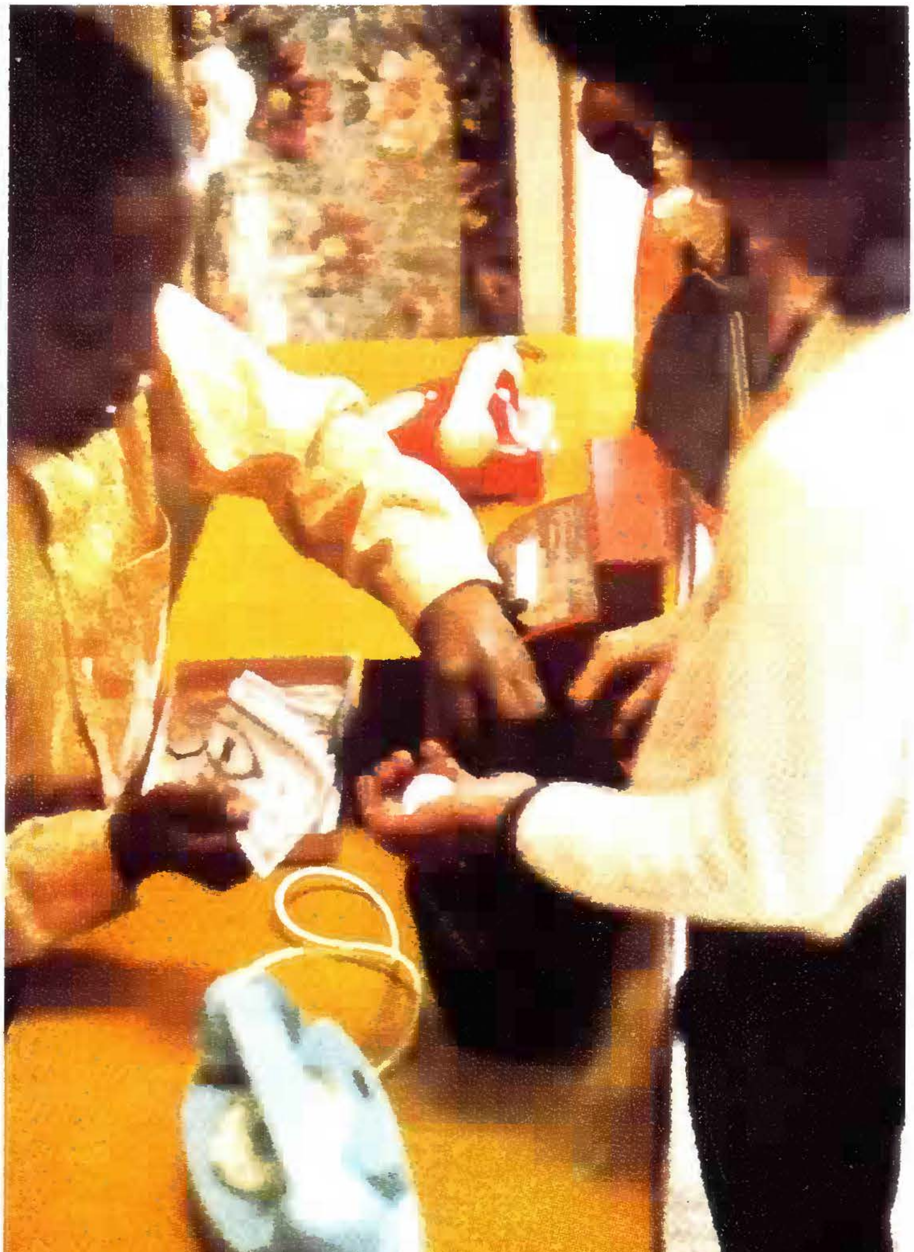
delta-k

Journal of the Mathematics Council of the Alberta Teachers' Association

Volume 50, Number 1 December 2012



Celebrating 50 years: 1962–2012

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. All manuscripts should be typewritten and properly referenced. All pages should be numbered.
4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author-date system or the American Psychological Association (APA) style manual.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB, T1S 2L4; e-mail gsterenberg@mtroyal.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

Cover photo courtesy of the Edmonton Public School Board archives.

delta-k

Volume 50, Number 1 December 2012

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 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB T1S 2L4; e-mail gsterenberg@mtroyal.ca. EDITORIAL AND PRODUCTION SERVICES: Document Production staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in CBCA Education. ISSN 0319-8367

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From the Editor's Desk

Gladys Sterenberg

This volume of *delta-K* celebrates a milestone for the Math Council of the Alberta Teachers' Association (MCATA): 50 years of publishing *delta-K*.

This milestone represents a significant stage in the development of our council. The word *mile* was originally used to describe the Roman unit of 1000 paces.[1] As I edited this issue, I was reminded of the paces we have taken from our inauguration. Henry Van Dyke (1920) asks, "Who will walk a mile with me . . .?" and in this volume of *delta-K,* you will read the writings of many who have walked alongside MCATA members throughout the years. Robert Brown Hamilton (1937) writes about learning from walking a mile with pleasure and sorrow, and as we read these articles, our memories of colleagues who have passed on may be tinged with pleasure and sorrow. Their insight and their dedication to MCATA continue to inspire me as I reflect on this milestone. Finally, Robert Frost (1969) reminds us that we have "miles to go before [we] sleep." It is my hope that the stories shared will provide both a retrospective and a current look at how the dedication of teachers and researchers in mathematics education encourages the continuing enhancement of mathematics teaching and learning.

Note

1. From the Oxford English Dictionary Online (www.oed.com).

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Hamilton, R B. 1937. "Along the Road." In *A Book of Personal Poems*, ed W R Bowlin. Chicago: Whitman.

Van Dyke, H. 1920. "A Mile with Me." In *The Poems of Henry Van Dyke*. New York: Scribner's.

Meet Your MCATA Executive

Back row (l–r): Daryl Chichak, Mark Mercer, Donna Chanasyk, Lisa Everitt, Alicia Burdess

Second row (l–r): Kris Reid, Carmen Wasylynuik, David Martin

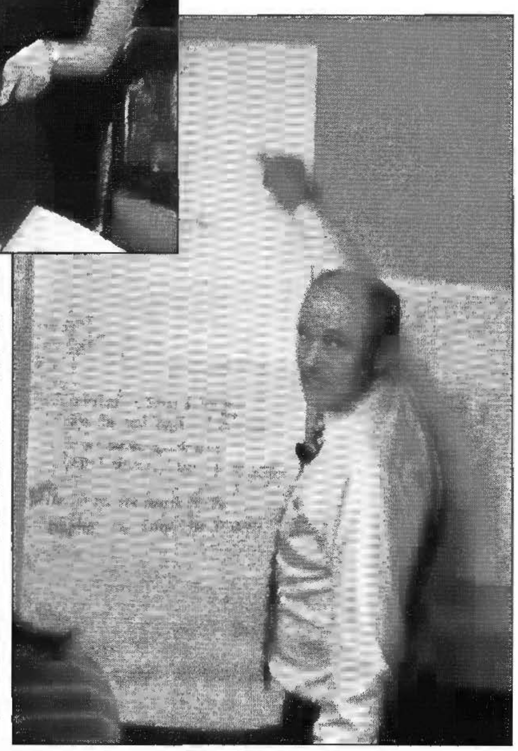
Third row (l–r): Rod Lowry, Debbie Duvall, Gladys Sterenberg, Olive Chapman, Robert Wong

Fourth row (l–r): Carol Henderson, Karen Bouwman, John Scammell

Front row (l–r): Marj Farris, Tancy Lazar, Indy Lagu

MCATA Spring Symposium

National Council of Teachers of Mathematics (NCTM) president Mike Shaughnessy was the speaker at the annual MCATA Spring Symposium, held May 4 in Calgary. The topic was infusing the classroom with reasoning and sense making.



Scholarly Teaching in School Mathematics

Erasmia Eliopoulos, Julie S Long, Richelle Marynowski, Lynn McGarvey and Gladys Sterenberg

In 2010/11, the Math Council of the Alberta Teachers' Association (MCATA), in partnership with researchers at the University of Alberta, invited members to participate in an online survey and follow-up interviews on scholarly teaching in mathematics education. The survey asked teachers to provide information about activities related to professional reading, inquiry into teaching and dissemination of knowledge to the profession. It is our hope that this report will provide insight into how we can enhance and support the scholarly activities of teacher leaders in mathematics education.

Background to the Inquiry

In schools, teachers' work has traditionally taken place behind classroom doors, hidden from view. More recently, the work of teachers has shifted to public forums as shared reflection and inquiry have become expectations of teachers' professional practice (Shulman 2004; Smith and Smith 2006). A similar shift is occurring in postsecondary institutions, supported by Scholarship of Teaching and Learning (SoTL) initiatives. The growth of SoTL comes, in part, from viewing teaching as scholarship, along with more traditional forms of scholarly activity, such as basic research (Boyer 1990).

From the SoTL perspective, excellent teaching and scholarly teaching are not the same. Teaching becomes scholarly when one engages in reflection and inquiry and when one studies, documents and analyzes issues of concern with the purpose of enhancing student learning and improving practice. The scholarship of teaching occurs when evidence-based inquiry is shared, is subject to critique and has the potential to contribute to existing knowledge on teaching within a discipline (Hutchings 2002; Hutchings and Shulman 1999; Kreber 2001).

The words *scholar, scholarly* and *scholarship* are typically associated with academe. Yet, the origin of *scholar* is the Latin *schola,* meaning "school."[1] Although originally used to refer to a student or someone taught in a school, *scholar* is now used most often in reference to the professoriate. Regardless of whether one is a student or a teacher, an apprentice or an expert, a scholar is someone who is devoted to learning, who acquires knowledge through focused inquiry and who communicates knowledge to others (Andresen 2000). This description points to three types of scholarly activity supported by the SoTL literature (Richlin 2001; Trigwell et al 2000):

- Engaging with the literature on teaching and learning to improve practice
- Engaging in critical inquiry into and reflection on practice
- Communicating results to contribute to the scholarship of teaching and learning

While the work of elementary and secondary school teachers is only occasionally referred to as scholarly (Diezmann 2005; Hatch 2006), we see many similarities between the scholarly activities identified in the SoTL literature and the emphasis on inquiry in the public school system. Although the impetus behind the SoTL movement was to bring "recognition and reward" to teaching in higher education, SoTL shares goals applicable to teaching at all levels, including improving student learning and enhancing the practice and profession of teaching.[2]

Our study was designed to identify the range of scholarly activities in which educators in school mathematics engage and the perceived impact of these activities on their practice and professional communities.

Our Study

Recruitment for the study was based on the assumption that educators are more likely to engage in scholarly activities if they are members of a professional or research association, if they have authored articles in research or professional journals, or if they have undertaken graduate studies focused on mathematics education. We used these criteria to identify currently or recently active schoolteachers and invited them to participate in the project.

Data collection for the project included both an online survey and semistructured interviews. Sixty participants completed the online survey, which consisted of both closed- and open-ended questions to collect demographic information and data on the three scholarly activities of reading, inquiring and disseminating. The open-ended questions were coded. Frequencies were generated, and descriptive statistics were used to analyze survey responses.

Telephone interviews with a subset of 14 self-selected survey participants occurred approximately two months after the close of the survey. During the interview, participants were asked to clarify and expand on their survey responses. Afterward, they received electronic copies of the interview transcript and were given an opportunity to make revisions. A thematic analysis (Boyatzis 1998; Braun and Clarke 2006) of the interview data was used to identify key assertions in relation to the participants' scholarly activities and the impact of those activities on their professional practice in mathematics teaching and learning.

Results

The 60 survey participants were primarily female (72 per cent) and over the age of 45 (50 per cent), with more than 15 years of teaching experience (43 per cent). Two-thirds (67 per cent) of them were employed as classroom teachers, with 31 per cent of that group teaching elementary, 24 per cent teaching junior high, 26 per cent teaching high school and the balance (19 per cent) teaching at more than one level. Participants identified themselves as curriculum leaders (17 per cent), consultants (17 per cent), school administrators (10 per cent) and coaches (5 per cent), often in addition to being classroom teachers.

The 14 interviewees from this sample represented a similar demographic in terms of gender, age, experience, teaching position and leadership roles.

Reading the Literature

Participants were asked what resources they used to inform their practice in mathematics teaching and learning. They identified specific websites (44 per cent), authors or books (44 per cent), journals (35 per cent) and textbooks (33 per cent) most often.

Online sources were a primary means of engaging with the literature. The types of websites mentioned most frequently were activity sites or question-and-answer forums for both teacher and student audiences (for example, virtual manipulatives, applets, and homework or test help) and sites with lesson ideas for teachers. One participant noted an online National Council of Teachers of Mathematics (NCTM) position statement on the use of calculators,[3] because "it's backed by research, and . . . you can share it with parents and you know you can trust it." Two interviewees referred to TED.com, a site devoted to inspiring speeches across all domains; one stated that "it makes beautiful connections to other subjects."

Participants also frequently referred to specific book titles and authors (Marilyn Burns, Marian Small, John Van de Walle and Grayson Wheatley, to name a few). Most of the books identified were teaching resources in which authors mediated current theories of learning and provided activity-based applications. Only one participant referred to literature about mathematics content (specifically in the areas of probability, number theory and the history of mathematics). Print journals (specifically, NCTM professional journals and the MCATA journal) were mentioned most frequently. These journals are peer-reviewed, and the articles primarily focus on applications of research and shared teaching experiences. One interviewee said, "I used ideas from *Teaching Children Mathematics*. I'd flip through those every once in a while." Another said, "A coworker of mine had some activities that he pulled from an older copy of the *Mathematics Teacher*, and I'd seen a couple of things from articles that were in there that I thought were really good." Only one participant mentioned a research journal.

Finally, authorized textbooks were identified frequently by survey participants, particularly by those teaching junior or senior high.

Inquiry into Practice

Survey participants were asked to complete a series of Likert scale questions to determine the impact of a variety of professional activities on their practice. The professional activities teachers rated as most significant were postsecondary coursework (95 per cent) and collaborative planning or lesson study (78 per cent). Other initiatives viewed as significant or very significant by the majority of respondents included school-based collaboration, workshops, self-initiated classroom research, mathematics study groups with colleagues, mentorship of student teachers or beginning teachers, coaching of colleagues, and research investigations.

In follow-up interviews, most participants mentioned collaborative inquiry with colleagues as being particularly meaningful. The inquiry usually had a practical aspect, such as marking provincial examinations or preparing lesson plans, but conversations around these practical tasks led them to reflect on their practice.

Participants also spoke about the ways student understandings of mathematical ideas spurred them to inquire into and improve their practice. Several participants pointed to observation of teaching coupled with conversation as being meaningful to their practice.

Dissemination to the Profession

In an open-ended question, survey participants were asked to describe how they shared their experiences in and knowledge of mathematics with others through presentations, publications, online dissemination and other forums.

Of the 87 per cent of participants who responded, 65 per cent indicated that they shared their knowledge through formal presentations at their school, in their district or at provincial conferences, and 54 per cent shared knowledge through formal or informal collaborations in their school or district. Other means of disseminating knowledge included contributing to print publications (15 per cent), participating in online discussions or lesson-sharing (13 per cent), coaching colleagues (12 per cent) and mentoring student teachers (12 per cent).

While most interview participants acknowledged that presentations at teachers' conferences were important, they consistently talked about the significance of collaborating with other colleagues. One stated,

> I've worked with small groups of teachers, have done some in-servicing at other schools. . . . When teachers have seen a presentation, often they'll email me and they'll ask me for more, and I invite them to come to my classroom. . . . And that's more powerful even than doing a session at convention.

Another participant mentioned recently doing an interactive whiteboard presentation and said, "I'm getting to the point where I could probably do something at a teachers' convention."

As in findings on reading and inquiry, most participants spoke about extending curriculum resources by creating and sharing teaching ideas. For example, one participant developed a project for kindergarten teachers that incorporated a variety of centre ideas from an authorized textbook. Only one interview participant mentioned publishing her work in a research journal.

Conclusion

The teachers engaged in all three forms of scholarly activity: reading current literature on teaching and learning in mathematics education, inquiring into their own practice and into the practice of colleagues through collaboration, and publicly communicating their ideas in both formal and informal ways.

However, contrary to much of the SoTL literature, the teachers' activities did not follow a reading–inquiry–dissemination cycle to a pre-specified issue. That is, the teachers' selection of literature was often based on recommendations from colleagues and was discussed and critiqued collaboratively, with the broad goal of challenging taken-for-granted notions of what it means to teach and learn.

The teachers in this study perceived the impact of their scholarly activities on their professional practice as being significant—but only when the activities were integrated and occurred in collaboration with others. It was the interactions among teachers in their "communities of conversation" (Shulman 1993, 6) that allowed them to analyze aspects of their own practice that had previously gone unexamined.

While the educators' aim appeared primarily to be improving students' learning experiences and their own professional practice, they did so by sharing knowledge and experiences in their local communities of practice. In this integrated process, they framed themselves both as teachers with an ongoing desire to learn and as scholars devoted to teaching and learning who engaged in active inquiry and communicated their knowing to others.

Recommendations

Teachers emphasized the importance of collaborative learning, in which knowledge is not necessarily owned by any one individual but, rather, is shared, taken up and modified within local contexts. Based on the survey and interview results, we believe that the following actions would contribute to the vitality of MCATA and teacher professional development in our schools:

- Online materials and professional books authored by workshop presenters were primary sources for teacher reading. We recommend that MCATA post on the council website current links to literature that explores emerging topics of interest. This could become a responsibility of one of the executive members.
- Teachers' inquiry into practice was affected most by participation in contextual, long-term lesson study with colleagues. We recommend that schools

and professional development consortia replace one-time workshop presentations with teacher release time devoted to collaborative development of lessons. We recommend that MCATA introduce study groups as part of the annual conference to provide teachers with opportunities to begin such study initiatives. We envision a space where teachers who are interested in the same topic could meet for a time of sharing and collaboration. We believe that MCATA could facilitate a connection between teachers with similar passions.

- Most teacher leaders in mathematics considered sharing their ideas as the result of encouragement from colleagues. We recommend that MCATA actively recruit conference speakers from within the membership. We believe that such encouragement will help teachers engage in scholarly activities. We believe that teachers should be recognized for the scholarly work they are doing in schools and that they should be supported in sharing this work.

Notes

This research was supported by a grant from the Alberta Advisory Committee for Educational Studies.

1. From the Oxford English Dictionary Online (www.oed.com).

2. See www.carnegiefoundation.org/scholarship-teaching-learning (accessed September 13, 2012).

3. See www.nctm.org/uploadedFiles/About_NCTM/Position_Statements/computation.pdf (accessed September 13, 2012).

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New Book Announcement

Selected Writings from the Journal of the Mathematics Council of the Alberta Teachers' Association

Celebrating 50 Years (1962–2012) of *delta-K*

Edited by Egan J Chernoff (University of Saskatchewan) and Gladys Sterenberg (University of Alberta)

The teaching and learning of mathematics in Alberta has a long and storied history. An integral part of the past 50 years of history has been *delta-K*, the journal of the Mathematics Council of the Alberta Teachers' Association (MCATA).

This monograph, which presents 50 memorable articles from the journal (10 from each of the past five decades), shares this rich history with a wide range of people interested in the teaching and learning of mathematics and mathematics education. The section for each decade begins with an introduction that provides the historical context and concludes with a commentary from a prominent member of the Alberta mathematics education community. As a result, the monograph provides both a historical account and a contemporary view of trends and issues (such as curriculum and technology) in the teaching and learning of mathematics. It is intended as a resource for a variety of people (including teachers of mathematics, mathematics teacher educators, mathematics education researchers, historians, and undergraduate and graduate students) and as a celebratory retrospective on the work of MCATA.

Selected Writings from the Journal of the Mathematics Council of the Alberta Teachers' Association: Celebrating 50 Years (1962–2012) of "delta-K" is a volume in the series The Montana Mathematics Enthusiast: Monograph Series in Mathematics Education, edited by Bharath Sriraman (University of Montana).

The monograph is in development and will be available from Information Age Publishing.

For more information, go to www.infoagepub.com/products/Mathematics-Council-of-the-Alberta-Teachers-Association or call the IAP office at 704-752-9125.

First Issue of the *Mathematics Council Newsletter* (January 1962)

THE ALBERTA TEACHERS' ASSOCIATION

MATHEMATICS COUNCIL NEWSLETTER

Vol. I, No. 1 January, 1962

Announcing the New Bulletin

At the inaugural conference of the Mathematics Council recommendation was made to the executive committee to provide a vehicle that would disseminate information to the members of the council. The executive committee met on November 4, 1961 and passed a motion authorizing the present publication. The intention is to publish a yearbook summarizing each annual conference and a series of news bulletins that will keep the members aware of current developments. These news bulletins are to be published in January, April, June, and November. It is proposed to include in these news bulletins, besides current news items that might be of interest to our council members, an article of theoretical nature.

L. W. Kunelius on Geometry

Editor's Note: At the Mathematics Council's inaugural conference last August, Mr. Kunelius, at the end of his address, was asked the question, "What is the place of geometry in our high schools?" What follows is his answer.

The place of geometry in high school has been defended on several grounds.

(1) It provides knowledge of important geometric relationships of which the educated adult has future need. In other words, it contributes to literacy.
(2) It provides the best means we presently have for teaching elementary ideas concerning the nature of proof.
(3) It has a peculiar or unique value for teaching habits of logical reasoning.

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(4) It provides an example of a logical mathematical system, the only example we have in our high school mathematics of a system founded upon undefined terms, definitions and postulates.

Let us look at each of these. If the main purpose is to teach geometric facts - in other words, knowledge of basic geometric constructions and relationships - this can be accomplished by the methods of informal or intuitive geometry in junior high. The pupil at the end of Grade IX knows that the angles of a triangle are together equal to two right angles or 180°, he knows the conditions for congruence of triangles, similarity and parallelism though he cannot give formal proofs for them. Furthermore the number of geometric facts which the average citizen should know and which are not now taught informally by the end of Grade IX is few. Clearly the first reason is not enough to justify demonstrative geometry in the high school.

It provides the pupils with a means of acquiring a glimpse of the nature of a logical mathematical system. But Euclid's geometry as it is taught, or can be taught at the high school level is not very logical; it has several weaknesses. Much simpler and better examples of postulational systems can be developed in algebra.

It provides a means for teaching pupils something of the nature of proof and for practice in logical or deductive reasoning. It can do these provided it is taught with that end in view - if the role and nature of undefined terms, definitions, assumptions and of theorems derived from them are brought out; if hidden assumptions are carefully analyzed in both mathematical and non-mathematical situations; if the character of analysis, of indirect proof, of inductive proof, are understood. To do this we don't need to study several books of Euclid. Nor is any crime committed if for example the congruence theorems are left unproven and treated as assumptions.

I therefore predict that though the mastery of the more significant theorems (significant because they have frequent applications in life) will continue to be recognized, this will be overshadowed by other objectives for high school geometry, viz, (a) that pupils come to appreciate the postulational structure of geometry, (b) that pupils come to appreciate that there are other geometrics each based on its own set of postulates, (c) that algebraic or analytic methods will be introduced more widely, and (d) that pupils come to appreciate the nature of proof and the types of reasoning involved in proof.

These changes will come gradually. I believe that within the framework of the present text some changes can be made which would provide the teacher with more time to give greater emphasis to the structure of geometry and the nature of proof. I would hope that alert teachers who are thinking along the lines I have outlined above will take the initiative and the courage to depart from the traditional approach to the text in Mathematics 10. There are some commendable features in the text which presently receive but scant notice due to pressure of time and lack of appreciation of their significance. I would say: telescope the work on theorems.

MCATA Committees

The ATA Accreditation Committee requested from our Council evaluation criteria in mathematics for accreditation. A committee consisting of the following was appointed: T. P. Atkinson, F. Tarlton, and J. K. Sherbanuk, all of Edmonton. This committee should complete its work in January.

J. Holditch of Edmonton has been appointed as MCATA library secretary.

John Cherniwchan and Eugene Wasylyk are appointed editors for MCATA.

Modern Mathematics by a Local Author

A book entitled, Modern Mathematics. Introductory Concepts and Their Implications, is to be published February 15, 1962 by W. J. Gage, Limited, Box 550, Scarborough, Ontario. The expected price is $4. A. B. Evenson, the author, formerly with the Curriculum Branch of the Department of Education, is now general supervisor of senior high schools in Edmonton.

This is a professional book written primarily for teachers who wish to become acquainted with the basic concepts, language and implications of modern mathematics. It is not aimed at any particular grade level. The material covered has implications for a program throughout all

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the grades. The main topics discussed are sets, number and numeration systems; conditions in one variable; relations, functions; elementary logic and proof, mathematical systems.

Experimentation at Junior High School Level

Experimentation with new materials in mathematics at junior high level is being conducted by a total of 33 teachers in the following administrative areas: Cities - Calgary, Edmonton, Red Deer, Jasper Place; Rural Areas - Clover Bar School Division, Pincher Creek School Division, Rocky Mountain School Division, County of Sturgeon, and Vermilion School Division. This experimentation is under the direction of the Junior High School Mathematics Subcommittee of the Alberta Department of Education.

Cuisenaire Materials at Primary Level

For the past three years members of the Grade I teaching staff of the Ponoka School have experimented with Cuisenaire materials in arithmetic. It is the opinion of this staff that the primary pupils grasped some of the more difficult concepts at this level more readily than they would have with traditional methods.

Seeing Through Arithmetic Inservice Work

Thirteen classrooms in Ponoka County are experimenting this year with the Seeing Through Arithmetic series. These experimental classes include every grade level from I through VI. The school committee has accepted the recommendation of H. L. Larson, assistant superintendent of the county, to introduce the STA series into all elementary classrooms in September 1, 1962.

In preparation for this introduction, Mr. Larson has organized a series of meetings dealing with the new approach to arithmetic teaching in the elementary grades. At meetings held in Ponoka and Rimbey on November 8 and 9, respectively, an excellent attendance was recorded and enthusiastic interest was exhibited. The first film of the Seeing Through Arithmetic series, put out by the Scott, Foresman and Co., was shown. We understand that this was the first showing of this film in Alberta.

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MCATA Constitution

The MCATA constitution as amended at the inaugural conference appears below.

Name. The name of this organization shall be the Mathematics Council of The Alberta Teachers' Association.
Object. The object of this organization shall be, to promote and advance the teaching of mathematics throughout the province, especially in elementary and secondary schools.
Membership. The following are eligible for membership: (a) any member of The Alberta Teachers' Association or non-member covered by the Teachers' Retirement Fund, (b) any certificated teacher in private schools, and (c) any member of the University of Alberta or of the Department of Education.
Fees. Membership fees may be established by resolution at the annual general meeting of this council.
Finances. The executive committee shall have power to collect fees and to make expenditures. A financial statement shall be submitted to the annual general meeting.
Officers. The officers of this council shall consist of a president, a vice-president, a past president, and a secretary-treasurer, to be elected for a term of one year at the annual general meeting, and a member appointed by the Executive Council of The Alberta Teachers' Association.
Executive Committee. The executive committee shall consist of the officers and one MCATA member from each of the elementary school, the junior high school, the high school, the Faculty of Education, and the Department of Mathematics of the University of Alberta, appointed by the officers.
Committees. The executive may appoint from time to time such commitees as are necessary to carry on the work of the council.
Liaison. Any communication regarding policy which this council wishes to make with any organization, government department, or other agency, within or without the province, shall be conducted through the Executive Council of The Alberta Teachers' Association or other regular channels of that Association.
Regional Councils. The executive committee of this council shall encourage the establishment of regional councils and shall have authority to determine regional boundaries and to establish

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regulations governing the organization of regional councils, consistent with this constitution.

Reports. The executive committee shall submit annually a written report of its activities to The Alberta Teachers' Association, prior to December 31.

Amendments. After three months' notice of motion to amend the constitution has been given to each member, this constitution may be amended by a two-thirds majority vote of the members present at any annual general meeting of this council, subject to ratification by the Executive Council of The Alberta Teachers' Association.

General Meetings. The Mathematics Council shall hold at least one general meeting a year. At least thirty days' notice shall be given for all general meetings.

Mathematics Publications, ATA Library

At the inaugural conference of the Mathematics Council, Dr. J. H. Hlavaty suggested a list of publications that might be useful as background reading on curriculum revision in mathematics. The ATA Library Committee was asked to consider adding these titles to the professional library, and the following is a list of the books which have been acquired -

Selections From Modern Abstract Algebra; Andree, Richard V.; Holt, Rinehart and Winston, New York

The Teaching of Arithmetic; Spitzer, Herbert F.; Houghton Mifflin Co.

New Thinking in School Mathematics, Report of Seminar held by Canadian Teachers' Federation at Ottawa, April 28-30, 1960 (four copies)

The Introduction to the Foundations and Fundamental Concepts of Mathematics; Eves, Howard, and Newsom, Carroll V.; Holt, Rinehart and Winston, New York.

"Patterns of Plausible Inference"; Polya, G.; Mathematics and Plausible Reasoning; Princeton University Press, Princeton, New Jersey, 1954, Vol. II.

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"Induction and Analogy in Mathematics"; Polya, G.; Mathematics and Plausible Reasoning, Princeton University Press, Princeton, New Jersey, 1954, Vol. I.

The Secondary Mathematics Curriculum, Report of the Secondary School Curriculum Committee, National Council of Teachers of Mathematics, Washington 6, D.C.

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- 7

MCATA Executive Meeting, November 4, 1961

At the first executive committee meeting of the MCATA, held at Barnett House, the following items were decided.

(1) The editorial committee was authorized to publish (a) a full report of the inaugural conference and (b) four newsletters a year.
(2) The executive committee approved in principle the idea that the MCATA had some responsibility in providing aid to elementary teachers in preparing themselves for mathematics curriculum changes.
(3) Steps are to be taken to investigate the possibility of holding a special seminar for local resource persons. A decision was reserved until the next executive meeting.
(4) The date for the next meeting was set for December 29.

Membership in the MCATA

Teachers of mathematics in Alberta are facing a major challenge in the implementation in the next few years of the proposed curriculum changes. The speed with which these changes can take place depends to a great extent on how ready the teachers will be to accept these changes. We believe the MCATA is part of the answer.

Don't delay. Join now. Application for membership in the MCATA may be sent to: Miss Olive Jagoe, 1431 - 26 Street, S.W., Calgary.

Wanted ...

We are scouting around for material that would be of interest to members of the MCATA. If you have some pet idea some teaching device, some strong convictions about some aspect of mathematics, we would be most pleased to hear from you. We are particularly interested in receiving comments on the present trend toward teaching modern mathematics in school systems. If you are handling an experimental unit, or a set of units, we are sure your comments would be useful to others.

Anything you care to submit will be gratefully received by John M. Cherniwchan, 276 Evergreen Street, Sherwood Park, or Eugene Wasylyk, Box 2, Thorhild.

8 -

"Baa, Baa, Black Sheep": A Delightful Memory

Werner Liedtke

Unpredictable, predictable and pleasant incidents are part of any classroom. When incidents are unpredictably pleasant, they become memorable (perhaps even more so with advanced age and more time for reflection).

A course dealing with diagnosis and intervention in elementary mathematics included three mature students. These women sat together at the back of the classroom, and the occasional whisper among them during a lecture was not disruptive. However, it was somewhat unusual that they always remained seated after the class had concluded. Their responses to my queries indicated that they did not have any questions or concerns about the course, and my worries were put to rest when they assured me that they were not involved in some sort of protest or secret plot.

At the beginning of the final class, one of the women requested that 15 minutes be set aside for a presentation. The request was granted. When the time arrived, lyric sheets were distributed to the other students, a ukulele appeared, and one of the women assumed the role of choir director.

On the lyric sheets, 10 verses and a refrain were written out in calligraphy. The class was led through a song to the familiar tune of "Baa, Baa, Black Sheep." Each verse referred to one or more key ideas of the course. These ideas were what the women had been discussing at the end of each class.

The lyrics are presented below, along with my comments.

How are they different?
How are they the same?
What do you think of when you see this?
Can you give it a new name?

The refrain captures the key ideas of the course. If we don't know how students think, effective intervention cannot take place. Sorting is a key strategy for many aspects of learning about mathematics, and the ability to think flexibly is an important goal of intervention.

Which is more and which is less?
And what do you do with zero?
If you can make up your own rule,
The prof will think you're a hero.

This hints at the confusion that can exist with regard to the role of zero, as well as the challenge to create settings where students can generate their own rules for generalizations about calculating answers for equations that include a zero.

You must get them to manipulate.
You must get them to simulate.
You must teach them to estimate.
Before you can remediate.

During diagnosis (as well as intervention), questions should be asked that engage the brain while objects are manipulated or while actions related to operations are simulated. It is very likely that in an intervention setting, estimation strategies will need to be taught or retaught.

Relate to experience.
Construct and record.
Rename and regroup.
You're colour-bound? Good Lord!

Being able to connect, to illustrate one's thinking and to think flexibly about numbers are important aspects of sense-making and conceptual understanding. Since students who experience difficulties may be easily distracted by the colour of objects, key manipulative materials used during diagnosis and intervention should be colourless or all of the same colour.

You don't understand place value?
We will help you, honey.
We'll relate to your experience,
And do it all with money.

In an intervention setting, money can play a role that goes beyond the ability to make connections. Key components of developing number sense (including visualizing numbers, thinking flexibly about numbers and relating numbers) can be accommodated.

"My students can't sketch or diagram,"
We hear the math tutor groan.
"So how can I get her to transfer
From the known to the unknown?"

Students who understand what they have learned are able to illustrate their thinking with manipulative material or diagrams. They can talk about the ideas and procedures they have learned in their own words. This understanding, as well as transfer to previous and ongoing learning, is a goal of intervention.

Working with subtraction
May cause you joy or sorrow.
Regroup, rename or trade
But never, never borrow!

How did the term *borrow* find its way into the mathematics classroom? The terminology used during intervention should foster visualization. It should also be consistent with the terminology that was used when numbers and numerals were examined.

Partitive or "quotative"?
Which method do you choose?
Division is so difficult.
No wonder we turn to booze!

Data collected over the years show that most requests for intervention are concerned with either basic multiplication facts or the division algorithm. Understanding the latter requires that students know about the two types of division—partition and measurement. At one time the term *quotitive* was used to describe measurement division. During a session on division, it was pointed out that this term does not help in any way with visualization, and it is also often misspelled (as indicated in the verse).

Always use the very best aids,
If you really don't want to fluff it.
Forget the abacus and number line.
Take the pocket chart and stuff it!

The ability to visualize numbers is a key indicator of the presence of number sense—the foundation of numeracy. Fostering the ability to visualize requires the use of a manipulative aid that clearly illustrates the properties of our numeration system. The aids mentioned in the verse lack this property.

When you are in an LA setting,
The conceptual domain be not forgetting.
There is one thing before we go.
Memorize this—"How do you know?"

This verse identifies the question that gives the greatest insight into what and how a student is thinking. Without an answer to this question, thorough diagnosis and effective intervention are impossible.

Understanding and basic facts,
As well as algorithm.
With the prof's notes under your arm,
Go forth and remediate with 'em!

It has been stated more than once that it is possible to "teach" anyone to repeat something. The challenge is to foster understanding—conceptual understanding. There are some who say that *remediation* has a negative connotation, and I am certain that if these ladies were to write the last line now it would read:

Go forth and plan your intervention with 'em!

A few ideas from the course that did not make it into the song were incorporated into the rectangular border of the lyric sheet. When students requiring diagnosis and intervention were discussed during the early sessions of the course, boys were most often identified (for example, by referring to *Johnny* or *he*). On more than one occasion, it was pointed out that this was unfair; thus, the reference to *honey* and *her* in the fourth and fifth verses. This reversal also became evident in two statements included in the border (with underlining for emphasis):

Yes, but how are her visualization skills?
Does she have a good memory bank?

Throughout the course, terminology was identified that does not foster students' ability to visualize, including the word *times* (the phrase *groups of* was recommended instead). The frame included the following statements:

Never say "times."
On Sunday morning we read the *Victoria Groups of Colonist*, or *The New York Groups Of.*

Two statements in the border referred to strategies employed during diagnosis and intervention:

Can you see an error pattern here?
And if that does not work, try something different.

The late-afternoon class must have made these ladies think about refreshments. One statement points out the advantage of living in a big country:

Somewhere in Canada it's cocktail hour right now.

The students who have these women as teachers will be very lucky, indeed. To Dale, Jennifer, Beth and the choir—a hearty thank you for the memorable moments.

Werner Liedtke is a professor emeritus at the University of Victoria. He has taught elementary school and courses in mathematics education and assessment. His main areas of interest are curriculum development and assessment strategies related to key aspects of early numeracy.

The Psychology of Learning in Teaching Mathematics

Marlow Ediger

The mathematics teacher has a plethora of ingredients to emphasize in teaching and learning situations. First of all, having breadth and depth of knowledge to use in the instructional arena is salient. Bringing in available technology to improve student learning is a must. These ingredients need blending with the psychology of learning in helping students succeed. It is vital, then, to secure the interest of learners in ongoing experiences. Without the interest factor, student involvement will be minimal. To achieve objectives, learning activities must be varied, challenging and appropriate to the ability levels of learners. It behooves the teacher to choose experiences that develop and maintain student interest. Additional criteria for teaching mathematics also need attention.[1]

Quality in Teaching Mathematics

Motivating students in attaining mathematics objectives means that their energy level increases. Motivated students are attentive and are eager to participate actively in the curriculum. Their attention span increases and they stay on task, rather than daydreaming or thinking about the weekend. The mathematics teacher must observe students to notice their achievement, as well as where diagnosis and remediation are necessary. Meticulous observation of learners in the classroom setting provides feedback for teaching and allows teachers to get to know student behaviour better while seeking innovative ways to motivate them. Creative approaches to the mathematics curriculum must be found in order to increase student motivation (Ediger and Rao 2011).

In each learning in mathematics, meaning must be emphasized. The facts, concepts and generalizations that are acquired serve as building blocks for subsequent lessons. If the learnings are hazy, students will likely experience difficulties with the new objectives. A professional teacher possesses a reservoir of mathematical knowledge and skills in guiding students to overcome problem areas. The content must make sense and be understood clearly. The teacher should pace the teaching of subject matter to facilitate achievement of the objectives. The goal is for students to master vital subject matter and abilities (such as critical and creative thinking, reasoning skills, and problem solving).

Purpose must be stressed and accepted by students in ongoing lessons. Thus, students acquire reasons for participating actively. With purpose, learners perceive a need for using the subject matter in school and in society. Mathematics is functional and precise, and it contains patterns for mastery. Too frequently, students see mathematics class as another day of drill and rote learning. Rather, mathematics must be perceived as exciting and challenging and as a process of discovery. Too often, learners in the classroom are perceived as lethargic and uninterested. It should be just the opposite, with active participation and use of manipulative materials that harmonize with the developmental levels of those in the classroom. It is possible to adapt the curriculum to where students are currently and then sequence learnings to make for continuous progress (Wiske 2004).

Resilience must be integrated into mathematics curricula. With resilience, students bounce back from difficulty and failure. A major problem in teaching is that learners give up too quickly in problem-solving activities, as well as when a new process is being emphasized. Securing student attention is salient, and carefully ordering steps in teaching new learnings is necessary. However, in school and in life, frustrating situations do occur. Students need help to deal with difficulties in a manner that stresses resilience. For example, the teacher can scaffold content and skills. A competent math teacher might then raise sequential questions whereby learners discover the answer or the knowledge needed to realize increasingly difficult objectives. Instead of taking an inductive approach, the teacher, deductively, might sequence from where the child is having difficulty to the desired level of accomplishment. This, too, can make for feelings of accomplishment and minimize frustration, increasing the chances of success in future endeavours.

Teachers must be accountable for their own behaviour. An off-the-cuff statement in a moment of weakness may cost a teacher a recommendation or even result in dismissal. Calling students names or losing one's temper is regrettable, and harassment and rudeness must be eliminated from the school setting. The consequences of such actions are high indeed. It behooves teachers to be in complete control. Teachers need emotional intelligence in order to control their behaviour and actions. Probably, we all have been in classes along the way from elementary school through the college years in which an instructor lost his or her temper for one reason or another. Teaching can be stressful, but teachers must be accountable for their behaviour, such as in the following situations:

- Students disrupting teaching in ongoing lessons
- Learners failing to "get it" during explanations of specific mathematical processes
- Students not being focused on the subject matter or skills presented
- A classroom atmosphere that does not promote learning

It is not easy to deal with one's emotions in difficult situations, but the responsibility for maintaining a positive learning environment rests with the teacher. Otherwise, learner achievement in mathematics will tend to go downhill. Teachers must learn about the customs, mores and folkways of a given community in order to understand the beliefs and motivational patterns of students. The total child is involved in learning, not just the academic and cognitive facets. Each student has diverse needs that must be fulfilled. Some of these needs the school can fulfill, such as providing breakfast and lunch. Some schools even serve dinner at the end of the school day. All teachers must attempt to involve learners in meeting their need to belong. Feeling isolated greatly hinders student progress in mathematics, as well as in other curriculum areas. Collaborative and committee endeavours in which learners work harmoniously together are a starting point.

In Conclusion

Mathematics teachers need to possess self-efficacy in knowledge to impart to students. Knowledge consists of relevant facts, concepts and generalizations, which are necessary for high-quality instruction. These are needed for a developmental approach in sequencing learnings for students. Subject-matter knowledge, alone, is not adequate. The mathematics teacher must also be able to use the best methodology possible to reach children's abilities in teaching and learning situations. High expectations but achievable objectives should be in the offing. Scaffolding should be used when helping learners realize optimal achievement. Appraisal of learner progress must be continuous, with the use of feedback to improve the curriculum. Inservice education is necessary in order to stay abreast of current developments in instruction, such as the ensuing common core objectives.

Note

1. See National Council of Teachers of Mathematics (1989).

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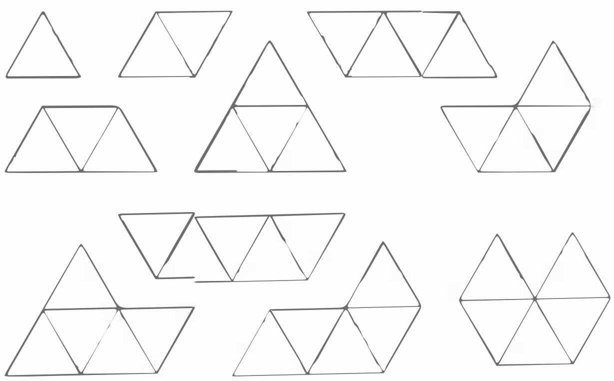
Polyiamond Compatibility

Richard Mah, Ryan Nowakowski and William Wei

A *polyiamond* is a connected plane figure formed by joining unit equilateral triangles edge to edge. Figure 1 shows the moniamond, the diamond, the triamond, three tetriamonds (in the shapes of A, I and V) and four pentiamonds (in the shapes of A, I, J and U).

Figure 1



A polyiamond X is said to be a *multiple* of another polyiamond Y if a copy of X can be assembled from copies of Y. A polyiamond is said to be a *common multiple* of two other polyiamonds if it is a multiple of both. If two polyiamonds have common multiples, they are said to be *compatible*.

Given two polyiamonds, even relatively small ones, it is not always easy to determine whether they are compatible. Here, we investigate the compatibility of polyiamonds up to the pentiamonds.

The moniamond is trivially compatible with every polyiamond. Figures 2a and 2b show that the diamond is compatible with the triamond, all three tetriamonds and all four pentiamonds.

Figure 2a

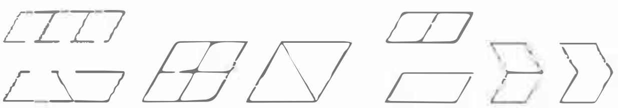


Figure 2b

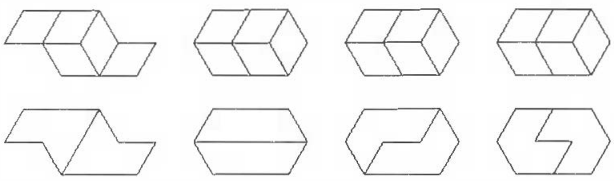


Figure 3 shows that the triamond is compatible with all three tetriamonds and four pentiamonds.

Figure 3

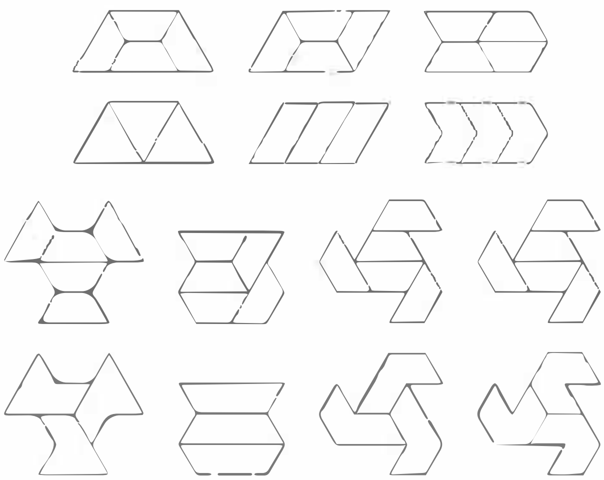
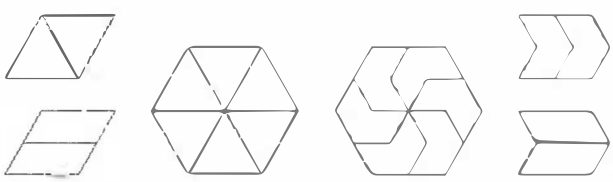


Figure 4 shows that the tetriamonds are compatible with one another.

Figure 4



The compatibility of the A-tetriamond with the pentiamonds is the most difficult case. We believe that it is not compatible with the U-pentiamond. However, we do not have a proof. On the other hand, Figure 5 shows that it is compatible with the other three pentiamonds.

Figure 5

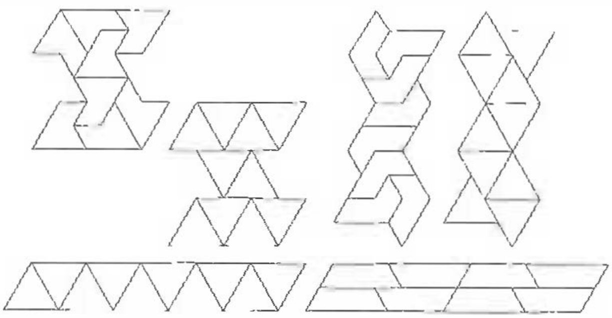


Figure 6 shows that the I-tetriamond is compatible with all four pentiamonds.

Figure 6

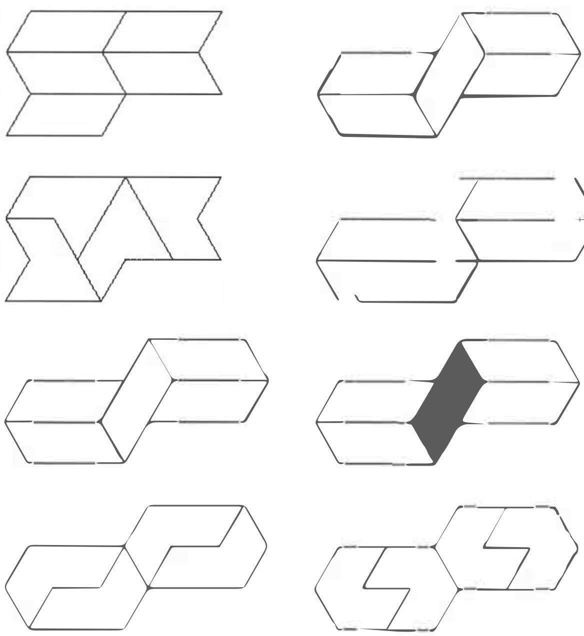


Figure 7 shows that the V-tetriamond is compatible with all four pentiamonds.

Figure 7

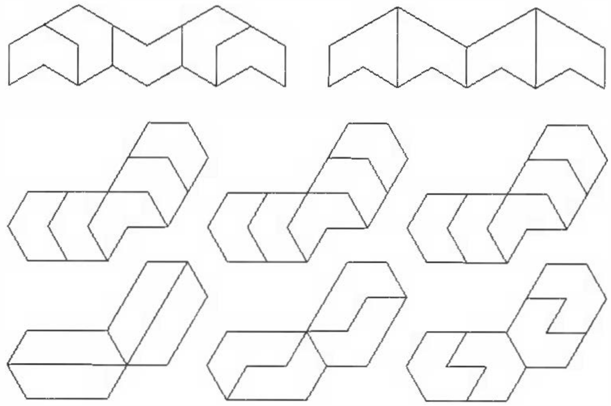
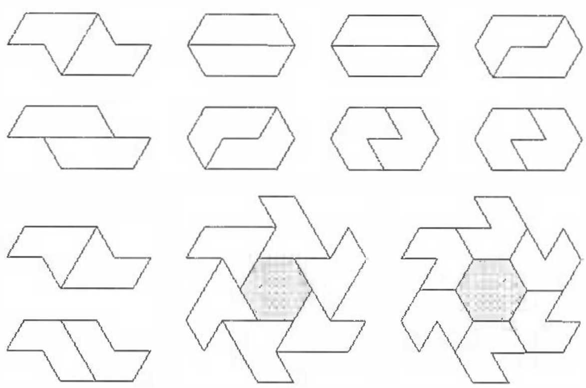


Figure 8 shows that the pentiamonds are compatible with one another.

Figure 8



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Ryan Nowakowski is a Grade 7 student at McKernan Junior High School, in Edmonton.

William Wei is a Grade 7 student at St Rose Junior High School, in Edmonton.

Creating Curved Art with Straight Lines and Perspective

Mark Mercer

Art creates a visual play between positive and negative spaces. Mary Boole's curve stitching—a precursor to string art and similar forms of line art—encourages this type of exploration. These forms address geometric concepts as artists count points and lines. In this challenge, students will create math objects that disappear after certain patterns appear—a tension between necessity and prominence.

Ultimately, this challenge encourages students to create beautiful images. As their skills and imagination develop negative spaces, the focus can shift to an exploration of art, revealing some beautiful mathematics.

Two problems are presented here. The individual problem encourages students to illustrate images using curves and presumes some developed skill. The group problem involves a quilting-type project in which students stitch together images illustrating a sort of dynamic. The group problem allows students to develop the skills needed for the individual problem. This fits nicely into current assessment paradigms, in which group work is needed for individual practice, solitary imagination and demonstration.

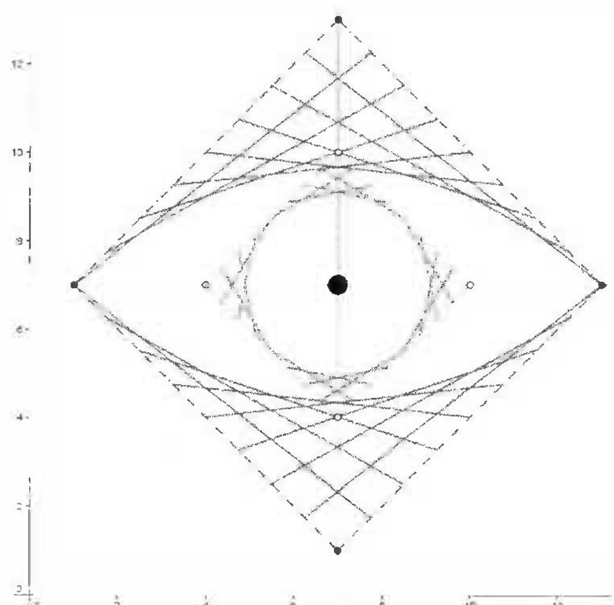
This article encourages a larger perspective in math education. For the teacher, this challenge requires an understanding of various math storytelling methods, as well as grade-specific knowledge about how to guide students through their mathematical progression—a shift in resource development.

Individual Problem

Create an image of simple shapes by using basic shape edges as axes and straight lines as grid patterns. The image should appear to have curved lines and artistic depth.

Consider the image below layered on a coordinate plane (Figure 1). It begins as two related squares rotated from an archetype position, and straight lines create the simple shapes of an oval and a circle, resembling an eye. Note that the image is created using only straight lines, and that the curved lines and depth are an optical illusion.

Figure 1



Group Problem

Create a set of images based on one simple shape, using a variety of patterns forming lines that result in an image that appears to have curves. These shapes and images narrate movement and a surprising fact about these curves, a visual proof. To view this visual proof, the images should be organized and presented with verbal or written explanations.

Because of the scope of this challenge, an example is given below, followed by material lists and other tips.

A Teacher's Preview

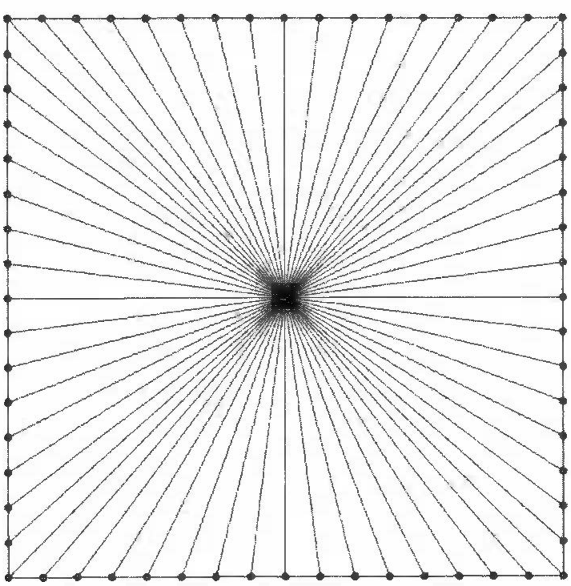
The following progression of images within a basic shape—a square—illustrates the group problem.[1] Taken together, the images reveal a pattern of change: a cross-pattern slowly opens at its centre (or intersection) to reveal a square that turns into a circle. Notice that this revealed square and circle are not accurate. The intersections and line segments creating the images are straight, not curved. As the intersections (or line segment distances) become closer, related to the original connecting dots, the lines of the revealed shape become more curved. Thus, the revealed shape approaches more accuracy. This illustrates some properties of a manufactured curve (made of intersections and line segments between segments).

Using a ruler is strongly recommended in order to be able to see endpoints, follow lines through various intersections and count how many line segments occur between connected points. (See the following section for an explanation of counting.) Working with this technique also provides experience when students struggle with visual patterns or are overwhelmed by visual stimuli (intersections and line segments).

This progression should also help develop students' geometric skills for the individual problem. Students are able to see the curved lines and depth while practising necessary line drawing and counting skills. With reflection, they also view others' attempts as meaningful and can stitch together a progression, enabling the mental imaging necessary for the individual problem.

Figure 2 shows the image created within a square when lines are drawn straight across, using a ruler.

Figure 2



In Figure 3, a ruler has been used to draw lines almost straight across—to the point on either side of the point that's straight across, making two lines. The intersections begin at the axes points and then repeat at decreasing intervals. The intersections and negative space form square patterns.

Figure 3

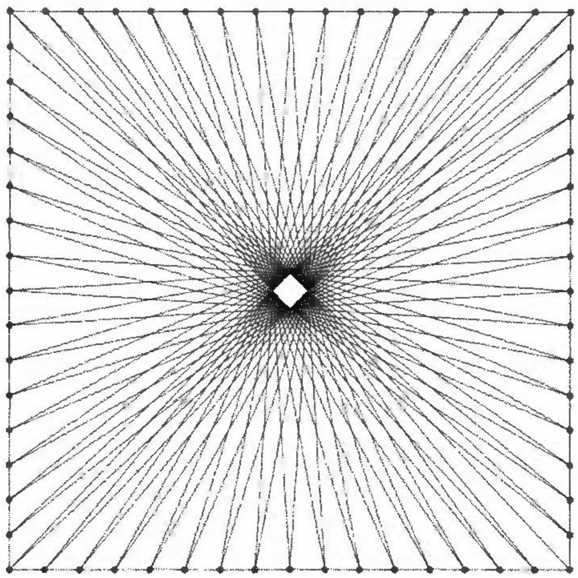
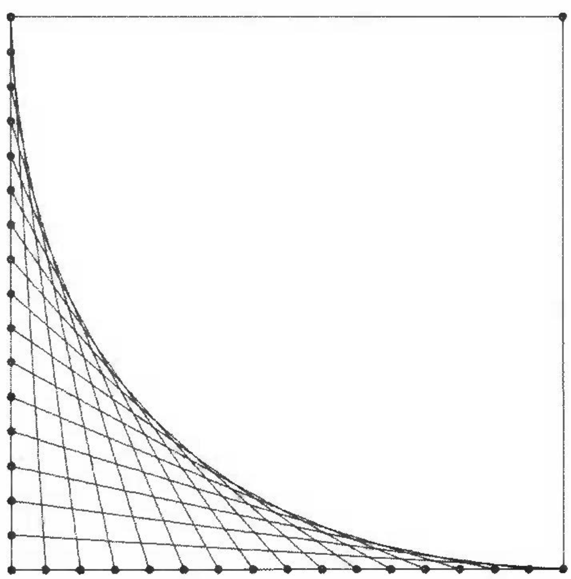


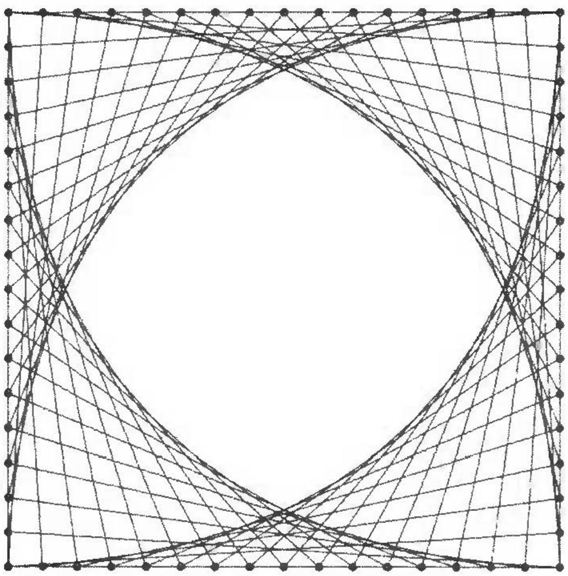
Figure 4 is an intermediary image in which the lines are connected to consecutive points on the perpendicular side of the square.

Figure 4



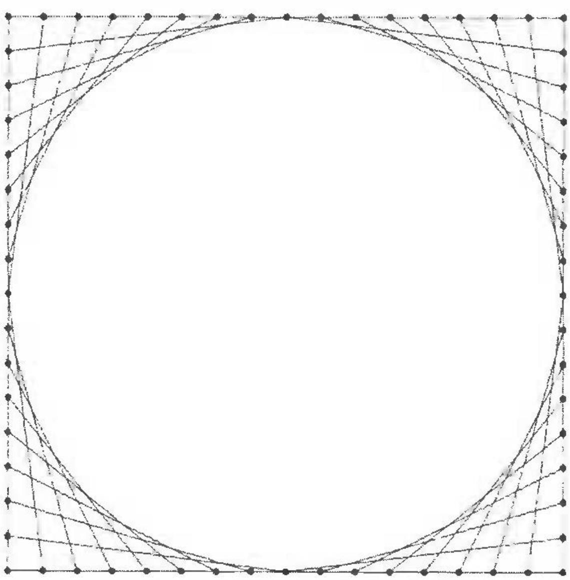
In Figure 5, the pattern from Figure 4 is repeated on each side of the square.

Figure 5



In Figure 6, the pattern from Figure 4 is followed—starting from the point in the middle of each side of the square.

Figure 6



This progression of images can be applied to other basic shapes—any closed regular or irregular polygon—to understand other progressions.

An Interlude: Counting

Figures 7 and 8 repeat Figures 2 and 6, respectively. The numbered line segments and emphasized lines serve as a visual for how to count in curve stitching. Notice that counting is made easier by attending to line segments rather than points, since endpoints are sometimes included and other times not.

Figure 7

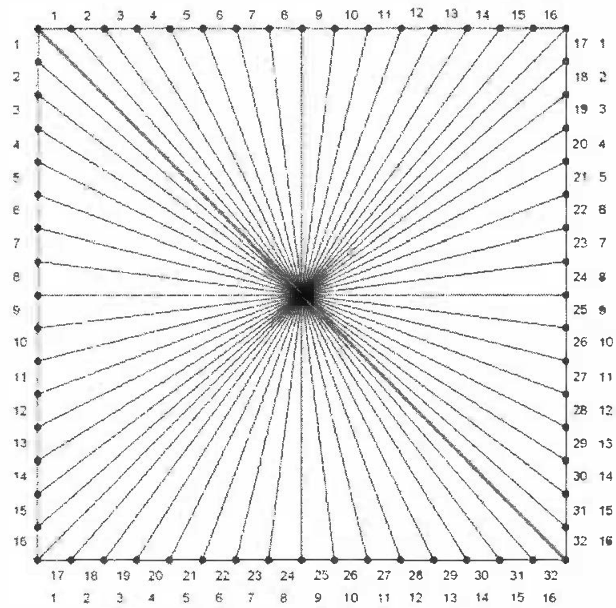
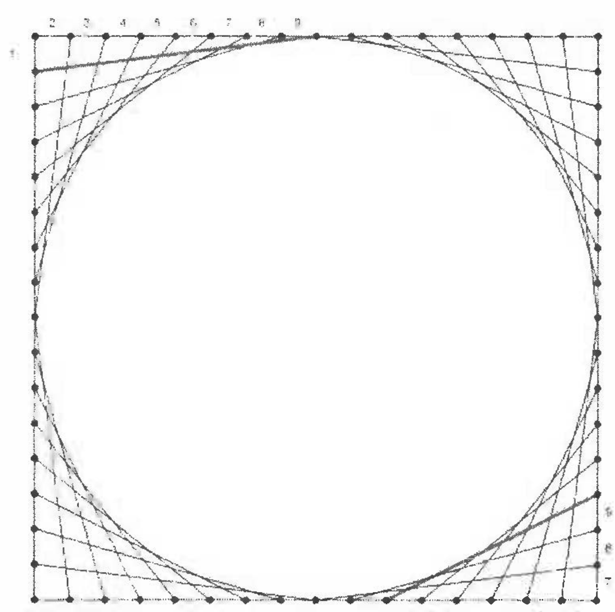


Figure 8



Variations and Extensions

Various shapes, ways of constructing curves and ways of adding illusions are possible. The Internet offers a plethora of images teachers can use to create progressions.

This article provides just one example of a progression that enables students' imaginations. What is possible and the kind of images needed to develop students' thinking and creativity depend on how the teacher facilitates this activity.

For younger or less experienced students, templates can be downloaded from various websites (see the section Useful Websites).

Older and more adventurous students, who can use a ruler with accuracy and precision, should be encouraged to create their own designs. Simple shapes can be made with points plotted and patterns. Students can also try connecting points by skip-counting (rather than counting every point). One recreational mathematician has shared some curve-stitch designs he created using technology (www.deimel.org/rec_math/curve_stitch.htm). These designs are complicated and are not intended for elementary students. I mention them here as an example of the detailed designs possible.

Classroom Set-Up

A group curve-stitching activity requires age-appropriate materials; student groupings based on leadership, cooperation and expertise; and a plan for progressing through the problem.

The following materials are needed:

- Templates on paper
- Pencils
- Ruler
- Coloured pencils (optional)

For students who are able to use a ruler, this challenge can be done with pencil and paper (no template). The challenge can also be adapted for the computer.

For younger students or those who require a more structured template, a template can be fixed to a flat piece of wood, with nails as points. String is then weaved around the nails. The string should be long enough for an adequate weave and can be wrapped around a small (pencil-sized) dowel. Each student should be given two or three choices of string.

For slightly older students, templates can be photocopied onto cardstock, with holes punched where points would be. The string is then sewn through the holes with plastic needles (available from the craft store) and affixed to the cardstock with tape.

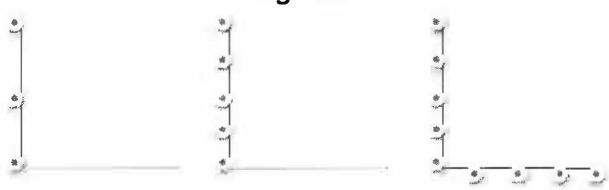
The appendix provides a sample template for student practice in which the intersections and line segments creating envelopes are clearly visible.

To progress through this challenge, prior student knowledge must be evaluated. Students should have experience creating simple curves or images (similar to those presented in this article or to professionally created templates). For specific ideas, refer to the section Beginning to Draw Curves. Then, students can begin creating their own templates—sharing ideas and working through difficulties creatively. Afterward, celebrate the finished artworks.

Beginning to Draw Curves

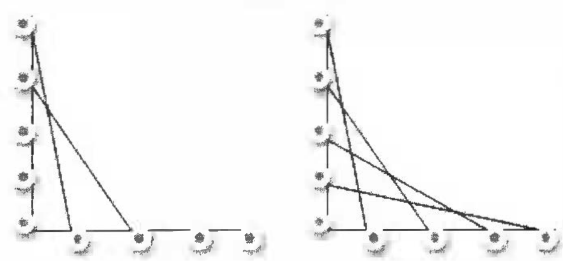
Once two axes are chosen, add equally spaced points. Plot a sequence of midpoints along each axis, as shown in Figure 9. Plotting midpoints assists with equal spacing. Different but equal spacing is also possible.

Figure 9



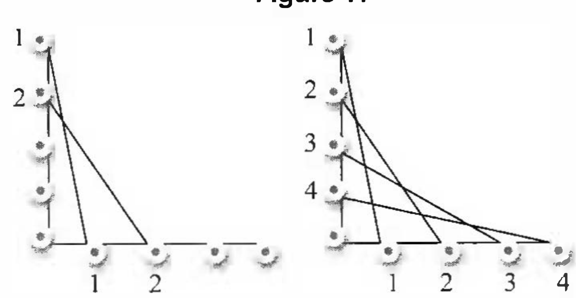
Add straight lines by following a predetermined pattern or rule. The pattern in Figure 10 creates the curve of the eyelid in Figure 1.

Figure 10



Some students have difficulty recognizing a visual pattern with many lines showing intersections and grids. Another way of counting, mimicking a ruler, is illustrated in Figure 11. The first point connects to the first point on the other axis. The second point connects to the second point, the third to the third, and so on. Notice that this does not follow a coordinate plane and may require explanation.

Figure 11



Mathematics in This Challenge

Separating the challenge into stages can help distinguish the mathematics used in this problem. Teachers can address concepts when preparing some space to create a curve, seeing manufactured curves, and generalizing curves through technology or various mathematical notations.

In the figures here, lines resembling coordinate plane axes prepare angled spaces. Some tick marks are counted by beginning with the endpoints and repeatedly finding midpoints. Others are counted by changing the distance of the spaces between the tick marks. The former method seems best for fixed line lengths drawn on paper, and the latter for programming software, where line lengths might consider other variables. "Midpointing" or measuring lines can be applied to various curricular outcomes. However, other ways of preparing spaces, like curved or circular lines, change this sort of vocabulary. Furthermore, searching for geometric art yields shapes as these prepared spaces. Understanding how to count in angled spaces seems a common math place to begin.

This article uses graphing terms such as *intersections* and *line segments*. Other terms are available, depending on the sort of visual mathematics developed. For example, in this article, points on lines (intercepts) are counted and lines are drawn to manufacture a curved line. This first set of mathematics is necessary for the construction of a curve, and its prominence immediately changes. Once this prominence changes, more mathematics can be used to observe and describe the result. Mathematics often works in the following way: ignoring some set of beginning mathematics and emphasizing another set, while existing as causal. Broadly speaking, students require these shifts in math contexts, especially when they are highly creative and intrinsically motivated, so that other shifts in math contexts are accessible.

Mathematical notations generalizing students' curves and art serve to attribute meaning to these mathematical lenses that are often studied purely. For example, the curves we see do not exist as smooth lines, ever. These might be tangents to an imaginary curved line that our brains process as a result of lower-level neuron processing. Thus, these curved lines can be explored and their properties viewed in relation to the process of creating art. Once viewed as tangents, calculus and postsecondary mathematics (like curve envelopes) enter what elementary students can develop with their sophistication of counting and tool usage. In between these imagined students' sophistication is mathematics requiring mathematical storytelling appropriate to whoever creates their art—what a teacher might do. This article encourages this type of facilitation.

Problems and challenges like this shift resource recording and development for these sorts of activities. Grade school teachers and other educators might become a community of storytellers, all possessing different perspectives but all responsible to the same plot. With the breadth and depth of this challenge, considering the sorts of mathematics available, various people contribute and someone, perhaps an editor, assembles this information for more natural methods of pedagogy.

Biology in This Challenge

This problem can also be connected to science. Like a piece of paper, the human retina is a two-dimensional space where three-dimensional sight begins. Before stimulating light receptors, light passes through layers of nerve cells. These nerve cells process changes of light between receptors into lines and overlapping shapes. These are the same shapes video game programmers and artists of various media use to create three-dimensional effects. This information is then received by the brain for further processing.

This means that the intersections creating curved lines, three-dimensional grids and other patterns in the positive and negative spaces can be known with experience instead of being simply more aesthetic. The nature of optical illusions can be explored.

Useful Websites

Curve-Stitch Designs
www.deimel.org/rec_math/curve_stitch.htm
Detailed curve-stitch designs created by a recreational mathematician using technology

GeoGebra
www.geogebra.org/cms/
A free drawing program that can be used for curve stitching

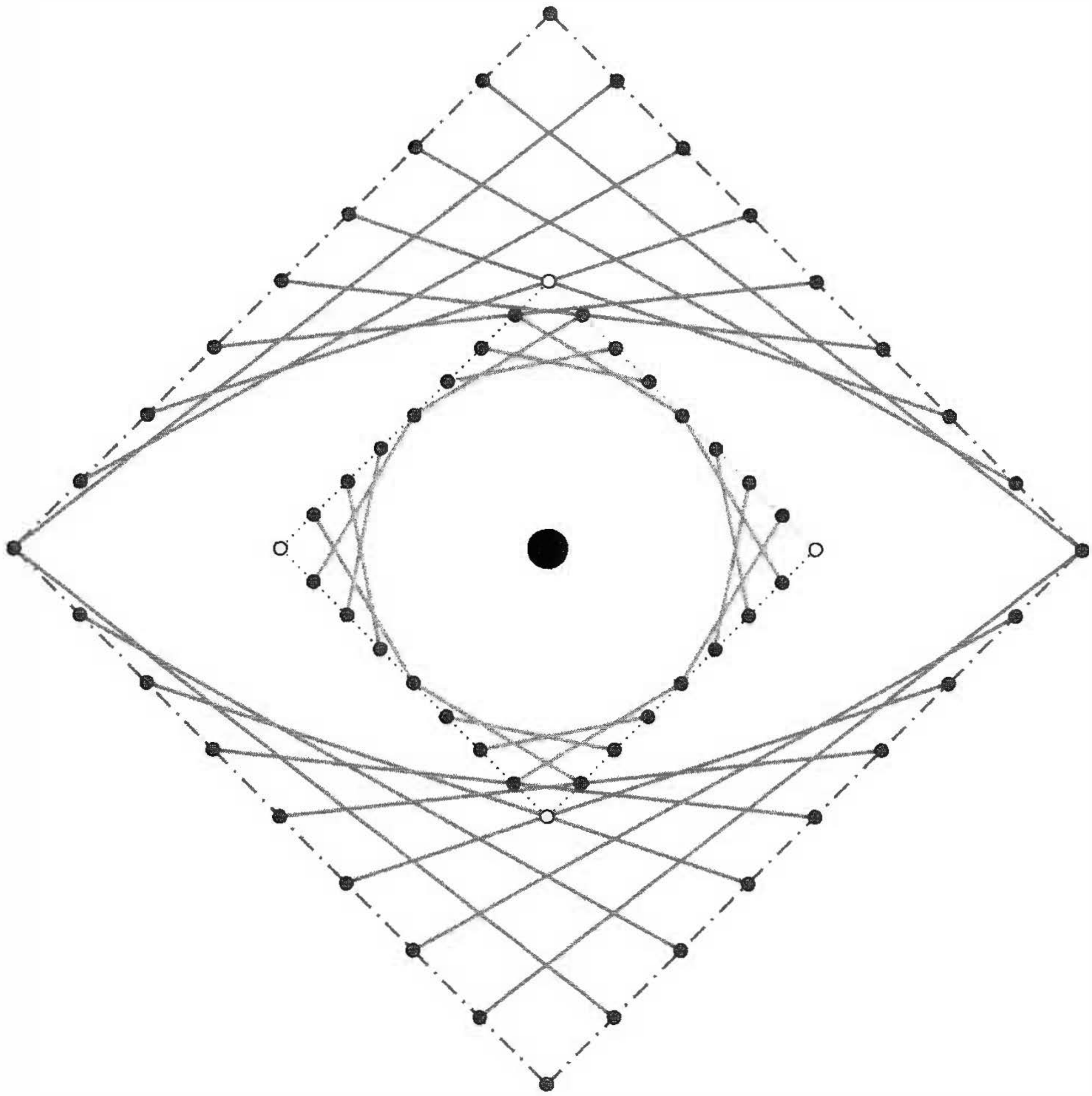
String Art Fun
www.stringartfun.com
Free templates, as well as templates for purchase

String Art Is My Craft
http://stringart.ismycraft.com
Free templates, templates for purchase and an opportunity to share information

String Art (Math Cats)
www.mathcats.com/crafts/stringart.html#patterns
Simple printable templates and a good explanation for getting started

Appendix

Sample Template



Note

1. These images were created with GeoGebra, a free drawing program available at www.geogebra.org.

Mark Mercer teaches math at Ross Sheppard High School, in Edmonton. He is interested in finding cross-curricular approaches to mathematics. Through art, he attempts to help students practise graphing strategies (physical practice) that logically connect to beautiful constructions (notions). With this play, he hopes to address misconceptions within the broad, creative space of a student's journey. If you have any questions or would like to share student work, e-mail Mark at mmercer@ualberta.ca.

Secondary Algebra: A Quadratic Case Study

Ed Barbeau

Jennifer Hyndman (professor and chair of the Department of Mathematics and Statistics, University of Northern British Columbia) recently had a very bright fourth-year student ask what doing research would mean for her. Even though we spend our time teaching students how to solve problems, which is one stage of doing research, we often do no teaching about how to create new problems, nor do we let students understand what research is. In this article, Ed Barbeau develops the idea that creativity in mathematics can be fostered at the undergraduate level through presentation of material that allows students to formulate questions.

Too often, the mathematics curriculum is seen solely in terms of delivering to the student standard topics to be mastered. However, there is a creative side that can be accessed by students still at school; not all new results require years of study of difficult and sophisticated areas. Geometry and combinatorics are two areas where students can enter on the ground floor, but as I shall indicate by an example, it is possible for a student to obtain an original algebraic result.

While the student in question is particularly strong, I wonder to what extent it is open to students in regular classes to formulate and prove their own results (even if they may be widely known), and how problems might be composed to encourage this to happen.

The Basic Problem

Let me first reconstruct the situation that led to the problem that I posed to students in a correspondence program and an undergraduate competition, and that inspired the original research of one of them. An oblong number is any product of two consecutive positive integers. If we examine the sequence {2, 6, 12, 20, 30, 42, 56, 72, …}, we might note that the product of two consecutive oblong numbers is also oblong. For example, 12×20 is equal to the oblong number $240 = 15 \times 16$. Adding 1 to each of the oblong numbers gives a sequence of positive integers of the form $k^2 + k + 1$, namely {3, 7, 13, 21, 31, 43, 57, 73, 91, …}, with the same property. These empirical observations might be made by an aware student sensitive to patterns. With a little effort, they can be established by deriving the identities:

$$[(x-1)x][x(x+1)] = (x^2-1)x^2$$

and

$$\begin{aligned}&[(x-1)^2 + (x-1) + 1][x^2+x+1] \\ &= (x^2-x+1)(x^2+x+1) \\ &= [(x^2+1)-x][(x^2+1)+x] \\ &= (x^2+1)^2 - x^2 \\ &= (x^2)^2 + x^2 + 1.\end{aligned}$$

Noting that both the forms $x(x+1)$ and x^2+x+1 are monic quadratic polynomials, we might ask whether these observations can be generalized to numbers of the form $f(x) = x^2 + bx + c$, where b and c are arbitrary integers and x takes consecutive integer values. For example, the product of two consecutive squares is again a square.

One way to approach this problem is to experiment with various examples, and then make an inspired guess as to the value of z generated by the equation $f(x)f(x+1) = f(z)$. Then the proof amounts to just checking algebraically that you are correct.

However, there is another way to approach the problem: transformationally. Suppose that $f(x)$ is an arbitrary monic quadratic polynomial in x. Then $g(t) = f(x+t)$ is a monic quadratic polynomial in t: $g(t) = t^2 + bt + c$. Then, briefly, noting that $g(y) = f(x = y)$ for each y,

$$\begin{aligned}f(x)f(x+1) &= g(0)g(1) = c(1+b+c) \\ &= c^2 + bc + c = g(c) = f(c+x) = f(g(0)+x) \\ &= f(f(x)+x).\end{aligned}$$

Since this sort of manipulation is foreign to most secondary students, let us consider the aspects of the situation that students ought to be made aware of.

First of all, there is change of perspective. The problem is posed as establishing a fact for a particular quadratic and any value of the argument; the realization needs to be made that, by means of a translation of the variable, one can rather prove it for any quadratic and a particular value of the variable, namely 0.

Secondly, there is the technical problem of mediating between the two perspectives. Thirdly, it is necessary to make some interpretations: the constant coefficient as the value of a polynomial at 0, and the expression $c(1 + b + c) = c^2 + bc + c$ as the value of $g(c)$. Finally, the evolution of the solution changes the problem. Because one can actually display $f(x)f(x + 1)$ as the composition of two quadratics with integer coefficients, the property that we are dealing with f values at integers is subsumed in the more general (and interesting) representation of $f(x)f(x + 1)$ as a composite.

The General Quadratic

What would the situation be for a quadratic with an arbitrary leading coefficient? Experimentation reveals that $f(x)f(x + 1)$ need not be a later value of the quadratic when it is evaluated at integers. However, the work we have done at the end of the last section suggests that we can refocus the problem, to see whether $f(x)f(x + 1) = g(h(x))$ for suitable quadratics g and h. This turns out to be true, and this result was given as a problem to students at both the secondary and tertiary levels.

PROBLEM. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which $f(x)f(x + 1) = g(h(x))$.

COMMENT. One attempt might be to reduce it to the monic case, an approach that would undoubtedly be difficult for a typical secondary student to consummate but when understood should signify a pretty deep understanding of algebraic relationships. Writing $f(x) = au(x)$, where $u(x)$ is monic, we have that

$$f(x)f(x+1) = a^2u(x)u(x+1) = a^2u(x+u(x))$$

so we can take $g(x) = a^2u(x)$ and $h(x) = x + u(r)$. When $f(x) = ax^2 + bx + c$, this leads to $g(x) = a^2x^2 + abx + ac$ and

$$h(x) = x^2 + \left(1 + \frac{b}{a}\right)x + \frac{c}{a}.$$

However, as the following solutions indicate, there are at least three other ways students might tackle this problem, depending on whether they conceive of the quadratic in factored form, in descending powers of x or in terms of completing the square. In the first solution, below, note how it contains the seeds of the generalization we will discuss later. The second solution uses the method of undetermined coefficients to obtain a set of five equations in six unknowns. While this may appear formidable, the situation is tractable when the solver realizes that only one solution is needed for an overdetermined system and makes a simplifying assumption. The final solution is an adept completion of the square manipulation. Each of the solutions requires a level of sophistication that we should be encouraging in students planning to go on in science and mathematics.

SOLUTION 1. [A Remorov] Let $f(x) = a(x - r)(x - s)$. Then,

$$\begin{aligned} f(x)f(x+1) &= a^2(x-r)(x-s+1)(x-r+1)(x-s) \\ &= a^2(x^2+x-rx-sx+rs-r) \\ &\quad (x^2+x-rx-sx+rs-s) \\ &= a^2[(x^2-(r+s-1)x+rs)-r] \\ &\quad [(x^2-(r+s-1)x+rs)-s] \\ &= g(h(x)), \end{aligned}$$

where $g(x) = a^2(x - r)(x - s) = af(x)$ and $h(x) = x^2 - (r + s - 1)x + rs$.

SOLUTION 2. Let $f(x) = ax^2 + bx + c$, $g(x) = px^2 + qx + r$ and $h(x) = ux^2 + vx + w$. Then,

$$\begin{aligned} f(x)f(x+1) &= a^2x^4 + 2a(a+b)x^3 \\ &\quad + (a^2+b^2+3ab+2ac)x^2 \\ &\quad + (b+2c)(a+b)x + c(a+b-c) \end{aligned}$$

and

$$\begin{aligned} g(h(x)) &= p(ux^2+vx+w)^2 + q(ux+vx+w) + r \\ &= pu^2x^4 + 2puvx^3 + (2puw + pv^2 + qu)x^2 \\ &\quad + (2pvw + qv)x + (pw^2 + qw + r). \end{aligned}$$

Equating coefficients, we find that $pu^2 = a^2$, $puv = a(a + b)$, $2puw + pv^2 + qu = a^2 + b^2 + 3ab + 2ac$, $(b + 2c)(a + b) = (2pw + q)v$ and $c(a + b + c) = pw^2 + qw + r$. We need to find just one solution of this system. Let $p = 1$ and $u = a$. Then, $v = a + b$ and $b + 2c = 2pw + q$ from the second and fourth equations. This yields the third equation automatically. Let $q = b$ and $w = c$. Then, from the fifth equation, we find that $r = ac$.

Thus, when $f(x) = ax^2 + bx + c$, we can take $g(x) = x^2 + bx + ac$ and $h(x) = ax^2 + (a + b)x + c$.

SOLUTION 3. [S Wang] Suppose that $f(x) = a(x + h)^2 + k = a(t - (½))^2 + k$, where $t = x + h + ½$. Then, $f(x + 1) = a(x + 1 + h)^2 + k = a(t + (½))^2 + k$, so that

$$\begin{aligned} f(x)f(x+1) &= a^2\left(t^2 - \frac{1}{4}\right)^2 + 2ak\left(t^2 + \frac{1}{4}\right) + k^2 \\ &= a^2t^4 + \left(-\frac{a^2}{2} + 2ak\right)t^2 + \left(\frac{a^2}{16} + \frac{ak}{2} + k^2\right). \end{aligned}$$

Thus, we can achieve the desired representation with

$$h(x) = t^2 = x^2 + (2h+1)x + \frac{1}{4}$$

and

$$g(x) = a^2x^2 + \left(\frac{-a^2}{2} + 2ak\right)x + \left(\frac{a^2}{16} + \frac{ak}{2} + k^2\right).$$

The Generalization

One student, James Rickards of Greely, Ontario, raised the situation to a higher level when he realized that he needed to know only that $f(x)f(x+1)$ was a quartic polynomial for which the sum of two of its roots was equal to the sum of the other two. This immediately suggested the generalization that if the quartic polynomial $f(x)$ has roots r_1, r_2, r_3, r_4 (not necessarily distinct), then $f(x)$ can be expressed in the form $g(h(x))$ for quadratic polynomials $g(x)$ and $h(x)$ if and only if the sum of two of r_1, r_2, r_3, r_4 is equal to the sum of the other two.

Let us run through the proof of this statement. Without loss of generality, suppose that $r_1 + r_2 = r_3 + r_4$. Let the leading coefficient of $f(x)$ be a. Define

$$h(x) = (x - r_1)(x - r_2)$$

and

$$g(x) = ax(x - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2).$$

Then,

$$\begin{aligned} g(h(x)) &= a(x - r_1)(x - r_2)[(x - r_1)(x - r_2) \\ &\quad - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2] \\ &= a(x - r_1)(x - r_2)[x^2 - (r_1 + r_2)x \\ &\quad - r_3^2 + r_1r_3 + r_2r_3] \\ &= a(x - r_1)(x - r_2)[x^2 - (r_3 + r_4)x \\ &\quad + r_3(r_1 + r_2 - r_3)] \\ &= a(x - r_1)(x - r_2)[x^2 - (r_3 + r_4)x \\ &\quad + r_3r_4] \\ &= a\,(x - r_1)(x - r_2)(x - r_3)(x - r_4) \end{aligned}$$

as required.

Conversely, assume that we are given quadratic polynomials $g(x) = b(x - r_5)(x - r_6)$ and $h(x)$, and that c is the leading coefficient of $h(x)$. Let $f(x) = g(h(x))$.

Suppose that

$$h(x) - r_5 = c(x - r_1)(x - r_2)$$

and that

$$h(x) - r_6 = c(x - r_3)(x - r_4).$$

Then,

$$f(x) = g(h(x)) = bc^2(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$

We have that

$$\begin{aligned} h(x) &= c(x - r_1)(x - r_2) + r_5 \\ &= cx^2 - c(r_1 + r_2)x + cr_1r_2 + r_5 \end{aligned}$$

and

$$\begin{aligned} h(x) &= c(x - r_3)(x - r_4) + r_6 \\ &= cx^2 - c(r_3 + r_4)x + cr_3r_4 + r_6, \end{aligned}$$

whereupon it follows that $r_1 + r_2 = r_3 + r_4$ and the desired result follows.

Let me allow Rickards to continue in his own words:

> I then wondered, how will this continue? What will the condition be for the composition of two third degree polynomials? I tried to construct a proof with only the assumption that the first symmetric polynomials agreed for some division into three groups of three roots of the whole ninth degree polynomial. While writing this out, it became apparent that I lacked something. I then saw that assuming the second symmetric polynomials agreed would be all that I needed. Thus I now had a good idea; I wrote out a proof for a polynomial of degree n^2. The next day or so, I realized that the fact the two polynomials being composed had the same degree was irrelevant; just minor modifications to make this as general as could be, a composition of degrees m and n.

Rickards figured out that the key property of the composite was that its roots could be partitioned into subsets for which all the symmetric polynomials agreed except for the product. This led him to a necessary and sufficient condition for a polynomial of degree mn to be the composite of polynomials of degrees m and n. He then addressed the determination of the composition factors of these degrees when the composite was given. He noted that while one could not generally know the actual roots of the polynomial, the coefficients of the composite factors depended only on knowing the value of the (equal) symmetric functions of roots in each of the partitioning sets, and these values could be retrieved from the coefficients of the given polynomial. It is convenient to give the proof for a monic polynomial, and then derive the general case; the details are found in Rickards (2011).

The relating of the monic to the general situation is a nice exercise for students. Suppose that $f(x)$ is a polynomial of degree nm and leading coefficient a, so that $f(x) = au(x)$ for some monic polynomial $u(x)$. Then we show that $f(x)$ is a composite of polynomials of degrees m and n if and only if $u(x)$ is so. Suppose that $f(x) = g(h(x))$, where $g(x)$ is of degree m with leading coefficient b and $h(x)$ is of degree n with leading coefficient c. Then, by comparison of leading coefficients, we have that $a = bc^m$. It can be checked that $u(x) = v(w(x))$, where $v(x) = (bc^m)^{-1}g(cx)$ and $w(x) = c^{-1}h(x)$.

On the other hand, suppose that $u(x) = v(w(x))$ for some monic polynomials $v(x)$ and $w(x)$ of respective degrees m and n. Then $f(x) = g(h(x))$ with $g(x) = au(x)$ and $h(x) = v(x)$.

Is Rickards's Result New?

I was enchanted by the elegance of Rickards's result. While the determination of a different criterion for a quartic to be the composite of two quadratics actually appears in my book (Barbeau 2003, problem 1.9.8, 44, 266), it is from the more pedestrian standpoint of a condition on the coefficients. Specifically, $ax^4 + bx^3 + cx^2 + dx + e$ is a composite of two quadratics if and only if $4abc - 8a^2d = b^3$. I was completely unaware of this new result, and a check of colleagues, the literature and the Internet did not reveal that it was previously known.

Whether it is actually new is open to question. While the composition of polynomials does not appear to have received much attention, it is conceivable that over the past 300 years, someone might have addressed the issue. However, such a result, if published, could have appeared in an obscure place and be impossible to track down. It seemed pretty enough to warrant appearing in a widely circulated current journal, regardless of its status.

Conclusion

It seems clear that if a curriculum is to be successful in preparing mathematics students for later study, it has to go beyond a straight presentation of results. Students require material that engages them, so that they acquire facility with the conventions and distinctions of mathematics and are able to make judgments about how a situation might be approached. Therefore, we need to be on the lookout for investigations and problems that encourage different perspectives and the search for connections.

I have presented one situation and mentioned issues that might arise. I hope that teachers may be able to present other examples, and that eventually exercises and problems that might lead to open-ended investigations by students might be more prominent in textbooks. As educators, we need to develop other case studies and then encourage teachers to try them out in their own settings. I have not had the opportunity to attempt this in a regular classroom situation. Its evolution is probably highly dependent on the context; it may happen that the discussion goes in a completely different direction and other questions emerge.

There are important issues pertinent to the preparation of students bound for science, technology and mathematics. Should such students be able to negotiate the subtleties of algebra usage illustrated by this example? If so, what are the implications for teacher training, the syllabus, the classroom experience and examinations? What preparation should be occurring all through the algebra sequence so that students attain both the perspective and the skills to manage it? What is the appropriate balance between presentation of such material by the teacher and investigation by individual students and groups? I invite teachers to try this example with their own classes and circles.

References

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Rickards. J. 2011. "When Is a Polynomial a Composition of Other Polynomials?" *The American Mathematical Monthly* 118, no 4 (April): 358–63.

Ed Barbeau is professor emeritus of mathematics at the University of Toronto.

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Alberta High School Mathematics Competition 2011/12

The Alberta High School Mathematics Competition is a two-part competition taking place in November and February of each school year. Book prizes are awarded for Part I, and cash prizes and scholarships for Part II. Presented here are the problems and solutions from the 2011/12 competition.

Part I

November 15, 2011

1. If $2^{2012} + 4^{1006} = 2^n$, then n is
   (a) 2013 (b) 2014 (c) 3018 (d) 4024
   (e) not an integer

2. Mini-marshmallows are cubes of 1 cm on each side, while giant marshmallows are cubes of 3 cm on each side. The number of mini-marshmallows whose combined surface area is the same as the surface area of one giant marshmallow is
   (a) 3 (b) 6 (c) 9 (d) 27 (e) 54

3. The number of customers in a restaurant on Tuesday is 20% more than the number on Monday, the number of customers on Wednesday is 50% more than the number on Monday, and the number of customers on Wednesday is n% more than the number on Tuesday. The value of n is
   (a) 20 (b) 25 (c) 30 (d) 50 (e) none of these

4. Sawa starts from point S and walks 1 km north, 2 km east, 3 km south and 4 km west. At this point, her distance, in kilometres, from S is
   (a) $\sqrt{5}$ (b) $2\sqrt{2}$ (c) 4 (d) 8 (e) 10

5. A millennium number is a positive integer such that the product of its digits is 1000. The number of six-digit millennium numbers is
   (a) 60 (b) 120 (c) 140 (d) 180 (e) 200

6. Let $x_1, x_2, \ldots$ be a sequence of positive rational numbers such that $x_1 = 16$, $x_2 = 32$ and
   $$x_n = \frac{x_{n-1} + x_{n-2}}{2}$$
   for all positive integers $n \geq 3$. Then the value of x_6 is
   (a) 24 (b) 25 (c) 26 (d) 27 (e) 28

7. The number of real solutions of the equation $2x^2 - 2x = 2x\sqrt{x^2 - 2x} + 1$ is
   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

8. Let $f(x)$ be a quadratic polynomial. If $f(1) = 2$, $f(2) = 4$ and $f(3) = 8$, then the value of $f(4)$ is
   (a) 12 (b) 14 (c) 15 (d) 16 (e) 18

9. A lucky number is a positive integer n such that 7 is the largest divisor of n that is less than n. The number of lucky numbers is
   (a) 1 (b) 2 (c) 3 (d) 4 (e) more than 4

10. The perimeter of a square lawn consists of four straight paths. Annabel and Bethany started at the same corner at the same time, running clockwise at constant speeds of 12 and 10 km/h, respectively. Annabel finished one lap around the lawn in one minute. During this minute, the number of seconds that Annabel and Bethany were on the same path was
    (a) 36 (b) 42 (c) 48 (d) 50 (e) none of these

11. ABCD is a quadrilateral with AD = BC and AB parallel to DC. It is only given that the lengths of AB and DC are 20 and 15 cm, respectively. Adrian puts n copies of this tile together so that the edge BC of each copy coincides with the edge AD of the next, and the edges DC of all copies together form a regular n-sided polygon. The value of n is
    (a) 6 (b) 8 (c) 12 (d) 20
    (e) not uniquely determined

12. The sum of 20 positive integers, not necessarily different, is 462. The largest possible value of greatest common divisor of these numbers is
    (a) 21 (b) 22 (c) 23 (d) 33 (e) 42

13. The largest real number m such that $(x^2 + y^2)^3 > m(x^3 + y^3)^2$ for any positive real numbers x and y is
    (a) ⅓ (b) ½ (c) 1 (d) 2 (e) none of these

14. Of the 49 squares of a 7 × 7 square sheet of paper, two are to be coloured black while the others remain white. Two colourings are called *distinct* if one cannot be obtained from the other by rotating the sheet of paper about its centre. The number of distinct colourings is
    (a) 288 (b) 294 (c) 296 (d) 300 (e) 588

15. The lengths of the sides of triangle ABC are consecutive positive integers. D is the midpoint of BC, and AD is perpendicular to the bisector of $\angle C$. The product of the lengths of the three sides is

(a) 24 (b) 60 (c) 120 (d) 210 (e) 336

16. For any real number r, $\lfloor r \rfloor$ is the largest integer less than or equal to r. For example, $\lfloor \pi \rfloor = 3$. Let n be a positive integer. Let $a_1 = n$, $a_2 = \left\lfloor \frac{a_1}{3} \right\rfloor$, $a_3 = \left\lfloor \frac{a_2}{3} \right\rfloor$ and $a_4 = \left\lfloor \frac{a_3}{3} \right\rfloor$. The number of positive integers n from 1 to 1000 inclusive such that none of a_1, a_2, a_3 and a_4 is divisible by 3 is

(a) 144 (b) 192 (c) 210 (d) 280
(e) none of these

Solutions

1. We have $2^{2012} + 4^{1006} = 2^{2012} + 2^{2012} = 2^{2013}$. The answer is **(a)**.
2. The surface area of a mini-marshmallow is 6 sq cm while that of a giant marshmallow is 54 sq cm. Thus, the desired number of mini-marshmallows is $54 \div 6 = 9$. The answer is **(c)**.
3. Suppose there are m customers on Monday. Then there are $1.2m$ on Tuesday and $1.5m$ on Wednesday. The increase of $0.3m$ from Tuesday to Wednesday is 25% of $1.2m$. The answer is **(b)**.
4. Sawa is 2 km south and 2 km west of S. Her distance from S, by the Pythagorean theorem, is $\sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ km from S. The answer is **(b)**.
5. Since $1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$, the digits can only be 1, 2, 4, 5 and 8. Three of them must be 5s, and they can be placed among the six digits in $\binom{6}{3} = 20$ ways. The product of the other three digits is 8, and they are (1, 1, 8), (1, 2, 4) or (2, 2, 2). They can be placed in three, six and one ways, respectively. Hence, the total number of six-digit millennium numbers is $20(3 + 6 + 1) = 200$. The answer is **(e)**.
6. We have $x_3 = 24$, $x_4 = 28$, $x_5 = 26$ and $x_6 = 27$. The answer is **(d)**.
7. Squaring both sides of $2x^2 - 2x - 1 = 2x\sqrt{x^2 - 2x}$, we have $4x^4 - 8x^3 + 4x + 1 = 4x^4 - 8x^3$, which simplifies to $4x + 1 = 0$. Hence, the only solution is $x = -¼$. Indeed,

$$2\left(-\frac{1}{4}\right)^2 - 2\left(-\frac{1}{4}\right) = \frac{5}{8}$$

and

$$2\left(-\frac{1}{4}\right)\sqrt{\left(-\frac{1}{4}\right)^2 - 2\left(-\frac{1}{4}\right) + 1} = \frac{5}{8}.$$

The answer is **(b)**.

8. Note that $g(x) = f(x + 1) - f(x)$ is a linear polynomial. Since $g(1) = f(2) - f(1) = 2$ and $g(2) = f(3) - f(2) = 4$, we have $g(3) = 6$. Hence, $f(4) = f(3) + g(3) = 8 + 6 = 14$. The answer is **(b)**.
9. Since a lucky number n is divisible by 7, it has the form $n = 7k$ for some positive integer k. If k is not a prime number, then it has a divisor h where $1 < h < k$, and $7h$ is a divisor of n larger than 7 but not equal to n. Hence, k must be a prime number. Moreover, it cannot be greater than 7. Hence, there are only four lucky numbers—namely, 14, 21, 35 and 49. The answer is **(d)**.
10. Annabel spent 15 seconds on each path, and Bethany 18 seconds. On the first path, Bethany was with Annabel all 15 seconds. On the second path, Bethany joined Annabel 3 seconds late, and was with her for 12 seconds. On the third path, Bethany was with Annabel for 9 seconds. On the fourth path, Bethany was with Annabel for 6 seconds. The total is $15 + 12 + 9 + 6 = 42$ seconds. The answer is **(b)**.
11. We can draw a regular polygon of any number of sides such that the side length is 20 cm. We can then draw a regular polygon of the same number of sides but with side length 15 cm, placed centrally inside the larger polygon. Then, a tile can be chosen that can pave the ring-shaped region inside the larger polygon but outside the smaller one. Hence, the answer is **(e)**.
12. Let $x_1, \ldots, x_{20}$ be the given numbers. If d is the greatest common divisor of these numbers, then

$$x_1 + \cdots + x_{20} = d\left(\frac{x_1}{d} + \cdots + \frac{x_{20}}{d}\right) = 462 = 21 \cdot 22.$$

The value $d = 22$ is obtained if $x_1 = \cdots = x_{19} = d$ and $x_{20} = 2d$. For each i, $x_i/d \geq 1$. Hence, $d \leq 462/20 = 23.1$. Since d divides 462, the largest value for d is indeed 22. The answer is **(b)**.
13. Dividing throughout by y, we have $(z^2 + 1)^3 > m(z^3 + 1)^2$, where $z = x/y$. This is equivalent to

$$(1 - m)z^6 + 3z^4 + (3 - 2m)z^3 + 1 - m > 0$$

for any positive real z. Hence, it is necessary to have $1 - m \geq 0$ (ie, $m \leq 1$). If we take $m = 1$, the inequality $(x^2 + y^2)^3 \geq (x^3 + y^3)^2$ is equivalent to

$$x^2y^2((x - y)^2 + 2x^2 + 2y^2) > 0,$$

which is clearly true. The answer is **(c)**.
14. There are $\binom{49}{2} = 1176$ colourings. The number of symmetrical colourings with respect to the middle square is

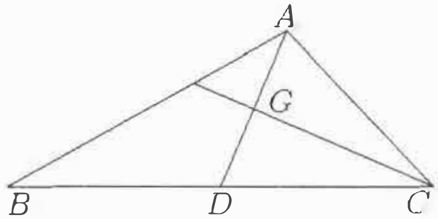
$$\frac{49 - 1}{2} = 24.$$

These colourings are counted twice. All the other colourings are counted four times. The desired number is

$$\frac{24}{2}+\frac{1176-24}{4}=300.$$

The answer is **(d)**.

15. Let AD intersect the bisector of ∠C at G. Then ∠CGA = 90° = ∠CGD, ∠GCA = ∠GCD and GD = GD. Hence, triangles GCA and GCD are congruent, so that AC = DC. It follows that we have BC = 2DC = 2AC. Now among three consecutive positive integers, one is double another. This is only possible if the integers are 1, 2 and 3, or 2, 3 and 4. The former does not yield a triangle. Hence, AC = 2, AB = 3 and BC = 4, so that AB · BC · CA = 24. The answer is **(a)**.

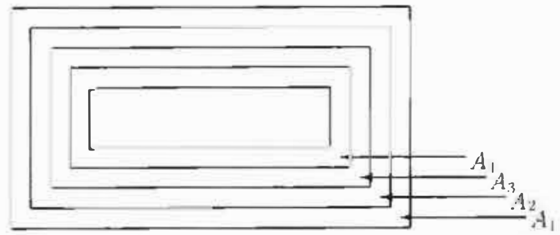


16. Note that if we write a positive integer m in base 3, then the base 3 representation of $\left\lfloor\frac{m}{3}\right\rfloor$ is simply the base 3 representation of m with the rightmost digit removed. Also, a positive integer m is divisible by 3 if and only if the rightmost digit of m is 0. Hence, in order that none of a_1, a_2, a_3 and a_4 is divisible by 3, the rightmost four digits of the base 3 representation of n are all non-0. Note that $1000 > 2(3^5 + 3^4 + 3^3 + 3^2 + 3 + 1)$. If n has at most six digits in its base 3 representation, the first two can be any of 0, 1 and 2, while the last four cannot be 0. There are $3^2 \times 2^4 = 144$ such numbers. Clearly, n cannot have more than seven digits as otherwise $n \geq 3^7 > 1000$. Suppose n has exactly seven digits. As before, the last four cannot be 0. Since $1000 < 3^6 + 3^5 + 3^3 + 3^2 + 3 + 1$, the first one must be 1; the second must be 0; and the third can be any of 0, 1 and 2. Hence, there are $3 \times 2^4 = 48$ such numbers. The total is 192. The answer is **(b)**.

Part II

February 1, 2012

1. A rectangular lawn is uniformly covered by grass of constant height. Andy's mower cuts a strip of grass 1 m wide. He mows the lawn using the following pattern. First he mows the grass in the rectangular "ring" A_1 of width 1 m running around the edge of the lawn. Then he mows the 1 m wide ring A_2 inside the first ring, then the 1 m wide ring A_3 inside A_2, and so on until the entire lawn is mowed. Andy starts with an empty grass bag. After he mows the first three rings, the grass bag on his mower is exactly full, so he empties it. After he mows the next four rings, the grass bag is exactly full again. Find, in metres, all possible values of the perimeter of the lawn.



2. In the quadrilateral ABCD, AB is parallel to DC. Prove that

$$\frac{PA}{PB}=\left(\frac{PD}{PC}\right)^2,$$

where P is a point on the side AB such that ∠DAB = ∠DPC = ∠CBA.

3. A positive integer is said to be *special* if it can be written as the sum of the square of an integer and a prime number. For example, 101 is special because 101 = 64 + 37. Here, 64 is the square of 8, and 37 is a prime number.
   (a) Show that there are infinitely many positive integers that are special.
   (b) Show that there are infinitely many positive integers that are not special.

4. In triangle ABC, AB = 2, BC = 4 and CA = $2\sqrt{2}$. P is a point on the bisector of ∠B such that AP is perpendicular to this bisector, and Q is a point on the bisector of ∠C such that AQ is perpendicular to this bisector. Determine the length of PQ.

5. Determine the smallest positive integer n for which there exist real numbers $x_1, \ldots, x_n$, $1 \leq x_i \leq 4$ for $i = 1, 2, \ldots, n$, which satisfy the following inequalities simultaneously:

$$x_1+x_2+\cdots+x_n \geq \frac{7n}{3}$$

and

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n} \geq \frac{2n}{3}.$$

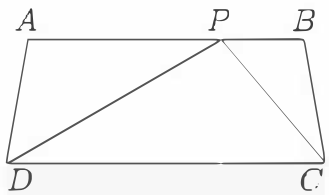
Solutions

1. Let the dimensions of the lawn (in metres) be a by b. The area of the first three rings is given by $ab - (a-6)(b-6) = 6(a+b) - 36$. Similarly, the area of the next four rings is given by $(a-6)(b-6) - (a-14)(b-14) = 8(a+b) - 160$. These two regions contain the same amount of grass, so they must be the same area. Thus, $6(a+b) - 36 = 8(a+b) - 160$. It follows that the only possible value of the perimeter of the lawn is $2(a+b) = 124$ m.

2. Since ∠DAB = ∠CBA and AB is parallel to DC, we have AD = BC. Since AB is parallel to DC, ∠BPC = ∠PCD. It follows that triangles BPC and PCD are similar. A similar argument shows that triangles ADP and PCD are also similar. Hence,

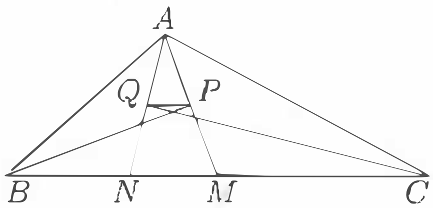
$$\left(\frac{\mathrm{PD}}{\mathrm{PC}}\right)^2 = \frac{\mathrm{BC}}{\mathrm{BP}}\cdot\frac{\mathrm{AP}}{\mathrm{AD}} = \frac{\mathrm{PA}}{\mathrm{PB}},$$

as desired.



3. (a) For any positive integer n, $n^2 + 2$ is special.
   (b) We claim that for infinitely many positive integers n, n^2 is not special. Suppose $n^2 = m^2 + p$ for some integer m and some prime number p. Then, $p = n^2 - m^2 = (n-m)(n+m)$. We must have $n - m = 1$ and $p = n + m = 2n - 1$. If we let $n = 3k + 2$ for any positive integer k, then $2n - 1 = 6k + 3$ is not a prime number. This justifies the claim.

4. Extend AP and AQ to cut BC at M and N, respectively. Then, ABM and ACN are isosceles, so that BM = 2 and NC = $2\sqrt{2}$. Hence, MN = $2\sqrt{2} - 2$. Now PQ is the segment joining the midpoints of AM and AN. Hence, PQ = MN/2 = $\sqrt{2} - 1$.



5. Suppose the real numbers $x_1, \ldots, x_n$, $1 \le x_i \le 4$ for $i = 1, 2, \ldots, n$, satisfy the two given inequalities. Then, $(x_i - 1)(x_i - 4) \le 0$ so that $x_i + 4/x_i \le 5$. Equality holds for $x_i = 1$ or $x_i = 4$. From these inequalities and the given ones, we obtain

$$5n = \frac{7n}{3} + \frac{8n}{3} \le x_1 + x_2 + \cdots + x_n + 4\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \le 5n.$$

Hence, $x_i = 1$ or 4, $i = 1, 2, \ldots, n$. Suppose $x_1 = x_2 = \cdots = x_k = 1$ and $x_{k+1} = x_{k+2} = \cdots = x_n = 4$ for some index k. Then $x_1 + x_2 + \cdots + x_n = k + 4(n-k) = 7n/3$. Hence, $5n = 9k$ so that 9 divides n. It follows that the smallest value of n is 9, with the numbers 1, 1, 1, 1, 1, 4, 4, 4 and 4.

Edmonton Junior High Math Contest 2012

Part A: Multiple Choice

1. The product of four positive integers is 1365. Which of the following could not be the sum of any three of the integers?
   (a) 15 (b) 21 (c) 23 (d) 24 (e) 25
2. Eight cards are placed face down. Each has one of the following numbers: 2, 3, 6, 7, 8, 9, 15 or 18. If you and your friend each turn over one card, what is the probability, to the nearest whole percentage point, that the sum of the pair of turned-up cards will be odd?
   (a) 47% (b) 50% (c) 57% (d) 60% (e) 67%
3. Five different integers have a sum of −6. The first integer is 2 greater than the third. The second and fifth integers are opposite. The fourth integer is 2 greater than the second, and it is double the third. Which of the following must be one of the integers?
   (a) 7 (b) 6 (c) 5 (d) 4 (e) 3
4. A cube is 3 cm × 3 cm × 3 cm. The complete surface is painted green. It is cut into congruent cubes that are 1 cm × 1 cm × 1 cm (unit cubes). What is the ratio of unit cubes with exactly two green faces to unit cubes with exactly three green faces?
   (a) 5:27 (b) 4:3 (c) 2:3 (d) 3:4 (e) 3:2
5. A square picture has a side length of 28 cm, and a circular picture has a diameter of 30 cm. They each have a uniform 5 cm border enclosing the exposed area of the photo. For which picture is the ratio of the exposed area of the photo to its border area the greatest, and what is that ratio?
   (a) The square picture, with a ratio of 0.70:1
   (b) The circular picture, with a ratio of 0.80:1
   (c) The square picture, with a ratio of 2.07:1
   (d) The circular picture, with a ratio of 2.27:1
   (e) Both have the same ratio of 0.75:1

Solutions

1. The prime factors of 1365 are 3, 5, 7 and 13.

$$3+5+7=15$$
$$3+7+13=23$$
$$3+5+13=21$$
$$5+7+13=25$$

It is not possible to make a sum of 24. The correct answer is **(d)**.

2. There are four even numbers and four odd numbers. Odd plus odd is even. Even plus even is even. Odd plus even is odd. There are 8 out of 14 pairs that are odd plus even. Therefore, the probability of an odd sum is 8/14, or 0.57—which is 57%. The correct answer is **(c)**.
3. Let x be the third number. Then, $x + 2$ is the first number, $2x$ is the fourth number, $2x - 2$ is the second number and $-2x + 2$ is the fifth number. Then,

$$x + x + 2 + 2x + 2x - 2 + -2x + 2 = -6$$
$$x = -2.$$

Therefore, the integers are 0, −6, −2, −4 and 6. The correct answer is **(b)**.

4. Of the 3^3, or 27, unit cubes, 12 have green on exactly two faces and 8 have green on exactly three faces. Therefore, 12:8 or 3:2 is the ratio. The correct answer is **(e)**.
5. A square with side length of 28 cm would have a square of 18 cm within since there is a border 5 cm wide around the inner square. The ratio would be

$$18^2 : 28^2 - 18^2$$
$$324 : 460$$
$$0.70 : 1.$$

A circle with diameter 30 cm would have a circle within with diameter 20 cm since there is a border 5 cm wide around the inner circle. The ratio would be

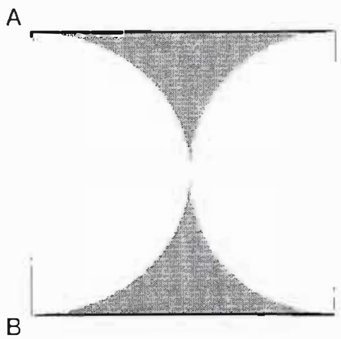
$$10^2 : 15^2 - 10^2$$
$$100 : 125$$
$$0.8 : 1.$$

Therefore, the circular picture has the greatest ratio, which is 0.8:1. The correct answer is **(b)**.

Part B: Short Answer

6. Connor walked an average of 6 km/h from his home to a park. When he returned along the same route, he averaged 6 km/h. At the halfway point from the park to home, he remembered that he'd left his watch behind and he ran back to the park at 9 km/h. Once he found his watch, he walked back home at 6 km/h. What was Connor's average speed, rounded to the nearest tenth, for the whole trip?

7. An artist mixes a shade of paint that is 33⅓% red, 15% yellow and the rest blue. He then mixes a second shade of paint, using only red and blue in a 3:1 ratio. If the artist then combines 300 mL of the first shade with 120 mL of the second shade and 85 mL of blue, what percentage of the resulting shade of paint will be blue? (Express your answer to the nearest whole percentage point.)
8. In a collection of 48 coins, one is counterfeit, and it has a mass that is slightly less than all of the other identical coins. What is the minimum number of mass comparisons required, using a balance scale, to ensure that the counterfeit coin is discovered?
9. The square root of 1/25 of a composite number is 4 less than the sixth prime number. What is the composite number?
10. What is the area of the shaded region in the diagram below, to the closest whole square centimetre, if the length of semicircle AB is 18.85 cm?



11. A pet shop had 1500 cats, dogs and birds. Of this total, 55% were cats and 20% were dogs. A group of cat lovers bought cats only, until just 40% of the remaining pets were cats. How many cats were bought by the cat lovers?
12. In the real number system of mathematics, how many of the following statements are true?
    - 1 is a prime number
    - $\sqrt{-25} = -5$
    - $\pi = 3.14$
    - $\frac{x^3}{x^0 - 1} \geq 0, x > 0$
    - $-n^4 = (-n)^4$
    - $\sqrt{x^{100}} = x^{10}$
    - If a three-dimensional object is a right square prism, it must be a cube
13. Wong's Fortune Cookies are sold in four package sizes. Compared to the largest size, the three smaller sizes contain, respectively, ⅓ as many cookies, ½ as many cookies and ¾ as many cookies. If you buy one of each of the four package sizes, you will get 62 cookies. How many cookies are in the second-smallest package?
14. What is the tens digit of the smallest multiple of 13 whose units digit is greater than 0 but less than its hundreds digit?

Solutions

6. Choose the distance from home to the park to be the lowest common multiple of 6 and 9, which is 18. Therefore, it took Connor 3 h to walk from his home to the park. At the halfway point, he walked for 1.5 h and ran for 1 h, after which he walked back home for 3 h. The total time spent was 3 + 1.5 + 1 + 3, or 8.5 h. The total distance was 3(18), or 54 km. So, Connor's average speed was 54/8.5, or approximately 6.4 km/h.
7. The first shade of paint is 100 – 33⅓ – 15, or 51⅔% blue. The second shade is 25% blue. The total amount of blue paint (in millilitres) is as follows:

$$\begin{aligned} &300(51\tfrac{2}{3}\%) + 120(25\%) + 85 \\ &= 155 + 30 + 85 \\ &= 270. \end{aligned}$$

The total amount of paint (in millilitres) is

$$300 + 120 + 85 = 505.$$

The percentage of the resulting shade of paint that is blue is

$$\frac{270}{505} = 53.465\%.$$

Rounding to the nearest whole percentage point gives us 53%.

8. Divide the 48 coins into three piles of 16 coins each. For the first weighing, put 16 coins on the left pan, 16 coins on the right pan and 16 coins on the table. Divide the 16 coins on the table into three piles. For the second weighing, put 5 coins on the left pan, 5 coins on the right pan and 6 coins on the table. For the third weighing, now there are piles of coins of 2, 2, 1 or 2, 2, 2. For the fourth weighing, 1 coin will be on each pan. Thus, a minimum of four weighings are necessary.
9. The sixth prime number is 13, so

$$\begin{aligned} \sqrt{\frac{x}{25}} &= 13 - 4 \\ \sqrt{\frac{x}{25}} &= 9 \\ \frac{x}{25} &= 81 \\ x &= 2025. \end{aligned}$$

The composite number is 2025.

10. Since the length of the semicircle is 18.85 cm, the circumference of the circle is 37.7 cm. Divide the circumference by π:

$$\frac{37.7}{\pi} = d$$
$$12 = d.$$

Therefore, the diameter of the circle is 12 cm. To find the area of the shaded region, subtract the area of the circle with radius 6 cm from the area of the square with side of 12 cm:

$$12^2 - (6^2)\pi = 30.9.$$

Rounding to the closest whole number, the answer is 31 cm.

11. For the cats, 55% of 1500 is 825. For the dogs, 20% of 1500 is 300. For the birds, 1500 – 825 – 300 = 375. Let n be the number of cats purchased. Then,

$$825 - n = 0.4(1500 - n)$$
$$n = 375.$$

There were 375 cats purchased.

12. All of the statements are false. Therefore, the answer is 0.

13. Let x represent the largest package size. Then, ⅓x represents the smallest size, ½x represents the second-smallest size and ¾x represents the third-smallest size. Then,

$$x + \frac{1}{3}x + \frac{1}{2}x + \frac{3}{4}x = 62$$
$$x = 24.$$

Therefore, the second-smallest size of package contains ½(24) or 12 cookies.

14. The hundreds digit must be at least 2. The first such multiple of 13 is 208, but the units digit is greater than 2. The next one is 221, which satisfies the conditions of the problem. The tens digit is 2.

Part C: Short Answer

15. John wrote a computer program that changes numbers. If he enters a number x, the program outputs the number

$$\frac{1}{1-x}.$$

For example, if John enters the number 3, the output is

$$\frac{1}{1-3} = -\frac{1}{2}.$$

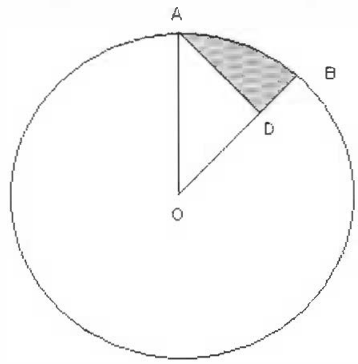
The first number John enters is 2, which results in an output of –1. John will then take this output and use it for his next input. What will be the computer's 2012th output?

16. How many positive integers up to 2012 have a ones (units) digit of either 1 or 8?

17. A level road of length 16 km links A and B. From B, the road goes uphill to C. Kelly and Kerry both ride their bikes at 8 km/h on level road, 5 km/h uphill and 10 km/h downhill. Kelly starts from A and goes toward C, while Kerry starts from C and goes toward A. They meet at a point on the level road 4 km from B. They continue to their respective destinations and turn around immediately. Where will they meet the second time?

18. Three regular polygons of equal side lengths fit perfectly around a point. If the first polygon has five sides and the second has ten sides, what is the number of sides of the third polygon?

19. In the diagram below, O is the centre of a circle of radius 1. A and B are two points on the circle, and ∠AOB = 45°. D is a point on radius OB so that segment AD is perpendicular to OB. Find the area of the shaded region. Write the answer in decimal form and round it to the nearest hundredth.



Solutions

15. The following table shows input values and output values for the first six inputs:

Number of inputs	Input value	Output value
1	2	–1
2	–1	½
3	½	2
4	2	–1
5	–1	½
6	½	2

The output values repeat for every three inputs. In 2010 inputs, there would be 670 repeats of the outputs of –1, ½ and 2. Therefore, the 2011th input would yield −1, and the 2012th input would yield ½.

16. The numbers that have a ones digit of 1 are 1, 11, 21, …, 2011. If we erase the 1 at the end, we obtain exactly the numbers 0, 1, 2, …, 201. Thus, there are exactly 201 + 1 = 202 such numbers (the extra number coming from the fact that we start our counting at 0). The numbers that have a ones digit of 8 are 8, 18, 28, …, 2008. If we erase the 8 at the end, we obtain exactly the numbers 0, 1, 2, …, 200. Thus, there are exactly 200 + 1 = 201 such numbers (the extra number coming from the fact that we start our counting at 0). Thus, in total there are 202 + 201 = 403 such numbers.

17. When Kelly and Kerry meet the first time, Kelly has travelled 12 km on level ground. Therefore, she has travelled for 12/8 or 1.5 h. The distance from B to C is unknown, but the fact that both Kelly and Kerry have travelled for 1.5 h can be used to find the distance from B to C:

$$1.5 = \frac{4}{8} + \frac{d}{10}$$
$$10 = d.$$

The distance from B to C is 10 km. The following chart can be used to determine when Kelly and Kerry will meet again.

	Time from starting points (h)	Distance travelled (km)	Location
Kelly	1	8	Halfway between A and B
Kerry	1	10	B
Kelly	2	8	B
Kerry	2	8	Halfway between B and A
Kelly	3	5	Halfway between B and C
Kerry	3	8	A
Kelly	4	5	C
Kerry	4	8	Halfway between A and B
Kelly	5	10	B
Kerry	5	8	B

Therefore, Kelly and Kerry will meet again at point B.

18. The measure of each interior angle of a regular polygon with ten sides is $180(n - 2)/n = 180(10 - 2)/10 = 144°$. The measure of each interior angle of a regular polygon with five sides is 108°. To fit perfectly around a point, the sum of the interior angles of the three polygons must be 360°: 360 − 144 − 108 = 108°. Therefore, the third polygon is a regular pentagon, which has five sides.

19. To find the area of circle O use πr^2, where $r = 1$; therefore, the area of circle O is π. Sector AOB represents ⅛ of the circle, so the area of sector AOB is $\pi/8$. Triangle AOD is an isosceles right triangle with hypotenuse 1, so the area of triangle AOD is ¼. The area of the shaded region is found by subtracting the area of triangle AOD from the area of sector AOB:

$$\frac{\pi}{8} - \frac{1}{4}$$

or

$$\frac{\pi - 2}{8}.$$

Written as a decimal, the answer is 0.14.

Calgary Junior High School Mathematics Contest 2012

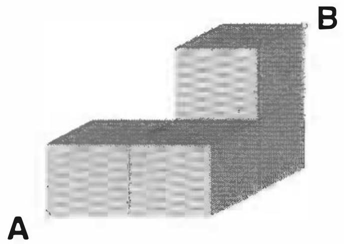
The Calgary Junior High School Mathematics Contest takes place every spring. The 90-minute exam is primarily for Grade 9 students; however, all junior high students in Calgary and surrounding districts are eligible. Participants write the exam in their own schools. School and individual prizes include trophies, medals, a cash award to the student achieving the highest mark, and the opportunity for the top students (and their teacher sponsors) to attend a banquet at the University of Calgary. The 36th annual contest took place on May 2, 2012.

Part A: Short Answer

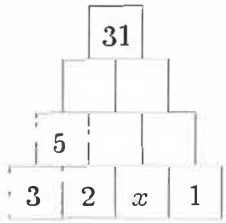
1. The sum of three different prime numbers is 12. What are the numbers?
2. Peter buys a pizza and eats half of it on the first day. On the second day he eats one-third of the remaining part. What fraction of the original pizza is still uneaten?
3. What whole number is equal to

$$(1\times 2)\left(\frac{1}{1}-\frac{1}{2}\right)+(2\times 3)\left(\frac{1}{2}-\frac{1}{3}\right)+(3\times 4)\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+(99\times 100)\left(\frac{1}{99}-\frac{1}{100}\right)?$$

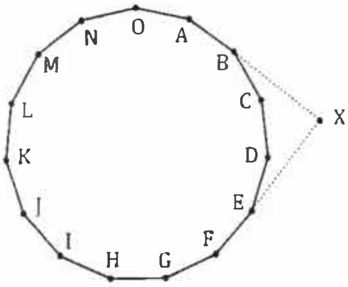
4. You have a giant spherical ball of radius 2 m sitting on level ground. You put a red dot on the top of the ball, and then you roll the ball 13π m north. How far from the ground (in metres) is the red dot?
5. The year 2012 is a leap year whose digits sum to 5 (2 + 0 + 1 + 2 = 5). Assume that leap years occur every four years. When will be the next leap year whose digits sum to 5?
6. Four identical cubes are stacked up, as in the diagram below. The length of each edge of each cube is 2 cm. The straight-line distance (in centimetres) from corner A to corner B can be written in the form $\sqrt{N}$, where N is a positive integer. What is N?



7. Andrew, Belinda, Cameron and Danielle gather every day for 30 days to play tennis. Each day, the four of them split into two teams of two to play a game, and one of the teams is declared the winning team. If Andrew, Belinda and Cameron are on the winning team for 12, 13 and 14 of the games, respectively, for how many of the games is Danielle on the winning team?
8. Each box in the diagram below contains a number, some of which are shown. The number in each box above the bottom row is obtained by adding the numbers in the two boxes connected to it in the row below. For example, 3 + 2 = 5. What number is in the box marked x?



9. The diagram below shows a regular 15-sided polygon ABCDEFGHIJKLMNO, so that all sides are equal and all angles are equal. Extend the sides AB and FE to meet at a point X. What is the size of ∠BXE (in degrees)?

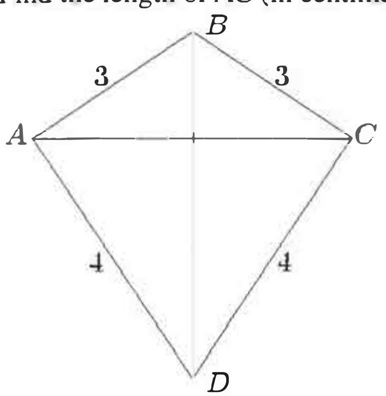


Answers

1. 2, 3, 7
2. ⅓
3. 99
4. 2
5. 2120
6. 48
7. 21
8. 7
9. 84

Part B: Long Answer

1. Matthew travelled 3 km in the following manner: he ran the first kilometre at 10 km/h, he biked the second kilometre at 12 km/h, and he drove the third kilometre at 60 km/h. How many minutes did it take Matthew to travel the 3 km?
2. Three tourists—weighing 45 kg, 50 kg and 80 kg, respectively—come to a riverbank. There is a boat there that any one of the tourists can operate, but the boat can carry 100 kg at most. Describe how all three tourists can get across the river by riding in the boat.
3. A teacher is marking math tests and keeping track of the average mark as she goes along. After she marks Geoff's test, the average of the tests she has marked so far increases by one mark (out of 100). Next she marks Bianca's test, and the average goes up by another mark. Geoff got 90 (out of 100) on the test. What is Bianca's mark?
4. ABCD is a quadrilateral with AB = BC = 3 cm and AD = DC = 4 cm and with ∠BAD = ∠BCD = 90°. Find the length of AC (in centimetres).



5. There was a basket containing marbles of four colours (red, orange, yellow and green). Alice, Bob and Cathy each counted the marbles in the basket and wrote down their results (see the table below). Unfortunately, each of them properly identified two of the colours but occasionally mixed up the other two colours: one person sometimes mixed up red and orange, another person sometimes mixed up orange and yellow, and the third person sometimes mixed up yellow and green. How many marbles of each colour were in the basket? Which colours did each of Alice, Bob and Cathy mix up?

	Red	Orange	Yellow	Green
Alice	2	5	7	9
Bob	2	4	9	8
Cathy	4	2	8	9

6. Notice that 338 = 294 + 44, and the two numbers 294 and 44 do not have any digits that are in 338. Also notice that 338 has just two different digits (3 and 8). Find positive integers A, B and C so that
   - $A = B + C$,
   - B and C do not have any digits used in A, and
   - A has more than two different digits.

   The larger the number of different digits A has, the higher your mark for this problem. (Earn a bonus mark if you can prove that your A has the largest possible number of different digits.)

Solutions

1. It took Matthew 1/10 of an hour, or 6 minutes, to run the first kilometre; 1/12 of an hour, or 5 minutes, to bike the second kilometre; and 1/60 of an hour, or 1 minute, to drive the third kilometre. So it took him 6 + 5 + 1 = 12 minutes to travel the entire 3 km.
2. First, the two lighter tourists (A and B) cross the river together, which is possible since 45 + 50 = 95 < 100. Then one of these tourists, say A, returns. The heaviest tourist (C) then goes across the river alone, and then tourist B returns alone. Finally, A and B again cross the river together, at which point all three tourists are on the other side of the river.
3. We can assume that all the tests marked before Geoff's have the same mark, all equal to the average A before Geoff's test is included. For the average to go up by one mark (to $A + 1$) when Geoff is included, Geoff's mark has to counterbalance all those A's, so it has to be N marks above $A + 1$, where N is the number of tests marked before Geoff's. So Geoff's mark must be $N + A + 1$. Similarly, when Bianca is included, the average goes up to $A + 2$, so Bianca's mark must be $N + 1$ marks above this average. Bianca's mark must be $N + 1 + A + 2$. Thus, Bianca's mark must be exactly two marks higher than Geoff's. If Geoff got 90, Bianca's mark must be 92.

   Note: Some contestants may get the right answer by doing only special cases. For example, a contestant might assume that only one test has been marked before Geoff's. Since Geoff got 90, and the average goes up by one mark when Geoff's test is marked, this means that the first student had to get 88 so that the average rises from 88 to 89 when Geoff's mark is included. Now, since the average rises one more mark to 90 when Bianca's mark is included, Bianca had to get 92, so that (88 + 90 + 92)/3 = 90. Such a special case should be worth only 3 marks out of 9. No matter how many special cases a contestant does, his or her mark on this question should not be more than 5 out of 9.

4. SOLUTION 1. By the Pythagorean theorem,

$$BD = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \text{ cm}.$$

Now we calculate the area of triangle ABD in two different ways. Thinking of AD as the base of the triangle and AB as the altitude, we get the area to be $(1/2)(4)(3) = 6 \text{ cm}^2$. Let E be the intersection of AC and BD. Then, thinking of BD as the base of triangle ABD, the altitude would be AE, so (1/2)(5)(AE) must equal the area 6, so $AE = 6 \times 2/5 = 2.4$ cm. Thus, AC = 2(2.4) = 4.8 cm.

SOLUTION 2. Once again, BD = 5 cm. Let E be the intersection of AC and BD. Triangles ABD and EBA are similar (because they are both right triangles with equal angle ABE). Thus,

$$\frac{AD}{BD} = \frac{AE}{AB},$$

so

$$AE = \frac{(AB)(AD)}{BD} = \frac{3 \times 4}{5} = 2.4 \text{ cm}.$$

Therefore, AC = 2AE = 2(2.4) = 4.8 cm.

5. Only one of the three people cannot identify the red colour, so the other two people must be correct about the number of red marbles; therefore, there must be two red marbles only. Thus, Cathy is not correct about red, so she must be mixing up red and orange. She must be correct about yellow and green, so there are eight yellow and nine green marbles. Therefore, the total of red and orange marbles is six, so there are 6 – 2 = 4 orange marbles. So, Alice mixes up orange and yellow, and Bob mixes up yellow and green.

6. The largest possible number of different digits in *A* is seven. There are lots of examples where *A* has seven different digits. Here are three such examples $B + C = A$:

353553355 + 55353355 = 408906710;
4888181 + 4184184 = 9072365; and
2325555 + 2353355 = 4678910.

Note that the first example does not use the digit 2, so both *B* and *C* use only two different digits (3 and 5). Nevertheless, *A* has only seven different digits.

Scoring: Give *no* marks if $A \neq B + C$ or if *B* or *C* contains a digit that is in *A*. Give 1 mark if a student gives a correct *A*, *B* and *C* in which *A* has three different digits. If *A* has four different digits, give 3 marks; if *A* has five different digits, give 5 marks; if *A* has six different digits, give 7 marks; and if *A* has seven different digits, give 9 marks. Give a bonus mark if a student gives a *clear, complete, correct* proof that having eight different digits is impossible for *A*.

Here is a proof that seven is the largest possible number of different digits in *A*. Suppose that there is a solution *A*, *B*, *C* where *A* has eight different digits. This would mean that *B* and *C* together could only have two different digits. Say that these digits are *b* and *c*. Imagine that *B* and *C* are put one below the other and then added in the usual way, one column at a time, right to left. Consider such a column containing two digits, each being either *b* or *c*. Then, the resulting digit in the sum *A* can only be one of the six possibilities $b + c$, $b + b$, $c + c$, $b + c + 1$, $b + b + 1$ or $c + c + 1$, where the +1's would result if there were a carry from the previous column. (Here by $b + c$, for example, we actually mean the units digit of $b + c$, if $b + c$ were 10 or greater.)

The remaining possibility is that a column contains only one digit, which would happen if one of *B* or *C* were longer than the other. We cannot allow the digits *b* and *c* to be in the sum *A*, but we could get (the units digit of) either $b + 1$ or $c + 1$ in *A*, if the previous column had a carry. This is how we can get seven different digits in *A*, using only two different digits in *B* and *C*.

To bump *A* up to eight different digits, we would need both of $b + 1$ and $c + 1$ to occur in the sum. But the only way this could happen is if one of *b* or *c* were 9, say $b = 9$. Then the number *B* could be two digits longer than *C*, where the first two digits of *B* were *c*9, and there was a carry in the third column. Then the second column would be $9 + 1 = 0$ and would create another carry in the first column, so we get the digit $c + 1$ in the sum from the first column. But in this case, the digit $b + b + 1 = 9 + 9 + 1$ would just be 9 again, so we would still get at most seven different digits in the sum. In fact, the digit $b + c + 1$ would be the same as *c*, so even seven different digits is not possible this way. An example giving six different digits in *A* would be 299229 + 9222 = 308451. Notice that *B* and *C* use only the digits $b = 9$ and $c = 2$, and both $b + 1 = 0$ and $c + 1 = 3$ occur in *A*, but *A* has only six different digits.

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MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

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