Calgary Junior High School Mathematics Contest 2012

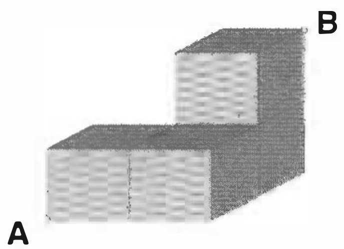
The Calgary Junior High School Mathematics Contest takes place every spring. The 90-minute exam is primarily for Grade 9 students; however, all junior high students in Calgary and surrounding districts are eligible. Participants write the exam in their own schools. School and individual prizes include trophies, medals, a cash award to the student achieving the highest mark, and the opportunity for the top students (and their teacher sponsors) to attend a banquet at the University of Calgary. The 36th annual contest took place on May 2, 2012.

Part A: Short Answer

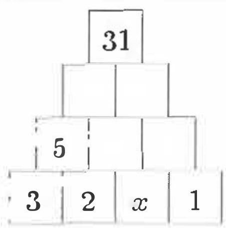
1. The sum of three different prime numbers is 12. What are the numbers?
2. Peter buys a pizza and eats half of it on the first day. On the second day he eats one-third of the remaining part. What fraction of the original pizza is still uneaten?
3. What whole number is equal to

$$(1 \times 2)\left(\frac{1}{1}-\frac{1}{2}\right) + (2 \times 3)\left(\frac{1}{2}-\frac{1}{3}\right) + (3 \times 4)\left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + (99 \times 100)\left(\frac{1}{99}-\frac{1}{100}\right)?$$

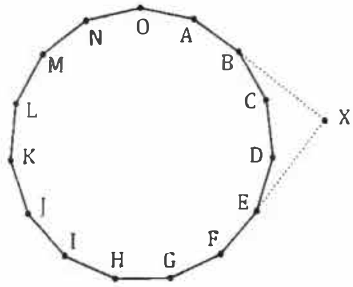
4. You have a giant spherical ball of radius 2 m sitting on level ground. You put a red dot on the top of the ball, and then you roll the ball 13π m north. How far from the ground (in metres) is the red dot?
5. The year 2012 is a leap year whose digits sum to 5 $(2 + 0 + 1 + 2 = 5)$. Assume that leap years occur every four years. When will be the next leap year whose digits sum to 5?
6. Four identical cubes are stacked up, as in the diagram below. The length of each edge of each cube is 2 cm. The straight-line distance (in centimetres) from corner A to corner B can be written in the form $\sqrt{N}$, where N is a positive integer. What is N?



7. Andrew, Belinda, Cameron and Danielle gather every day for 30 days to play tennis. Each day, the four of them split into two teams of two to play a game, and one of the teams is declared the winning team. If Andrew, Belinda and Cameron are on the winning team for 12, 13 and 14 of the games, respectively, for how many of the games is Danielle on the winning team?
8. Each box in the diagram below contains a number, some of which are shown. The number in each box above the bottom row is obtained by adding the numbers in the two boxes connected to it in the row below. For example, 3 + 2 = 5. What number is in the box marked x?



9. The diagram below shows a regular 15-sided polygon ABCDEFGHIJKLMNO, so that all sides are equal and all angles are equal. Extend the sides AB and FE to meet at a point X. What is the size of $\angle BXE$ (in degrees)?

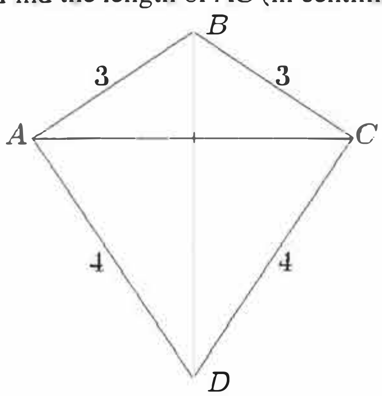


Answers

1. 2, 3, 7
2. ⅓
3. 99
4. 2
5. 2120
6. 48
7. 21
8. 7
9. 84

Part B: Long Answer

1. Matthew travelled 3 km in the following manner: he ran the first kilometre at 10 km/h, he biked the second kilometre at 12 km/h, and he drove the third kilometre at 60 km/h. How many minutes did it take Matthew to travel the 3 km?
2. Three tourists—weighing 45 kg, 50 kg and 80 kg, respectively—come to a riverbank. There is a boat there that any one of the tourists can operate, but the boat can carry 100 kg at most. Describe how all three tourists can get across the river by riding in the boat.
3. A teacher is marking math tests and keeping track of the average mark as she goes along. After she marks Geoff's test, the average of the tests she has marked so far increases by one mark (out of 100). Next she marks Bianca's test, and the average goes up by another mark. Geoff got 90 (out of 100) on the test. What is Bianca's mark?
4. ABCD is a quadrilateral with AB = BC = 3 cm and AD = DC = 4 cm and with ∠BAD = ∠BCD = 90°. Find the length of AC (in centimetres).



5. There was a basket containing marbles of four colours (red, orange, yellow and green). Alice, Bob and Cathy each counted the marbles in the basket and wrote down their results (see the table below). Unfortunately, each of them properly identified two of the colours but occasionally mixed up the other two colours: one person sometimes mixed up red and orange, another person sometimes mixed up orange and yellow, and the third person sometimes mixed up yellow and green. How many marbles of each colour were in the basket? Which colours did each of Alice, Bob and Cathy mix up?

	Red	Orange	Yellow	Green
Alice	2	5	7	9
Bob	2	4	9	8
Cathy	4	2	8	9

6. Notice that 338 = 294 + 44, and the two numbers 294 and 44 do not have any digits that are in 338. Also notice that 338 has just two different digits (3 and 8). Find positive integers A, B and C so that
   - $A = B + C$,
   - B and C do not have any digits used in A, and
   - A has more than two different digits.

   The larger the number of different digits A has, the higher your mark for this problem. (Earn a bonus mark if you can prove that your A has the largest possible number of different digits.)

Solutions

1. It took Matthew 1/10 of an hour, or 6 minutes, to run the first kilometre; 1/12 of an hour, or 5 minutes, to bike the second kilometre; and 1/60 of an hour, or 1 minute, to drive the third kilometre. So it took him 6 + 5 + 1 = 12 minutes to travel the entire 3 km.
2. First, the two lighter tourists (A and B) cross the river together, which is possible since 45 + 50 = 95 < 100. Then one of these tourists, say A, returns. The heaviest tourist (C) then goes across the river alone, and then tourist B returns alone. Finally, A and B again cross the river together, at which point all three tourists are on the other side of the river.
3. We can assume that all the tests marked before Geoff's have the same mark, all equal to the average A before Geoff's test is included. For the average to go up by one mark (to $A + 1$) when Geoff is included, Geoff's mark has to counterbalance all those A's, so it has to be N marks above $A + 1$, where N is the number of tests marked before Geoff's. So Geoff's mark must be $N + A + 1$. Similarly, when Bianca is included, the average goes up to $A + 2$, so Bianca's mark must be $N + 1$ marks above this average. Bianca's mark must be $N + 1 + A + 2$. Thus, Bianca's mark must be exactly two marks higher than Geoff's. If Geoff got 90, Bianca's mark must be 92.

   Note: Some contestants may get the right answer by doing only special cases. For example, a contestant might assume that only one test has been marked before Geoff's. Since Geoff got 90, and the average goes up by one mark when Geoff's test is marked, this means that the first student had to get 88 so that the average rises from 88 to 89 when Geoff's mark is included. Now, since the average rises one more mark to 90 when Bianca's mark is included, Bianca had to get 92, so that (88 + 90 + 92)/3 = 90. Such a special case should be worth only 3 marks out of 9. No matter how many special cases a contestant does, his or her mark on this question should not be more than 5 out of 9.

4. SOLUTION 1. By the Pythagorean theorem,

$$BD = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \text{ cm}.$$

Now we calculate the area of triangle ABD in two different ways. Thinking of AD as the base of the triangle and AB as the altitude, we get the area to be $(1/2)(4)(3) = 6 \text{ cm}^2$. Let E be the intersection of AC and BD. Then, thinking of BD as the base of triangle ABD, the altitude would be AE, so $(1/2)(5)(AE)$ must equal the area 6, so $AE = 6 \times 2/5 = 2.4$ cm. Thus, $AC = 2(2.4) = 4.8$ cm.

SOLUTION 2. Once again, BD = 5 cm. Let E be the intersection of AC and BD. Triangles ABD and EBA are similar (because they are both right triangles with equal angle ABE). Thus,

$$\frac{AD}{BD} = \frac{AE}{AB'}$$

so

$$AE = \frac{(AB)(AD)}{BD} = \frac{3 \times 4}{5} = 2.4 \text{ cm}.$$

Therefore, $AC = 2AE = 2(2.4) = 4.8$ cm.

5. Only one of the three people cannot identify the red colour, so the other two people must be correct about the number of red marbles; therefore, there must be two red marbles only. Thus, Cathy is not correct about red, so she must be mixing up red and orange. She must be correct about yellow and green, so there are eight yellow and nine green marbles. Therefore, the total of red and orange marbles is six, so there are $6 - 2 = 4$ orange marbles. So, Alice mixes up orange and yellow, and Bob mixes up yellow and green.

6. The largest possible number of different digits in A is seven. There are lots of examples where A has seven different digits. Here are three such examples $B + C = A$:

353553355 + 55353355 = 408906710;
4888181 + 4184184 = 9072365; and
2325555 + 2353355 = 4678910.

Note that the first example does not use the digit 2, so both B and C use only two different digits (3 and 5). Nevertheless, A has only seven different digits.

Scoring: Give *no* marks if $A \neq B + C$ or if B or C contains a digit that is in A. Give 1 mark if a student gives a correct A, B and C in which A has three different digits. If A has four different digits, give 3 marks; if A has five different digits, give 5 marks; if A has six different digits, give 7 marks; and if A has seven different digits, give 9 marks. Give a bonus mark if a student gives a *clear, complete, correct* proof that having eight different digits is impossible for A.

Here is a proof that seven is the largest possible number of different digits in A. Suppose that there is a solution A, B, C where A has eight different digits. This would mean that B and C together could only have two different digits. Say that these digits are b and c. Imagine that B and C are put one below the other and then added in the usual way, one column at a time, right to left. Consider such a column containing two digits, each being either b or c. Then, the resulting digit in the sum A can only be one of the six possibilities $b + c$, $b + b$, $c + c$, $b + c + 1$, $b + b + 1$ or $c + c + 1$, where the +1's would result if there were a carry from the previous column. (Here by $b + c$, for example, we actually mean the units digit of $b + c$, if $b + c$ were 10 or greater.)

The remaining possibility is that a column contains only one digit, which would happen if one of B or C were longer than the other. We cannot allow the digits b and c to be in the sum A, but we could get (the units digit of) either $b + 1$ or $c + 1$ in A, if the previous column had a carry. This is how we can get seven different digits in A, using only two different digits in B and C.

To bump A up to eight different digits, we would need both of $b + 1$ and $c + 1$ to occur in the sum. But the only way this could happen is if one of b or c were 9, say $b = 9$. Then the number B could be two digits longer than C, where the first two digits of B were $c9$, and there was a carry in the third column. Then the second column would be $9 + 1 = 0$ and would create another carry in the first column, so we get the digit $c + 1$ in the sum from the first column. But in this case, the digit $b + b + 1 = 9 + 9 + 1$ would just be 9 again, so we would still get at most seven different digits in the sum. In fact, the digit $b + c + 1$ would be the same as c, so even seven different digits is not possible this way. An example giving six different digits in A would be $299229 + 9222 = 308451$. Notice that B and C use only the digits $b = 9$ and $c = 2$, and both $b + 1 = 0$ and $c + 1 = 3$ occur in A, but A has only six different digits.