiences and environment manipulatives provide.

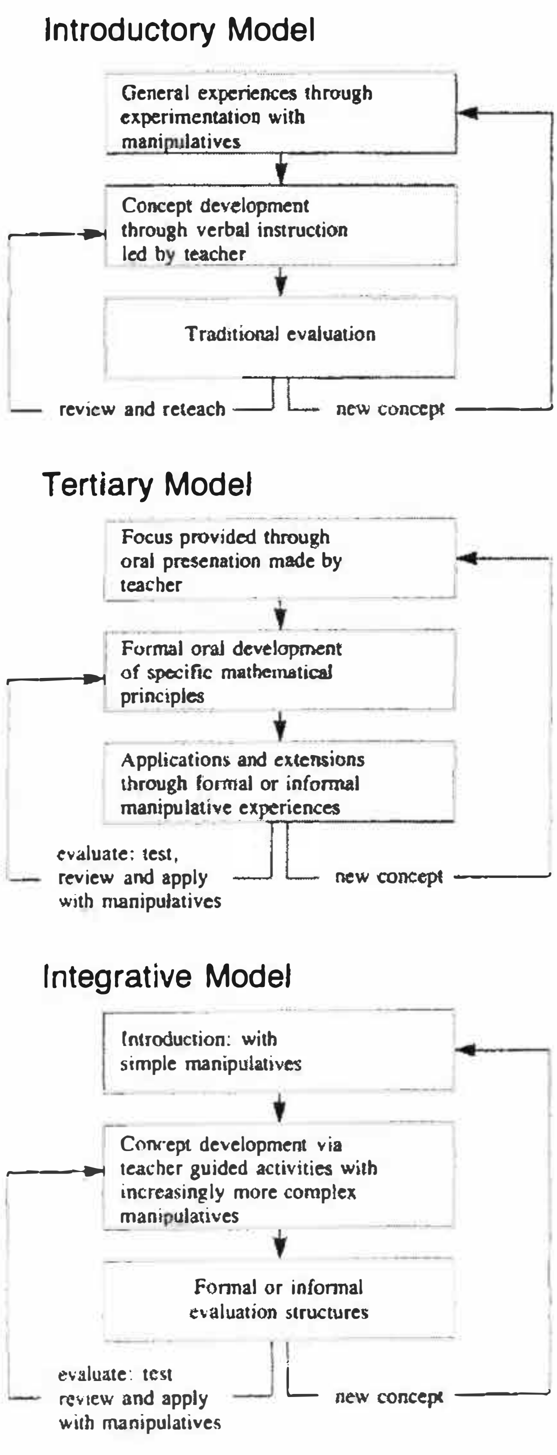
The final model is the *integrative model* (see Diagram 3). In this model, manipulatives are used continually throughout instruction; knowledge and skills are introduced, developed, reviewed and evaluated through concrete experiences with physical representations of mathematical concepts. By using manipulatives at all points during instruction it is hoped that high motivation and interest levels will be maintained throughout the entire instructional cycle. Using a manipulative for all phases of instruction eliminates the need to introduce more than one set of materials. The major assumptions of this model are that students learn better and retain longer what is learned in a familiar context, and that all phases of instructional cycle may be delivered easily using manipulatives.

No one model is correct or better than another. Instead, the teacher should use the model that best suits the material to be taught, the needs of the students and his or her own instructional style. The teacher may wish to consider the mathematics skills and motivation levels of the students, the students' learning styles, the synthesis and generalization skills of the students, the ease with which students master and apply learned concepts, and the complexity of the mathematics concepts to be taught. Each model possesses its own advantages, disadvantages and assumptions. The teacher must select the model in which the disadvantages are minimized, the assumptions appear realistic and the advantages are exploited. When these conditions exist, the purpose of the manipulative is maximized and effective implementation is achieved and measured by improved student learning.

## Some Factors Influencing Effective Implementation

Manipulatives may be evaluated according to a variety of criteria. Hynes (1986) has suggested that manipulatives may be evaluated according to both their pedagogical and physical attributes. With respect to pedagogical attributes, manipulatives must provide a clear representation of a mathematical concept, be appropriate for the student level, interest the students, be versatile, contribute to the building of a mathematical concept, assist in developing vocabulary, improve spatial visualization, promote problem solving, provide a sense of proof and promote creativity. With respect to physical attributes, the manipulative must be durable, simple, attractive, manageable, cost effective and reasonable in terms of the quantity required. Not all manipulatives exemplify all of these

**Diagram 3**
**Implementing Models**




attributes, but generally, the better the manipulative, the more conditions it will satisfy.

The attributes that Hynes describes are valuable when discussing the relative differences between manipulatives, but the true value of manipulatives lies in how effectively they may be employed in teaching and learning situations. The most versatile, motivating and attractive manipulative will not be effective unless properly employed. Therefore, the manner in which the activity is conducted may be just as important as the materials themselves.

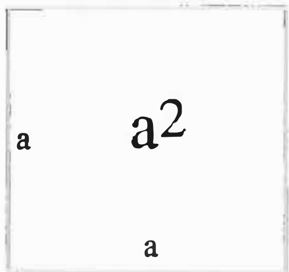
The first implementational consideration is the degree to which the student has control over concept development. If given time to simply experiment and play with the objects, will students develop the desired concept on their own? To what extent must the students' interaction with the manipulatives be guided by the teacher? To allow students to discover and develop concepts independently is often too time consuming, and there is no guarantee that the concept will ever be clearly or correctly formalized; however, concepts that are developed independently are more likely to be retained and treasured. The teacher must decide which form of manipulative is preferable based upon such considerations as students' past experiences with discovery learning, students' learning styles, the time available for the development of a concept, the motivation level of the students and the ease with which the concept may be summarized from the play experience.

The second implementational consideration pertains to the degree to which the student may control the manipulative. Is it desirable that each student has his or her own set of manipulatives, or is it sufficient that the teacher manipulate one set for the benefit of all? When the teacher manipulates the materials for the students, then the visual experience is substituted for the tactile. When working with individuals or small groups, the discovery learning approach is possible, and this approach necessitates a tactile experience. When working with a large group, a visual learning experience is more practical. In essence, the teacher defines his or her own role as that of consultant or group leader. The choice of role dictates the degree to which the teacher intervenes in and controls the concept development process.

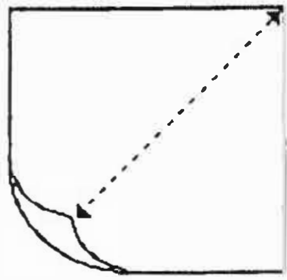
The third implementational consideration is the degree to which the mathematical concept embodied within the manipulative is obvious to the students. When the concept is obvious, then the materials are appropriate for developing mathematical relationships or facts. When the concept is less obvious, then the manipulative serves as a data-keeping tool. When the manipulative is used as a data-keeping tool, the

## Diagram 4

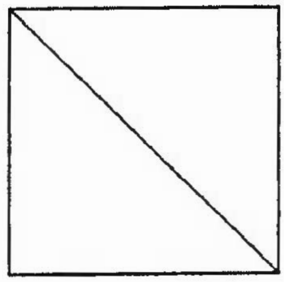
Illustrating the difference of squares: $a^2 - b^2 = (a-b)(a+b)$



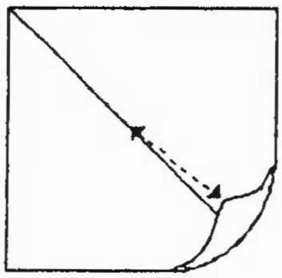

1. Begin with a square piece of paper.



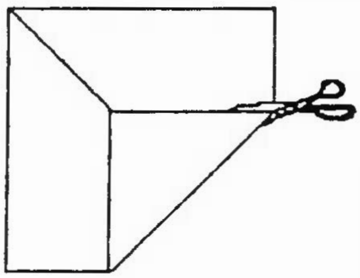
2. Fold diagonally.



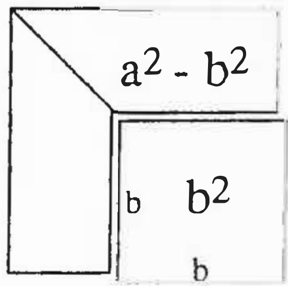
3. Crease.



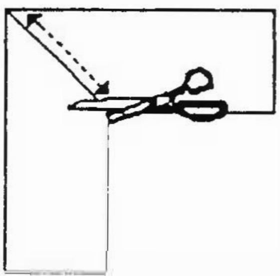
4. Fold one corner on the diagonal crease to any point part way along the crease.



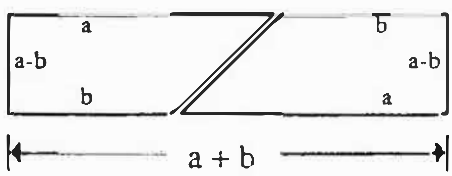
5. Cut along the paper's edge.




6. Remove $b^2$.



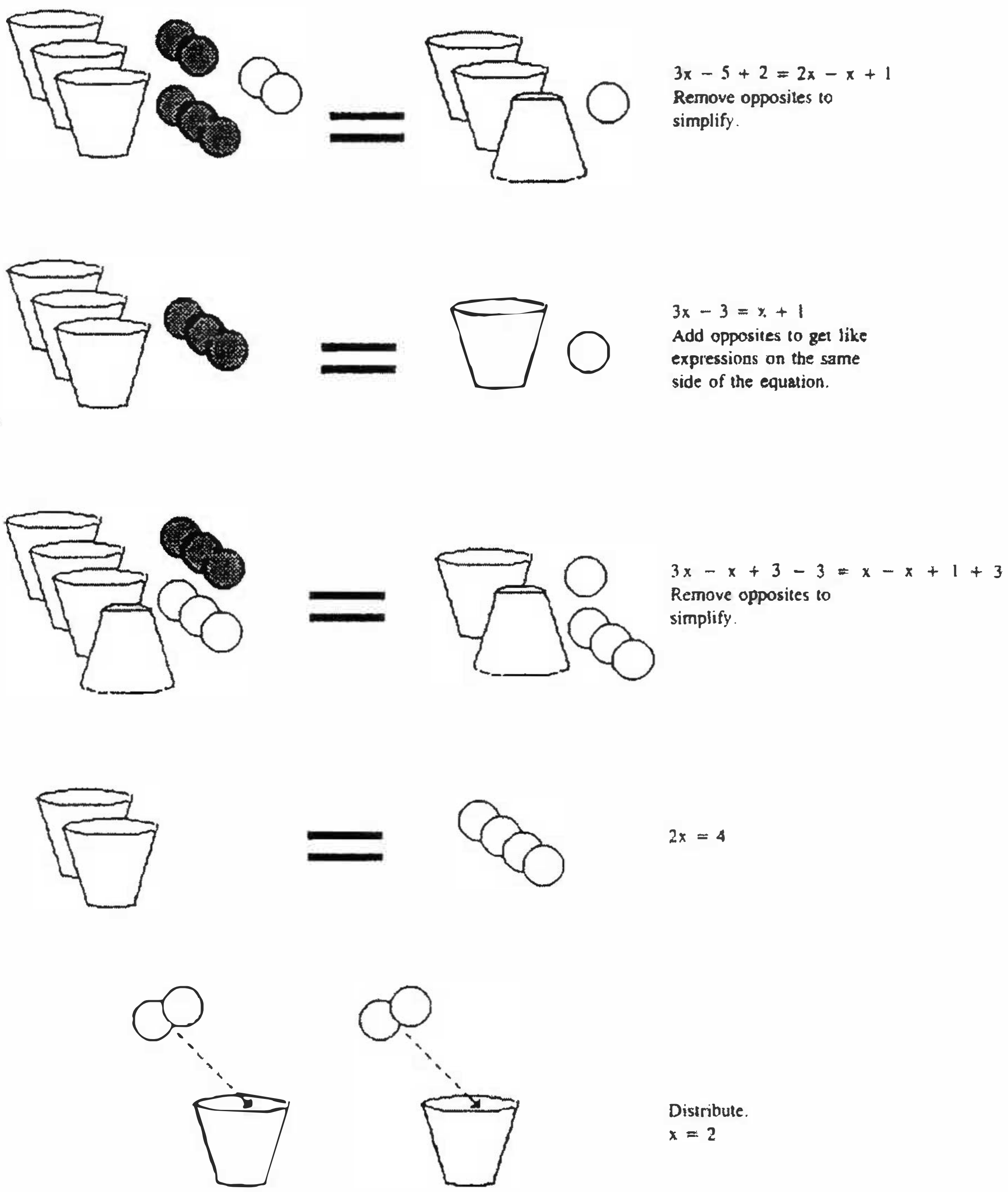
7. Cut along the remaining diagonal fold.




8. Rearrange.

## Diagram 5

Modeling the process of solving algebraic equations with paper cups and colored paper chips.



student is relieved of data-keeping functions and may concentrate more fully on process skills. The following examples clarify the data-keeping and concept-development natures of manipulatives.

The first example is illustrated in Diagram 4. In this example, paper is folded and cut (Bober and Percevault 1987) in such a way as to illustrate why $a^2 - b^2 = (a - b)(a + b)$. Once the activity is completed, an inherent sense of proof or obviousness makes it difficult to contest that the relationship is true.

The second example is illustrated in Diagram 5. In this example, the students model the process of solving simple algebraic equations through the manipulation of ordinary objects such as paper cups and circles cut from coloured construction paper. In this exercise, the process of solving equations is emphasized, and the materials serve the purpose of keeping track of the various symbols and quantities found on each side of an algebraic equation. Certain key relationships, such as $-x + x = 0$ and $-1 + 1 = 0$, are not made more obvious through this exercise.

Experience with these manipulatives simply provides the students with an alternate way of conceptualizing and remembering a process; it does not necessarily impart a greater understanding as to why the inherent relationships within the process are true.

The manipulatives are useful in learning the process because it is easier to remember how to distribute paper chips equally to paper cups than it is to correctly divide both sides of an equation by a constant value. The teacher must decide under what circumstances it is preferable to select manipulatives that facilitate a deeper or more complete understanding of a concept as opposed to manipulatives that simply promote an alternate conceptualization of a process. Both alternatives have value depending upon the students with whom the teacher is working and the objectives to be met.

The fourth implementational consideration is the number of concepts a manipulative supports within an instructional unit. If many concepts within an instructional unit are to be taught using manipulatives, then it is desirable to use similar materials for each topic within the unit. Using similar materials helps students link and relate these topics, relieves the need to constantly introduce and familiarize students with new materials, and provides a sense of continuity and coherence to the unit. However, a manipulative can be effective even if it supports only one concept, especially when used to review a concept or provide a brief extension to a previously developed concept. The manipulative must fit the instructional purposes and processes that the teacher has designed.

The fifth implementational consideration is the degree of familiarity students need with the materials in order to use them properly. How much time must be spent introducing the materials to the students and developing necessary vocabulary? If students are not properly familiarized with the materials, they will spend less time focusing on mathematical principles and more time trying to remember the manipulative procedure. Furthermore, if students are not familiar with the materials, they will not possess the vocabulary or language necessary to ask questions of themselves and others or to summarize their new knowledge. Certain materials require a longer introduction time, and generally, materials that require more introduction are less desirable. In order to justify a longer introduction time, the teacher must consider how well the manipulative embodies the mathematical concept, the number of concepts that may be taught using the materials, the required degree and extent of teacher–student interaction and whether students will work with their own sets of materials.

Well-constructed manipulative materials do not guarantee effective instruction. Even good manipulative materials will only be as effective as the process through which they are employed, and this process requires careful thought and reflection by those who understand the mathematics curriculum as well as children's thinking processes, capabilities, needs and interests.

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# A Collection of Connections for Junior High Western Canadian Protocol Mathematics

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*Note: This article first appeared in* delta-K *volume 35, number 1, pages 29–37 (1998). Minor changes have been made in accordance with current ATA style.*

We have put together "A Collection of Connections" that consists of 12 uses of junior high school mathematics. These activities support the communication and connections strands of the Western Canadian Protocol mathematics curriculum. In using them, teachers may adapt them extensively. They can serve as a basis of one to three mathematics periods. We have also found that teachers need to plan if they intend to incorporate them into their teaching units. They can be used as end-of-unit activities or as focal activities in the development of a unit. We would encourage teachers to use the activities as a means of bringing mathematics to life. These contexts provide an opportunity to enhance a student's view of mathematics. As one Grade 9 girl said at the conclusion of one activity, "That just proves that mathematics is everywhere."

The following are samples from the measurement and algebra strand:

*Measurement*
Why Do Sled Dogs Curl Up?
Why Do Sled Dogs Curl Up? Student Activities

*Algebra*
Managing an Elk Herd
Managing an Elk Herd Student Activities

## Why Do Sled Dogs Curl Up?

### Intent of the Lesson

An important mathematical idea developed here is that surface area and volume are independent. The sled dog is able to change its surface area although its volume remains constant. Formulas for surface area (and volume), estimating and visualizing are used in this lesson.

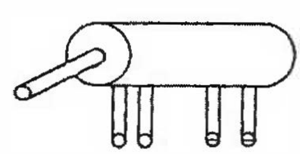
### General Question

Zoologists and veterinarians are interested in studying the behaviour of animals. They found that on cold winter nights sled dogs curled up. In fact, all dogs curl up when they are cold. However, dogs sleeping in the house in front of an open fire usually sleep stretched out.

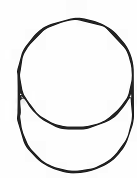
What is the reason for such behaviour? It is a known fact that heat loss from an animal is related to the amount of surface area of the animal that is exposed to the surroundings. A dog would, therefore, want to decrease its surface area exposed to the cold. We should note that a dog cannot change its volume, so it must change its shape. Can we calculate the surface area of a dog when it is stretched out and when it is curled up?

#### *Discussion Questions*

A discussion of the problem at this point should raise several points. A point of mathematical interest is that a dog stretched out can be thought of as five small cylinders (legs and tail) and one large cylinder (body and head). However, when it is curled up it represents one very large but flat cylinder.



stretched-out dog



curled-up dog

Answering this question is going to require approximations. In the first place, the parts of the dog's body mentioned above are only approximately cylinders. When a dog curls up, the curled-up shape is only approximately a cylinder.

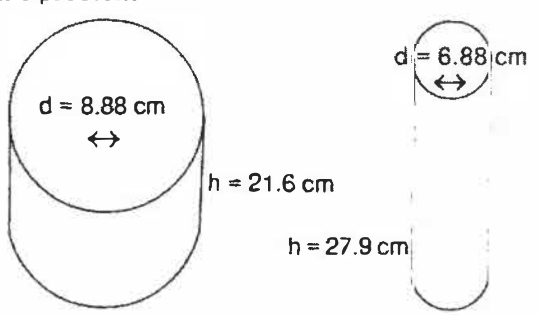
- What shape(s) does a dog represent when it is stretched out? (Long narrow cylinder.)
- What shape does a dog represent when it is curled up? (Flat wide cylinder.)

- What kind of approximations would we need to make when measuring the dog? (Ignore legs and tail.)
- What happens to the legs and tail when it is curled up? (They are tucked inside.)
- Does snow have insulating properties? (Yes.) How does this make a difference to a curled up dog? (It wants to have as large a surface area as possible in contact with the snow.)

# Preliminary Activities

## The Paper Cylinder

Take a sheet of paper (say 8½×11). Make a cylinder out of it lengthwise and find its volume. Then make a cylinder out of it widthwise. (See diagram.) What are the surface areas of the two cylinders? What is the volume of each of the two cylinders? The surface areas stay the same (because they are the same sheet of paper) but the volumes differ. Students may do this as a problem to solve. We should notice that when the 8½-inch side is made into a cylinder its diameter is 8½ divided by p. The volume can then be calculated with this diameter and the known height (11 inches). Students can make good use of a calculator in solving this problem.




large flat cylinder    long narrow cylinder

Sample calculations (using metric measurements of 27.9 cm by 21.6 cm):
For the large flat cylinder:
circumference (c) = 27.9 cm, height (h) = 21.6 cm
diameter (d) = $\frac{c}{\pi} = \frac{27.9 \text{ cm}}{\pi}$ = 8.88 cm
radius (r) = $\frac{d}{2} = \frac{8.88 \text{ cm}}{2}$ = 4.44 cm
Volume (V) = $\pi r^2 h = \pi(4.44 \text{ cm})^2(21.6 \text{ cm}) = 1337.7 \text{ cm}^3$
For the long narrow cylinder:
circumference (c) = 21.6 cm, height (h) = 27.9 cm
diameter (d) = $\frac{c}{\pi} = \frac{21.6 \text{ cm}}{\pi}$ = 6.88 cm
radius (r) = $\frac{d}{2} = \frac{6.88 \text{ cm}}{2}$ = 3.44 cm
Volume (V) = $\pi r^2 h = \pi(3.44 \text{ cm})^2(27.9 \text{ cm}) = 1037.2 \text{ cm}^3$

We know from this investigation that the volume is independent of the surface area. That is, these two cylinders have the same surface area but their volumes differ. The taller cylinder has less volume. We might also note that, in general, a long skinny object has less volume than a short fat one. In fact, the volume of a cylinder of fixed surface area is greatest when the diameter is approximately equal to the height of the cylinder.

### *Discussion Questions*

- How can we determine the diameter of these cylinders in two ways? (Measurement and dividing the circumference by $\pi$.)
- Why are the surface areas of these two cylinders not exactly equal? (If we add the area of the top and bottom, the shorter cylinder will have slightly greater surface area.)
- How does this demonstration relate to a dog curling up? What remains constant when a dog curls up? (Volume remains constant.)
- Would we expect the [surface area] SA of a curled-up dog to be greater or less than that of a stretched-out dog? (We are going from five relatively long and skinny cylinders to one short and fat one.)

### *The Snake*

A snake curls up for much the same reason as the dog—to conserve its heat in the cold desert nights. In fact, a snake has to warm up every morning before it can begin to move. Because of this, it is important to the snake not to lose too much heat. Mathematically, the snake is much nicer than the dog. The snake is an obvious cylinder, both stretched out and curled up. Stretched out it is a very long cylinder and curled up it is a large flat cylinder. (For this experiment, a snake can be simulated by a piece of rope with one-inch diameter or a child's play snake or [use] a real snake if you happen to have one.)

In addition to finding the surface areas we should find the volumes of the stretched out shape and the curled-up one. These volumes should be approximately the same. This calculation is an important check on our approximate measurements. Note: because the snake is curled up we cannot find its diameter by dividing its length by $\pi$.

### *Teaching Suggestion*

Although the lesson is centred on the behaviour of sled dogs, the preliminary activities lend themselves to much better calculations. In this lesson, the preliminary activities are more important than "answering the general question." This latter section, because of the difficulty in dealing with a real dog, is more a matter of estimation and discussion. In some ways the teacher can treat the sled dog question as the motivational device while the preliminary activities become the mathematical aspects of the lesson. With respect to the approximations in the next section of the lesson, we have found that students are eager to participate in these approximations. They do not treat the lesson as less significant just because it contains a lot of approximations.

## Answering the General Question

Having a real husky dog and taking actual measurements would be ideal. A student in the class might have a suitable pet and could be encouraged to provide measurements for the class. Approximation is fundamental to this activity. We need realistic measurements to begin with but approximation is still necessary. One approximation we might make is simply to measure the body and head of the dog, ignoring the legs and the tail. The outside of the fur should be the surface area rather than the actual skin surface because it is the outside that is losing the heat. A rolled-up blanket, a cylindrical pillow, a flexible toy dog or a slinky toy could be used in making approximate measurements.

### *Discussion Questions*

- Why is the diameter of the dog's body when it is stretched out equal to the height of the flat cylinder of the curled-up dog?
- By ignoring the legs and tail in our calculations, on which side of the comparison of surface areas are we erring? (The legs and tail contribute a large surface area to the stretched out dog and essentially no surface area to the curled-up dog.)
- What happens to the legs and tail of the curled-up dog? Do they lose more heat in the stretched out position or in a curled-up position?
- What is the effect of the snow insulating the dog in both positions? In other words, which position makes best use of the insulating properties of snow?
- Which shape (long or short) of dog benefits most from the curling up to conserve heat?
- Does your knowledge of surface area increase your understanding of the behaviour of dogs?

We might consider the curled-up dog to be a sphere. A discussion among students could settle this issue and calculations for the sphere could be made for comparison with a cylinder. That is, what happens to surface area if we consider it a sphere?

Volume of cylinder = $\pi r^2 h$
Surface area of cylinder = $2\pi rh + 2\pi r^2$
Volume of sphere = $(4/3)\pi r^3$
Surface area of sphere = $4\pi r^2$]

## Materials

Coloured sheets of 8½×11 paper, transparent tape, toy snake or 1-inch rope (4 feet long), long cushion (to simulate a dog), stuffed toys and a slinky toy. Although we have not tried it, a piece of dryer venting hose could be used to simulate a dog.

## Modifications

### *Other Animals*

Once the situation of the dog has been discussed, students can bring stuffed toys from home that could be measured in the stretched-out position and curled-up position. As mentioned earlier, a Slinky toy has possibilities.

When a fat teddy bear is curled up, its shape becomes a sphere rather than a cylinder. Some stuffed toys have very large legs that cannot be ignored in the calculation of the surface area before the toy is curled up. The legs can be treated as cylinders with one end.

The idea of the proportion of the surface in contact with the snow being much larger in the curled-up position than the stretched-out position is worth discussing. In general, the snow is a better insulator than the air, especially because of the wind blowing (wind chill) on the exposed side. In fact, the dog loses only about one-tenth as much heat through the snow as through the air. This makes the curling-up behaviour even more understandable. If the dog is considered to be a flat cylinder, using your measurements, what per cent of the surface is next to the snow? In curling up, not only does a dog reduce his or her surface area but also a good percentage of this smaller surface area is better insulated.

A survival suggestion for humans in cold water is to curl up into the fetal position. Students can approximate the surface area of a classmate in stretched-out and curled-up positions.

# Why Do Sled Dogs Curl Up? Student Activities

### *General Question*

Zoologists and veterinarians are interested in studying the behaviour of animals. They found that on cold winter nights sled dogs curled up. In fact, all dogs curl up when they are cold. However, dogs sleeping in the house in front of an open fire usually sleep stretched out.

What is the reason for such behaviour? It is a known fact that heat loss from an animal is related to the amount of surface area of the animal that is exposed to the surroundings. A dog would, therefore, want to decrease its surface area exposed to the cold. We should note that a dog cannot change its volume, so it must change its shape. How much more is the surface area of a dog when it is stretched out than when it is curled up? [See illustration on page 71.]

### Activities

1. (a) With an 8½×11 sheet of paper, we can make a cylinder in two ways. Once the cylinder is made, we can find the diameter of the cylinder in two ways. What are they?
   (b) How do you convert inches to centimetres? What is 8½ inches in centimetres?
2. (a) Why are the curved surfaces of both cylinders that you made with the paper (question 1) equal?
   (b) To what other surface area are they both equal?
   (c) Why are the *total* surface areas of both cylinders not the same? By how much are they different?
3. (a) A tall cylinder has less volume than a short cylinder of the same surface area. However, if a tall cylinder and a short cylinder have the same volume, which will have the greater surface area?
   (b) How does this conclusion relate to the stretched-out dog when he curls up?
4. (a) Using your measurements for a "snake," show the calculations for the snake stretched out and the snake curled up.
   (b) How much surface area does the snake lose by curling up?
   (c) What percentage is this loss of the stretched-out surface area?
   (d) What per cent surface area does the snake gain when he goes from being curled up to stretching out?
   (e) Why are these percentages not the same? If you wanted to impress someone with the importance of the reduction of surface area, which of these percentages would you use?
5. (a) Take approximate measurements from a real dog, a toy animal or a piece of tubing and determine the surface area of the "animal" in both positions.
   (b) Using the same measurements, make the calculation to determine the volume of the "animal" in both positions. Are the volumes nearly equal? Why?
   (c) Again with approximate measurement, determine the surface areas of the legs and tail. Go back to question 5(a) and add these surface areas to the surface of the stretched-out animal. How much does this increase the differences in surface areas?
   (d) Why do we not add the surface area of the legs and tail to the curled-up surface area? What assumptions are we making?
6. (a) What shape do you think polar bears become when they curl up?
   (b) Why do you think the polar bear's fur is much thicker on the back and sides than on the belly?

## Managing an Elk Herd

### Intent of the Lesson

The linear equation is used to simulate the growth of a population, a common problem in wildlife management. The mathematics includes linear equations, percentages, probability and graphing. It is possible to use a spreadsheet to illustrate the solution.

### General Question

All wildlife areas, even in a country as large as Canada, are limited in size. Because of this limitation, the wildlife that the area will support is limited. This is especially true in an area such as Jasper National Park. Even in larger areas where hunting is allowed, wildlife managers need information about herd sizes. Each year, decisions are made about how many animals may be taken by hunters. In making this decision, managers need to be able to predict the growth of herds.

Consider a typical problem for a wildlife manager—a herd of animals is 10,000. How large will that herd be in 5 years or 10 years? The answer to the problem can be found with the help of mathematics. First, information about survival (death) rates of the animals from year to year must be known, as well as birth rates. Very simply, if the death rate is 20 per cent and the birth rate is 40 per cent, the herd size will grow. Under such conditions, the herd will soon become too large for the area. Animals will need to be moved to new areas and/or hunted. The question is can we predict the size of the herd from year to year?

#### *Teaching Suggestions*

The teacher should be prepared to fully discuss the context of this problem. This may include discussion of biological processes, ethical considerations of managing any wildlife population or hunting.

#### *Discussion Questions*

- What do you think *survival rate* means?
- Will the survival rate be different for males and females? For fawns? (Will be higher for males than females and higher for adults than for fawns.)
- If the herd is growing too fast and a harvest is needed, what options do wildlife managers have?

(Allowing hunting, or capturing and moving animals to other areas.)

- What kind of herd growth is desirable in most wildlife areas? (No population growth.)

## Preliminary Activity

### *Exam Marks*

Interestingly, the mathematics needed for the elk herd problem is the same mathematics that students need to figure out final marks:

If the midterm mark (M) counts for 25 per cent of the course mark (C) and the final exam (F) counts for 75 per cent, what will the course mark be if your midterm mark was 72 per cent and your final exam mark was 92 per cent?

To solve this we take 25 per cent of 72, which is 18, and 75 per cent of 92, which is 69, and add them together for a final score of 87. We should observe that the 87 is much closer to 92 than to 72. Why is that? The reason is that the final exam mark is much more important than the midterm mark and is worth more. Now, what is the mathematical equation to determine the course mark?

$0.25M + 0.75F = C$

If we think about these as fractions, ¼ of the midterm and ¾ of the final exam are added together.

In another class, the course mark (C) is determined by taking 15 per cent of assignments (A), 25 per cent of the midterm (M) and 60 per cent of the final exam (F). What is the equation to determine the mark in this class?

$0.15A + 0.25M + 0.60F = C$

In this class the final exam is four times more important than the mark in assignments.

Ask students to do the calculation for given A, M and F.

## Answering the General Question

### *The Management of an Elk Herd*

A wildlife officer has the following information on an elk herd in her region:

| | | |
|---|---|---|
| Survival rates for | males | 95% |
| | females | 90% |
| | male fawns | 50% |
| | female fawns | 45% |
| Birth rate for | male fawns | 48% of adult females |
| | female fawns | 42% of adult females |

### *Discussion Questions*

- How do we get these numbers? (Biologists and wildlife managers.)
- Why are survival rates for male and female fawns different? (Male elk are bigger.)
- Why is birthrate only dependent on adult females? (In a herd of 100 males and 10 females, what is the maximum number of fawns that can be born?)
- What does the male fawn survival rate of 50 per cent mean? (One half live until next year.)

Using these rates, we can predict from year to year how large a herd will be. Very simply, a herd in year one with 100 males and 100 females will have 95 males and 95 females surviving in year two. The number of fawns will be 48 male fawns and 42 female fawns. The total herd will be $95 + 95 + 48 + 42 = 280$. This number is based on a birth rate determined from the number of females in the previous year.

The answer will be different if only the surviving females give birth, that is, if the birthrate is based on the number of females surviving from the year before. Initially, let's consider that the birth rate is based on the number in the herd in year one. Now we should write the equation, just as we did for the course mark:

$0.95M$ (males) $+ 0.90F$ (females) $+ 0.48F + 0.42F$ = Herd in year two

(Using "year one" and "year two" terminology leads to less confusion in the lesson.)

Now we introduce a harvest of the elk herd. The harvest may be gathering them up and moving them or it may mean hunting them. If, in the case above, we wanted to keep the herd at 200, we would need to harvest about 80 animals in year two. (In this equation we are assuming that the count is made after the harvest. In other words, the count is the last thing that happens in the year.) The equation would look like this:

$0.95M + 0.90F + 0.48F + 0.42F - H$ (harvest) = Herd in year two

Now we can proceed to figure out the herd size in year three. Notice that we have four types of elk:

| | |
|---|---|
| M(ale) adults - | 95% survival rate |
| F(emale) adults - | 90% survival rate |
| m(ale) fawns - | 50% survival rate |
| f(emale) fawns - | 45% survival rate |

To do our calculations for year two, we really need to know not the total size of the herd, but how many of each type of animal we have. Also, because females are more important to the herd survival, we will want a system that harvests more males than females. Therefore, we will have a male harvest (Hm) and a female harvest (Hf). Of course, only adults are harvested. Why is this? We need several equations, one for each type of animal. What, for example, is the equation to determine the number of males in any year? Since the adult male population is made up of surviving adults plus surviving fawns less the harvest, the equation for males in any year is

0.95M + 0.50m – Hm = Number of males

Because we have so many elements to attend to, a table is convenient.

### *Using a Table*

The use of a table will simplify our calculations.

| Column A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|
| Year | Males (adult) | Females (adult) | males (fawns) | females (fawns) | Total | Hm = Hf = |
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |

We need equations to go from year to year:

| | | |
|---|---|---|
| Male adults: | 0.95M + 0.5m - Hm | where M and m are males from previous year |
| Female adults: | 0.90F + 0.45f - Hf | where F and f are females from previous year |
| males: | 0.48F | where F is female adults from previous year |
| females: | 0.42F | |

Harvests (Hm and Hf) are constants from year to year which are subtracted. We should note the assumptions that are being made about the how the herd functions.

### *Discussion Questions*

- Why does 0.5m become part of the adult males? (We are assuming that fawns become adults after one year.)
- What does 0.48F mean? (For every 100 females that were alive the year before, 48 male calves were born.)
- If we assume that only surviving females give birth, what does the 0.48F become? (It becomes 0.48 (0.90F) because 90 per cent of the females survive and 48 per cent of those have male fawns.)
- What does the 0.48F become if first we count our F, then a harvest occurs (Hf), then 90 per cent survive and then 48 per cent of these have male calves? (The 0.48F becomes (F - Hf) × 0.90 × 0.45.)

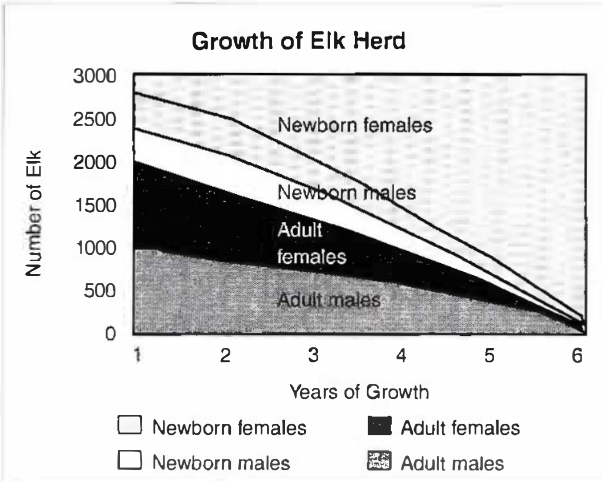
The point of these questions is that our model is a simplified version of what probably happens in an elk herd. Once we understand how the simplified version works, we can make it more complicated. A student assignment can be to determine how equations differ under different assumptions about herd growth.

Although we can begin the herd with any numbers, let us start with 400 male and 400 female fawns, 1,000 each of males and females and a harvest of 300 males and 300 females. Note: our equations determine the number of adult males, for example, in any year by calculating survivors (both adults and fawns) subtracting the harvest. So, in the example below the 1,000 males and females is a count after the harvest. Realistically, the count could have been taken around Christmas, after the harvest.

| 1 | A | B | C | D | E | F | G | H | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | Year | Adult males | Adult females | Newborn males | Newborn females | Total herd | Harvest males | Harvest females | | |
| 3 | 1 | 1,000 | 1,000 | 400 | 400 | 2,800 | 300 | 300 | Male survival rate | 95% |
| 4 | 2 | 850 | 780 | 480 | 420 | 2,530 | 300 | 300 | Female survival rate | 90% |
| 5 | 3 | 748 | 591 | 374 | 328 | 2,041 | 300 | 300 | Male birthrate | 48% |
| 6 | 4 | 597 | 379 | 284 | 248 | 1,509 | 300 | 300 | Female birthrate | 42% |
| 7 | 5 | 409 | 153 | 182 | 159 | 904 | 300 | 300 | Newborn male survival rate | 50% |
| 8 | 6 | 180 | –91 | 73 | 64 | 227 | 300 | 300 | Newborn female survival rate | 45% |
| 9 | | | | | | | | | Harvest males | 300 |
| 10 | | | | | | | | | Harvest females | 300 |

Year 2 Males = 0.95(1000) + 0.50(400) – 300 = 850
Year 2 Females = 0.90(1000) + 0.45(400) – 300 = 780
Year 2 male fawns = 0.48(1000) = 480
Year 2 female fawns = 0.42(1000) = 420

The graph for each category for each year is shown below:




*Teaching Suggestion*

The simplified system of predicting the herd is the most appropriate for the curriculum because of the clarity of the resulting weighted variables in the equations. That is, in any given year, the increase in the total male population, for example, is the result of a certain percentage of adults surviving, a certain percentage of fawns surviving and a certain birthrate. The growth of the total male population is a linear equation with three variables and a constant term.

Students may not be concerned with the actual seasonal happening of the herd. Realistically, the sequence of events is probably a fall herd being harvested, the count being taken, the remaining animals surviving and then fawns being born to surviving females. Some of these elements may be introduced as exercises.

*Using Our System*

Ten groups of students could be given a common herd size, say 1,000, but different harvest rates. For example, if, after five years, we want the herd to be about 1,500, what harvesting policy should we have? That is, which group ends up closest to 1,500 after five years? Because females are more important to herd growth assign different harvest rates for males and females. We can see how the wildlife manager would be able to make predictions.

Another possibility is to have a harvesting rate that varies from year to year.

*Teaching Suggestion*

A table on the blackboard can be used to record the results from the different groups. Each group would record its harvest rate and the herd size in each of the five years. Based on an examination of the pattern, an estimate of the appropriate harvest (for a herd size after five years of 1,500) can be discussed.

## Materials

Although no special materials are needed for this lesson, a spreadsheet program may be useful.

## Modifications

*Using a Spreadsheet*

This problem lends itself well to use of the spreadsheet. Equations can be entered in the cells and the harvest can be varied for different runs. Graphs which show adult males, adult females, male fawns and female fawns in different colours are quite dramatic.

Students can easily be taught to write equations for a spreadsheet. Referring back to the table, the equations for cell $B_4$ is $B_3(0.95) + D_3(0.45) - Hm$. Similarly, the equation for $B_5$ is $B_4(0.95) + D_4(0.45) - Hm$ and so on. Once students understand the problem, filling in cells in a spreadsheet is simple and the graphing results are very motivating.

On page 78 are a spreadsheet and graph of a harvest policy that keeps the herd constant for 20 years.

# Managing an Elk Herd Student Activities

## General Question

All wildlife areas, even in a country as large as Canada, are limited in size. Because of this limitation, the wildlife that the area will support is limited. This is especially true in an area such as Jasper National Park. Even in larger areas where hunting is allowed, wildlife managers need information about herd sizes. Each year, decisions are made about how many animals may be taken by hunters. In making this decision, managers need to be able to predict the growth of herds.

Consider a typical problem for a wildlife manager. A herd of 10,000 animals has 5,000 males and 5,000 females. Given the following information, how large will that herd be in 5 years or 10 years?

| | | |
|---|---|---|
| Survival rates for | males | 95% |
| | females | 90% |
| | male fawns | 50% |
| | female fawns | 45% |
| Birth rate for | male fawns | 48% of adult females |
| | female fawns | 42% of adult females |

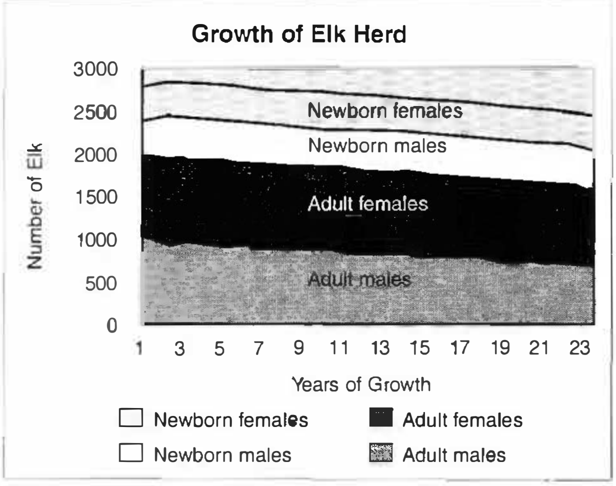
The answer to the problem can be found with the help of mathematics. First, information about survival (death) rates of the animals from year to year must be known, as well as birth rates. Very simply, if the

| 1 | A | B | C | D | E | F | G | H | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | Year | Adult males | Adult females | Newborn males | Newborn females | Total herd | Harvest males | Harvest females | | |
| 3 | 1 | 1,000 | 1,000 | 400 | 400 | 2,800 | 200 | 90 | Male survival rate | 95% |
| 4 | 2 | 950 | 990 | 480 | 420 | 2,840 | 200 | 90 | Female survival rate | 90% |
| 5 | 3 | 943 | 990 | 475 | 416 | 2,824 | 200 | 90 | Male birthrate | 48% |
| 6 | 4 | 933 | 988 | 475 | 416 | 2,812 | 200 | 90 | Female birthrate | 42% |
| 7 | 5 | 924 | 986 | 474 | 415 | 2,800 | 200 | 90 | Newborn male survival rate | 50% |
| 8 | 6 | 915 | 985 | 473 | 414 | 2,787 | 200 | 90 | Newborn female survival rate | 45% |
| 9 | 7 | 906 | 983 | 473 | 413 | 2,774 | 200 | 90 | Harvest males | 200 |
| 10 | 8 | 897 | 980 | 472 | 413 | 2,761 | 200 | 90 | Harvest females | 90 |
| 11 | 9 | 888 | 978 | 471 | 412 | 2,748 | 200 | 90 | | |
| 12 | 10 | 879 | 975 | 469 | 411 | 2,734 | 200 | 90 | | |
| 13 | 11 | 869 | 973 | 468 | 410 | 2,720 | 200 | 90 | | |
| 14 | 12 | 860 | 970 | 467 | 409 | 2,705 | 200 | 90 | | |
| 15 | 13 | 851 | 967 | 466 | 407 | 2,690 | 200 | 90 | | |
| 16 | 14 | 841 | 963 | 464 | 406 | 2,674 | 200 | 90 | | |
| 17 | 15 | 831 | 960 | 462 | 405 | 2,658 | 200 | 90 | | |
| 18 | 16 | 820 | 956 | 461 | 403 | 2,640 | 200 | 90 | | |
| 19 | 17 | 810 | 952 | 459 | 401 | 2,622 | 200 | 90 | | |
| 20 | 18 | 799 | 947 | 457 | 400 | 2,602 | 200 | 90 | | |
| 21 | 19 | 787 | 942 | 455 | 398 | 2,582 | 200 | 90 | | |
| 22 | 20 | 775 | 937 | 452 | 396 | 2,560 | 200 | 90 | | |
| 23 | 21 | 762 | 931 | 450 | 394 | 2,537 | 200 | 90 | | |
| 24 | 22 | 749 | 925 | 447 | 391 | 2,513 | 200 | 90 | | |

death rate is 20 per cent and the birth rate is 40 per cent, the herd size will grow. Under such conditions, the herd will soon become too large for the area. Animals will need to be moved to new areas and/or hunted. How can we predict the size of the herd from year to year?

## Activities

1. (a) A teacher wants to determine your final grade by giving equal weight to your assignments and midterm test and double that to your final. What would be the equation that she could use?
   (b) What would your mark be if you got 80 per cent on both the assignments and the midterm and 60 per cent on your final?
2. (a) When we use the equation $0.95F + 0.45f - Hf$ to determine the number of adult females in a certain year, what sequence of events are we assuming? (Remember that we have four events: the count, the survival, the birth and the harvest.)
   (b) What would this equation be if adult female survival was only 80 per cent and the survival from fawn to adult for females was 30 per cent?



3. (a) If the survival rate for the adult elk is applied after the yearly harvest, what does the equation for adult males become? In this scenario, the young are born, then the count is made giving the M and F values for the year, then the adults are harvested and then 95 per cent of them survive the winter.
   (b) In the scenario in question 3(a) what would be the equation for the number of females able to have calves during the next summer?
4. (a) In our mathematical system, the number of fawns depends only on the number of females. Suppose that a biologist tells you that a herd cannot grow normally unless at least 20 of the adults are male. How would you change the mathematical system to take this into consideration?
   (b) Another biologist tells you that the rule is 20 per cent of the adults are male. How would you account for that?
5. (a) Make up what you think is the most likely sequence of events in the growth of a herd. Assume the year goes from January to January; in other words, the *count* is made in January. Remember four things happen: births, survival, harvest and count.
   (b) Make the equation for adult males for your sequence from question 5(a).
   (c) Make the equation for female birthrates for your sequence from question 5(a).
6. If you know the survival rates and birth rate of a herd, how could you estimate what the harvest should be to keep the herd size stable?
7. How would the equations have to be changed for a population of wolves who pair with a mate for life? Which equation(s) would have to change?
8. Find a spreadsheet and write up the first four lines of it for the elk herd.

**Authors' Note**

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# Getting Conversation into the Mathematics Classroom: Essential, but Formidable—Perhaps "A Bridge Too Far"?

*Werner Liedtke*

The parts of this conclusion include selected comments about the importance of conversation; a glance at students' abilities to communicate mathematically in the past; observational data and selected opinions intended to illustrate possible difficulties that may be encountered while trying to get conversation into the classroom to foster ability to communicate and reason mathematically; and an attempt to look back from the future.

## Why Is the Ability to Learn How to Communicate Mathematically a "Must"?

"Students are expected to communicate in order to learn and express their understanding" is identified as one of the "critical components that students must encounter in a mathematics program in order to reach the goals of mathematics education and encourage lifelong learning in mathematics" (Western and Northern Canadian Protocol 2006, 6). One of the main goals of mathematics education is to prepare students to "communicate and reason mathematically" (p 4). Learning how to talk about what has been learned and is learned in one's own words and being able to illustrate and explain one's thinking with diagrams or objects are key indicators of conceptual understanding: the understanding that facilitates or transfers to future learning and perhaps to lifelong learning of mathematics. Without opportunities to communicate, share and compare ideas about skills and procedures, students will not internalize the language that is related to these ideas or develop conceptual understanding.

The presentation of a theoretical framework and the pedagogical content knowledge that is part of such a framework is beyond the scope of this brief conclusion. *Getting into the Mathematics Conversation: Valuing Communication in Mathematics Classrooms* (Elliot and Garnett 2008), published by the National Council of Teachers of Mathematics (NCTM), includes a theoretical framework and pedagogical content knowledge matrices for listening, speaking, reading, writing and multiple communication forms, and many practical ideas for all grade levels.

## Students' Ability to Communicate Mathematically: Looking Back

In *Learning Mathematics: The Cognitive Science Approach to Mathematics Education,* Davis (1986) makes reference to "disaster studies" and talks about "students who have used the human ability to imitate, and use a new language, in order to learn a form of knowledge which has allowed them to pass tests on the formal apparatus of mathematics and physics, without acquiring the 'assembled' experiential metaphors that would give meaning to this work" (p 370). It is sad that this statement applies to many of us who have completed mathematics courses in the past. Several years ago, the president of the University of Victoria was quoted in the local newspaper as stating that "even the mathematics majors know less mathematics than they think they do because they are unable to talk about the mathematics they have learned in their own words."

Anyone who has conducted diagnostic and/or assessment interviews will have collected data in support of the two previous quotes. Excerpts from interviews I have conducted show that many students will repeat verbatim the words used by their teachers in the steps of algorithmic procedures. During interviews some students will utter statements that do not make sense and they will give credit for these to their teachers.

Responses collected during interviews indicate that as part of their mathematics learning students have practised procedures that they did not understand. These students do not know the reasons for the moves they make with numerals as they record them on paper. Many explanations for algorithmic procedures begin with, "First I start with the ones." The question, "Why do you start there?" is sometimes met with a shrug of the shoulders or with "I have to." During one interview one subject answered with, "So I don't get confused." The question, "Confused with what?" was met with, "I don't know, but I don't get confused."

When one subject, who had recorded a row of zeros for the second partial product as part of her calculations of the product for two three-digit factors where the second factor had a zero in the tens place, was asked, "Why did you record the row of zeros here?" she stated, "To keep me in the rhythm." The majority of subjects will record a row of zeroes—indicators of lack of understanding and lack of number sense.

During interviews, examples of unique and unexpected "mathematical reasoning" will surface that can be quite amusing. In defence of these students, it is unlikely that the ability to communicate mathematically, the development of number sense and practising with understanding were part of these students' mathematics teaching and learning experiences.

## Developing Students' Ability to Communicate Mathematically: Looking Ahead

The development of abilities to communicate and reason mathematically is dependent upon the environment created by teachers in the classroom and on how the strategies from the classroom setting are supported in the home.

### The Mathematics Classroom

If students are to develop the ability to communicate and reason mathematically, opportunities are needed for students to engage in conversations in the mathematics classroom. The inclusion of these opportunities requires the preparation of plans that include high-order thinking and/or open-ended questions and creating an environment that provides students with "opportunities to develop and present new procedures; listen to the shared procedures of others, including their teachers and peers; discuss why different procedures work; and practice procedures they understand" (Hiebert 2000, 437). During conversations, all different types of comments and responses have to be accommodated and integrated into ongoing discussions. It is more than likely that the classroom environment that is required differs from the settings the majority of teachers experienced when they learned mathematics. It may also be that the training teachers received did not prepare them to create and manage these types of opportunities for their students.

In many elementary classrooms settings mathematics learning is a solitary activity. The focus for students is on completing activity sheets. One possible reason for this could be that their teachers learned about mathematics in this fashion. The other main reason could be related to the texts provided by publishers for the students. Many publishers have based the designs of references for students on the assumption of "two practice pages of tasks for every school day." According to publishers, that is what some teachers ask for. How many of these teachers end up talking about *curriculum pressure*, the pressure to cover the material?

Many years ago, while teaching in the largest elementary school in a city (there were four classes of each grade, 1 to 6), the intermediate grade teachers exchanged subjects. These exchanges included music, art, science and physical education. Not one of the teachers was willing to give up and exchange mathematics, and the reason for this was obvious. Assigning pages from a prescribed text requires very little or no planning beyond explaining examples, assigning tasks and marking, a setting that is void of conversations that include high-order thinking and/or open-ended questions. The emphasis was on speed. "Being good in mathematics" was defined by many teachers as being able to arrive at or to recite answers quickly. Many people and some teachers of mathematics still believe this to be true.

Data from research indicate that teachers who during their training were prepared to implement approaches other than those they experienced as they learned mathematics eventually tend to end up employing strategies that they experienced as part of their learning of mathematics. This could be a result of some sort of peer pressure in the school or it could be due to the fact that "teachers, especially elementary school teachers with limited knowledge of and experience with mathematics generally, tend to feel more comfortable with and capable of teaching lower level knowledge and skills rather than more complex knowledge and processes" (Romberg 1995, 76).

For the majority of teachers, the implementation of a mathematics program that gets mathematics conversation into the classroom and is conducive to reaching the goals of developing the abilities to communicate and reason mathematically requires professional development. In times of restraint, monies for this necessity are not available and without this in-service it seems fair to assume that any data based on observations from looking back in the future will not be any different from any previous looking-back observations.

## Beyond the Mathematics Classroom

As a member of the University of Victoria Speakers Bureau, I have contact with many different groups of parents of children in preschool and the primary grades throughout the year. These parents pose questions during presentations and some will exchange ideas after the session. The comments that follow are selected from conversations with these parents. Some statements are based on reactions by parents and relatives as they observe me interact with my grandchildren. A few generalizations come from observing parents interact with their children at sports venues and in settings that involve waiting for an appointment of some sort.

The Irish poet Yeats told us that "education is not about filling a bucket but lighting a fire." I believe that statement to be true for mathematics education. As a fire is lit, or as attempts are made to do so, the "bucket" will be filled! I encounter parents who disagree with Yeats—they believe that the "bucket" needs to be filled with rote facts and procedures. I have been put in the position to respond to, "Why spend time on understanding the number 5 (developing number sense)? Why not teach them how to add and subtract?" One parent of a preschool child asked me what I thought about her child enrolling in Kumon Math. Another parent was attracted to Montessori because young children get to work with big numbers. Teachers of preschool children have told me that there are parents who urge them to teach their children about addition and subtraction, rather than spend time on readiness activities.

Open-ended questions and problems can put children into a state somewhere between what they know and understand and potential knowledge and understanding. Tasks of this type provide for valuable teachable moments. However, these moments are negated every time an adult interjects a child's developing thought process with not only an answer, but also the thinking strategy the adult thinks appropriate for arriving at the answer. I find it frustrating to see children deprived of talking about ideas in their own words and, as a result, missing the opportunity to internalize the meanings of the words they say. Providing a child with the words that parents want to hear when they notice indicators of cognitive conflict brings to a sudden halt the emergence of possible informative and interesting ideas. This action is not a developmental shortcut.

Most parents define a *teachable moment in mathematics* to mean correcting an answer that is wrong, as well as telling or showing how to arrive at the answer. At times this procedure begins with the comment, "No, that is not it," uttered while a child is in the process of attempting to explain his or her thinking. Parents fail to see any possible advantages of having children try to explain the thinking used to arrive at an incorrect answer. Consider the following incident. It was reported to me by a teacher enrolled in a course I taught. His son, who was in Grade 2, arrived at home, looked at his digital watch and stated, "Dad, it is 5:41—21 minutes to *Scooby Doo*." The dad, a teacher, took charge of this "teachable moment" with the questions "How many minutes is it from 41 to 50?" "How many minutes is it from 50 to 60?" and "What is the answer for 9 plus 10?" The son followed the answer, "19," with "But dad, there are 2 minutes of commercials first."

Over the years I have had opportunities to conduct diagnostic interviews at all elementary school grade levels. At one time many of these were videorecorded to be used for instructional purposes on campus and as part of distance education courses. Several things became obvious to students who viewed excerpts from these recordings. Children who used their own words to describe calculation procedures were in the minority. Most children were not confident enough to look away from the objects or diagrams they were facing as they talked to the interviewer. I was amazed at the fact that children trusted a complete stranger and were willing to talk mathematics. However, challenges existed. I did make the point to a group of students in my course before they ventured to conduct their own interviews that, based on my experience, a special effort may be required when interviews are conducted with children whose parents' backgrounds are Asian or First Nation.

One of the students in the course, a Chinese-Canadian, interviewed a boy with a Chinese background. The following conclusion about the interview and an addendum that were submitted as part of the assignment are presented verbatim:

> While the boy needs remediation in some basic concepts, the major problem is not with the boy but with the instructional strategy he has experienced. Incidentally, the boy is Chinese. This point

is peripherally relevant in that I know of no Chinese families, except ones where the parents are completely enculturated or are university-educated teachers, that would not emphasize speed and correctness of basic facts and getting a page full of computations correct, regardless of understanding. The emphasis for school, and life, is on learning the rules quickly, accepting and tolerating them, and applying them with deadly accuracy. For those who are allowed few or no mistakes while learning, admitting to a mistake at home, and at school, would be a humiliating experience. There is a lot of pressure in terms of adapting and excelling and maintaining the stereotype of the hardworking immigrant and being socioeconomically "better" than the parents were. It would be pretty hard to explain to parents why their child, being schooled here in the land of opportunity, is using blocks in Grade 4 when students in Hong Kong, in an extremely competitive society of high "academic" standards, stopped using them in kindergarten. I hope there's some cultural validity in what I just said and I hope it helps clarify your comment at the beginning about students with a Chinese background being afraid to provide anything but the right answer. (And then there's me, who was my father's greatest disappointment when I decided to go to university for, of all the low-status, low-paying jobs, teaching … once I became a principal, I found I had a father again).

Over the years this explanation has served me well as I tried to talk to more students and to parents of these students in schools and in the neighbourhood.

## Attempting a Look Back from the Future

I believe that the results of looking back from the future with respect to ability to communicate and reason mathematically will depend on the answers to four main questions.

1. How much inservice did teachers receive to enable them to get mathematics conversation into the classroom? Professional development for teachers is essential and without it the results of looking back in the future may be predictable.
2. How well were parents informed about types of conversation about mathematics learning that are supportive of what is done in the classroom? Parents need information about conversing with their children about the mathematics they are learning. This information cannot be of a general nature. Educational jargon or "edu-speak" needs to be avoided. The language used needs to be clear and specific. Mathematical ideas and terms need to be illustrated with examples. A need for specificity and clarity becomes clear when some documents from the Ministry [of Education] and articles in journals are examined. For example, a ministry pamphlet for parents entitled *Math for Families: Helping Your Child with Math at Home* (Achieve BC, nd) includes many statements that are difficult and even impossible to interpret (even for someone with a mathematics education background). Consider these selected examples from the first two pages, followed by my questions:
   - "Look for toys that encourage your child to think creatively." How do parents define *thinking creatively*?
   - "Talk about ideas related to numbers, space, time and money." How will parents know what to say and ask?
   - "Use logical thinking: 'There are four kids coming to the party. How many treats do we need?'" How is *logical thinking* different from *thinking*? Is it logical thinking because the hosts need to be considered?

   An article in *Maclean's*, "Have You Finished Your Homework, Mom?" (Reynolds 2012) mentions professors and parents who are experiencing difficulties while trying to help their children with homework. Reading the article makes it clear to me that these parents are unfamiliar with the critical components and the goals for students presented in the new curriculum. Number sense, the key for numeracy, was not mentioned once; neither were confidence building and equipping students with strategies to get unstuck. As with many articles about issues in education, "edu-speak" or jargon-heavy concepts are included, which are meaningless or will result in different interpretations unless they are defined for the readers. For example, phrases used include *traditional method*, *complex problems*, *highly conceptual approach*, *if they don't know their facts they won't be able to do fractions*, *solid grounding in math* and *mastering*.

   Parents need detailed and specific information that includes examples for engaging in conversations with their children that complement and supplement classroom goals related to fostering the ability to communicate and reason mathematically. Without the required specific information, it will be difficult to reach these goals, since parents will be unable to go beyond what they experienced themselves.

3. How often did children engage in any type of meaningful conversation and conversations about mathematical ideas or notions outside the classroom? The task of involving children in conversations may be becoming more difficult.
   Children who at one time were involved in conversing with parents and peers are now focused on a gadget in their hands. I recall that last Christmas one advertisement announced that there was an electronic gadget available for every age group; some gadgets claimed to be educational. In every group setting I find myself, and that includes living rooms, there are adults as well as children who are totally occupied with a gadget in their hands. At one time, sport practices or waiting-room delays involved parents and younger siblings interacting while they watched an older brother or sister in action or while they waited to be called. Now many or most of the younger siblings that I observe are engaged in a solitary setting with an electronic gadget of some sort. I have seen many parents responding to a request from a young child without looking away from what they were holding in their hands. I believe this solitary silent existence must have some sort of detrimental effect on children's ability to communicate that could contribute to making the task of bringing conversation into the classroom more challenging.
4. What other obstacles stood in the way of making it easy or even possible to reach the goals related to fostering the development of ability to communicate and reason mathematically? Possible obstacles include the following:
   - Many or most parents, including math professors (Reynolds 2012), did not understand the new goals of the mathematics curriculum. Many people believed that what was appropriate for us is appropriate for our children.
   - Many people, including teachers, believed that having students repeat rules and practise without understanding was appropriate for students (Reynolds 2012). Some people believed that certain types of tasks that involve calculation procedures, ie, "algorithmic problems" (Norris 2012), were unsuitable for engaging students in conversations or in the types of discussions described earlier (Hiebert 2000).
   - There were people, including mathematicians, who assumed that there exists a best way of teaching and learning about mathematics or that mathematics should be learned the way they learned mathematics.
   - The curriculum used by teachers was void of specific guidelines and examples related to bringing conversations into the classroom (Chorney 2012). This meant that authors of reference texts for students and teachers lacked a framework for the inclusion of specific examples related to communicating and reasoning mathematically.

What might be a hypothetical conclusion about the abilities of future students to communicate and reason mathematically? In the past I have had the opportunity to read articles by authors who did take a look back at the outcomes related to mathematics teaching and learning and concluded that nothing had changed. Will it be possible to overcome the obstacles, bring conversation into the mathematics classroom, and develop the ability to communicate and reason mathematically? For the sake of our students, let's hope so. Are there perhaps too many obstacles to overcome? Are you willing to speculate?

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